On Deformations of Compressible Hyperelastic Material

by

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To my family

Preface

This thesis was submitted to the University of Glasgow in accordance with the requirements for the degree of Doctor of Philosophy.

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Summary

We consider the character of several finite deformations of compressible isotropic, nonlinear hyperelastic materials, specifically azimuthal shear of a thick-walled circular cylindrical tube, the bending deformation of a rectanglar block and axial shear of a thick-walled circular cylindrical tube. For each problem the equilibrium equations are applied to the special case of isochoric deformation, and explicit necessary and sufficient conditions on the strain-energy function for the material to admit such a deformation are obtained. These conditions are examined for several strain-energy functions and in each case complete solutions of the equilibrium equations are obtained. The predictions of the shear response for different strain-energy functions are compared using numerical results to show the dependence of the applied shear stress on the resulting macroscopic deformation. It is then shown how consideration of isochoric deformations in *compressible* elastic materials provides a means of generating classes of strain-energy functions for which closed-form solutions can be found for *incompressible* materials. For the problem of bending deformation we find that isochoric deformation is not possible in a compressible material. The conditions for a non-isochoric bending deformation to be admitted by the equilibrium equations are then examined for each of three classes of compressible isotropic ma-

terials. Explicit solutions for each case are then derived. Finally, we consider an incremental displacement superimposed on the azimuthal shear of a circular cylindrical tube. Numerical results are obtained to show the incremental displacement and nominal stresses for a special material when the internal boundary is subject to an incremental displacement.

Chapter 1

Introduction

This thesis is concerned mainly with the solution of boundary-value problems for compressible isotropic, nonlinear hyperelastic materials. In particular, the plane strain character of several finite deformations of the materials is considered. We restrict our attention to boundary conditions corresponding to prescribed boundary displacement. Under certain assumptions the resulting two governing ordinary differential equations of equilibrium are combined to yield necessary or necessary and sufficient conditions on the strain-energy function for the material to admit each deformation. Some existing results are recoverd as special cases and some new results for particular strain-energy functions are determined and discussed. In the final chapter we consider an incremental displacement superimposed on the azimuthal shear of a circular tube and numerical results for the incremental displacement and nominal stress are obtained.

In Chapter 2, following the approach of standard texts such as Ogden (1984) and Atkin and Fox (1980), we introduce the notation that will be used throughout the thesis as well as setting up the equations of equilibrium and constitutive relations used in finite elasticity for compressible materials. We then introduce the corresponding equations for incremental deformation superimposed on a finitely deformed state for the case of plane strain.

In Chapter 3 we study the *azimuthal shear* deformation (also known as *torsional, circular* or *circumferential shear*) of a circular cylindrical tube. In this (plane strain) deformation the outer circular cylindrial boundary is rotated at constant radius relative to the (fixed) inner boundary about their common axis. Examples of closed-form solutions for this problem are those described by Isherwood and Ogden (1977a, b), Ogden and Isherwood (1978) and Carroll and Murphy (1994), while numerical results have been provided by Mioduchowski and Haddow (1979), Ertepinar (1985, 1990), Haughton (1993), and Wineman and Waldron (1996). For slightly compressible materials the small volume change associated with azimuthal shear has been calculated by Ogden (1979). Certain aspects of the azimuthal shear problem for compressible materials have also been examined by Simmonds and Warne (1992), while a basic formulation of the problem and some general formulas are contained in the earlier paper by Adkins (1955).

A recent development in compressible nonlinear elasticity has concerned the search for problems in which isochoric deformations are possible, as instigated in a series of papers by Polignone and Horgan (1991, 1992, 1994). Work by both Haughton (1993) and Polignone and Horgan (1994) has focussed on this issue for the azimuthal shear problem. In both these papers it was found that pure azimuthal shear (that is azimuthal shear with no radial displacement) is possible for certain strain-energy functions, and examples of solutions were given in each case. Haughton (1993) also obtained a set of necessary and sufficient conditions on the material parameters for the existence of a solution to the equilibrium equations.

In the corresponding problem for incompressible isotropic elastic materials an exact solution was first given by Rivlin (1949) for the Mooney-Rivlin form of strain-energy function (see also Green and Zerna, 1968, and Ogden, 1984). Aspects of the problem for incompressible materials have also been examined by Ogden (1978) and Ogden, Chadwick and Haddon (1973), while some new exact solutions have been obtained recently by Tao, Rajagopal and Wineman (1992) for certain members of a class of generalized neo-Hookean materials.

Chapter 3 has several purposes. First, to use the plane strain character of the azimuthal shear deformation to express the strain energy as a function of two independent deformation invariants. This allows the equilibrium equations to be expressed, in Section 3.2, in a form which when specialized to pure azimuthal shear leads naturally to an explicit necessary condition on the strain-energy function to admit the deformation in question (Section 3.3) and several existing results are recovered. Second, in Section 3.4, to apply this necessary condition to a certain class of strain-energy functions and to obtain solutions of the boundary-value problem for particular members of that class, thereby showing how the results obtained previously for several strain-energy functions are embraced within a single framework. Thirdly, in Section 3.5, to obtain solutions to the pure azimuthal shear problem for a separate class of strain-energy functions. Fourthly, to illustrate the theoretical results with numerical calculations highlighting the dependence on the azimuthal shear stress applied to the outer boundary. This includes comparison of results for several different strain-energy functions. Finally, in Section 3.6, the corresponding problem for incompressible materials is considered briefly and it is shown that the solutions obtained for compressible materials are also valid in incompressible case. In particular, it is shown how the results of Tao *et al* (1992) for a generalized form of (incompressible) neo-Hookean strain-energy function are related to those obtained in Section 3.5.

It is emphasized that the method used in Chapter 3 of examining isochoric deformation in compressible elastic materials is applicable more generally than to the specific problem considered here and it leads to the generation of classes of strain-energy functions for which closed-form solutions can be found for both compressible and incompressible materials.

In Chapter 4 the finite *bending* deformation of an isotropic compressible nonlinearly elastic rectangular block subject to prescribed bending displacement is discussed. For the finite bending deformation of an isotropic compressible nonlinearly elastic rectangular block, which is deformed into a sector of a circular cylindrical tube, some examples of the closed-form solutions for this problem have been described by Ogden (1984), Carroll (1988), and Carroll and Horgan (1990). Recently a kind of semilinear material model with a special case of constant modified stretches has been discussed by Aron and Wang (1995, 1996) for this problem. In general, it is not easy to find closed-form solutions of the boundary-value problem in the nonlinear theory of compressible elasticity. Deformations are usually different for different strain-energy functions. In this chapter, we try to find a simple method to obtain closed-form solutions for the bending deformation. Considering the inherent characteristic of the bending deformation that there is a linear relation between the principal stretches, we obtain, in Section 4.2, an explicit necessary and sufficient condition on the strain-energy function for the existence of solutions of the discussed deformation. Several different forms of the necessary and sufficient condition are given.

In particular, we discuss the isochoric bending deformation of compressible materials in Section 4.3. After using the necessary and sufficient conditions, we deduce the interesting result that isochoric bending deformation of a block is not admissible in a compressible elastic material but is only possible in an incompressible elastic material.

With the help of the necessary and sufficient conditions, the equation for a Blatz-Ko material given by Carroll and Horgan (1990) is recovered in Section 4.4, and the solution of the equation is obtained implicitly. We also introduce three examples from each of three classes of compressible isotropic elastic materials. The necessary and sufficient conditions lead to closed-form solutions for the boundary-value problem in each case.

In Chapter 5 we consider the analogous problem associated with the *axial shear* deformation of a thick-walled circular cylindrical tube of homogeneous compressible isotropic nonlinearly elastic material. Axial shear is also known as *telescopic shear*. When the radius is unchanged by the deformation the deformation is isochoric and we refer to it as *pure axial shear*. The terminology (axisymmetric) *anti-plane shear* was used by Knowles (1976, 1977) and adopted by several subsequent authors.

Different aspects of the axial shear problem for compressible materials have been examined by Mioduchowski and Haddow (1974), Knowles (1976), Agarwal (1979) and Ertepinar and Erarslanoğlu (1990), the latter being concerned with the numerical solution of the governing equations. More recently, theoretical results have been obtained by Jiang and Knowles (1991), Polignone and Horgan (1992), Jiang and Beatty (1995) and Beatty and Jiang (1996, 1997). Analysis of the problem for incompressible materials dates back to Rivlin (1949) but the only (relatively) recent contribution appears to be that by Knowles (1976).

In Section 5.1 the axial shear deformation problem is formulated and the governing differential equations derived. The specializations appropriate for pure axial shear are then noted. With these specializations the two governing ordinary differential equations of equilibrium are combined, in Section 5.2, to obtain a necessary and sufficient condition on the strain-energy function for the material to be capable (in principle) of sustaining the considered deformation. This condition is given in relatively simple form but it is noted that it can be shown to be equivalent to a condition given by Jiang and Beatty (1995). The condition may be regarded as a necessary condition for the material to admit pure axial shear since additional conditions are required to guarantee the existence of a solution for any material satisfying the necessary condition.

In Section 5.3 we consider the form of solution obtained by several authors for different forms of strain-energy function and comment, within the framework developed in Section 5.3, on the common feature of these functions which leads to this solution. We also note the form of the most general strain-energy function for which this particular solution arises.

In Section 5.4 we use the necessary and sufficient condition derived in Section 5.3 to generate some specific forms of strain-energy function which yield new exact solutions of the pure axial shear problem, and these solutions are given explicitly. The results are illustrated graphically for comparison by plotting the axial shear stress on the outer boundary of the tube against the corresponding axial shear displacement in suitable dimensionless form.

The corresponding problem for incompressible materials is discussed in Section 5.5. We take note of the governing equations for the axial shear problem for an incompressible material and indicate how the solutions obtained in Section 5.4 can be used to generate forms of incompressible, isotropic strain-energy function for which solutions may be given in a similar way to that described for the azimuthal shear problem in Chapter 3.

In Chapter 6 we study azimuthal shear of an *eccentric* annulus of compressible elastic material with the centre circle displaced by a small amount and the outer surface fixed. This is treated as an incremental problem with a small (in-plane) displacement superimposed on a known pure azimuthal shear. In Section 6.2 the superimposed incremental deformation problem is formulated, and the incremental equilibrium equations in the absence of body forces are reduced to four ordinary differential equations. The specializations appropriate for the known underlying pure azimuthal shear deformation are then noted in Section 6.3.

In order to obtain a quantitative idea of the nature of the solutions to the incremental equations, we choose the simplest strain-energy function discussed in Chapter 3 to study the considered deformation in Section 6.4. Unfortunately, due to the complicated nature of the four ordinary differential equations, closed-form solutions are not obtained and numerical solutions are therefore obtained. Results for the displacement, as well as the incremental nominal stresses are obtained and illustrated graphically in Section 6.5.

The results of Chapter 3 have been published in the Quarterly Journal of Mechanics and Applied Mathematics (1998). The material of Chapter 5 has been accepted by the International Journal of Non-Linear Mechanics.

Chapter 2

The Basic Theory

2.1 Preliminaries

An elastic body undergoing motion occupies different regions of three dimensional Euclidean space at different times. It is convenient to choose a fixed region, $\mathcal{B}_0$ say, as reference, and to identify points of the body with their position vectors X in $\mathcal{B}_0$, which is then called the *reference configuration.* At time t we denoted by $\mathcal{B}_t$ the configuration then occupied by the body. This is called the *current configuration.*

2.1.1 Deformation and Strain

We consider the mapping from the reference configuration $\mathcal{B}_0$ to a current configuration $\mathcal{B}_t$ as a deformation or motion of the body. It carries each point X in the reference configuration $\mathcal{B}_0$ into a point in the current configuration $\mathcal{B}_t$. The point in $\mathcal{B}_0$ is denoted by the position vector X and the point in $\mathcal{B}_t$ by x, relative to arbitrarily chosen origins. The deformation may be regarded as a one parameter mapping χ_t: $\mathcal{B}_0 \to \mathcal{B}_t$. We write

$$\mathbf{x} = \chi_t(\mathbf{X}) \qquad \mathbf{X} \in \mathcal{B}_0. \tag{2.1.1}$$

For a fixed time, it is convenient to suppress the dependence on t and write the deformation of the body in the form

$$\mathbf{x} = \chi(\mathbf{X}) \qquad \mathbf{X} \in \mathcal{B}_0, \tag{2.1.2}$$

where χ is assumed to be at least differentiable with respect to $\mathbf{X}$.

On taking the differential of equation (2.1.2), we obtain

$$d\mathbf{x} = \mathrm{Grad}\chi d\mathbf{X}, \tag{2.1.3}$$

where Grad is the gradient operator in $\mathcal{B}_0$. The second-order tensor Gradχ is known as the *deformation gradient tensor*, which we denote here by $\mathbf{A}$. A deformation with $\mathbf{A}$ constant is *homogeneous*. In general, $\mathbf{A}$ depends on $\mathbf{X}$. We choose basis vectors $\mathbf{E}_i$ and $\mathbf{e}_i$ $(i = 1,\ 2,\ 3)$ in the reference and current configurations respectively. When the bases $\mathbf{E}_i$ in $\mathcal{B}_0$ are Cartesian rectangular coordinates, we may express $\mathbf{A}$ in the form

$$\mathbf{A} = \left(\mathbf{E}_1\frac{\partial}{\partial X_1} + \mathbf{E}_2\frac{\partial}{\partial X_2} + \mathbf{E}_3\frac{\partial}{\partial X_3}\right)\mathbf{x}, \tag{2.1.4}$$

for example. When the bases $\mathbf{E}_i$ in $\mathcal{B}_0$ are cylindrical polar coordinates $\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z$, we have

$$\mathbf{A} = \left(\mathbf{E}_R\frac{\partial}{\partial R} + \mathbf{E}_\Theta\frac{1}{R}\frac{\partial}{\partial \Theta} + \mathbf{E}_Z\frac{\partial}{\partial Z}\right)\mathbf{x}. \tag{2.1.5}$$

We adopt the conventional assumption that the Jacobian

$$J = \det \mathbf{A} \tag{2.1.6}$$

is positive at each point of $\mathcal{B}_0$. Physically, this implies, in particular, that interpenetration of matter is excluded (for example, see Atkin and Fox (1980)).

According to Nanson's formula, we have

$$\mathbf{n}\mathrm{d}s = J\mathbf{A}^{-T}\mathbf{N}\mathrm{d}S, \tag{2.1.7}$$

where $\mathrm{d}\mathbf{S} = \mathbf{N}\mathrm{d}S$ and $\mathrm{d}\mathbf{s} = \mathbf{n}\mathrm{d}s$ represent infinitesimal vector area elements in $\mathcal{B}_0$ and $\mathcal{B}_t$ respectively, $\mathbf{N}$ is the positive unit normal to the surface $\mathrm{d}\mathbf{S}$ and $\mathbf{n}$ to the surface $\mathrm{d}\mathbf{s}$. The corresponding volumes $\mathrm{d}V$ in $\mathcal{B}_0$ and $\mathrm{d}v$ in $\mathcal{B}_t$ are related by

$$\mathrm{d}v = J\mathrm{d}V. \tag{2.1.8}$$

For this reason, if the volume in $\mathcal{B}_0$ is unchanged during the deformation, we have $J = 1$ and the deformation is said to be *isochoric.* Furthermore, a material for which the volume of any region in $\mathcal{B}_0$ is unchanged during any deformation is said to be *incompressible,* and it is said to be *compressible* if there is no such constraint on the body during any possible motion. Here we mostly consider compressible materials.

Since the tensor $\mathbf{A}$ is non-singular by the assumption (2.1.6), there are the unique decompositions from the *Polar Decomposition Theorem* of the form

$$\mathbf{A} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \tag{2.1.9}$$

where $\mathbf{U}$ and $\mathbf{V}$ are symmetric, positive definite tensors and $\mathbf{R}$ is proper orthogonal. The tensors $\mathbf{U}$ and $\mathbf{V}$ are called the *right* and *left stretch tensors* respectively, and $\mathbf{R}$ satisfies

$$\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}, \qquad \det\mathbf{R} = 1, \tag{2.1.10}$$

where $\mathbf{I}$ is the identity tensor and the superscript T denotes the transpose of a tensor. With the help of equation (2.1.10) we have $\det \mathbf{A} = \det \mathbf{U}$. If $\mathbf{R} = \mathbf{I}$, we have $\mathbf{A} = \mathbf{U} = \mathbf{V}$ and the deformation is known as a *pure strain*. We use the notation $\mathbf{B} = \mathbf{A}\mathbf{A}^T$ and $\mathbf{C} = \mathbf{A}^T\mathbf{A}$ for the *left* and *right Cauchy-Green deformation tensors* respectively. They are then easily related to $\mathbf{U}$ and $\mathbf{V}$ through

$$\mathbf{B} = \mathbf{V}^2, \qquad \mathbf{C} = \mathbf{U}^2. \tag{2.1.11}$$

Since $\mathbf{U}$ is symmetric and positive definite, its principal values λ_i $(i = 1, 2, 3)$ are positive. Let $\mathbf{u}^{(i)}$ $(i = 1, 2, 3)$ be the principal axes of $\mathbf{U}$. Then, we have the spectral decomposition

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}. \tag{2.1.12}$$

The λ_i are also the principal values of $\mathbf{V}$ corresponding to principal axes $\mathbf{v}^{(i)} = \mathbf{R}\mathbf{u}^{(i)}$ $(i = 1, 2, 3)$, and we therefore have

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}. \tag{2.1.13}$$

We refer to λ_i as the *principal stretches*, $\mathbf{u}^{(i)}$ and $\mathbf{v}^{(i)}$ as *Lagrangian* and *Eulerian principal axes* of the deformation respectively. From equation (2.1.11), we deduce that $\mathbf{B}$ has the same principal axes as $\mathbf{V}$, and $\mathbf{C}$ as $\mathbf{U}$, and

$$\mathbf{B} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{2.1.14}$$

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}. \tag{2.1.15}$$

The principal stretches $\lambda_1, \lambda_2, \lambda_3$ are scalar quantities which are independent of the choice of the coordinate system. The principal invariants of $\mathbf{U}$ (or $\mathbf{V}$) are defined as

$$
\begin{aligned}
i_1 &= \lambda_1 + \lambda_2 + \lambda_3 = \text{tr } \mathbf{U}, \\
i_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = \tfrac{1}{2}(\text{tr } \mathbf{U})^2 - \tfrac{1}{2}\text{tr } \mathbf{U}^2, \qquad (2.1.16) \\
i_3 &= \lambda_1\lambda_2\lambda_3 = \det \mathbf{U}.
\end{aligned}
$$

Similarly, the principal invariants of $\mathbf{B}$ (or $\mathbf{C}$) are denoted by

$$
\begin{aligned}
I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \text{tr } \mathbf{B}, \\
I_2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 = \tfrac{1}{2}(\text{tr } \mathbf{B})^2 - \tfrac{1}{2}\text{tr } \mathbf{B}^2, \qquad (2.1.17) \\
I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2 = \det \mathbf{B}.
\end{aligned}
$$

2.1.2 Analysis of stress and equation of equilibrium

Let v be the region occupied by an arbitrary part of the body in the current configuration $\mathcal{B}_t$, and s be the closed surface bounding v, the outward unit normal to which is denoted by $\mathbf{n}$. We use the notation $\mathbf{t}$ as the traction per unit area of s, and $\mathbf{b}$ as the body force per unit volume of v. From Cauchy's law that $\mathbf{t}$ depends linearly on $\mathbf{n}$, we may deduce that there exists a second order tensor $\mathbf{T}$, which is symmetric and independent of $\mathbf{n}$, such that

$$\mathbf{t}(\mathbf{x}, \mathbf{n}) = \mathbf{T}(\mathbf{x})\mathbf{n}, \qquad (2.1.18)$$

for all $\mathbf{x}$ in $\mathcal{B}_t$ and each unit vector $\mathbf{n}$. Thus, $\mathbf{T}$ is said to be the *Cauchy stress tensor*. In Cauchy elasticity, the stress response of a material is determined solely by the deformation gradient tensor, and we may write

$$\mathbf{T} = \mathbf{H}(\mathbf{A}), \qquad (2.1.19)$$

where $\mathbf{H}$ is called the *response function* of the material.

Material objectivity requires that $\mathbf{T}$ is objective (see, for example Ogden (1984)). A consequence of objectivity is that the response function $\mathbf{H}$ must satifies

$$\mathbf{H}(\mathbf{QA}) = \mathbf{Q}\ \mathbf{H}(\mathbf{A})\mathbf{Q}^T, \tag{2.1.20}$$

where $\mathbf{Q}$, denoting a general rotation, is an arbitrary proper orthogonal tensor.

Furthermore, if the material is isotropic, we must have

$$\mathbf{H}(\mathbf{A}) = \mathbf{H}(\mathbf{AP}), \tag{2.1.21}$$

where $\mathbf{P}$ is an arbitrary proper orthogonal tensor. We may choose $\mathbf{P}$ to have the value $\mathbf{R}^T$ given by (2.1.9), and from equation (2.1.21) we may deduce that

$$\mathbf{H}(\mathbf{A}) = \mathbf{H}(\mathbf{AR}^T) = \mathbf{H}(\mathbf{VRR}^T) = \mathbf{H}(\mathbf{V}). \tag{2.1.22}$$

Finally, we obtain

$$\mathbf{H}(\mathbf{QVQ}^T) = \mathbf{QH}(\mathbf{V})\mathbf{Q}^\mathbf{T}. \tag{2.1.23}$$

A consequence of the objectivity and isotropy is that the stress $\mathbf{T}$ must be coaxial with the Eulerian principal axes $\mathbf{v}^{(i)}$. Thus, if $t_1,\ t_2,\ t_3$ denote the principal values of $\mathbf{T}$, then we may write

$$\mathbf{T} = \sum_{i=1}^{3} t_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}. \tag{2.1.24}$$

In some situations it is more convenient to introduce a concept of *nominal stress tensor* $\mathbf{S}$, as the traction per unit area of the reference configuration, which can be expressed as

$$\mathbf{t}ds = \mathbf{Tn}ds = \mathbf{S}^T\mathbf{N}dS,$$

where $\mathbf{N}$ again denotes the outward unit normal of the area S in $\mathcal{B}_0$, The area elements $\mathrm{d}S$ and $\mathrm{d}s$ on S and s are related by Nanson's formula (2.1.7). We then have

$$\mathbf{S} = J\mathbf{A}^{-1}\mathbf{T}. \tag{2.1.25}$$

With the help of the symmetry of $\mathbf{T}$, it follows that

$$\mathbf{A}\mathbf{S} = \mathbf{S}^T\mathbf{A}^T, \tag{2.1.26}$$

so that $\mathbf{S} \neq \mathbf{S}^T$ in general. If we introduce the notation

$$\mathbf{T}^{(1)} = \frac{1}{2}(\mathbf{S}\mathbf{R} + \mathbf{R}^T\mathbf{S}^T), \tag{2.1.27}$$

as the symmetric part of tensor $\mathbf{SR}$, $\mathbf{T}^{(1)}$ is said to be the *Biot stress tensor*. For an isotropic material $\mathbf{SR}$ is a symmetric tensor (see Ogden (1984)) and we have

$$\mathbf{S} = \mathbf{T}^{(1)}\mathbf{R}^T, \tag{2.1.28}$$

where $\mathbf{T}^{(1)}$ is conjugate to the right stretch $\mathbf{U}$. Let $t_1^{(1)}$, $t_2^{(1)}$, $t_3^{(1)}$ denote the principal values of $\mathbf{T}^{(1)}$. We then have

$$\mathbf{T}^{(1)} = \sum_{i=1}^{3} t_i^{(1)}\mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}. \tag{2.1.29}$$

From equation (2.1.29) it follows that

$$\mathbf{S} = \sum_{i=1}^{3} t_i^{(1)}\mathbf{u}^{(i)} \otimes \mathbf{v}^{(i)}. \tag{2.1.30}$$

The (Eulerian) equation of motion is given by

$$\mathrm{div}\ \mathbf{T} + \rho\mathbf{b} = \rho\ddot{\mathbf{x}}, \tag{2.1.31}$$

where ρ denotes the density per unit volume in $\mathcal{B}_t$, $\mathbf{b}$ represents the body force per unit volume in $\mathcal{B}_t$ and a superposed dot represents material differentiation with respect to time t. In the static case, we have $\ddot{\mathbf{x}} = \mathbf{0}$ and equation (2.1.31) is referred to as the *equilibrium equation.* If, furthermore, there is no body force, the equilibrium equation reduces to

$$\text{div } \mathbf{T} = \mathbf{0}. \tag{2.1.32}$$

In rectangular Cartesian coordinates, equation (2.1.32) can be written in component form as

$$\begin{aligned}
\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} &= 0, \\
\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} &= 0, \\
\frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} &= 0.
\end{aligned} \tag{2.1.33}$$

In cylindrical polar coordinates, equation (2.1.32) can be expressed in component form as

$$\begin{aligned}
\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{\partial T_{zr}}{\partial z} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) &= 0, \\
\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{z\theta}}{\partial z} + \frac{1}{r}(T_{r\theta} + T_{\theta r}) &= 0, \\
\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r}T_{rz} &= 0.
\end{aligned} \tag{2.1.34}$$

Let ρ_0 denotes the density per unit volume in $\mathcal{B}_0$. Then, the mass conservation equation may be expressed in the form

$$J = \rho_0/\rho. \tag{2.1.35}$$

The corresponding Lagrangian equation of the motion is given by

$$\text{Div } \mathbf{S} + \rho_0 \mathbf{b} = \rho_0 \ddot{\mathbf{x}}, \tag{2.1.36}$$

where Div is the divergence operator in $\mathcal{B}_0$. Similarly, the component forms of the equilibrium equation, in the absence of body force, may be written in Cartesian rectangular coordinates as

$$\begin{aligned}
\frac{\partial S_{11}}{\partial X_1}+\frac{\partial S_{12}}{\partial X_2}+\frac{\partial S_{13}}{\partial X_3}&=0,\\
\frac{\partial S_{21}}{\partial X_1}+\frac{\partial S_{22}}{\partial X_2}+\frac{\partial S_{23}}{\partial X_3}&=0,\\
\frac{\partial S_{31}}{\partial X_1}+\frac{\partial S_{32}}{\partial X_2}+\frac{\partial S_{33}}{\partial X_3}&=0,
\end{aligned} \tag{2.1.37}$$

and in cylindrical polar coordinates as

$$\begin{aligned}
\frac{\partial S_{Rr}}{\partial R}+\frac{1}{R}\frac{\partial S_{\Theta r}}{\partial \Theta}+\frac{\partial S_{Zr}}{\partial Z}+\frac{1}{R}(S_{Rr}-S_{\Theta\theta})&=0,\\
\frac{\partial S_{R\theta}}{\partial R}+\frac{1}{R}\frac{\partial S_{\Theta\theta}}{\partial \Theta}+\frac{\partial S_{Z\theta}}{\partial Z}+\frac{1}{R}(S_{R\theta}+S_{\Theta r})&=0,\\
\frac{\partial S_{Rz}}{\partial R}+\frac{1}{R}\frac{\partial S_{\Theta z}}{\partial \Theta}+\frac{\partial S_{Zz}}{\partial Z}+\frac{1}{R}S_{Rz}&=0.
\end{aligned} \tag{2.1.38}$$

2.1.3 Strain-energy function

We consider the stress power per unit volume for a Cauchy elastic material, which may be represented by $\mathrm{tr}(\mathrm{S}\mathrm{d}\mathrm{A}/\mathrm{d}t)$, see Ogden (1984). If there exists a scalar function $W(\mathrm{A})$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}W(\mathrm{A})=\mathrm{tr}\left\{\mathrm{S}\frac{\mathrm{dA}}{\mathrm{d}t}\right\}, \tag{2.1.39}$$

such an elastic material is said to be a *Green elastic material* (or *hyperelastic material*). Since the function $W(\mathrm{A})$ may measure the energy stored in the material during deformation, we refer to it as the *strain-energy function.* When such a W exists, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}W(\mathrm{A})=\mathrm{tr}\left\{\frac{\partial W(\mathrm{A})}{\partial \mathrm{A}}\frac{\mathrm{dA}}{\mathrm{d}t}\right\}. \tag{2.1.40}$$

Comparison of equations (2.1.39) and (2.1.40) shows that the nominal stress tensor may be expressed as

$$\mathrm{S}=\frac{\partial W(\mathrm{A})}{\partial \mathrm{A}}, \tag{2.1.41}$$

and hence, by equation (2.1.25), the Cauchy stress tensor as

$$\mathbf{T} = J^{-1}\mathbf{A}\frac{\partial W(\mathbf{A})}{\partial \mathbf{A}}. \tag{2.1.42}$$

For a homogeneous objective, isotropic hyperelastic material, $W(\mathbf{A})$ satisfies

$$W(\mathbf{A}) = W(\mathbf{QA}) = W(\mathbf{AP}), \tag{2.1.43}$$

for arbitrary rotations $\mathbf{Q}$ and $\mathbf{P}$. From equation (2.1.43), it is easily shown that

$$W(\mathbf{QVQ}^T) = W(\mathbf{V}), \tag{2.1.44}$$

which holds for arbitrary $\mathbf{Q}$. By choosing $\mathbf{Q} = \mathbf{R}$ we deduce that

$$W(\mathbf{A}) = W(\mathbf{U}) = W(\mathbf{V}). \tag{2.1.45}$$

Noting equation (2.1.28), we then have

$$\mathbf{T}^{(1)} = \frac{\partial W(\mathbf{U})}{\partial \mathbf{U}}, \tag{2.1.46}$$

and equation (2.1.40) may be rewritten as

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \mathrm{tr}\left(\mathbf{T}^{(1)}\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\right). \tag{2.1.47}$$

According to equation (2.1.12), the strain-energy function may be regarded as a function of the principal stretches $\lambda_1,\ \lambda_2,\ \lambda_3$, namely

$$W = W(\lambda_1,\ \lambda_2,\ \lambda_3), \tag{2.1.48}$$

and must satisfy the symmetry requirement

$$W(\lambda_1,\ \lambda_2,\ \lambda_3) = W(\lambda_2,\ \lambda_3,\ \lambda_1) = W(\lambda_3,\ \lambda_1,\ \lambda_2). \tag{2.1.49}$$

Thus, $\mathbf{T}^{(1)}$ is given by

$$\mathbf{T}^{(1)} = \sum_{i=1}^{3}\frac{\partial W}{\partial \lambda_i}\mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{2.1.50}$$

when expressed in spectal form. Correspondingly, the principal components t_i of Cauchy stress $\mathbf{T}$ are

$$t_i = J^{-1}\lambda_i \frac{\partial W}{\partial \lambda_i}, \qquad i = 1,\ 2,\ 3 \tag{2.1.51}$$

where no summation is implied, and

$$\mathbf{T} = J^{-1}\sum_{i=1}^{3} \lambda_i \frac{\partial W}{\partial \lambda_i} \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}. \tag{2.1.52}$$

When a natural configuration is taken as the reference configuration, we may assume that W vanishes in such a configuration, and set

$$W(1,\ 1,\ 1) = 0. \tag{2.1.53}$$

Analogously, the value of the stress is zero in the natural state, that is

$$\frac{\partial W}{\partial \lambda_i}(1,\ 1,\ 1) = 0, \qquad i = 1,\ 2,\ 3. \tag{2.1.54}$$

Additionally, for compatibility with the classical theory in the linear approximation, we must have

$$\begin{aligned} \frac{\partial^2 W}{\partial {\lambda_i}^2}(1,\ 1,\ 1) &= \kappa + \frac{4}{3}\mu \\ \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j}(1,\ 1,\ 1) &= \kappa - \frac{2}{3}\mu \qquad i \neq j \end{aligned} \tag{2.1.55}$$

with $i,\ j = 1,\ 2,\ 3$, where μ is the shear modulus and κ the bulk modulus in the natural configuration. (see Ogden 1984)

In particular, we may treat W as a function of the principal invariants $i_1,\ i_2,\ i_3$ given by equation (2.1.16),

$$W(\mathrm{A}) = \bar{W}(i_1,\ i_2,\ i_3). \tag{2.1.56}$$

Therefore, equations (2.1.53) and (2.1.54) lead to

$$\bar{W}(3,\ 3,\ 1) = 0,$$

$$\bar{W}_1(3,\ 3,\ 1) + 2\bar{W}_2(3,\ 3,\ 1) + \bar{W}_3(3,\ 3,\ 1) = 0, \tag{2.1.57}$$

and equations (2.1.55) to

$$\bar{W}_2(3,\ 3,\ 1) + \bar{W}_3(3,\ 3,\ 1) = -2\mu,$$

$$\bar{W}_{11}(3,\ 3,\ 1) + 4\bar{W}_{21}(3,\ 3,\ 1) + 2\bar{W}_{31}(3,\ 3,\ 1) + 4\bar{W}_{22}(3,\ 3,\ 1)$$

$$+ 4\bar{W}_{23}(3,\ 3,\ 1) + \bar{W}_{33}(3,\ 3,\ 1) = \kappa + \frac{4}{3}\mu, \tag{2.1.58}$$

where $\bar{W}_q = \partial\bar{W}/\partial i_q$, $q = 1,\ 2,\ 3$ and $\bar{W}_{pq} = \partial^2\bar{W}/\partial i_p\partial i_q$, $p,\ q = 1,\ 2,\ 3$.

We may also regard W as a function of any three other independent invariants of the deformation. For example, we refer to W as a function of $I_1,\ I_2,\ I_3$ given by equation (2.1.17), namely

$$W(\mathbf{A}) = \tilde{W}(I_1,\ I_2,\ I_3). \tag{2.1.59}$$

Analogously to (2.1.57) - (2.1.58), the following equations should hold

$$\tilde{W}(3,\ 3,\ 1) = 0,$$

$$\tilde{W}_1(3,\ 3,\ 1) + 2\tilde{W}_2(3,\ 3,\ 1) + \tilde{W}_3(3,\ 3,\ 1) = 0, \tag{2.1.60}$$

and

$$\tilde{W}_2(3,\ 3,\ 1) + \tilde{W}_3(3,\ 3,\ 1) = -\frac{1}{2}\mu,$$

$$\tilde{W}_{11}(3,\ 3,\ 1) + 4\tilde{W}_{21}(3,\ 3,\ 1) + 2\tilde{W}_{31}(3,\ 3,\ 1) + 4\tilde{W}_{22}(3,\ 3,\ 1)$$

$$+ 4\tilde{W}_{23}(3,\ 3,\ 1) + \tilde{W}_{33}(3,\ 3,\ 1) = \frac{1}{4}\kappa + \frac{1}{3}\mu, \tag{2.1.61}$$

where $\tilde{W}_q = \partial\tilde{W}/\partial I_q$, $q = 1,\ 2,\ 3$ and $\tilde{W}_{pq}(3,\ 3,\ 1) = \partial^2\tilde{W}/\partial I_p\partial I_q$, $p,\ q = 1,\ 2,\ 3$.

2.2 Equations of incremental elasticity

2.2.1 Incremental deformation

Recalling the deformation described in Section 2.1.1

$$x = \chi(X), \tag{2.2.1}$$

we consider that a displacement $\mathbf{u}$ is superimposed on the deformation for each $\mathbf{X} \in \mathcal{B}_0$, and then

$$u = \delta x = \delta\chi(X), \tag{2.2.2}$$

where the symbol δ indicates a small increment in the quantity concerned.

Supposing that the displacement is small enough to be able to ignore terms of second order in $\mathbf{u}$, $\mathbf{u} = \delta x$ is said to be an *incremental deformation.* The exact deformation gradient corresponding to this displacement is given by

$$\delta A = \delta(\text{Grad } x) = \text{Grad } (x + \mathbf{u}) - \text{Grad } x = \text{Grad } \mathbf{u}. \tag{2.2.3}$$

However the corresponding increment of J has the linear approximation

$$\delta J = J\text{tr } \{(\delta A)A^{-1}\}. \tag{2.2.4}$$

If the reference configration is updated to the basic deformed configration (2.2.1), the deformation gradient relating to the current configuration is given by

$$\delta A_0 = \text{grad } \mathbf{u} = \delta A A^{-1}, \tag{2.2.5}$$

where *grad* refer to the gradient operation taken with respect to x.

2.2.2 Incremental constitutive law and equilibrium equation

For a compressible material, the nominal stress increment to the first order may be written as

$$\delta \mathrm{S} = \mathcal{A}\delta \mathrm{A}, \tag{2.2.6}$$

where

$$\mathcal{A} = \frac{\partial \mathrm{S}}{\partial \mathrm{A}}, \tag{2.2.7}$$

is a fourth-order tensor which is called the *tensor of first-order elastic moduli* associated with the conjugate pair (**S**, **A**).

If we now let a superposed dot instead of δ represent the increment in a quantity concerned, so that $\dot{\mathrm{A}} = \delta \mathrm{A}$ for example, and equations (2.2.4) - (2.2.6) may be written as

$$\dot{J} = J\mathrm{tr}\,\{\dot{\mathrm{A}}\mathrm{A}^{-1}\}, \tag{2.2.8}$$

$$\dot{\mathrm{A}}_0 = \dot{\mathrm{A}}\mathrm{A}^{-1}, \tag{2.2.9}$$

$$\dot{\mathrm{S}} = \mathcal{A}\dot{\mathrm{A}}. \tag{2.2.10}$$

The nominal stress increment relating to the current configration is given by

$$\dot{\mathrm{S}}_0 = \mathcal{A}_0\dot{\mathrm{A}}_0, \tag{2.2.11}$$

where $\mathcal{A}_0$ is called the tensor of *instantaneous moduli.* With the help of (2.1.7) we have

$$\dot{\mathrm{S}}_0 = J^{-1}\mathrm{A}\dot{\mathrm{S}}. \tag{2.2.12}$$

On substituting equations (2.2.10) and (2.2.12) into equation (2.2.11), and comparing with equation (2.2.9), we may find the relationship between $\mathcal{A}$ and $\mathcal{A}_0$ in the component form

$$\mathcal{A}_{oijkl} = J^{-1} A_{is} A_{kt} \mathcal{A}_{sjtl}. \tag{2.2.13}$$

The equilibium equation without body force in the current configuration is

$$\text{div } \dot{\mathbf{S}}_0 = \mathbf{0}. \tag{2.2.14}$$

For a hyperelastic material, we have

$$\mathcal{A} = \frac{\partial^2 W}{\partial \mathbf{A}^2}, \tag{2.2.15}$$

with the help of equations (2.1.41) and (2.2.7). Furthermore, if the material is homogeneous and isotropic, the nonzero components of $\mathcal{A}_0$ on the Eulerian principal axes, from equation (2.2.11), may be given by

$$\mathcal{A}_{oiijj} = J^{-1} \lambda_i \lambda_j W_{ij}, \tag{2.2.16}$$

$$\mathcal{A}_{oijij} = J^{-1} \frac{(\lambda_i W_i - \lambda_j W_j)\, \lambda_i^2}{\lambda_i^2 - \lambda_j^2} \qquad i \neq j, \tag{2.2.17}$$

$$\mathcal{A}_{oijji} = \mathcal{A}_{ojiij} = \mathcal{A}_{oijij} - J^{-1} \lambda_i W_i \quad i \neq j \tag{2.2.18}$$

with $i,\ j = 1,\ 2,\ 3$, where $W_i = \partial W / \partial \lambda_i$ and $W_{ij} = \partial^2 W / \partial \lambda_i \partial \lambda_j$, and no summation convention applies here.

The details of this section can be seen Ogden (1984, Chapter 6).

Chapter 3

Azimuthal shear of a circular cylindrical tube

3.1 Background

We consider a compressible nonlinearly elastic thick-walled circular cylindrical tube whose cross-section in its natural (unstressed) configuration is defined by

$$0 < A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \tag{3.1.1}$$

where (R, Θ, Z) are polar coordinates. Attention is restricted to plane deformations in which there is no extension along the axis of the cylinder and the deformation of a cross-section is independent of the axial coordinate, Z say. To maintain plane-strain conditions appropriate axial loading is required on the ends of the tube, but these will not be needed explicitly.

An azimuthal shear deformation is defined by

$$r = r(R), \quad \theta = \Theta + g(R), \quad z = Z, \tag{3.1.2}$$

where (r, θ, z) are cylindrical polar coordinates associated with the deformed configuration.

We take the boundary conditions as

$$r(A) = A, \quad r(B) = B, \tag{3.1.3}$$

$$g(A) = 0, \quad g(B) = \psi, \tag{3.1.4}$$

in the cross-section of the tube, ψ being the angle through which the boundary $R = B$ is rotated.

Referred to cylindrical polar coordinates the deformation gradient tensor $\mathbf{A}$ has components

$$\mathbf{A} = \begin{bmatrix} r' & 0 & 0 \\ rg' & r/R & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.1.5}$$

where the prime indicates differentiation with respect to R, and its inverse is

$$\mathbf{A}^{-1} = \begin{bmatrix} 1/r' & 0 & 0 \\ -Rg'/r' & R/r & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.1.6}$$

The principal invariants I_1, I_2, I_3 of the deformation tensor $\mathbf{B} = \mathbf{A}\mathbf{A}^{\mathrm{T}}$ are given by

$$\begin{aligned} I_1 &= r'^2 + r^2 g'^2 + r^2/R^2 + 1, \\ I_2 &= r^2 g'^2 + r^2/R^2 + r'^2 + r^2 r'^2/R^2, \\ I_3 &= r^2 r'^2/R^2. \end{aligned} \tag{3.1.7}$$

For an arbitrary strain-energy function $W = W(I_1, I_2, I_3)$ which satisfies equations (2.1.60) and (2.1.61), we calculate, with the help with equations (2.1.17)

$$\begin{aligned}\frac{\partial W}{\partial \lambda_i} &= 2\lambda_i\left(W_1\frac{\partial I_1}{\partial \lambda_i}+W_2\frac{\partial I_2}{\partial \lambda_i}+W_3\frac{\partial I_3}{\partial \lambda_i}\right)\\ &= 2\lambda_i W_1+2\lambda_i^{-1}\left(I_2W_2+I_3W_3\right)-2\lambda_i^{-3}W_3,\end{aligned} \tag{3.1.8}$$

where $W_i = \partial W/\partial I_i$ $(i = 1, 2, 3)$. Equation (2.1.52) is then rewritten

$$\mathbf{T} = 2I_3^{-\frac{1}{2}}\sum_{i=1}^{3}\left[I_2W_2+I_3W_3+\lambda_i^2W_1-\lambda_i^{-2}W_2\right]\mathbf{v}^{(i)}\otimes\mathbf{v}^{(i)}. \tag{3.1.9}$$

According to equation (2.1.14), it follows that

$$\mathbf{T} = 2I_3^{-\frac{1}{2}}(I_2W_2+I_3W_3)\mathbf{I}+2I_3^{-\frac{1}{2}}W_1\mathbf{B}-2I_3^{\frac{1}{2}}W_2\mathbf{B}^{-1}. \tag{3.1.10}$$

Since $\mathbf{T} = I_3^{-1}\mathbf{AS}$, the equations (2.1.38) of equilibrium in the absence of body forces, therefore lead to two nonlinear ordinary differential equations, which may be written as

$$\begin{aligned}&\frac{d}{dR}\left[\frac{Rr'}{r}W_1+\left(\frac{Rr'}{r}+\frac{rr'}{R}\right)W_2+\frac{rr'}{R}W_3\right]+\left(\frac{Rr'^2}{r^2}-\frac{1}{R}-Rg'^2\right)W_1\\ &+\left(\frac{R^3}{r^4}-\frac{R}{r^2r'^2}-\frac{R^3g'^2}{r^2r'^2}\right)W_2=0,\\ &2g'R(r^2W_1+\frac{R^2}{r'^2}W_2)=K,\end{aligned} \tag{3.1.11}$$

where K is a constant, the second of these having been obtained by integration with respect to R.

A boundary value problem is specified by equations (3.1.11) together with the boundary conditions (3.1.3) and (3.1.4), the existence and form of solutions for $r(R)$ and $g(R)$ being dependent on the choice of the strain-energy function of the material. In general, closed form solutions of boundary-value problems in finite elasticity are usually rather difficult to obtain. For the circular shear problem, the special case in which the radial displacement is always zero, which is usually called the pure azimuthal shear (or torsional), has been considered by several authors. Some explicit results have been obtained for some special forms of strain-energy functions for compressible materials. Here we describe such strain-energy functions and the associated solutions.

For pure azimuthal shear, we set $r = R$ in equations (3.1.7). Then,

$$I_1 = I_2 = 3 + R^2 g'^2, \quad I_3 = 1, \tag{3.1.12}$$

and equations (3.1.11) become

$$\frac{d}{dR}(W_1 + 2W_2 + W_3) - Rg'^2(W_1 + W_2) = 0,$$
$$R^3 g'(W_1 + W_2) = K. \tag{3.1.13}$$

The boundary conditions (3.1.3) are then satisitied automatically and (3.1.4) are

$$g(A) = 0, \quad g(B) = \psi, \tag{3.1.14}$$

where $\psi > 0$ is a given constant.

Ertepinar (1990) has considered the polynomial form of strain-energy function proposed by Levinson and Burgess (1971) in the form

$$W = \frac{1}{2}\mu\left[\alpha(I_1 - 3) + (1-\alpha)\left(\frac{I_2}{I_3} - 3\right) + 2(1-2\alpha)\left(I_3^{\frac{1}{2}} - 1\right) + (2\alpha + \beta)\left(I_3^{\frac{1}{2}} - 1\right)^2\right]. \tag{3.1.15}$$

When $\alpha = \frac{3}{4}$, the solution for g(R) is found to be

$$g(R) = c_1 R^{-2} + c_2, \tag{3.1.16}$$

where c_1 and c_2 are constants which are determined by the boundary conditions (3.1.4).

Similarly the generalized Blatz-Ko material (1962), which has the form

$$\begin{aligned} W &= \frac{1}{2}\mu f\left[I_1 - 1 - \frac{1}{\nu} + \frac{(1-2\nu)}{\nu} I_3^{-\frac{(1-2\nu)}{\nu}}\right] \\ &+ \frac{1}{2}\nu(1-f)\left[\frac{I_2}{I_3} - 1 - \frac{1}{\nu} + \frac{(1-2\nu)}{\nu} I_3^{\frac{(1-2\nu)}{\nu}}\right], \end{aligned} \tag{3.1.17}$$

leads to the solution (3.1.16) when $f = \frac{3}{4}$, which was obtained by Polignone and Horgan (1994), where μ and ν are the shear modulus and Poisson's ratio and $\mu > 0, \quad -1 < \nu \leq \frac{1}{2}$.

Haughton (1993) obtained the same solution for $g(R)$ for the class of strain energy function considered by Agarwal (1979). This has the form

$$W = \frac{1}{2}\mu[(I_1 - 3)H_1(I_3) + (I_2 - 3)H_2(I_3) + H_3(I_3)], \tag{3.1.18}$$

where $H_i(I_3)$ $(i = 1,\ 2,\ 3)$ are functions which satisfy

$$H_3(1) = 0, \quad H_1'(1) + 2H_2'(1) + H_3'(1) = 0, \tag{3.1.19}$$

and

$$H_1(1) = H_2(1). \tag{3.1.20}$$

A necessary condition for the solution to exist was found to be

$$H_1(1) + H_2(1) + 4(H_1'(1) + H_2'(1)) = 0. \tag{3.1.21}$$

Carroll (1988) considered three distinct classes of strain-energy function, defined as follows.

First are the harmonic materials, which have strain-energy function

$$W = f(i_1) + c_2(i_2 - 3) + c_3(i_3 - 1), \tag{3.1.22}$$

where i_1, i_2, i_3 are given by equations (2.1.16), for which pure circular shearing is impossible. However, a circular shearing solution with $r \neq R$ was found by Ogden (1978) for (3.1.22).

Second are the compressible Varga materials with strain energy function

$$W = c_1(i_1 - 3) + c_2(i_2 - 3) + h(i_3). \tag{3.1.23}$$

The solution for $g(R)$ from equation (3.1.13) is

$$g(R) = \cos^{-1}(C/R^2) + E, \tag{3.1.24}$$

where C and E are constants determined by the boundary conditions.

The third class of strain-energy functions is given by

$$W = c_1(i_1 - 3) + s(i_2) + c_3(i_3 - 1), \tag{3.1.25}$$

for which, again, circular shearing is not possible. We note here that c_1, c_2 and c_3 are arbitrary constants and f, h and s are arbitrary functions which must ensure that equations (2.1.60) and (2.1.61) are satisfied.

Haughton (1993) also considered a class of strain energy functions given by

$$W(I_1, I_2, I_3) = \Phi(I_1) + \Phi(I_2) + \Psi(I_3), \tag{3.1.26}$$

where

$$2\Phi(3) + \Psi(1) = 0, \quad 3\Phi'(3) + \Psi'(1) = 0. \tag{3.1.27}$$

In the special case, when Φ and Ψ are defined as

$$\Phi(x) = \alpha x^{\frac{1}{2}}, \tag{3.1.28}$$

with

$$\Psi(1) = -2\alpha\sqrt{3},$$

$$\Psi'(1) = -\alpha\sqrt{3}/2, \tag{3.1.29}$$

$$\Psi''(1) > 13\alpha/12\sqrt{3},$$

the solutions for $g(R)$ from (3.1.13) is

$$g(R) = \frac{\sqrt{3}}{2}\cos^{-1}(c_1/R^2) + c_2, \qquad 0 < c_1 < A^2, \tag{3.1.30}$$

where c_1 and c_2 are constants.

3.2 New formulation of the azimuthal shear problem

In this section, we will give a special characteristic of the circular shearing of elastic materials and then, the system of the two second-order ordinary differential equations (3.1.11) for $r(R)$ and $g(R)$ will be simplified.

Noting equations (3.1.7), we observe immediately the connection

$$I_2 = I_1 + I_3 - 1, \tag{3.2.1}$$

between the principal invariants. We note that (3.2.1) holds not only for the deformation (3.1.2) but for every plane strain deformation.

With the restriction to plane strain only two of the invariants I_1, I_2, I_3 are independent, and the strain energy $W(I_1, I_2, I_3)$ per unit reference volume of a compressible isotropic elastic material may then be regarded as a function of two invariants. Accordingly, we define $\bar{W}(I_1, I_3)$ by

$$\bar{W}(I_1, I_3) = W(I_1, I_1 + I_3 - 1, I_3) \tag{3.2.2}$$

when (3.2.1) holds identically. We then have

$$\bar{W}_1 = W_1 + W_2, \qquad \bar{W}_3 = W_2 + W_3, \tag{3.2.3}$$

and

$$\begin{aligned} \bar{W}_{11} &= W_{11} + 2W_{12} + W_{22}, \\ \bar{W}_{13} &= W_{12} + W_{13} + W_{22} + W_{23}, \\ \bar{W}_{33} &= W_{22} + 2W_{23} + W_{33}. \end{aligned} \tag{3.2.4}$$

Equations (2.1.60) and (2.1.61) become

$$\begin{aligned} &\bar{W}(3,1) = 0, \quad \bar{W}_1(3,1) = -\bar{W}_3(3,1) = \frac{1}{2}\mu, \\ &\bar{W}_{11}(3,1) + 2\bar{W}_{13}(3,1) + \bar{W}_{33}(3,1) = \frac{1}{4}\kappa + \frac{1}{3}\mu, \end{aligned} \tag{3.2.5}$$

where $\bar{W}_i = \partial\bar{W}/\partial I_i$, $i = 1,3$ and $\bar{W}_{ij} = \partial^2\bar{W}/\partial I_i \partial I_j$, $i,\ j = 1,3$.

Furthermore, we calculate

$$\begin{aligned} \frac{\partial W}{\partial \lambda_i} &= W_1 \frac{\partial I_1}{\partial \lambda_i} + W_2 \frac{\partial (I_1 + I_3 - 1)}{\partial \lambda_i} + W_3 \frac{\partial I_3}{\partial \lambda_i} \\ &= (W_1 + W_2)\frac{\partial I_1}{\partial \lambda_i} + (W_2 + W_3)\frac{\partial I_3}{\partial \lambda_i} \\ &= \bar{W}_1 \frac{\partial I_1}{\partial \lambda_i} + \bar{W}_3 \frac{\partial I_3}{\partial \lambda_i} \\ &= 2\lambda_i \bar{W}_1 + 2\lambda_i^{-1} I_3 \bar{W}_3. \end{aligned} \tag{3.2.6}$$

According to equation (2.1.30) and (2.1.50), the nominal stress tensor $\mathbf{S}$ is given by

$$\begin{aligned} \mathbf{S} &= \sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i} \mathbf{u}^{(i)} \otimes \mathbf{v}^{(i)} \\ &= 2\sum_{i=1}^{3} \left(\lambda_i \bar{W}_1 + \lambda_i^{-1} I_3 \bar{W}_3\right) \mathbf{u}^{(i)} \otimes \mathbf{v}^{(i)} \\ &= 2\bar{W}_1 \mathbf{A}^T + 2 I_3 \bar{W}_3 \mathbf{A}^{-1}, \end{aligned} \tag{3.2.7}$$

and, making use of equations (3.1.5) and (3.1.6), the physical components of $\mathbf{S}$ are

$$
\begin{aligned}
S_{Rr} &= 2r'\bar{W}_1 + \frac{2r^2r'}{R^2}\bar{W}_3,\\
S_{\Theta\theta} &= 2\frac{r}{R}\bar{W}_1 + \frac{2rr'^2}{R^2}\bar{W}_3,\\
S_{Zz} &= 2\bar{W}_1 + \frac{2r^2r'^2}{R^2}\bar{W}_3,\\
S_{R\theta} &= 2rg'\bar{W}_1, \quad \mathrm{S}_{\Theta r} = -\frac{2r^2r'g'}{R^2}\bar{W}_3,\\
S_{Rz} &= S_{Zr} = S_{\Theta z} = S_{Z\theta} = 0.
\end{aligned} \tag{3.2.8}
$$

The corresponding Cauchy stress tensor $\mathbf{T} = I_3^{-\frac{1}{2}}\mathbf{AS}$ is given by

$$
\mathbf{T} = 2I_3^{-\frac{1}{2}}\bar{W}_1\mathbf{B} + 2I_3^{\frac{1}{2}}\bar{W}_3\mathbf{I}. \tag{3.2.9}
$$

Substitution from $\mathbf{B} = \mathbf{AA}^T$ in (3.2.9) yields the physical components of $\mathbf{T}$, namely

$$
\begin{aligned}
T_{rr} &= 2\frac{Rr'}{r}\bar{W}_1 + \frac{2rr'}{R}\bar{W}_3\\
T_{\theta\theta} &= 2\left(\frac{rRg'^2}{r'} + \frac{r}{Rr'}\right)\bar{W}_1 + \frac{2rr'}{R}\bar{W}_3\\
T_{zz} &= \frac{2R}{rr'}\bar{W}_1 + \frac{2rr'}{R}\bar{W}_3\\
T_{r\theta} &= 2Rg'\bar{W}_1,\\
T_{rz} &= T_{\theta z} = 0.
\end{aligned} \tag{3.2.10}
$$

After substitution of the components of $\mathbf{S}$ from (3.2.8) with (3.2.1) and (2.1.17) into (2.1.38), the equilibrium equations Div $\mathbf{S} = \mathbf{0}$, two equations are obtained. The radial equation may be written

$$
\frac{d}{dR}\left(\frac{Rr'}{r}\bar{W}_1 + 2\frac{rr'}{R}\bar{W}_3\right) + \frac{r'}{r}\left(\frac{Rr'}{r} - \frac{Rrg'^2}{r'} - \frac{r}{Rr'}\right)\bar{W}_1 = 0, \tag{3.2.11}
$$

and the azimuthal equation is

$$
\frac{d}{dR}(Rg'\bar{W}_1) + \frac{2Rr'g'}{r}\bar{W}_1 = 0. \tag{3.2.12}
$$

From (3.2.12) it leads to

$$\frac{1}{r^2}\frac{d}{dR}(r^2 Rg'\bar{W}_1) = 0. \tag{3.2.13}$$

and hence, on integration, to

$$r^2 Rg'\bar{W}_1 = \bar{K}, \tag{3.2.14}$$

where $\bar{K}$ is a constant of integration. Noting equations $(3.2.8)_4$ and $(3.2.10)_4$, we then obtain

$$rRS_{R\theta} = 2\bar{K}, \qquad r^2 T_{r\theta} = 2\bar{K}. \tag{3.2.15}$$

Assuming the constant τ is the common value of the azimuthal shear stress $T_{r\theta} = S_{R\theta}$ at the outer boundary $r = b = B$, it follows

$$2\bar{K} = b^2\tau. \tag{3.2.16}$$

Therefore, equations (3.2.11) and (3.2.12) can be rewritten

$$\frac{d}{dR}(Rr'\bar{W}_1) + r\frac{d}{dR}\left(\frac{rr'}{R}\bar{W}_3\right) - \frac{r}{R}\bar{W}_1 - rRg'^2\bar{W}_1 = 0,$$

$$rRS_{R\theta} \equiv r^2 T_{r\theta} = 2r^2 Rg'\bar{W}_1 = b^2\tau. \tag{3.2.17}$$

3.3 Pure azimuthal shear

Pure azimuthal shear is the isochoric specialization of the deformation (3.1.2) corresponding to $r = R$. Then (3.1.7) reduce to

$$I_1 = 3 + r^2 g'^2, \quad I_3 = 1, \tag{3.3.1}$$

and, locally, the deformation is a simple shear with amount of shear rg', the azimuthal direction being the direction of shear.

With the specialization (3.3.1) equations (3.2.17) reduce to

$$\frac{d}{dr}(\bar{W}_1 + \bar{W}_3) - rg'^2\bar{W}_1 = 0, \tag{3.3.2}$$

$$2r^3g'\bar{W}_1 = b^2\tau. \tag{3.3.3}$$

Let $\gamma = rg'$ denote the amount of shear. Then $\gamma > 0$ is associated with $\tau > 0$ (shearing in the position θ-direction with $g(r) > 0$ for $r > a$) and $\gamma < 0$ corresponds to $\tau < 0$. By defining

$$\hat{W}(\gamma) = \bar{W}(3 + \gamma^2, 1), \tag{3.3.4}$$

then, we have

$$\hat{W}' = 2\gamma\bar{W}_1,$$
$$\hat{W}'' = 2\bar{W}_1 + 4\gamma^2\bar{W}_{11} = 2\left[\bar{W}_1 + 2(I_1 - 3)\bar{W}_{11}\right]. \tag{3.3.5}$$

Furthermore, we can rewrite (3.3.3) as

$$T_{r\theta} \equiv \hat{W}'(\gamma) = \frac{b^2\tau}{r^2} \tag{3.3.6}$$

with $\hat{W}'(\gamma) > 0$ (< 0) for $\gamma > 0$ (< 0). An equation analogous to (3.3.6) in the incompressible theory was given by Ogden (1979).

Increasing shear γ corresponds to increasing shearing stress $T_{r\theta}$ provided

$$\hat{W}''(\gamma) > 0, \tag{3.3.7}$$

and we therefore impose (3.3.7) for all γ. The monotonicity of $\hat{W}'(\gamma)$ implied by (3.3.7) ensures that, in principle, (3.3.6) can be inverted to give $\gamma = rg'$ uniquely as a function of r and hence g is determined by integration. Note that from (3.3.6) and (3.3.7) it follows that $rg'' + g' < 0$ (> 0) when $\gamma > 0$ (< 0).

From equation (3.3.5) the above requirements on $\hat{W}'(\gamma)$ and $\hat{W}''(\gamma)$ are equivalent to

$$\bar{W}_1(I_1, 1) > 0, \quad 2(I_1 - 3)\bar{W}_{11}(I_1, 1) + \bar{W}_1(I_1, 1) > 0, \tag{3.3.8}$$

and it can be shown that these inequalities are equivalent to $(3.6)_2$ given by Haughton (1993) and (3.1) given by Beatty and Jiang (1997).

With these conditions holding we may replace r by I_1 as the independent variable in (3.3.2) by using (3.3.1) and (3.3.3). First, we rewrite equations (3.3.2) and (3.3.3) as

$$r\frac{d}{dr}(\bar{W}_1 + \bar{W}_3) = (I_1 - 3)\bar{W}_1, \tag{3.3.9}$$

$$2r^2\sqrt{I_1 - 3}\bar{W}_1 = b^2\tau, \tag{3.3.10}$$

and then differentiate equation (3.3.10) with respect to r to obtain

$$4r\sqrt{I_1 - 3}\bar{W}_1 + 2r^2\frac{d}{dr}\left(\sqrt{I_1 - 3}\bar{W}_1\right) = 0,$$

and, hence

$$r\frac{d}{dr}\left(\sqrt{I_1 - 3}\bar{W}_1\right) = -2\sqrt{I_1 - 3}\bar{W}_1. \tag{3.3.11}$$

Substituting equation (3.3.11) into equation (3.3.9) leads to

$$2\frac{d}{dr}(\bar{W}_1 + \bar{W}_3) + \left(\sqrt{I_1 - 3}\right)\frac{d}{dr}\left(\sqrt{I_1 - 3}\bar{W}_1\right) = 0. \tag{3.3.12}$$

On elimination of differentiation with respect to r in favour of I_1 it leads to the key condition

$$2(I_1 - 1)\bar{W}_{11}(I_1, 1) + 4\bar{W}_{13}(I_1, 1) + \bar{W}_1(I_1, 1) = 0 \tag{3.3.13}$$

on the strain-energy function. Substituting (3.2.4) into equation (3.3.13) leads the equation (3.2) given by Beatty and Jiang (1997). It can also be shown that (3.3.13) is equivalent to, but simpler than equation $(3.6)_1$ given by Haughton (1993).It is emphasized that equations (3.3.8) and (3.3.13) together are *sufficient conditions* for the strain-energy function $\bar{W}$ to admit a pure azimuthal shear deformation for all τ. With (3.3.13) holding it follows that $(3.3.8)_2$ is equivalent to

$$\bar{W}_{11}(I_1, 1) + \bar{W}_{13}(I_1, 1) < 0. \tag{3.3.14}$$

On the other hand, whilst (3.3.13) is also a *necessary condition* the inequality $(3.3.8)_2$ is not in general necessary only if we require the existence of a unique solution. The latter inequality can be relaxed, if need be, to allow for shear softening in which the shear stress exhibits a maximum as a function of γ (with consequent loss of ellipticity). In these circumstances non-uniqueness of solution arises. Existence and uniqueness of solution is guaranteed if (3.3.7) holds. To ensure existence of solution for all τ when the strain energy satisfies (3.3.13) and when (3.3.7) does not hold, the (weaker) requirement is that $\hat{W}'(\gamma)$ be continous and unbounded. If the latter has a finite global maximum then there will be values of τ for which solutions do not exist, and this point is illustrated by one of the examples considered in Section 3.4.

Using (3.2.5) we may integrate (3.3.13) with respect to I_1 to obtain

$$2(I_1 - 1)\bar{W}_1(I_1, 1) + 4\bar{W}_3(I_1, 1) - \bar{W}(I_1, 1) = 0, \tag{3.3.15}$$

which is equivalent to equation (3.3.13).

Now let us check the results given in Section 3.1. Using equation (3.2.1), the form of the strain-energy function (3.1.15) becomes

$$
\begin{aligned}
\bar{W} &= \frac{1}{2}\mu[\alpha(I_1-3)+(1-\alpha)\left(\frac{I_1}{I_3}-2\right) \\
&\quad -(1-\alpha)I_3^{-1}+2(1-2\alpha)(I_3^{\frac{1}{2}}-1)+(2\alpha+\beta)(I_3^{\frac{1}{2}}-1)^2],
\end{aligned} \tag{3.3.16}
$$

and equation (3.1.17) is

$$
\begin{aligned}
\bar{W} &= \frac{1}{2}\mu f\left[I_1-1-\frac{1}{\nu}+\frac{(1-2\nu)}{\nu}I_3^{-\frac{(1-2\nu)}{\nu}}\right] \\
&\quad +\frac{1}{2}\nu(1-f)\left[\frac{I_1}{I_3}-I_3^{-1}-\frac{1}{\nu}+\frac{(1-2\nu)}{\nu}I_3^{\frac{(1-2\nu)}{\nu}}\right].
\end{aligned} \tag{3.3.17}
$$

The materials (3.1.18) with equation (3.2.1) can be rewritten

$$
\bar{W}=\frac{1}{2}\mu[(I_1-3)h_1(I_3)+h_3(I_3)], \tag{3.3.18}
$$

where

$$
h_1(I_3)=H_1(I_3)+H_2(I_3),
$$

$$
h_3(I_3)=(I_3-1)H_2(I_3)+H_3(I_3).
$$

The three classes of strain-energy functions considered by Carroll in (3.1.23), (3.1.24) and (3.1.26) may be expressed as follows.

For the harmonic materials

$$
\bar{W}=f\left(\sqrt{I_1+2I_3^{\frac{1}{2}}-1}+1\right)+c_2\left(\sqrt{I_1+2I_3^{\frac{1}{2}}-1}+I_3^{\frac{1}{2}}-3\right)+c_3\left(I_3^{\frac{1}{2}}-1\right), \tag{3.3.19}
$$

for the Varga materials

$$
\bar{W}=c_1\left(\sqrt{I_1+2I_3^{\frac{1}{2}}-1}-2\right)+c_2\left(\sqrt{I_1+2I_3^{\frac{1}{2}}-1}+I_3^{\frac{1}{2}}-3\right)+h\left(I_3^{\frac{1}{2}}\right), \tag{3.3.20}
$$

and for the third class of materials

$$\bar{W} = c_1\left(\sqrt{I_1+2I_3^{\frac{1}{2}}-1}-2\right)+c_2\left(\sqrt{I_1+2I_3^{\frac{1}{2}}-1}+I_3^{\frac{1}{2}}\right)+c_3\left(I_3^{\frac{1}{2}}-1\right). \qquad (3.3.21)$$

It is easy to see that for the materials (3.3.16)-(3.3.18), for which pure azimuthal shearing is admitted, $\bar{W}$ is linear in I_1. In this case we have $\bar{W}_1$ = constant and $\bar{W}_{11} = 0$. Equation (3.2.5) leads $\bar{W}_1 \equiv \frac{1}{2}\mu > 0$, and, therefore, inequalities (3.3.8) hold automatically. On use of these materials with equation (3.3.15), we may recover the same results with the help of equation (3.3.13), similarly for the strain-energy function given by Haughton in equation (3.1.26).

In Section 3.4 we apply (3.3.15) to a certain class of strain-energy functions. Together with inequalities (3.3.8) equation (3.3.15) then determines a subclass of materials for which the pure azimuthal shear deformation is possible. We then use (3.3.3) to determine $g(r)$ subject to the boundary conditions (3.1.4) for several members of the subclass.

3.4 Solutions for a class of strain-energy functions

We now consider the class of strain-energy functions for which $\bar{W}$ is given in the form

$$\bar{W}(I_1, I_3) = f(I_1)h_1(I_3) + h_2(I_3), \qquad (3.4.1)$$

where f is to be determined using (3.3.15) while the functions h_1, h_2 are to be consistent with (3.2.5). The motivation for considering (3.4.1) is that for strain-energy functions considered previously for which pure azimuthal shear solutions have been found $W(I_1, I_2, I_3)$ is linear in I_1 and I_2 and hence, by (3.2.1), $\bar{W}(I_1, I_2)$

is linear in I_1 while $h_1(I_3)$, $h_2(I_3)$ have very specific forms. Thus, (3.4.1) provides a more general class of strain-energy functions for which pure azimuthal shear might be possible, and, as will be seen below, the special case in which $f(I_1)$ is linear in I_1 enables results of Polignone and Horgan (1994) and Haughton (1993) to be recovered.

We have

$$\bar{W}_1 = f'(I_1)h_1(I_3), \qquad \bar{W}_3 = f(I_1)h_1'(I_3) + h_2'(I_3), \tag{3.4.2}$$
$$\bar{W}_{11} = f''(I_1)h_1(I_3), \quad \bar{W}_{13} = f'(I_1)h_1'(I_3), \quad \bar{W}_{33} = f(I_1)h_1''(I_3) + h_2''(I_3),$$

where the prime on function f indicates differentiating with respect to I_1, and the prime on functions h_i, $(i = 1,\ 2)$ differentuation with respect to I_3.

Without loss of generality we take $h_1(1) = 1$. Then, with (3.4.2), (3.2.5) gives

$$f(3) + h_2(1) = 0,\ f'(3) = \frac{1}{2}\mu,\ f(3)h_1'(1) + h_2'(1) = -\frac{1}{2}\mu,$$
$$f''(3) + \mu h_1'(1) + f(3)h_1''(1) + h_2''(1) = \frac{1}{4}\kappa + \frac{1}{3}\mu, \tag{3.4.3}$$

while (3.3.15) yields

$$2(I_1 - 1)f'(I_1) + 2kf(I_1) + 4h_2'(1) - h_2(1) = 0, \tag{3.4.4}$$

where the constant k is defined by

$$2k = 4h_1'(1) - 1. \tag{3.4.5}$$

By absorbing the constant particular solution of (3.4.4) for $f(I_1)$ times $h_1(I_3)$ into $h_2(I_3)$, we may, without loss of generality, set

$$4h_2'(1) - h_2(1) = 0. \tag{3.4.6}$$

It then follows that, in order to satisfy (3.4.4) and $(3.4.3)_1$, $f(I_1)$ must have the form

$$f(I_1) = -\frac{\mu}{k}2^k(I_1 - 1)^{-k}, \tag{3.4.7}$$

from which the values of $f(3)$, $f''(3)$ may be read off.

From (3.4.3), (3.4.5) and (3.4.6) we then have

$$\begin{aligned} h_2(1) &= \frac{\mu}{k}, \qquad h_2'(1) = \frac{\mu}{4k}, \\ h_2''(1) - \frac{\mu}{k}h_1''(1) &= \frac{1}{4}\kappa + \frac{1}{3}\mu - \frac{1}{4}\mu k, \end{aligned} \tag{3.4.8}$$

in terms of the parameter k.

The differential equation (3.3.6) may now be written as

$$\hat{W}'(\gamma) \equiv \mu 2^{k+1}\gamma(2+\gamma^2)^{-(k+1)} = \frac{b^2\tau}{r^2}. \tag{3.4.9}$$

It is easy to show that (3.3.7) holds for all γ if and only if

$$k \leq -\frac{1}{2}, \tag{3.4.10}$$

while, if $2k+1 > 0$ the inequality (3.3.7) only holds for $\gamma^2 < 2/(2k+1)$. Here we restrict attention to values of k satisfying (3.4.10) so that γ is determined uniquely as a function of r, but it may be appropriate to relax this constraint in order to examine shear softening effects. Therefore, pure azimuthal shear is applicable only for a very restricted class of strain-energy functions in the form (3.4.1), which may now be written

$$\bar{W}(I_1, I_3) = -\frac{\mu}{k}2^k(I_1 - 1)^{-k}h_1(I_3) + h_2(I_3), \tag{3.4.11}$$

with $h_1(1) \equiv 1$ and equations (3.4.8) and (3.4.10) holding.

In what follows we obtain solutions for $g(r)$ from (3.4.9) with $\gamma = rg'(r)$ for specific values of k.

Case(i): $k = -\frac{1}{2}$

With $h_1'(1) = 0$ this case yields

$$\bar{W}(I_1, I_3) = \sqrt{2}\mu(I_1 - 1)^{\frac{1}{2}} + h_2(I_3), \tag{3.4.12}$$

where

$$h_2(1) = -2\mu, \quad h_2'(1) = -\frac{1}{2}\mu, \quad h_2''(1) = \frac{1}{4}\kappa + \frac{11\mu}{24}. \tag{3.4.13}$$

Equation (3.4.9) simplifies to

$$\gamma(2+\gamma^2)^{-\frac{1}{2}} = \frac{b^2 s}{r^2}, \tag{3.4.14}$$

where the notation

$$s = \frac{\tau}{\mu\sqrt{2}} \tag{3.4.15}$$

is introduced as a dimensionless measure of the shearing stress on $r = b$.

It is easily shown that the solution of (3.4.14) for $g(r)$ satisfying the boundary conditions (3.1.4) is

$$g(r) = \frac{1}{\sqrt{2}}\left[\sin^{-1}(\eta s) - \sin^{-1}\left(\frac{b^2 s}{r^2}\right)\right] \tag{3.4.16}$$

and hence

$$\psi \equiv g(b) = \frac{1}{\sqrt{2}}[\sin^{-1}(\eta s) - \sin^{-1} s], \tag{3.4.17}$$

where $\eta = b^2/a^2$. We note that (3.4.17) has limited validity in that it yields a real value of ψ only if $|s| \leq \eta^{-1}$, and hence an upper bound is placed on the permissible values of the shearing stress, i.e

$$|\tau| \leq \sqrt{2}\mu\frac{a^2}{b^2}. \tag{3.4.18}$$

In this sense the strain-energy function (3.4.12) has limited utility. Numerical illustration of the relationship between ψ and s will be provided below.

Case(ii): $k = -1$

In this case we have

$$\bar{W}(I_1, I_3) = \frac{1}{2}\mu(I_1 - 1)h_1(I_3) + h_2(I_3), \tag{3.4.19}$$

with

$$h_1'(1) = -\frac{1}{4}, \quad h_2(1) = -\mu, \quad h_2'(1) = -\frac{1}{4}\mu, \tag{3.4.20}$$

$$\mu h_1''(1) + h_2''(1) = \frac{1}{4}\kappa + \frac{7\mu}{12}. \tag{3.4.21}$$

Equation (3.4.9) yields

$$\mu\gamma = \frac{b^2\tau}{r^2} \tag{3.4.22}$$

and the solution for $g(r)$ satisfying (3.1.4) is then simply

$$g(r) = \frac{s}{\sqrt{2}}\left(\eta - \frac{b^2}{r^2}\right), \tag{3.4.23}$$

so that

$$\psi \equiv g(b) = \frac{s}{\sqrt{2}}(\eta - 1), \tag{3.4.24}$$

s again being defined by (3.4.15).

Equation (3.4.22) is precisely the result obtained in the incompressible theory for the neo-Hookean (or Mooney-Rivlin) form of strain-energy function, ψ being linear in s. Equation (3.4.19) is a compressible version of the neo-Hookean strain-energy for plane deformations. The solution (3.4.23) is valid for any functions $h_1(I_3)$, $h_2(I_3)$ satisfying $h_1(1) = 1$ and the conditions (3.4.20) and (3.4.21). For plane deformations (3.4.19) includes the specialized forms of strain-energy functions due to

Burgess and Levinson (1971), Blatz and Ko (1962) and Agarwal (1979) which were discussed by Haughton (1993) and Polignone and Horgan (1994) for the problem under consideration.

Case(iii): $k = -\frac{3}{2}$

Here we have

$$\bar{W}(I_1, I_3) = \frac{\mu}{3\sqrt{2}}(I_1 - 1)^{\frac{3}{2}} h_1(I_3) + h_2(I_3), \tag{3.4.25}$$

with (3.4.7) appropriately specialized as

$$\begin{aligned} &h_1'(1) = -\frac{1}{2}, \quad h_2(1) = -\frac{2}{3}\mu, \quad h_2'(1) = -\frac{1}{6}\mu, \\ &\frac{2}{3}\mu h_1''(1) + h_2''(1) = \frac{1}{4}\kappa + \frac{17\mu}{24}, \end{aligned}$$

and (3.4.9) becomes

$$\gamma(2+\gamma^2)^{\frac{1}{2}} = \frac{2b^2 s}{r^2}. \tag{3.4.26}$$

Equation (3.4.26) coupled with the boundary conditions (3.1.4) can be solved to give

$$g(r) = m(a) - m(r) + \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{m(r)}{\sqrt{2}}\right] - \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{m(a)}{\sqrt{2}}\right], \tag{3.4.27}$$

where $m(r)$ is defined as

$$m(r) = \left[\left(1 + \frac{4b^4 s^2}{r^4}\right)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}}, \tag{3.4.28}$$

and

$$\psi \equiv g(b) = m(a) - m(b) + \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{m(b)}{\sqrt{2}}\right] - \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{m(a)}{\sqrt{2}}\right]. \tag{3.4.29}$$

Case(iv): $k = -\frac{3}{4}$

For this case we have

$$\bar{W}(I_1, I_3) = \frac{2^{\frac{5}{4}}}{3}\mu\,(I_1 - 1)^{\frac{3}{4}}\,h_1(I_3) + h_2(I_3), \tag{3.4.30}$$

with

$$h_1'(1) = -\frac{1}{8}, \quad h_2(1) = -\frac{4}{3}\mu, \quad h_2'(1) = -\frac{1}{3}\mu,$$
$$\frac{3}{4}\mu h_1''(1) + h_2''(1) = \frac{1}{4}\kappa + \frac{25\mu}{48},$$

and equation (3.4.9) becomes

$$\gamma(2+\gamma^2)^{-\frac{1}{4}} = 2^{\frac{1}{4}}\frac{b^2 s}{r^2}. \tag{3.4.31}$$

The required solution of (3.4.31) with (3.1.4) is

$$g(r) = \frac{\sqrt{2}}{4}\left[\tan^{-1} n(r) - \tan^{-1} n(a) + \frac{1}{n(a)} - \frac{1}{n(r)}\right], \tag{3.4.32}$$

where

$$n(r) = \left\{\left[\left(1 + \frac{4r^8}{b^8 s^4}\right)^{\frac{1}{2}} - 1\right]/2\right\}^{\frac{1}{2}}, \tag{3.4.33}$$

and then

$$\psi \equiv g(b) = \frac{\sqrt{2}}{4}\left[\tan^{-1} n(b) - \tan^{-1} n(a) + \frac{1}{n(a)} - \frac{1}{n(b)}\right]. \tag{3.4.34}$$

3.5 A second class of strain-energy functions

Here we consider an alternative representation for the strain-energy function W in terms of the principal invariants i_1, i_2, i_3 of the stretch tensor $\mathbf{V} = \mathbf{B}^{\frac{1}{2}}$. In general, I_1, I_2, I_3 are given in terms of i_1, i_2, i_3 by

$$I_1 = i_1^2 - 2i_2, \quad I_2 = i_2^2 - 2i_1 i_3, \quad I_3 = i_3^2. \tag{3.5.1}$$

For the plane deformation considered here these may be reduced using (3.2.1) and, analogously to (3.2.1), we have

$$i_2 = i_1 + i_3 - 1. \tag{3.5.2}$$

Thus, we may write

$$\bar{W}(I_1, I_3) = \tilde{W}(i_1, i_3), \tag{3.5.3}$$

and then we have

$$\begin{aligned}
&\tilde{W}_1 = 2(i_1 - 1)\bar{W}_1, \qquad \tilde{W}_3 = -2\bar{W}_1 + 2i_3\bar{W}_3,\\
&\tilde{W}_{11} = 2\bar{W}_1 + 4(i_1 - 1)^2\bar{W}_{11}, \quad \tilde{W}_{13} = 4(i_1 - 1)(-\bar{W}_{11} + i_3\bar{W}_{13}),\\
&\tilde{W}_{33} = 4\bar{W}_{11} - 8i_3\bar{W}_{13} + 4i_3^2\bar{W}_{33} + 2\bar{W}_3,
\end{aligned} \tag{3.5.4}$$

where $\tilde{W}_p = \partial\tilde{W}/\partial i_p$, $p = 1, 3$ and $\tilde{W}_{pq} = \partial^2\tilde{W}/\partial i_p \partial i_q$, $p,\ q = 1, 3$. Therefore, $\tilde{W}(i_1, i_3)$ must satisfy

$$\begin{aligned}
&\tilde{W}(3,1) = 0, \quad \tilde{W}_1(3,1) = -\tilde{W}_3(3,1) = 2\mu,\\
&\tilde{W}_{11}(3,1) + 2\tilde{W}_{13}(3,1) + \tilde{W}_{33}(3,1) = \kappa + \frac{4}{3}\mu,
\end{aligned} \tag{3.5.5}$$

from equation (3.2.5).

In terms of $\tilde{W}$ the condition (3.3.15) becomes

$$(i_1 - 1)\tilde{W}_1(i_1, 1) + 2\tilde{W}_3(i_1, 1) - \tilde{W}(i_1, 1) = 0. \tag{3.5.6}$$

Making use of equations (3.3.1) and (3.5.6), $\hat{W}$ can, here, be expressed

$$\hat{W}(\gamma) = \tilde{W}(1 + \sqrt{4 + \gamma^2}, 1). \tag{3.5.7}$$

This prompts consideration of the class of strain-energy functions defined by

$$\tilde{W} = \tilde{f}(i_1)\tilde{h}_1(i_3) + \tilde{h}_2(i_3), \tag{3.5.8}$$

in parallel with (3.4.1). Without loss of generality we also take $\tilde{h}_1(1) = 1$. Equation (3.5.5) then gives

$$\tilde{f}(3) + \tilde{h}_2(1) = 0, \quad \tilde{f}'(3) = 2\mu, \quad \tilde{f}(3)\tilde{h}_1'(1) + \tilde{h}_2'(1) = -2\mu,$$
$$\tilde{f}''(3) + 4\mu\tilde{h}_1'(1) + \tilde{f}(3)\tilde{h}_1''(1) + \tilde{h}_2''(1) = \kappa + \frac{4}{3}\mu. \tag{3.5.9}$$

Equation (3.5.6) yields

$$(i_1 - 1)\tilde{f}'(i_1) + \tilde{k}\tilde{f}(i_1) + 2\tilde{h}_2' - \tilde{h}_2 = 0, \tag{3.5.10}$$

where $\tilde{k} = 2\tilde{h}_1' - 1$.

Following the procedure used in Section 3.4 we set

$$2\tilde{h}_2' - \tilde{h}_2 = 0, \tag{3.5.11}$$

and obtain

$$\tilde{f}(i_1) = -\frac{4\mu}{\tilde{k}}2^{\tilde{k}}(i_1 - 1)^{-\tilde{k}}, \tag{3.5.12}$$

with

$$\tilde{h}_1(1) = 1, \qquad \tilde{h}_1'(1) = \frac{1}{2}(\tilde{k} + 1),$$
$$\tilde{h}_2(1) = \frac{4\mu}{\tilde{k}}, \qquad \tilde{h}_2'(1) = \frac{2\mu}{\tilde{k}},$$
$$-\frac{4\mu}{\tilde{k}}\tilde{h}_1''(1) + \tilde{h}_2''(1) = \kappa + \frac{4}{3}\mu - (\tilde{k} + 1)\mu. \tag{3.5.13}$$

Using equation (3.5.7) the equilibrium equation (3.3.6) becomes

$$\hat{W}'(\gamma) \equiv \mu 2^{\tilde{k}+2}\gamma(4 + \gamma^2)^{-(\tilde{k}+2)/2} = \frac{b^2\tau}{r^2}, \tag{3.5.14}$$

and (3.3.7) is satisfied for all γ if and only if

$$\tilde{k} \leq -1, \tag{3.5.15}$$

while if $\tilde{k}+1>0$ (3.3.7) requires $\gamma^2<4/(\tilde{k}+1)$. Therefore, pure azimuthal shear is applicable only for a very restricted form of strain-energy function in the form (3.4.1), namely

$$\tilde{W}(i_1,i_3)=-\frac{4\mu}{\tilde{k}}2^{\tilde{k}}(i_1-1)^{-\tilde{k}}\tilde{h}_1(i_3)+\tilde{h}_2(i_3), \tag{3.5.16}$$

with $\tilde{h}_1(1)=1$ and equations (3.5.13) and (3.5.15) holding.

If $\tilde{k}=-1$ then (3.5.16) becomes

$$\tilde{W}(i_1,i_3)=2\mu(i_1-1)\tilde{h}_1(i_3)+\tilde{h}_2(i_3), \tag{3.5.17}$$

with (3.5.13) appropriately specialized, and (3.5.14) reduces to

$$\gamma(4+\gamma^2)^{-\frac{1}{2}}=\frac{b^2s}{\sqrt{2}r^2}, \tag{3.5.18}$$

where s is again defined by (3.4.17).

Equation (3.5.18) is solved for $g(r)$ using $\gamma=rg'(r)$ to give[1]

$$g(r)=\sin^{-1}\left(\frac{\eta s}{\sqrt{2}}\right)-\sin^{-1}\left(\frac{b^2s}{\sqrt{2}r^2}\right), \tag{3.5.19}$$

and the twist-shearing stress relationship is therefore

$$\psi\equiv g(b)=\sin^{-1}\left(\frac{\eta s}{\sqrt{2}}\right)-\sin^{-1}\left(\frac{s}{\sqrt{2}}\right). \tag{3.5.20}$$

The solution (3.5.19) is equivalent to that obtained by Haughton (1993) for the compressible Varga form of strain-energy function, for which $\tilde{h}_1(i_3)\equiv 1$.

Equation (3.5.19) should be compared with (3.4.16) arising in Case (i) in Section 4. As with (3.4.16) the solution (3.5.19) is restricted to a finite range of values of s, in this case $s\le\sqrt{2}\eta^{-1}$.

[1]Comparing (3.5.19) with (3.4.16) shows that (3.5.19) may be obtained from (3.4.16) by making the transformation $g\to g/\sqrt{2}$, $s\to s/\sqrt{2}$ in (3.4.16).

If $\tilde{k} = -2$ equation (3.5.14) yields, apart from differences in notation, the same solution as in Case (ii) in Section 4 and (3.4.1) and (3.5.8) represent the same (compressible neo-Hookean) form of strain-energy function.

Solutions for certain other values of k may also be obtained explicitly. It suffices here to mention briefly two such solutions.

For $\tilde{k} = -\frac{3}{2}$ we have

$$\tilde{f}(i_1) = \frac{2\sqrt{2}}{3}\mu\,(i_1 - 1)^{\frac{3}{2}}, \tag{3.5.21}$$

and the differential equation (3.5.14) may be written

$$\gamma(4 + \gamma^2)^{-\frac{1}{4}} = \frac{sb^2}{r^2}. \tag{3.5.22}$$

Comparison of (3.5.22) with (3.4.31) shows that the solution of (3.5.22) may be obtained from (3.4.32) with (3.4.33) by making the transformations $g \to g/\sqrt{2}$, $s \to s/\sqrt{2}$ in those equations. Then, we have

$$g(r) = \frac{1}{2}\left[\tan^{-1} n_1(r) - \tan^{-1} n_1(a) + \frac{1}{n_1(a)} - \frac{1}{n_1(r)}\right], \tag{3.5.23}$$

where

$$n_1(r) = \left\{\left[\left(1 + \frac{16r^8}{b^8 s^4}\right)^{\frac{1}{2}} - 1\right]/2\right\}^{\frac{1}{2}}, \tag{3.5.24}$$

and

$$\psi \equiv g(b) = \frac{1}{2}\left[\tan^{-1} n_1(b) - \tan^{-1} n_1(a) + \frac{1}{n_1(a)} - \frac{1}{n_1(b)}\right], \tag{3.5.25}$$

Similarly, for $\tilde{k} = -3$ we have

$$\tilde{f}(i_1) = \frac{1}{6}\mu\,(i_1 - 1)^3, \tag{3.5.26}$$

and (3.5.14) becomes

$$\gamma(4+\gamma^2)^{\frac{1}{2}} = 2\sqrt{2}\frac{sb^2}{r^2}. \tag{3.5.27}$$

Comparison of (3.5.27) with (3.4.26) enables the solution of (3.5.27) to be read off from (3.4.27) with (3.4.28) again by using the transformations $g \to g/\sqrt{2}$, $s \to s/\sqrt{2}$. Thus,

$$g(r) = \sqrt{2}(m_1(a) - m_1(r)) + \tan^{-1}\left[\frac{m_1(r)}{\sqrt{2}}\right] - \tan^{-1}\left[\frac{m_1(a)}{\sqrt{2}}\right], \tag{3.5.28}$$

where $m_1(r)$ is defined as

$$m_1(r) = \left[\left(1 + \frac{2b^4s^2}{r^4}\right)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}}, \tag{3.5.29}$$

and

$$\psi \equiv g(b) = \sqrt{2}[m_1(a) - m_1(b)] + \tan^{-1}\left[\frac{m_1(b)}{\sqrt{2}}\right] - \tan^{-1}\left[\frac{m_1(a)}{\sqrt{2}}\right]. \tag{3.5.30}$$

More generally, the transformations $k \to \tilde{k}/2$, $\gamma \to \gamma/\sqrt{2}$, $\tau \to \tau/\sqrt{2}$ take equation (3.4.9) into equation (3.5.14), thus establishing a direct relationship between the solutions of equations (3.4.11) and (3.5.16) for all k in the range of values under consideration.

In Sections 3.4 and 3.5 we have obtained solutions by considering two distinct pairs of deformation invariants, namely (in plane strain) (I_1, I_3) and (i_1, i_3). Further solutions may be obtained by using different pairs of invariants or the stretches, λ_1, λ_2 say, with $\lambda_3 = 1$. Note that a suitable starting point is to recast (3.3.15), or equivalently (3.5.6), in terms of the stretches. Thus, we may write

$$\bar{W}(I_1, I_3) = w(\lambda_1, \lambda_2, 1), \tag{3.5.31}$$

with $\lambda_1\lambda_2 = 1$. Then we have $I_1 = \lambda_1^2 + \lambda_2^2 + 1,\ I_3 = 1$ and

$$w_1 = 2\lambda_1 \bar{W}_1 + 2\lambda_2 \bar{W}_3, \qquad w_2 = 2\lambda_2 \bar{W}_1 + 2\lambda_1 \bar{W}_3, \tag{3.5.32}$$

where $w_1 = \partial w/\partial\lambda_1,\ w_2 = \partial w/\partial\lambda_2$. It follows that

$$\lambda_1 w_1 + \lambda_2 w_2 = 2(\lambda_1^2 + \lambda_2^2)\bar{W}_1 + 4\bar{W}_3. \tag{3.5.33}$$

This leads to

$$\lambda_1 \frac{\partial w}{\partial\lambda_1} + \lambda_2 \frac{\partial w}{\partial\lambda_2} - w = 0, \tag{3.5.34}$$

evaluated for $\lambda_1\lambda_2 = 1$ from equation (3.3.15).

In Figure 3.1 and 3.2 the angle of twist ψ is plotted against the dimensionless azimuthal shear stress s (> 0) applied to the boundary $r = b$ for each of the solutions obtained in Sections 3.4 and 3.5 in order to compare the predictions of the different strain-energy functions. Results are shown for a relatively thin-walled annulus with $\eta(= b^2/a^2) = \sqrt{2}$ in Figure 3.1 and for a relatively thick-walled annulus with $\eta(= b^2/a^2) = 4$ in Figure 3.2. For each strain-energy function considered we have $ds/d\psi > 0$, which is consistent with the assumption (3.3.7), while it can be seen that $d^2s/d\psi^2$ is either positive, negative or zero for all s, subject to the restrictions $|s| \leq \eta^{-1}(\sqrt{2}\eta^{-1})$ for $k = -\frac{1}{2}$ ($\tilde{k} = -1$). As Figure 3.1 and Figure 3.2 illustrate larger values of η admit larger angles of twist for a given value of s.

3.6 The incompressible problem

Results for incompressible materials may be deduced from those for pure azimuthal shear in a compressible material by considering the incompressible material to have

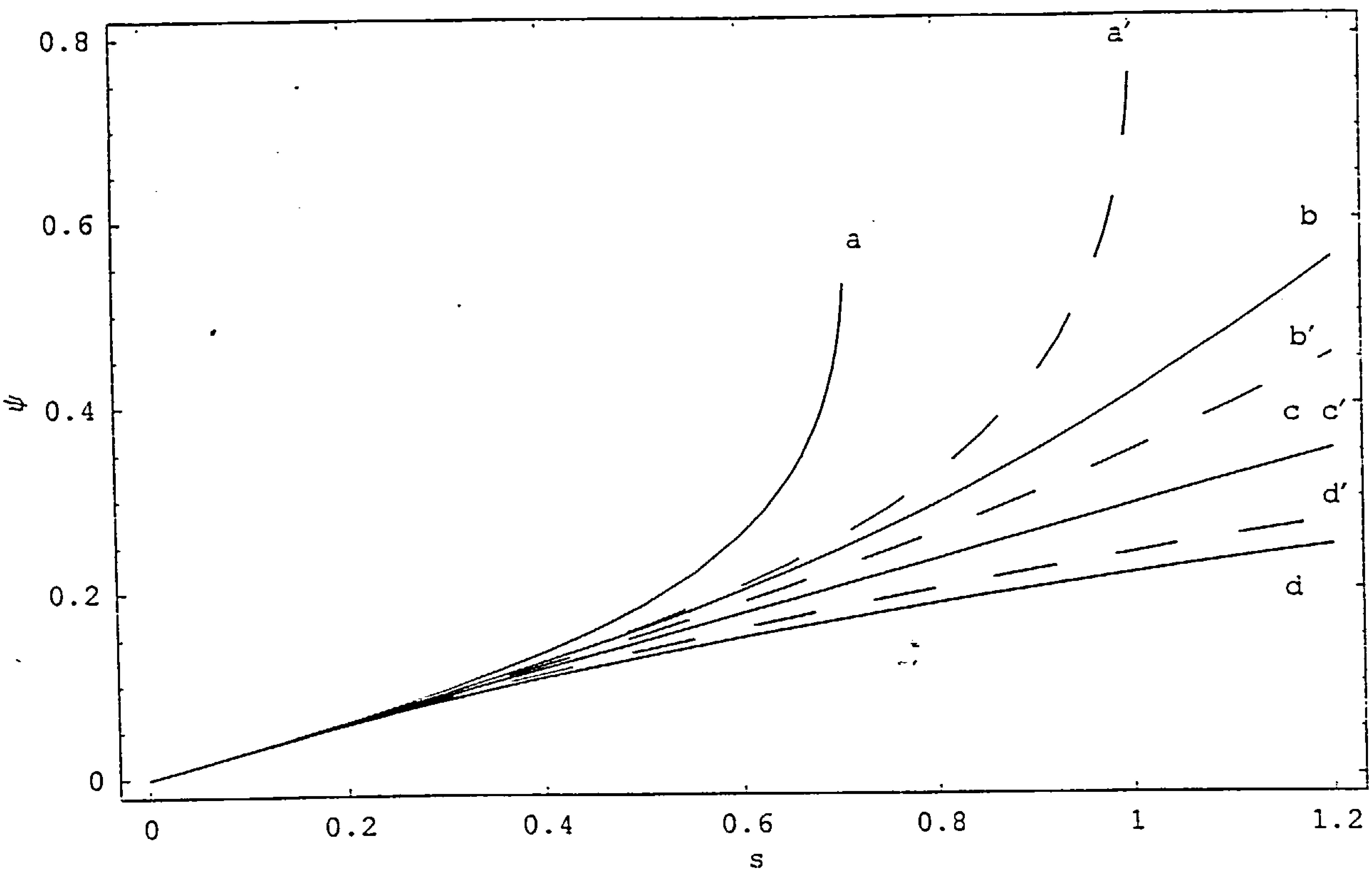


Figure 3.1: Plot of the angle of twist ψ against the dimensionless shear stress s for the strain-energy functions with $k = -\frac{1}{2}, -\frac{3}{4}, -1, -\frac{3}{2}$ (a, b, c, d respectively) and $\bar{k} = -1, -\frac{3}{2}, -2, -3$ (a', b', c', d' respectively) with $\eta = \sqrt{2}$.

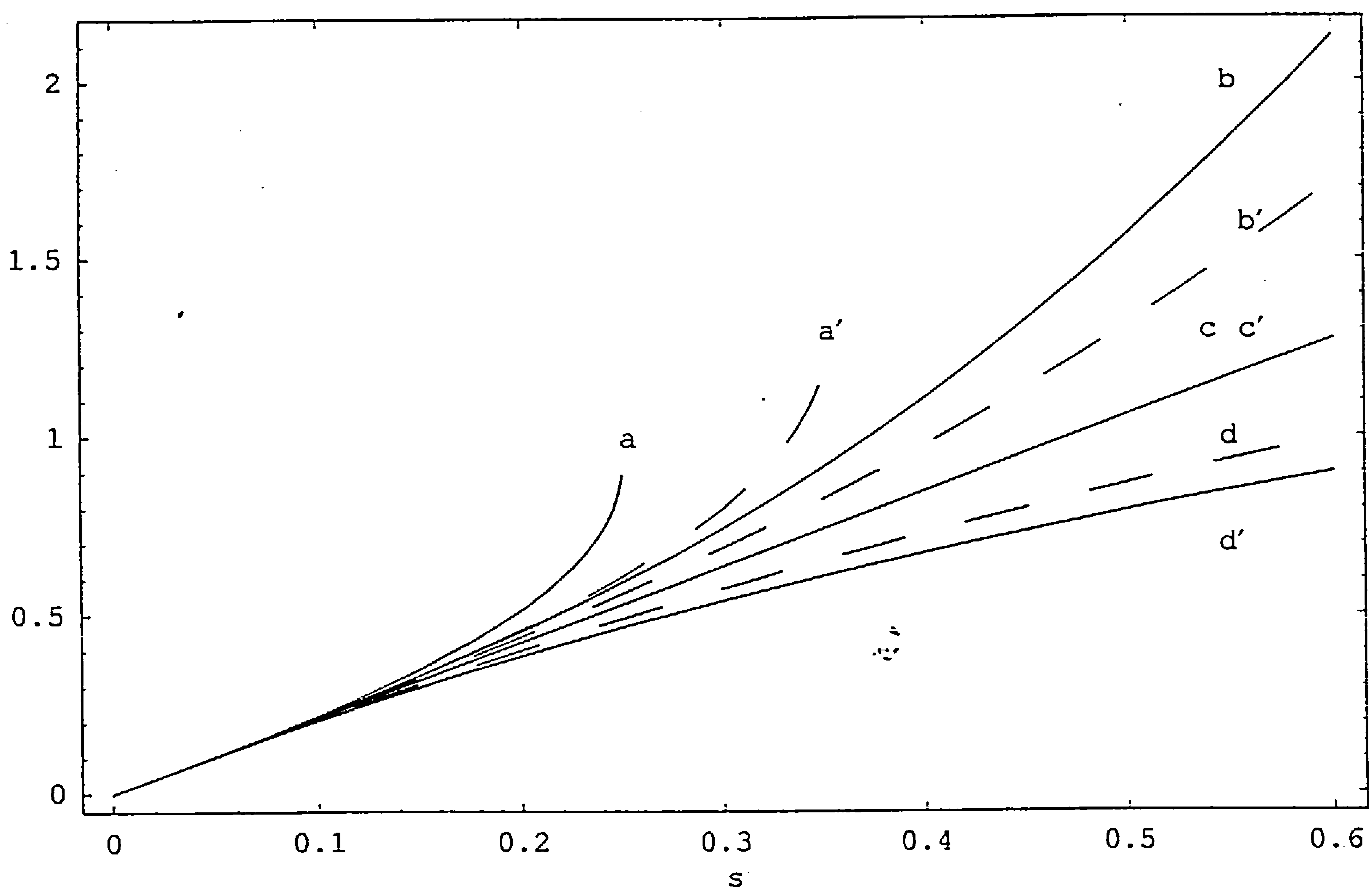


Figure 3.2: Plot of the angle of twist ψ against the dimensionless shear stress s for the strain-energy functions with $k = -\frac{1}{2}, -\frac{3}{4}, -1, -\frac{3}{2}$ (a, b, c, d respectively) and $\tilde{k} = -1, -\frac{3}{2}, -2, -3$ (a', b', c', d' respectively) with $\eta = 4$.

strain-energy function (in plane strain), w say, defined by

$$w(I_1) = \bar{W}(I_1, 1), \tag{3.6.1}$$

where $\bar{W}(I_1, I_3)$ is defined by (3.2.2).

Correspondingly, in terms of the amount of shear γ, we define $\hat{w}(\gamma)$ by

$$\hat{w}(\gamma) = w(3 + \gamma^2). \tag{3.6.2}$$

Then, equation (3.3.6), which serves to determine γ, is unchanged but now written

$$\hat{w}'(\gamma) = \frac{b^2\tau}{r^2} \tag{3.6.3}$$

(see also Ogden(1978,1984)).

For an incompressible material the Cauchy stress tensor $\mathbf{T}$ in (3.2.9) is replaced by

$$\mathbf{T} = 2w'(I_1)\mathbf{B} - p\mathbf{I}, \tag{3.6.4}$$

where p is the arbitrary hydrostatic pressure associated with the incompressibility constraint. With $\mathbf{B} = \mathbf{A}\mathbf{A}^T$ calculated from (3.1.5) for $r = R$, the radial equation of equilibrium may be written

$$r\frac{dT_{rr}}{dr} = T_{\theta\theta} - T_{rr} \equiv \gamma\hat{w}'(\gamma). \tag{3.6.5}$$

The role of this equation in the incompressible theory is different from that of its counterpart (3.3.2) in the compressible theory. Equation (3.6.5) serves to determine T_{rr} (or, equivalently, p) once γ is found using (3.6.3).

Solutions of equations (3.6.3) and (3.6.5) involve no restriction on the form of strain-energy function other than that imposed by the incompressibility constraint and the inequality (3.3.7). But, by adapting the strain-energy functions discussed in Sections 3.4 and 3.5 to the incompressible situation the solution obtained there are seen to be equally valid for incompressible materials. To illustrate this point, we take

$$w(I_1) = f(I_1) - f(3) \tag{3.6.6}$$

with $f(I_1)$ given by (3.4.7). Hence

$$\hat{w} = \frac{\mu}{k}\left[1 - 2^k(2+\gamma^2)^{-k}\right], \tag{3.6.7}$$

and

$$\hat{w}' \equiv \mu 2^{k+1}\gamma(2+\gamma^2)^{-(k+1)} = \frac{b^2\tau}{r^2} \tag{3.6.8}$$

as in (3.4.9).

Similarly, in view of the connection $I_1 = i_1^2 - 2i_1$ obtained from (3.5.1) and (3.5.2) with $i_3 = 1$, we may consider

$$w(i_1) = \tilde{f}(i_1) - \tilde{f}(3) \tag{3.6.9}$$

with (3.5.12) and obtain

$$\hat{w} = \frac{4\mu}{\tilde{k}}\left[1 - 2^{\tilde{k}}(2+\gamma^2)^{-\tilde{k}/2}\right]. \tag{3.6.10}$$

Then we have

$$\hat{w}' \equiv \mu 2^{\tilde{k}+2}\gamma(2+\gamma^2)^{-(\tilde{k}+2)/2} = \frac{b^2\tau}{r^2} \tag{3.6.11}$$

as in (3.5.14).

The solutions given in Sections 3.4 and 3.5 for specific values of k and $\tilde{k}$ can now be applied in the incompressible situation and equation (3.6.5) may be used to calculate the stress distribution. Details are not given here.

It is of interest to relate the present work to that of Tao *el al* (1992), who considered a generalized form of neo-Hookean strain-energy function which we write here as

$$w(I_1) = \frac{\mu}{2\alpha n}\left\{[1+\alpha(I_1-3)]^n - 1\right\}, \tag{3.6.12}$$

where α and n are constants and μ has the same interpretation in (3.1.15). In respect of (3.6.12) equation (3.6.3) becomes

$$\hat{w}'(\gamma) \equiv \mu\gamma(1+\alpha\gamma^2)^{n-1} = \frac{b^2\tau}{r^2}. \tag{3.6.13}$$

Results for the strain-energy function (3.6.12) can be read off from those for (3.6.6), for which (3.6.3) becoms

$$\hat{w}'(\gamma) \equiv \mu 2^{k+1}\gamma(2+\gamma^2)^{-(k+1)} = \frac{b^2\tau}{r^2}, \tag{3.6.14}$$

by making the transformations $k \to -n$, $\gamma \to \sqrt{2\alpha\gamma}$, $\tau \to \sqrt{2\alpha\gamma}$. Specific solutions were obtained in Tao *el al* (1992) for $n = 1.5,\ 1,\ 0.75,\ 0.5$, and these correspond to the four values of k considered in Section 4.

The approach adopted here of deducing results for incompressible materials from solutions for isochoric deformations in compressible materials clearly has wider applicability than for the azimutal shear problem considered here. The method enables specific strain-energy functions for which solutions can be found to be identified.

Chapter 4

Bending deformation of a rectangular block

4.1 Preliminaries

We consider the finite bending deformation of an isotropic compressible nonlinearly elastic rectangular block defined by

$$-A \leq X_1 \leq A, \quad -B \leq X_2 \leq B, \quad -C \leq X_3 \leq C \tag{4.1.1}$$

in some unstressed reference configuration, where (X_1, X_2, X_3) are rectangular Cartestian coordinates. We suppose that the block is deformed into a sector of a circular cylindrical tube, and the deformation is assumed to be symmetric about the X_1-axis and defined by

$$r = f(X_1), \quad \theta = g(X_2), \quad z = \lambda X_3, \tag{4.1.2}$$

where λ is a constant, (r, θ, z) are cylindrical polar coordinates associated with the deformed configuration, and $g(-X_2) = -g(X_2)$.

Since the deformation may be achieved by subjecting the outer surface to a prescribed bending displacement we may write

$$\begin{aligned} f(-A) &= a, \quad f(A) = b, \\ g(-B) &= -\psi, \quad g(B) = \psi, \end{aligned} \tag{4.1.3}$$

where a, b and ψ are constants and $b > a$. We take (4.1.3) as the boundary conditions for the deformation (4.1.2) in this thesis.

Referred to a reference rectangular Cartesian basis $\{\mathbf{E}_i\}$ and a current cylindrical polar basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$, the deformation gradient tensor $\mathbf{A}$ has components

$$\mathbf{A} = \begin{bmatrix} f' & 0 & 0 \\ 0 & fg' & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \tag{4.1.4}$$

where the primes attached to f and g indicate differentiation with respect to X_1 and X_2 respectively.

Noting equation (2.1.9), the deformation gradient can be represented uniquely in polar decomposed form, so that

$$\mathbf{A} = \mathbf{U}, \quad \mathbf{R} = \mathbf{I} \tag{4.1.5}$$

and the principal stretches are therefore given by

$$\lambda_1 = f'(X_1), \quad \lambda_2 = f(X_1)g'(X_2), \quad \lambda_3 = \lambda. \tag{4.1.6}$$

Let $t_1^{(1)}$, $t_2^{(1)}$, $t_3^{(1)}$ denote the principal values of the Biot stress $\mathbf{T}^{(1)}$ defined by (2.1.29). Using equation (2.1.30) we may write the nominal stress as

$$\mathbf{S} = t_1^{(1)}\,\mathbf{E}_1 \otimes \mathbf{e}_r + t_2^{(1)}\,\mathbf{E}_2 \otimes \mathbf{e}_\theta + t_3^{(1)}\,\mathbf{E}_3 \otimes \mathbf{e}_z, \tag{4.1.7}$$

where

$$\mathbf{e}_r = \cos\theta\ \mathbf{E}_1 + \sin\theta\ \mathbf{E}_2, \quad \mathbf{e}_\theta = -\sin\theta\ \mathbf{E}_1 + \cos\theta\ \mathbf{E}_2, \quad \mathbf{e}_z = \mathbf{E}_3, \tag{4.1.8}$$

with $\theta = g(X_2)$.

On substitution of (4.1.7) with (4.1.8) into (2.1.37), the equilibrium equation (2.1.37) in the absence of body forces reduces to the two scalar equations

$$\frac{\partial t_1^{(1)}}{\partial X_1} - t_2^{(1)} g'(X_2) = 0, \tag{4.1.9}$$

$$\frac{\partial t_2^{(1)}}{\partial X_2} = 0. \tag{4.1.10}$$

(see, for example, Ogden 1984).

Noting that λ_1 is independent of X_2, we see from (4.1.10) that

$$\frac{\partial t_2^{(1)}}{\partial \lambda_2} f(X_1) g''(X_2) = 0.$$

To avoid trivial cases, we make the assumption that $f(X_1) \neq 0$ and $\partial t^{(1)}/\partial\lambda_2 \neq 0$. It follows that $g''(X_2) = 0$, which leads to

$$g(X_2) = kX_2, \tag{4.1.11}$$

where k is a constant of integration, so that the equation (4.1.10) holds automatically.

Therefore, the principal stretches can be rewritten as

$$\lambda_1 = f'(X_1), \quad \lambda_2 = kf(X_1), \quad \lambda_3 = \lambda. \tag{4.1.12}$$

From equations (4.1.11) and (4.1.9) respectively we deduce that

$$\lambda_2' = k\lambda_1, \qquad \frac{\partial t_1^{(1)}}{\partial X_1} = kt^{(2)}. \tag{4.1.13}$$

4.2 Equilibrium equation in terms of the strain-energy function

In this section we consider the strain-energy function per unit reference volume of a compressible isotropic elastic material, which is given by

$$W = W(\mathbf{U}) = W(\lambda_1, \lambda_2, \lambda_3), \tag{4.2.1}$$

where the function W must satisfy the conditions (2.1.49), (2.1.53), (2.1.54) and (2.1.55).

Recalling equations (4.1.12), λ_1, λ_2 are functions of X_1 only, and λ_3 is a constant. It follows that the strain-energy function W may be expressed as a function of X_1. From equation (2.1.50) we have $t_i^{(1)} = \partial W/\partial \lambda_i$, and equation (4.1.9) with (4.1.11) therefore leads to

$$\frac{dW_1}{dX_1} - kW_2 = 0, \tag{4.2.2}$$

where $W_i = \partial W/\partial \lambda_i$. Then we have the result that *a rectangular block of homogeneous compressible isotropic hyperelastic material with strain-energy function* $W = W(\lambda_1, \lambda_2, \lambda_3)$ *can undergo a deformation of the form (4.1.2) with (4.1.11) if and only if the equation*

$$W(X_1) = \lambda_1(X_1)W_1(X_1) \tag{4.2.3}$$

holds, where $W_1 = \partial W/\partial \lambda_1$. We now establish the results (4.2.3).

Since $W = W(\lambda_1, \lambda_2, \lambda_3)$ depends on X_1 only, it may be differentiated to give

$$\frac{dW}{dX_1} = W_1\frac{d\lambda_1}{dX_1} + W_2\frac{d\lambda_2}{dX_1}. \tag{4.2.4}$$

From equation (4.1.13) we have

$$k\lambda_1 = \frac{d\lambda_2}{dX_1}. \tag{4.2.5}$$

Then, equation (4.2.4) reduces to

$$\frac{dW}{dX_1} = W_1\frac{d\lambda_1}{dX_1} + k\lambda_1 W_2. \tag{4.2.6}$$

As we know, the deformation (4.1.2) with (4.1.11) is possible if and only if there is a $f(X_1)$ to ensure that equation (4.2.2) holds. Assuming such a function $f(X_1)$ satisfying equation (4.2.2) exists, the latter equation may be rewritten

$$W_2 = \frac{1}{k}\frac{dW_1}{dX_1}. \tag{4.2.7}$$

Substituting (4.2.7) into (4.2.6) then leads to

$$\frac{dW}{dX_1} = W_1\frac{d\lambda_1}{dX_1} + \lambda_1\frac{dW_1}{dX_1}. \tag{4.2.8}$$

Using equation (2.1.53) and (2.1.54) we may integrate (4.2.8) wth respect to X_1 to obtain condition (4.2.3).

Alternatively, assuming equation (4.2.3) holds, it follows on differentiation that equation (4.2.8) holds. Using (4.2.5) we may compare equation (4.2.8) with equation (4.2.6) to obtain equation (4.2.2). Hence, the necessary and sufficient condition that the solution of equation (4.2.2) exists must be that equation (4.2.3) holds. Equation (4.2.3) provides a way for us to derive solutions by solving a first-order nonlinear ordinary differential equation for λ_1 or λ_2 and is a first integral of the equilibrium equation.

Equation (4.2.3) may be re-expressed in a number of different forms. For example, it can be rewritten as

$$W = \lambda_1 t_1^{(1)}. \tag{4.2.9}$$

For incompressible materials, we have $\lambda_1\lambda_2\lambda_3 = 1$, and

$$t_1^{(1)} = \frac{\partial W}{\partial \lambda_1} - p\lambda_1^{-1}, \qquad t_2^{(1)} = \frac{\partial W}{\partial \lambda_2} - p\lambda_2^{-1},$$

where p is an arbitrary hydrostatic pressure. Since equations (4.1.9), (4.1.11) and (4.1.13) hold, we then have

$$\begin{aligned} \frac{d\lambda_1 t_1^{(1)}}{dX_1} &= \frac{d\lambda_1}{dX_1} t_1^{(1)} + \lambda_1 \frac{dt_1^{(1)}}{dX_1} \\ &= \frac{d\lambda_1}{dX_1} t_1^{(1)} + \frac{d\lambda_2}{dX_1} t_2^{(1)} \\ &= \lambda_1'(\frac{\partial W}{\partial \lambda_1} - p\lambda_1^{-1}) + \lambda_2'(\frac{\partial W}{\partial \lambda_2} - p\lambda_2^{-1}) \\ &= \frac{dW}{dX_1}. \end{aligned} \tag{4.2.10}$$

Thus, equation (4.2.9) applies for either compressible or incompressible materials.

An equivalent equation based on the principal invariants $I_1,\ I_2,\ I_3$ given by (2.1.17) may be expressed in terms of the strain-energy function

$$W = \bar{W}(I_1,\ I_2,\ I_3), \tag{4.2.11}$$

where $\bar{W}$ should satisfy (2.1.60) and (2.1.61). The condition (4.2.3) then becomes

$$\bar{W}(X_1) = 2f'^2(X_1)\Big[\bar{W}_1(X_1) + (k^2f^2(X_1)+\lambda^2)\bar{W}_2(X_1) + k^2\lambda^2 f^2(X_1)\bar{W}_3(X_1)\Big], \tag{4.2.12}$$

where $\bar{W}_p,\ p = 1,\ 2,\ 3$, indicates differentiation with respect to I_p evaluated for

$$\begin{aligned} I_1 &= f'^2 + k^2f^2 + \lambda^2, \\ I_2 &= \lambda^2 f'^2 + \lambda^2 k^2 f^2 + k^2 f^2 f'^2, \\ I_3 &= \lambda^2 k^2 f^2 f'^2. \end{aligned} \tag{4.2.13}$$

Similarly, if we regard W as a function of the principal invariants $i_1,\ i_2,\ i_3$ defined by (2.1.16), then we write

$$W = \check{W}(i_1,\ i_2,\ i_3), \tag{4.2.14}$$

where (2.1.57) and (2.1.58) hold. The condition (4.2.3) can then be rewritten as

$$\breve{W}(X_1) = f'(X_1)\left[\breve{W}_1(X_1) + (kf(X_1)+\lambda)\breve{W}_2(X_1) + k\lambda f(X_1)\breve{W}_3(X_1)\right], \quad (4.2.15)$$

evaluated for

$$\begin{aligned} i_1 &= f' + kf + \lambda, \\ i_2 &= \lambda f' + \lambda kf + kff', \\ i_3 &= \lambda kff', \end{aligned} \quad (4.2.16)$$

where the subscripts denote derivatives with respect to $i_1,\ i_2,\ i_3$.

Equations (4.2.12) and (4.2.15) can also be simplified in the case $\lambda = 1$. The deformation (4.1.2) with (4.1.11) is given by

$$r = f(X_1), \quad \theta = kX_2, \quad z = X_3. \quad (4.2.17)$$

For $\lambda = 1$ equations (4.2.13) simplify to

$$\begin{aligned} I_1 &= f'^2 + k^2 f^2 + 1, \\ I_2 &= f'^2 + k^2 f^2 + k^2 f^2 f'^2, \\ I_3 &= k^2 f^2 f'^2. \end{aligned} \quad (4.2.18)$$

It follows immediately, just as for the azimuthal shear problem discussed in Chapter 3, that

$$I_2 = I_1 + I_3 - 1. \quad (4.2.19)$$

The strain energy $W = \bar{W}(I_1, I_2, I_3)$ per unit reference volume of a compressible isotropic elastic material may then be regarded as a function of two invariants, and we define $\widetilde{W}(I_1, I_3)$ by

$$\widetilde{W}(I_1, I_3) = \bar{W}(I_1, I_1 + I_3 - 1, I_3), \quad (4.2.20)$$

as in (3.2.2). In this case, $\widetilde{W}(I_1, I_3)$ must satisfy (3.2.5) and equation (4.2.12) can be simplified to

$$\widetilde{W}(X_1) = 2f'^2(X_1)\widetilde{W}_1(X_1) + 2I_3(X_1)\widetilde{W}_3(X_1), \tag{4.2.21}$$

where the subscripts denote derivatives with respect to I_1, I_3.

Analogously, for $\lambda = 1$ equations (4.2.16) yield

$$i_2 = i_1 + i_3 - 1. \tag{4.2.22}$$

Equation (4.2.22) holds not only for the deformation (4.2.17) but for every plane strain deformation. With the restriction to plane strain only two of the invariants i_1, i_2, i_3 are independent, and the strain energy $W = \check{W}(i_1, i_2, i_3)$ per unit reference volume of a compressible isotropic elastic material may then be regarded as a function of two invariants. Accordingly, we define $\tilde{W}(i_1, i_3)$ by

$$\tilde{W}(i_1, i_3) = \check{W}(i_1, i_1 + i_3 - 1, i_3), \tag{4.2.23}$$

where $\check{W}$ should satisfy (3.5.5). Equation (4.2.16) may then be expressed as

$$\tilde{W}(X_1) = f'(X_1)\tilde{W}_1(X_1) + i_3(X_1)\tilde{W}_3(X_1), \tag{4.2.24}$$

where the subscripts denote derivatives with respect to i_1, i_3.

The relationship between i_1, i_3 and λ_1, λ_2 may be inverted to give λ_1, λ_2 uniquely as functions of i_1, i_3. We write this in the form

$$\lambda_1 = \begin{cases} \frac{i_1-1}{2} + \sqrt{\left(\frac{i_1-1}{2}\right)^2 - i_3} & \text{when} \quad \lambda_1 > \lambda_2, \\ \frac{i_1-1}{2} & \text{when} \quad \lambda_1 = \lambda_2, \\ \frac{i_1-1}{2} - \sqrt{\left(\frac{i_1-1}{2}\right)^2 - i_3} & \text{when} \quad \lambda_1 < \lambda_2, \end{cases} \tag{4.2.25}$$

and

$$\lambda_2 = \begin{cases} \frac{i_1-1}{2} - \sqrt{\left(\frac{i_1-1}{2}\right)^2 - i_3} & \text{when} \quad \lambda_1 > \lambda_2, \\ \frac{i_1-1}{2} & \text{when} \quad \lambda_1 = \lambda_2, \\ \frac{i_1-1}{2} + \sqrt{\left(\frac{i_1-1}{2}\right)^2 - i_3} & \text{when} \quad \lambda_1 \leq \lambda_2. \end{cases} \tag{4.2.26}$$

Equation (4.2.24) can then be rewritten as

$$\tilde{W} = \begin{cases} \left(\frac{i_1-1}{2} + \sqrt{\left(\frac{i_1-1}{2}\right)^2 - i_3}\right) \tilde{W}_1 + i_3\tilde{W}_3 & \text{when} \quad \lambda_1 > \lambda_2, \\ \left(\frac{i_1-1}{2} - \sqrt{\left(\frac{i_1-1}{2}\right)^2 - i_3}\right) \tilde{W}_1 + i_3\tilde{W}_3 & \text{when} \quad \lambda_1 \leq \lambda_2. \end{cases} \tag{4.2.27}$$

4.3 Isochoric deformation

In this section we consider the deformation to be isochoric, so that

$$\lambda_1\lambda_2\lambda = 1. \tag{4.3.1}$$

For simplicity, we take $\lambda = 1$ here. Equation (4.3.1) then gives

$$kff' = 1. \tag{4.3.2}$$

Equation (4.3.2) can be integrated to give

$$f^2 = \frac{2X_1}{k} + c, \quad g = kX_2, \tag{4.3.3}$$

where

$$k = \frac{\psi}{B}, \quad b^2 = a^2 + \frac{4AB}{\psi}, \quad c = \frac{1}{2}(a^2 + b^2). \tag{4.3.4}$$

At this point a general strain-energy function can be considered, and we set $q(\lambda_1) = W(\lambda_1, \lambda_1^{-1}, 1)$. Using (2.1.53), we require $q(1) = 0$, and furthermore, for compatibility with the classical theory, we must have $\frac{dq}{d\lambda_1}(1) = 0, \quad \frac{d^2q}{d\lambda_1^2}(1) = 4\mu$.

Since $d\lambda_1/dX_1 = f''(X_1) \neq 0$, λ_1 can be used as the independent variable in place of X_1. Therefore, the different cases of the condition (4.2.3), (4.2.9), (4.2.12), (4.2.15), (4.2.21), (4.2.24) and (4.2.27) can be expressed as equations with respect to λ_1 for isochoric bending of a block. Here we consider $q(\lambda_1) = \tilde{W}(\lambda_1 + \lambda_1^{-1} + 1, 1)$, and the condition (4.2.24) leads to

$$\tilde{W}(\lambda_1 + \lambda_1^{-1} + 1, 1) = \lambda_1 \tilde{W}_1(\lambda_1 + \lambda_1^{-1} + 1, 1) + \tilde{W}_3(\lambda_1 + \lambda_1^{-1} + 1, 1), \qquad (4.3.5)$$

where the subscrips of W denote derivatives with respect to i_1, i_2. Differentiating this with respect to λ_1, we obtain

$$(\lambda_1^2 - 1)(\lambda_1 \tilde{W}_{11} + \tilde{W}_{13}) + \tilde{W}_1 = 0. \qquad (4.3.6)$$

But, in the reference configuration, we must have

$$\tilde{W}_1(3, 1) = 2\mu, \qquad (4.3.7)$$

from (3.5.5). Hence, in the limit $\lambda_1 \to 1$, equation (4.3.6) implies that either $\lambda_1 \breve{W}_{11} + \breve{W}_{13} \to \infty$ or $\mu = 0$ which is not sensible. We have, therefore, only one choice, namely that $\lambda_1 \breve{W}_{11} + \breve{W}_{13} \to \infty$, which means that the bulk modulus $\kappa \to \infty$, and the material must be incompressible. We therefore have the interesting conclusion that *the considered isochoric bending of a block is only possible for an incompressible material.*

4.4 Examples

In this section we discuss several examples of non-isochoric compressible materials by using the conditions given in Section 4.2.

4.4.1 A special Blatz-Ko material

The strain-energy function proposed by Blatz and Ko (1962) is

$$W = \frac{1}{2}\mu\left(\frac{I_2}{I_3} + 2I_3^{\frac{1}{2}} - 5\right). \tag{4.4.1}$$

For $\lambda = 1$ it may be simplified to

$$\widetilde{W} = \frac{1}{2}\mu(\frac{I_1}{I_3} - I_3^{-1} + 2I_3^{\frac{1}{2}} - 4) \tag{4.4.2}$$

on elimination of I_2 using (4.2.19).

In this case it follows from (4.2.21) that

$$f'^2 + 3k^2f^2 - 4k^2f^2f'^2 = 0, \tag{4.4.3}$$

which leads to

$$\left(\frac{df}{dX_1}\right)^2 = \frac{3k^2f^2}{4k^2f^2 - 1}. \tag{4.4.4}$$

This equation was given by Carroll and Horgan (1990). Using the boundary condition (4.1.3), the solution for $f(X_1)$ is given implicitly by

$$\sqrt{4k^2f^2(X_1) - 1} - \arctan\sqrt{4k^2f^2(X_1) - 1} = \sqrt{3}k(X_1 + d_1), \tag{4.4.5}$$

where the constant d_1 is defined by

$$d_1 = \frac{1}{2\sqrt{3}k}\left[\sqrt{4k^2a^2 - 1} + \sqrt{4k^2b^2 - 1} - \left(\arctan\sqrt{4k^2a^2 - 1} + \arctan\sqrt{4k^2b^2 - 1}\right)\right].$$

We note here that the solution is only valid for the boundary data a, b, ψ satisfying

$$2\sqrt{3}kA = \sqrt{4k^2b^2 - 1} - \sqrt{4k^2a^2 - 1} - \left(\arctan\sqrt{4k^2b^2 - 1} - \arctan\sqrt{4k^2a^2 - 1}\right),$$

where $k = \psi/B$.

Next, we consider some special strain-energy functions from the three classes of strain-energy function discussed by Carroll (1988).

4.4.2 Harmonic materials

In general, a harmonic material has strain-energy function

$$\breve{W}(i_1, i_2, i_3) = F(i_1) + c_2(i_2 - 3) + c_3(i_3 - 1), \tag{4.4.6}$$

where F is a function of i_1, c_2 and c_3 are constants, and for consistency with (2.1.57) and (2.1.58), we must have

$$\begin{aligned} F(3) &= 0, \qquad c_2 + c_3 = -2\mu, \\ F'(3) &= 2\mu - c_2, \qquad F''(3) = \kappa + \frac{4}{3}\mu. \end{aligned} \tag{4.4.7}$$

In this case condition (4.2.15) yields

$$F(i_1) - \lambda_1 F'(i_1) + c_2\lambda_2\lambda - 3c_2 - c_3 = 0, \tag{4.4.8}$$

where $F'(i_1) = dF(i_1)/di_1$. Differentiation of (4.4.8) with respect to X_1 yields

$$\frac{dF'(i_1)}{dX_1} - kF'(i_1) - c_2 k\lambda = 0, \tag{4.4.9}$$

which integrates to give

$$F'(i_1) = -c_2\lambda + \alpha e^{kX_1}, \tag{4.4.10}$$

where α is a constant of integration. If, additionally, $F''(i_1) > 0$ equation (4.4.10) may be inverted to give i_1 uniquely as a function of X_1 since the right-hand side of (4.4.10) is a monotonic function of X_1, and we may write

$$i_1 = (F')^{-1}\left(-c_2\lambda + \alpha e^{kX_1}\right) \equiv S(X_1), \tag{4.4.11}$$

where -1 means inverse.

Furthermore, equation $(4.2.13)_1$ yields

$$f'(X_1) + kf(X_1) + \lambda = S(X_1), \tag{4.4.12}$$

and integration of which gives

$$f(X_1) = e^{-kX_1} \int^{X_1} (S(x) - \lambda) e^{kx} dx + \beta e^{-kX_1}, \tag{4.4.13}$$

where β is a constant. A similar result was obtained by Ogden (1984) for the strain-energy function given by

$$\breve{W}(I_1, I_2, I_3) = F(I_1) + c_2(I_2 - 3) + c_3(I_3 - 1).$$

In what follows we obtain solutions for $f(X_1)$ from (4.4.13) for special forms of $F(i_1)$. For example. We consider $\breve{W}$ given by

$$\breve{W}(i_1, i_2, i_3) = \mu[(i_1 - 3)^2 + 2(i_2 - 3) - 4(i_3 - 1)], \tag{4.4.14}$$

and $\kappa = 2\mu/3$ from equation (4.4.7).

In respect of (4.4.14), equation (4.4.8) yields

$$f'(X_1)^2 - (kf(X_1) + 2\lambda - 3)^2 + 3\lambda^2 - 6\lambda + 2 = 0. \tag{4.4.15}$$

The solution of $f(X_1)$ is given by

$$f(X_1) = \frac{1}{2k}\left[e^{kX_1 + d_2} + \left(\frac{3}{2}\lambda^2 - 3\lambda + 1\right) e^{-(kX_1 + d_2)} - 2\lambda + 3\right], \tag{4.4.16}$$

where the constant d_2 is

$$d_2 = kA + \ln\left[ka + 2\lambda - 3 + \sqrt{k^2a^2 + 2(2\lambda - 3)ka + \lambda^2 - 6\lambda + 7}\right],$$

and the boundary data a, b, ψ should satisfy

$$e^{2kA} = \frac{kb + 2\lambda - 3 + \sqrt{k^2b^2 + 2(2\lambda - 3)kb + \lambda^2 - 6\lambda + 7}}{ka + 2\lambda - 3 + \sqrt{k^2a^2 + 2(2\lambda - 3)ka + \lambda^2 - 6\lambda + 7}}, \tag{4.4.17}$$

where $k = \psi/B$.

We may obtain special solution for $f(X_1)$ for certain values of λ. For example, set

$$3\lambda^2 - 6\lambda + 2 = 0,$$

and then $\lambda_{1,\,2} = 1 \pm \frac{1}{\sqrt{3}}$. In the case $\lambda = 1 + \frac{1}{\sqrt{3}}$, equation (4.4.15) becomes

$$f'(X_1) - kf(X_1) + 1 - \frac{2}{\sqrt{3}} = 0. \tag{4.4.18}$$

The solution with the boundary condition (4.1.3) is

$$f(X_1) = \frac{e^{kX_1}}{k\sqrt{(ka - 1 + 2/\sqrt{3})(kb - 1 + 2/\sqrt{3})}} + \frac{1}{k}\left(1 - \frac{2}{\sqrt{3}}\right), \tag{4.4.19}$$

where $a,\ b, \psi$ are limited by

$$kb - 1 + \frac{2}{\sqrt{3}} = \left(ka - 1 + \frac{2}{\sqrt{3}}\right) e^{2kA}.$$

4.4.3 Varga materials

In this case, the material is given by

$$\check{W}(i_1, i_2, i_3) = c_1(i_1 - 3) + c_2(i_2 - 3) + H(i_3), \tag{4.4.20}$$

where H is a function of i_3, c_1 and c_2 are constants. To ensure that equations (2.1.57) and (2.1.58) are satisfied, we must have

$$\begin{aligned} &H(1) = 0, \qquad c_1 + c_2 = 2\mu, \\ &H'(1) = -2\mu - c_2, \qquad H''(1) = \kappa + \frac{4}{3}\mu. \end{aligned} \tag{4.4.21}$$

Condition (4.2.15) then leads to

$$H(i_3) - H'(i_3)i_3 + (c_1 + c_2\lambda)\lambda_2 + c_1\lambda - 3c_1 - 3c_2 = 0, \tag{4.4.22}$$

where $H'(i_3) = dH(i_3)/di_3$. Differentiation of (4.4.22) with respect to X_1 yields

$$\lambda f(X_1)\frac{d'H(i_3)}{dX_1} = c_1 + c_2\lambda, \tag{4.4.23}$$

and then

$$f(X_1) = \frac{c_1 + c_2\lambda}{\lambda}\left[\frac{dH'(i_3)}{dX_1}\right]^{-1}. \tag{4.4.24}$$

Here we consider

$$\breve{W}(i_1, i_2, i_3) = \mu[4(i_1 - 3) - 2(i_2 - 3) + (i_3 - 1)^2], \tag{4.4.25}$$

and equation (4.4.21) gives again $\kappa = \frac{2}{3}\mu$.

Equation (4.4.22) reduces to

$$-k^2 f'(X_1)^2 f^2(X_1) + 2(2-\lambda)kf(X_1) + 4\lambda - 5 = 0. \tag{4.4.26}$$

The solution of equation (4.4.26) with the boundary condition (4.1.3) is given by

$$\frac{1}{3}y^3 - (4\lambda - 5)y = \frac{2(\lambda-2)^2}{\lambda}kX_1 + d_3, \tag{4.4.27}$$

where

$$y = \sqrt{2(2-\lambda)kf(X_1) - 5 + 4\lambda}, \tag{4.4.28}$$

and d_3 is defined by

$$d_3 = \frac{1}{3}\sqrt{2(2-\lambda)ka - 5 + 4\lambda}[(2-\lambda)ka - 4\lambda + 5]$$
$$+\frac{1}{3}\sqrt{2(2-\lambda)kb - 5 + 4\lambda}[(2-\lambda)kb + -4\lambda + 5],$$

and a, b, ψ should satisfy

$$\frac{6(\lambda-2)^2}{\lambda}kA = \sqrt{2(2-\lambda)kb - 5 + 4\lambda}[(2-\lambda)kb - 4\lambda + 5] \tag{4.4.29}$$

$$-\sqrt{2(2-\lambda)ka - 5 + 4\lambda}[(2-\lambda)ka - 4\lambda + 5],$$

where $k = \psi/B$.

When we take $\lambda = 5/4$, the solution has the form

$$f(X_1) = \left[\frac{3}{5}\sqrt{\frac{6}{k}}X_1 + \frac{1}{2}\left(a^{\frac{3}{2}} + b^{\frac{3}{2}}\right)\right]^{\frac{2}{3}}, \tag{4.4.30}$$

where a, b, ψ satisfies

$$\frac{6\sqrt{6}}{5}A = \sqrt{k}\left(b^{\frac{3}{2}} - a^{\frac{3}{2}}\right).$$

If we set $\lambda = 2$, we have a solution for $f(X_1)$

$$f(X_1) = \sqrt{\frac{6}{k}X - 1 + b^2 + a^2},$$

where a, b, ψ must have

$$b^2 - a^2 = \frac{12}{k}A.$$

4.4.4 Carroll's Class *III* materials

The third class of materials considered by Carroll has strain-energy function $\breve{W}$ given by

$$\breve{W}(i_1, i_2, i_3) = c_1(i_1 - 3) + G(i_2) + c_3(i_3 - 1), \tag{4.4.31}$$

where G is a function of i_2, c_1 and c_3 are constants, which, for consistency with equations (2.1.57) and (2.1.58), should satisfy

$$G(3) = 0, \qquad c_1 - c_3 = 4\mu,$$
$$G'(3) = -(c_3 + 2\mu), \qquad G''(3) = \frac{1}{4}\kappa + \frac{1}{3}\mu. \tag{4.4.32}$$

On substitution of (4.4.31) into (4.2.15), we obtain

$$G(i_2) - (\lambda_1\lambda_2 + \lambda_1\lambda)G'(i_2) + c_1(\lambda_2 + \lambda) - (3c_1 + c_3) = 0. \tag{4.4.33}$$

Differentiation of (4.4.33) respect to X_1 yields

$$(\lambda + kf(X_1))\frac{dG'(i_2)}{dX_1} + k\lambda G'(i_2) + c_2 k = 0, \tag{4.4.34}$$

and then

$$f(X_1) = (\lambda G'(i_2) - c_2)\left[\frac{dG'(i_2)}{dX_1}\right]^{-1} - \frac{\lambda}{k}. \tag{4.4.35}$$

Now we consider the special case when the strain-energy function is given by

$$\breve{W}(i_1, i_2, i_3) = \mu[2(i_1 - 3) + (i_2 - 3)^2 - 2(i_3 - 1)], \tag{4.4.36}$$

which corresponds to $\kappa = 20\mu/3$. The general solution for $f(X_1)$ in this case is very complicated, and we do not give the details here. For example, we may set

$$2\lambda^3 - 5\lambda^2 + 6\lambda - 1 = 0.$$

Equation (4.4.33) then reduces to

$$\lambda^2\left(kf(X_1) + \frac{1 - 3\lambda}{\lambda^2}\right)^2 - f'(X_1)^2(kf(X_1) + \lambda)^2 = 0. \tag{4.4.37}$$

Using the boundary condition (4.1.3), the solution is given implicitly by

$$kf(X_1) + \left(\lambda - \frac{1 - 3\lambda}{\lambda^2}\right)\ln\left(\left|kf(X_1) + \frac{1 - 3\lambda}{\lambda^2}\right|\right) = k\lambda X_1 + d_4, \tag{4.4.38}$$

where d_4 is defined by

$$d_4 = \frac{1}{2}k(a + b) + \frac{\lambda^3 + 3\lambda - 1}{2\lambda^2}\ln\left|ka + \frac{1 - 3\lambda}{\lambda^2}\right|\left|kb + \frac{1 - 3\lambda}{\lambda^2}\right|.$$

The solution is only valid for a, b, ψ satisfying

$$2k\lambda A = k(b - a) + \frac{\lambda^3 + 3\lambda - 1}{\lambda^2}\ln\frac{|kb\lambda^2 + 1 - 3\lambda|}{|ka\lambda^2 + 1 - 3\lambda|},$$

where $k = \psi/B$.

Chapter 5

Axial shear of a circular cylindrical tube

5.1 Axial shear and the governing equations

We consider a thick-walled right circular cylindrical tube of homogeneous compressible isotropic nonlinearly elastic material which, in its (undeformed, stress-free) natural configuration, is defined by

$$A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L,$$

where (R, Θ, Z) are cylindrical polar coordinates. The *axial shear* deformation of the right circular cylindrical tube is defined by

$$r = g(R), \quad \theta = \Theta, \quad z = Z + \omega(R), \tag{5.1.1}$$

where (r, θ, z) denote cylindrical polar coordinates associated with the deformed configuration, $g(R)$ describes the radial deformation and $\omega(R)$ denotes the axial displacement. For the special case in which $g(R) = R$ there is no radial displacement and the deformation is referred to as *pure axial shear.*

Referred to unit basis vectors $(\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z)$ and $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ associated respectively with coordinates (R, Θ, Z) and (r, θ, z) the deformation gradient tensor, denoted $\mathbf{A}$, may be written

$$\mathbf{A} = g'(R)\mathbf{e}_r \otimes \mathbf{E}_R + \frac{1}{R}g(R)\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \omega'(R)\mathbf{e}_z \otimes \mathbf{E}_R + \mathbf{e}_z \otimes \mathbf{E}_Z, \qquad (5.1.2)$$

where the prime denotes differentiation with respect to R and $\det\mathbf{A} = g(R)g'(R)/R$. See, for example, Ogden (1984).

Let $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$, $\mathbf{u}^{(3)}$ be the unit Lagrangian principal axes associated with this deformation, that is the principal axes of $\mathbf{A}^T\mathbf{A}$. Then, we may express $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$, $\mathbf{u}^{(3)}$ in terms of the vectors $(\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z)$ in the form

$$\mathbf{u}^{(1)} = \cos\psi\mathbf{E}_R + \sin\psi\mathbf{E}_Z, \quad \mathbf{u}^{(2)} = \mathbf{E}_\Theta, \quad \mathbf{u}^{(3)} = -\sin\psi\mathbf{E}_R + \cos\psi\mathbf{E}_Z, \quad (5.1.3)$$

where ψ defines the orientation of the axes $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(3)}$ in the plane of $\mathbf{E}_R$ and $\mathbf{E}_Z$.

Next, we note that the polar decomposition (2.1.9) of $\mathbf{A}$ defines a unique rotation tensor $\mathbf{R}$ and a unique positive definite symmetric (right) stretch tensor, which may used to give

$$\mathbf{A}^T\mathbf{A} = \mathbf{U}^2, \qquad (5.1.4)$$

where the spectral decomposition of $\mathbf{U}$ is given by (2.1.12).

Then, substitution of (5.1.3) and (2.1.12) into equation (5.1.4) shows that the principal stretches λ_1 and λ_3 satisfy the connections

$$\lambda_1^2\cos^2\psi + \lambda_3^2\sin^2\psi = g'(R)^2 + \omega'(R)^2, \qquad (5.1.5)$$

$$\lambda_1^2 \sin^2\psi + \lambda_3^2 \cos^2\psi = 1, \tag{5.1.6}$$

$$(\lambda_1^2 - \lambda_3^2)\sin\psi\cos\psi = \omega'(R), \tag{5.1.7}$$

from which it may be deduced that

$$\lambda_1\lambda_3 = g'(R), \tag{5.1.8}$$

while from det$\mathbf{A} = \lambda_1\lambda_2\lambda_3$, it then follows that λ_2 is given by

$$\lambda_2 = \frac{1}{R}g(R). \tag{5.1.9}$$

From equations (5.1.5)-(5.1.6) we obtain

$$\lambda_1^2 + \lambda_3^2 = \omega'(R)^2 + g'(R)^2 + 1. \tag{5.1.10}$$

and then, using (5.1.5)-(5.1.7) with (5.1.10), we have

$$\begin{aligned} \sin^2\psi &= \frac{1-\lambda_3^2}{\lambda_1^2-\lambda_3^2}, \\ \cos^2\psi &= \frac{\lambda_1^2-1}{\lambda_1^2-\lambda_3^2}, \\ \sin\psi\cos\psi &= \frac{\omega'}{\lambda_1^2-\lambda_3^2}. \end{aligned} \tag{5.1.11}$$

For an isotropic elastic material the constitutive law is expressed in terms of the strain-energy function $W(\lambda_1, \lambda_2, \lambda_3)$ per unit reference volume, where W should satisfy equations (2.1.49) and (2.1.53)-(2.1.55).

On use of (2.1.50) we denote the principal Biot stresses as

$$t_i^{(1)} = \frac{\partial W}{\partial \lambda_i}, \quad i \in \{1, \ ,2, \ 3\}, \tag{5.1.12}$$

and the expression for $\mathbf{R}$ may be obtained from (2.1.9) in the form $\mathbf{R} = \mathbf{A}\mathbf{U}^{-1}$ together with (5.1.3) and the inverse of (2.1.12). Thus,

$$\begin{aligned} \mathbf{R} = {} & (\lambda_1^{-1}g'(R)\cos\psi\mathbf{e}_r + \lambda_1^{-1}\omega'(R)\cos\psi\mathbf{e}_z + \lambda_1^{-1}\sin\psi\mathbf{e}_z) \otimes \mathbf{u}^{(1)} + \frac{1}{R}\lambda_2^{-1}g(R)\mathbf{e}_\theta \otimes \mathbf{u}^{(2)} \\ & + \ (-\lambda_3^{-1}g'(R)\sin\psi\mathbf{e}_r - \lambda_3^{-1}\omega'(R)\sin\psi\mathbf{e}_z + \lambda_3^{-1}\cos\psi\mathbf{e}_z) \otimes \mathbf{u}^{(3)}. \end{aligned} \tag{5.1.13}$$

An expression for the nominal stress tensor $\mathbf{S}$ is then deduced from (2.1.28) and after some manipulations using (5.1.8), (5.1.11) and (5.1.3) we find that the non-vanishing components of $\mathbf{S}$ referred to the two sets of cylindrical polar coordinates are given by

$$S_{Rr} = \lambda_1\lambda_3\frac{\lambda_1 t_1^{(1)} - \lambda_3 t_3^{(1)}}{\lambda_1^2 - \lambda_3^2} - \frac{\lambda_3 t_1^{(1)} - \lambda_1 t_3^{(1)}}{\lambda_1^2 - \lambda_3^2}, \tag{5.1.14}$$

$$S_{Rz} = \frac{\lambda_1 t_1^{(1)} - \lambda_3 t_3^{(1)}}{\lambda_1^2 - \lambda_3^2}\omega'(R), \quad S_{\Theta\theta} = t_2^{(1)}, \quad S_{Zr} = \frac{\lambda_3 t_1^{(1)} - \lambda_1 t_3^{(1)}}{\lambda_1^2 - \lambda_3^2}\omega'(R), \tag{5.1.15}$$

$$S_{Zz} = \frac{\lambda_1 t_1^{(1)} - \lambda_3 t_3^{(1)}}{\lambda_1^2 - \lambda_3^2} - \lambda_1\lambda_3\frac{\lambda_3 t_1^{(1)} - \lambda_1 t_3^{(1)}}{\lambda_1^2 - \lambda_3^2}, \tag{5.1.16}$$

where, in terms of the principal stretches,

$$\omega'(R)^2 = (\lambda_1^2 - 1)(1 - \lambda_3^2). \tag{5.1.17}$$

For the problem at hand the equilibrium equations (2.1.38) in the absence of body forces reduce to two ordinary differential equations with R as the independent variable since the principal stretches λ_1, λ_2, λ_3 are functions of R only from equations (5.1.5)-(5.1.7) with (5.1.11). We may write these, by specializing (2.1.38), in the form (Ogden 1984)

$$\frac{d(RS_{Rz})}{dR} = 0, \tag{5.1.18}$$

$$\frac{dS_{Rr}}{dR} + \frac{1}{R}(S_{Rr} - S_{\Theta\theta}) = 0, \tag{5.1.19}$$

with the stress components given by (5.1.14) and (5.1.16). Note that S_{Zr} and S_{Zz} do not appear in these equations and merely describe the components of traction on the ends of the cylinder required to maintain the given deformation.

We take the boundary conditions in the form

$$g(A) = A, \qquad g(B) = B, \tag{5.1.20}$$

$$\omega(A) = 0, \qquad \omega(B) = d, \tag{5.1.21}$$

d being the axial displacement of the outer surface relative to the inner one.

Equation (5.1.18) may be integrated in the form

$$\frac{\lambda_1 t_1^{(1)} - \lambda_3 t_3^{(1)}}{\lambda_1^2 - \lambda_3^2}\omega'(R) \equiv S_{Rz} = \frac{\tau B}{R}, \tag{5.1.22}$$

where τ is the axial shear stress on the outer surface per unit reference area that produces the displacement d.

5.2 Equations for pure axial shear

In the specialization to pure axial shear we have $r \equiv g(R) = R$ and equations (5.1.8) and (5.1.9) reduce to

$$\lambda_1 \lambda_3 = 1, \qquad \lambda_2 = 1. \tag{5.2.1}$$

and (5.1.2) to

$$\mathbf{A} = \mathbf{I} + \omega'(r)\mathbf{e}_z \otimes \mathbf{E}_Z, \tag{5.2.2}$$

where $\mathbf{I}$ is the identity tensor. Thus, the deformation is *isochoric*. Moreover, it is locally a simple shear of amount $\gamma \equiv \lambda - \lambda^{-1}$, where $\lambda_1 = \lambda$, $\lambda_3 = \lambda^{-1}$. Without loss of generality we take $\lambda \geq 1$ and reduce (5.1.17) to

$$\gamma = \lambda - \lambda^{-1} = \omega'(r). \tag{5.2.3}$$

In view of (5.2.1) and the connection (5.2.3) between λ and γ we may regard the strain energy as a function of γ defined by

$$\hat{W}(\gamma) = W(\lambda, 1, \lambda^{-1}). \tag{5.2.4}$$

Equation (5.1.22) then takes on the simple form

$$\hat{W}'(\gamma) = \frac{\tau b}{r}, \tag{5.2.5}$$

while (5.1.19) specializes to

$$\frac{d}{dr}\left(\frac{t_1^{(1)} + t_3^{(1)}}{\lambda_1 + \lambda_3}\right) + \frac{1}{r}\left(\frac{t_1^{(1)} + t_3^{(1)}}{\lambda_1 + \lambda_3} - t_2^{(1)}\right) = 0 \tag{5.2.6}$$

evaluated for $\lambda_1 = \lambda$, $\lambda_2 = 1$, $\lambda_3 = \lambda^{-1}$ with $t_1^{(1)}$, $t_2^{(1)}$, $t_3^{(1)}$ given by (5.1.12). For pure axial shear the boundary conditions (5.1.20) are satisfied automatically. Thus, we require to solve equations (5.2.5) and (5.2.6) in conjunction with (5.1.21).

Since the assumed form of deformation is isochoric, equations (5.2.5) and (5.2.6) provide two equations for $\omega(r)$ for any given form of strain-energy function. In general, the two equations are not compatible and the assumed deformation is not supportable. For some forms of strain-energy function, however, the equations are mutually consistent and their combination yields a necessary and sufficient condition on the strain-energy function for this to be the case. This condition can then be used to generate forms of strain-energy function for which solutions can be found.

Provided that the function $\hat{W}'$ is suitably well behaved equation (5.2.5) can be inverted (at least implicitly) to give γ as a function of r, and ω is then determined from (5.2.3) by integration. Possible sufficient conditions for this are that $\hat{W}'(\gamma)$ is continuous and $\to \infty$ as $\gamma \to \infty$. If, additionally, $\hat{W}'(\gamma)$ is a monotone increasing function of γ then

$$\hat{W}''(\gamma) > 0 \tag{5.2.7}$$

for all γ and (5.2.5) yields a unique solution for γ for all τ. Alternative sufficient conditions, which, for example, allow for shear softening, so that (5.2.7) need not hold for all γ, may also be considered but they are not discussed here.

Under the assumption that (5.2.7) holds, differentiation of (5.2.5) yields

$$r\frac{d}{dr} = -\frac{\hat{W}'(\gamma)}{\hat{W}''(\gamma)}\frac{d}{d\gamma}, \tag{5.2.8}$$

and making use of (5.1.12) this is then used to recast (5.2.6) in the form

$$\hat{W}'(\gamma)\frac{d}{d\gamma}\left(\frac{W_1 + W_3}{\lambda + \lambda^{-1}}\right) = \hat{W}''(\gamma)\left(\frac{W_1 + W_3}{\lambda + \lambda^{-1}} - W_2\right), \tag{5.2.9}$$

where W_1, W_2, W_3 are evaluated for $\lambda_1 = \lambda$, $\lambda_2 = 1$, $\lambda_3 = \lambda^{-1}$ and we note the connection (5.2.3) between λ and γ.

Thus, (5.2.8) has been used to replace the independent variable r in (5.2.6) by γ in (5.2.9). Equation (5.2.9) is a necessary and sufficient condition for the strain-energy function to be capable of supporting the pure axial shear deformation. This does not in general mean that the deformation exists for any strain-energy function which satisfies (5.2.9). For such a strain-energy it is necessary to solve (5.2.5) for γ, which requires appropriate sufficient conditions, as discussed above.

Equation (5.2.9) may be re-expressed in a number of different forms. For example, in terms of the derivatives of $W(\lambda_1, \lambda_2, \lambda_3)$ evaluated for $\lambda_1 = \lambda$, $\lambda_2 = 1$, $\lambda_3 = \lambda^{-1}$, from (5.1.22) with (5.1.12) we have

$$r\frac{d}{d\lambda}\left(\frac{\lambda W_1 - \lambda^{-1}W_3}{\lambda + \lambda^{-1}}\right)\frac{d\lambda}{dr} = -\frac{\lambda W_1 - \lambda^{-1}W_3}{\lambda + \lambda^{-1}}. \tag{5.2.10}$$

Therefore, (5.2.10) may be used to replace the independent variable r in (5.2.6) by λ, which gives

$$\frac{\lambda W_1 - \lambda^{-1} W_3}{\lambda + \lambda^{-1}} \frac{d}{d\lambda}\left(\frac{W_1 + W_3}{\lambda + \lambda^{-1}}\right) + W_2 \frac{d}{d\lambda}\left(\frac{\lambda W_1 - \lambda^{-1} W_3}{\lambda + \lambda^{-1}}\right)$$
$$= \frac{W_1 + W_3}{\lambda + \lambda^{-1}} \frac{d}{d\lambda}\left(\frac{\lambda W_1 - \lambda^{-1} W_3}{\lambda + \lambda^{-1}}\right). \tag{5.2.11}$$

Equation (5.2.11) may be rewritten

$$(\lambda W_1 - W_2)\left(W_{13} - \lambda^{-2} W_{33} - \frac{W_1 + W_3}{\lambda + \lambda^{-1}}\right)$$
$$+ (W_2 - \lambda^{-1} W_3)\left(\lambda^2 W_{11} - W_{13} + \frac{W_1 + W_3}{\lambda + \lambda^{-1}}\right) = 0. \tag{5.2.12}$$

Alternatively, if we regard W as a function of the principal invariants I_1, I_2, I_3 defined by equation (2.1.17), then, by writing $W = \bar{W}(I_1, I_2, I_3)$, equation (5.2.9) becomes

$$(\bar{W}_1 + \bar{W}_2)(2\bar{W}_{11} + 6\bar{W}_{12} + 2\bar{W}_{13} + 4\bar{W}_{22} + 2\bar{W}_{23} + W_2)$$
$$+2\gamma^2 \bar{W}_2(\bar{W}_{11} + 2\bar{W}_{12} + \bar{W}_{22}) = 0, \tag{5.2.13}$$

where $\bar{W}_i$, $\bar{W}_{ij}$, i, $j = 1, 2, 3$ indicate differentiation with respect to the arguments evaluated for $I_1 = I_2 = \gamma^2 + 3$, $I_3 = 1$.

Equation (5.2.13) was obtained by Jiang and Beatty (1997). An equivalent equation based on the principal invariants i_1, i_2, i_3 defined by equation (2.1.16) with the strain energy given by

$$\tilde{W}(i_1, i_2, i_3) = \bar{W}(i_1^2 - 2i_2, i_2^2 - 2i_1, i_3^2), \tag{5.2.14}$$

from equation (3.5.1) is

$$(\tilde{W}_{11} + 2\tilde{W}_{12} + \tilde{W}_{22})[\tilde{W}_1 + (i-1)\tilde{W}_2] + (\tilde{W}_{12} + \tilde{W}_{22} + \tilde{W}_{13} + \tilde{W}_{23})(\tilde{W}_1 + \tilde{W}_2)$$

$$-2(\tilde{W}_1 + \tilde{W}_2)(\tilde{W}_1 - \tilde{W}_2)/(i^2 - 1) = 0, \tag{5.2.15}$$

evaluated for $i_1 = i_2 = i \equiv \lambda + \lambda^{-1} + 1$, $i_3 = 1$, where the subscripts denote derivatives with respect to i_1, i_2, i_3.

5.3 Results for a special class of strain-energy functions

Pure axial shear of a circular cylindrical tube of compressible elastic material appears to have been studied first by Mioduchowski and Haddow (1974). They considered a strain-energy function of the form

$$\bar{W} = \frac{1}{2}\mu[H_1(I_3)(I_1 - 1) + H_3(I_3)], \tag{5.3.1}$$

where $H_1(1) = 1$, $H_3(1) = 0$, $H_3'(1) = -1$, and showed that this led to a solution of the form (in the present notation)

$$\omega = d\frac{\log r - \log a}{\log b - \log a} \tag{5.3.2}$$

provided $H_1'(1) = 0$ when the boundary conditions (5.1.21) are applied. They also obtained numerical results for (non-isochoric) axial shear in respect of the Levinson-Burgess and Blatz-Ko forms of strain-energy function (Levinson and Burgess (1971); Blatz and Ko (1962)). These are both special cases of the strain-energy function

$$\bar{W} = \frac{1}{2}\mu[H_1(I_3)(I_1 - 1) + H_2(I_3)(I_2 - 1) + H_3(I_3)], \tag{5.3.3}$$

where the function H_1, H_2, H_3 satisfy appropriate conditions for consistency with (2.1.60) and (2.1.61).

Further conditions on the latter functions for (5.3.3) to yield pure axial shear were given by Polignone and Horgan (1992) and Jiang and Beatty (1995) and in each case a solution of the form (5.3.2) was obtained (with the interpretation of d dependent on the choice of boundary conditions). The strain-energy function (5.3.3) and certain of its specializations, including the Levinson-Burgess and Blatz-Ko forms, had been considered earlier by Agarwal (1979), who also obtained the solution (5.3.2) in some particular cases. Numerical results for the Levinson-Burgess material were given by Ertepinar and Erarslanoğlu (1990). For further discussion of the pure axial shear problem we refer to the detailed analysis of Polignone and Horgan (1992) and Jiang and Beatty (1995). In the latter paper it was shown that conditions on the strain-energy function for pure axial shear to be admitted given by Knowles (1977) are sufficient but not necessary.

Thus far, equation (5.3.2) appears to provide the only closed-form solution available for pure axial shear. In the remainder of this section we obtain conditions on the strain-energy function for which (5.3.2) is admitted as a solution.

From (5.3.2) and the definition (5.2.3) we have

$$\gamma = \omega'(r) = \frac{d}{r(\log b - \log a)}. \tag{5.3.4}$$

Using (5.3.4) we may write (5.2.5) in the form

$$\hat{W}'(\gamma) = \gamma\tau b(\log b - \log a)/d. \tag{5.3.5}$$

But, bearing in mind the connection $\hat{W}(\gamma) = \bar{W}(I_1, I_2, I_3) = \bar{W}(\gamma^2 + 3, \gamma^2 + 3, 1)$, we have

$$\hat{W}'(\gamma) = 2\gamma(\bar{W}_1 + \bar{W}_2), \tag{5.3.6}$$

so that $\bar{W}_1 + \bar{W}_2 \equiv$ constant is only valid for the form of (5.3.5), and then, for consistency with (2.1.60) and (2.1.61), we must have

$$\bar{W}_1(\gamma^2+3,\gamma^2+3,1)+\bar{W}_2(\gamma^2+3,\gamma^2+3,1)=\frac{1}{2}\mu. \qquad (5.3.7)$$

Equation (5.3.6) becomes

$$\hat{W}'(\gamma)=\mu\gamma. \qquad (5.3.8)$$

Combining equations (5.3.5) and (5.3.8) we deduce the (linear) relationship

$$\tau=\frac{\mu d}{b(\log b-\log a)} \qquad (5.3.9)$$

between τ and d, which is identical to that obtained by Mioduchowski and Haddow (1974) for compressible materials. The same formula is also obtained in respect of the incompressible Mooney-Rivlin strain-energy function (see, for example, Green and Zerna (1968)).

Furthermore, differentiation of (5.3.7) with respect to γ then yields

$$\bar{W}_{11}+2\bar{W}_{12}+\bar{W}_{22}=0 \qquad (5.3.10)$$

evaluated for $I_1=I_2=\gamma^2+3,\ I_3=1$.

The latter equation characterizes a material with a *constant shear response function*, as described by Jiang and Beatty (1995); see also Polignone and Horgan (1992). Clearly, by integration of (5.3.10), after multiplication by 2γ, and use of (2.1.60) and (2.1.61) equation (5.3.7) is recovered and hence (5.3.7) and (5.3.10) are equivalent, and also equivalent to (5.3.8). Jiang and Beatty (1995) proved that the shear response function is constant if and only if (5.3.10) holds. Furthermore, they showed that whenever (5.3.10) holds the deformation must have the form (5.3.2). *Here* *we have shown, conversely, that when the deformation has the form (5.3.2) the strain-energy function must be such that (5.3.8), and hence (5.3.10), holds.* Specific materials for which (5.3.10) holds have been discussed by Polignone and Horgan (1992) and Jiang and Beatty (1995).

5.4 Some new solutions

5.4.1 Example 1

In Section 3.4 we obtained closed-form solutions for the pure azimuthal shear problem for members of the class of strain-energy functions defined, in the present notation, by

$$\bar{W} = f(I_1)h_1(I_3) + h_3(I_3). \tag{5.4.1}$$

Without loss of generality we take $h_1(1) = 1$ and the functions f, h_1, h_3 in (5.4.1) should satisfy the specialization of (2.1.60) and (2.1.61) to this form of strain-energy function. Notice that (5.4.1) is independent of I_2 but that, in the specialization to pure axial shear, $I_2 = I_1$ and $I_3 = 1$.

Substitution of (5.4.1) into (5.2.13) yields

$$f'(I_1)(f''(I_1) + kf'(I_1)) = 0, \tag{5.4.2}$$

where the constant k is defined by

$$k = h_1'(1). \tag{5.4.3}$$

Therefore, we may deduce that

$$f(I_1) = -\frac{\mu}{2k}\left[e^{-k(I_1-3)} - 1\right]. \tag{5.4.4}$$

With $I_3 = 1$ and $I_1 = \gamma^2 + 3$ it follows from (5.2.5) that

$$\hat{W}'(\gamma) \equiv \mu\gamma e^{-k\gamma^2} = \frac{\tau b}{r} \tag{5.4.5}$$

and we note that (5.2.7) is satisfied for all γ if and only if $k \leq 0$.

In the limiting case $k = 0$ the strain-energy function is linear in I_1 and the solution for this case was given in Section 5.3. For other values of k equation (5.4.5) cannot be inverted explicitly to give γ as a function of r. Thus, unlike the situation for pure azimuthal shear, the strain-energy function (5.4.1) does not yield new closed-form solutions for pure axial shear. Equation (5.4.5) may be solved numerically but we do not pursue this here.

5.4.2 Example 2

As in Section 3.5 and analogously to (5.4.1) we next consider the strain-energy function defined by

$$\tilde{W} = \tilde{f}(i_1)\tilde{h}_1(i_3) + \tilde{h}_3(i_3), \tag{5.4.6}$$

where the functions f, h_1 and h_2 satisfy the specialization of (2.1.57) and (2.1.58). Without loss of generality, we here set $\tilde{h}_1(1) = 1$ again.

Equation (5.2.15) then yields

$$\tilde{f}'(i_1)\left[\tilde{f}''(i_1) + \left(\tilde{k} - \frac{2}{i_1^2 - 1}\right)\tilde{f}'(i_1)\right] = 0, \tag{5.4.7}$$

where $\tilde{k} = \tilde{h}_1'(1)$. We then deduce that

$$\tilde{f}'(i_1) = 4\mu\frac{i_1 - 1}{i_1 + 1}e^{-\tilde{k}(i_1 - 3)}. \tag{5.4.8}$$

In this case equation (4.2.5) becomes

$$\frac{4\mu\gamma}{2 + \sqrt{\gamma^2 + 4}}e^{-\tilde{k}(\sqrt{\gamma^2+4}-2)} = \frac{\tau b}{r}. \tag{5.4.9}$$

As with (5.4.5) equation (5.4.9) cannot in general be inverted. Exceptionally, for $\tilde{k} = 0$, equation (5.4.6) yields

$$\tilde{f}(i_1) = 4\mu[i_1 - 2\log(i_1 + 1)] \tag{5.4.10}$$

apart from an additive constant, which can be accommodated through $\tilde{h}_3(i_3)$. Equation (5.4.9) then leads to

$$\gamma = \frac{4\tilde{\tau}br}{r^2 - \tilde{\tau}^2 b^2}, \tag{5.4.11}$$

where $\tilde{\tau}$ is defined by

$$\tilde{\tau} = \tau / 4\mu. \tag{5.4.12}$$

From (5.4.11) and (5.2.3) the axial displacement ω is obtained in the form

$$\omega = 2\tilde{\tau}\log\left(\frac{r^2 - \tilde{\tau}^2 b^2}{a^2 - \tilde{\tau}^2 b^2}\right). \tag{5.4.13}$$

It follows that

$$d = 2\tilde{\tau}\log\left(\frac{(1 - \tilde{\tau}^2)\tilde{\eta}^2}{1 - \tilde{\tau}^2\tilde{\eta}^2}\right), \tag{5.4.14}$$

where the geometrical parameter $\tilde{\eta}$ is defined by

$$\tilde{\eta} = b/a. \tag{5.4.15}$$

We note, in particular, that for this solution the upper bound $|\tilde{\tau}| < \tilde{\eta}^{-1}$ is placed on the allowable axial shear stress, with $d \to \infty$ as $|\tilde{\tau}| \to \tilde{\eta}^{-1}$ from below.

5.4.3 Example 3

In this example we consider $\tilde{W}$ to be linear in i_1 and i_2 so that

$$\tilde{W} = (i_1 - 3)h_1(i_3) + (i_2 - 3)h_2(i_3) + h_3(i_3), \tag{5.4.16}$$

where h_1, h_2 and h_3 are functions of i_3, which, for consistency with (2.1.57) and (2.1.58), must satisfy

$$h_3(1) = 0, \qquad h_1(1) + h_2(1) = 2\mu,$$
$$h_2(1) + h_3'(1) = -2\mu, \qquad 2h_1'(1) + 4h_2'(1) + h_3''(1) = \kappa + \frac{4}{3}\mu. \quad (5.4.17)$$

Subsituting (5.4.16) into (5.2.15) yields

$$h_1'(1) + h_2'(1) - 2\frac{h_1(1) - h_2(1)}{i_1^2 - 1} = 0. \quad (5.4.18)$$

Therefore, (5.4.17) and (5.4.18) force (5.4.16) to have form

$$\tilde{W} = \mu(i_1 + i_2 - 6) + h_3(i_3), \quad (5.4.19)$$

and (5.4.17) reduces to

$$h_3(1) = 0, \qquad h_3'(1) = -3\mu, \qquad h_3''(1) = \kappa + \frac{4}{3}\mu. \quad (5.4.20)$$

In respect of (5.4.19) equation (5.2.5) becomes

$$\hat{W}'(\gamma) \equiv 2\mu\gamma(4 + \gamma^2)^{-\frac{1}{2}} = \frac{\tau b}{r}, \quad (5.4.21)$$

and hence

$$\gamma = 2\tilde{\tau}b/\sqrt{r^2 - \tilde{\tau}^2 b^2}, \quad (5.4.22)$$

where $\tilde{\tau}$ is defined by $\tilde{\tau} = \tau/2\mu$. Integration of (5.4.22) yields the solution

$$\omega = 2\tilde{\tau}b\left[cosh^{-1}\left(\frac{r}{\tilde{\tau}b}\right) - cosh^{-1}\left(\frac{a}{\tilde{\tau}b}\right)\right], \quad (5.4.23)$$

so that

$$d = 2\tilde{\tau}b\left[cosh^{-1}\left(\frac{1}{\tilde{\tau}}\right) - cosh^{-1}\left(\frac{1}{\tilde{\tau}\tilde{\eta}}\right)\right], \quad (5.4.24)$$

As for the solution in Example 2 the solution (5.4.23) is only valid for $|\tilde{\tau}| < \tilde{\eta}^{-1}$.

5.4.4 Example 4

Here we consider the class of strain-energy functions defined by

$$\bar{W} = g(I_2)h_2(I_3) + h_3(I_3), \tag{5.4.25}$$

analogously to (5.4.1), where $h_2(1) = 1$ and equations (2.1.60) and (2.1.61) hold. On substitution of (5.4.25) into (5.2.13) we deduce that

$$g'(I_2)[(2\gamma^2 + 4)g''(I_2) + kg'(I_2)] = 0, \tag{5.4.26}$$

where k is defined by

$$k = h_2'(1) + \frac{1}{2}. \tag{5.4.27}$$

Apart from an additive constant, the function g must have the form

$$g(I_2) = \begin{cases} \mu(I_2 - 1)^{1-k}/2^{1-k}(1-k) & k \neq 1, \\ \mu \log(I_2 - 1) & k = 1. \end{cases} \tag{5.4.28}$$

In this case equation (5.2.5) becomes

$$\hat{W}'(\gamma) \equiv \mu\gamma(2+\gamma^2)^{-k}2^k = \frac{\tau b}{r}, \tag{5.4.29}$$

and (5.2.7) is satisfied for all γ if and only if $k \leq 1/2$.

Note that for $k = 0$ the results discussed in Section 4 are reproduced. We now obtain solutions for the three values of k in order to illustrate the results.

Case (a): $k = 1/2$

Here the resulting axial displacement function is very similar to that in Example 3. In this case, (5.4.25) becomes

$$\bar{W} = \sqrt{2}\mu\sqrt{I_2 - 1}h_2(I_3) + h_3(I_3), \tag{5.4.30}$$

where the functions h_1 and h_2 should satisfy

$$h_2(1) = 1, \qquad h_3(1) = -2\mu, \qquad 2\mu h_2'(1) + h_3'(1) = -\mu \tag{5.4.31}$$

$$2\mu h_2'(1) + 2\mu h_2''(1) + h_3''(1) = \frac{1}{4}\kappa - \frac{1}{6}\mu.$$

In respect of (5.4.30) equation (5.4.29) leads to

$$\frac{\sqrt{2}\mu\gamma}{\sqrt{\gamma^2+2}} = \frac{\tau b}{r}, \tag{5.4.32}$$

which may be solved to give

$$\omega = 4\tilde{\tau}b\left[\cosh^{-1}\left(\frac{r}{2\sqrt{2}\tilde{\tau}b}\right) - \cosh^{-1}\left(\frac{a}{2\sqrt{2}\tilde{\tau}b}\right)\right], \tag{5.4.33}$$

where $\tilde{\tau}$ is again defined by (5.4.12), and hence

$$d = 2\tilde{\tau}b\left[\cosh^{-1}\left(\frac{1}{2\sqrt{2}\tilde{\tau}}\right) - \cosh^{-1}\left(\frac{1}{2\sqrt{2}\tilde{\tau}\tilde{\eta}}\right)\right]. \tag{5.4.34}$$

As in Example 3 an upper bound is placed on the allowable shearing stress $\tilde{\tau}$, in this case $|\tilde{\tau}| < 1/2\sqrt{2}\tilde{\eta}$.

Case (b): $k = 1/4$

The strain-energy function is given by

$$\bar{W} = \frac{1}{3}2^{\frac{5}{4}}\mu(I_2 - 1)^{\frac{3}{4}}h_2(1) + h_3(1), \tag{5.4.35}$$

where the functions h_1 and h_2 must satisfy

$$h_2(1) = 1, \qquad h_3(1) = -\frac{4}{3}\mu, \qquad \frac{4}{3}\mu h_2'(1) + h_3'(1) = -\mu, \tag{5.4.36}$$

$$2\mu h_2'(1) + \frac{4}{3}\mu h_2''(1) + h_3''(1) = \frac{1}{4}\kappa + \frac{7}{12}\mu.$$

From the specialization of (5.4.29) we have

$$2^{\frac{1}{4}}\mu\gamma(\gamma^2+2) = \frac{\tau b}{r}, \tag{5.4.37}$$

and hence

$$\gamma^2 = 64\left(\frac{\tilde{\tau}b}{r}\right)^4 + 16\left(\frac{\tilde{\tau}b}{r}\right)^2\sqrt{16\left(\frac{\tilde{\tau}b}{r}\right)^4 + 1}. \tag{5.4.38}$$

After integration of $\omega'(r) = \gamma$, we obtain the solution in the form

$$\begin{aligned}\omega = {} & 2\tilde{\tau}b\left[\log\frac{1+m(r)}{1-m(r)} - \log\frac{1+m(a)}{1-m(a)}\right] \\ + {} & 2\tilde{\tau}b\left[\frac{2}{m(a)} - \frac{2}{m(r)} - 2\tan^{-1}m(r) + 2\tan^{-1}m(a)\right],\end{aligned} \tag{5.4.39}$$

where $m(r)$ is defined by

$$m(r) = \left[\left(\frac{16\tilde{\tau}^4b^4}{r^4} + 1\right)^{\frac{1}{2}} - \frac{4\tilde{\tau}^2b^2}{r^2}\right]^{\frac{1}{2}} \tag{5.4.40}$$

and $\tilde{\tau}$ is once more given by (5.4.12), while

$$\begin{aligned}d = {} & 2\tilde{\tau}b\log\frac{[1+m(b)][1-m(a)]}{[1-m(b)][1+m(a)]} \\ + {} & 2\tilde{\tau}b\left[\frac{2}{m(a)} - \frac{2}{m(b)} - 2\tan^{-1}m(b) + 2\tan^{-1}m(a)\right].\end{aligned} \tag{5.4.41}$$

Case (c): $k = -1/2$

The strain-energy function has the form

$$\bar{W} = \frac{\mu}{3\sqrt{2}}\mu(I_2 - 1)^{\frac{3}{2}}h_2(1) + h_3(1), \tag{5.4.42}$$

where the functions h_1 and h_2 satisfy

$$h_2(1) = 1, \qquad h_3(1) = -\frac{2}{3}\mu, \qquad \frac{2}{3}\mu h_2'(1) + h_3'(1) = -\mu, \tag{5.4.43}$$

$$2\mu h_2'(1) + \frac{2}{3}\mu h_2''(1) + h_3''(1) = \frac{1}{4}\kappa - \frac{1}{6}\mu.$$

In respect to (5.4.42) equation (5.4.29) yields

$$\frac{1}{\sqrt{2}}\mu\gamma(\gamma^2 + 2)^{\frac{1}{2}} = \frac{\tau b}{r}. \tag{5.4.44}$$

Similarly to Case (b) specialization of (5.4.44) in this case leads to

$$\gamma^2 = \left(1 + \frac{32\tilde{\tau}^2 b^2}{r^2}\right)^{\frac{1}{2}} - 1, \tag{5.4.45}$$

and, after integration of $\omega'(r) = \gamma$, we find the resulting solution in the form

$$\omega = 4\tilde{\tau}b\left[n(r) - n(a) + \frac{1}{2}\log\frac{1+n(r)}{1-n(r)} - \frac{1}{2}\log\frac{1+n(a)}{1-n(a)}\right], \tag{5.4.46}$$

where $n(r)$ is defined by

$$n(r) = \sqrt{2}\left[\frac{1}{\left(1 + \frac{32\tilde{\tau}^2 b^2}{r^4}\right)^{\frac{1}{2}} + 1}\right]^{\frac{1}{2}}. \tag{5.4.47}$$

and $\tilde{\tau}$ is still given by (5.4.12), while

$$d = 4\tilde{\tau}b\left[n(b) - n(a) + \frac{1}{2}\log\frac{1+n(b)}{1-n(b)} - \frac{1}{2}\log\frac{1+n(a)}{1-n(a)}\right]. \tag{5.4.48}$$

In Fig. 5-1 we illustrate the relationship between d and τ in dimensionless form by plotting d/a against $\tilde{\tau}$ in respect of $k = 1/2,\ k = 1/4,\ k = -1/2$ and $k = 0$, the latter being the linear relationship obtained from (5.3.9) in the form

$$\frac{d}{a} = 4\tilde{\tau}\tilde{\eta}\log\tilde{\eta}. \tag{5.4.49}$$

The value 2 is taken as representative of the ratio $\tilde{\eta} = b/a$. The behaviour of (5.4.14) and (5.4.24) is similar to that for curve (a) in Fig. 5-1.

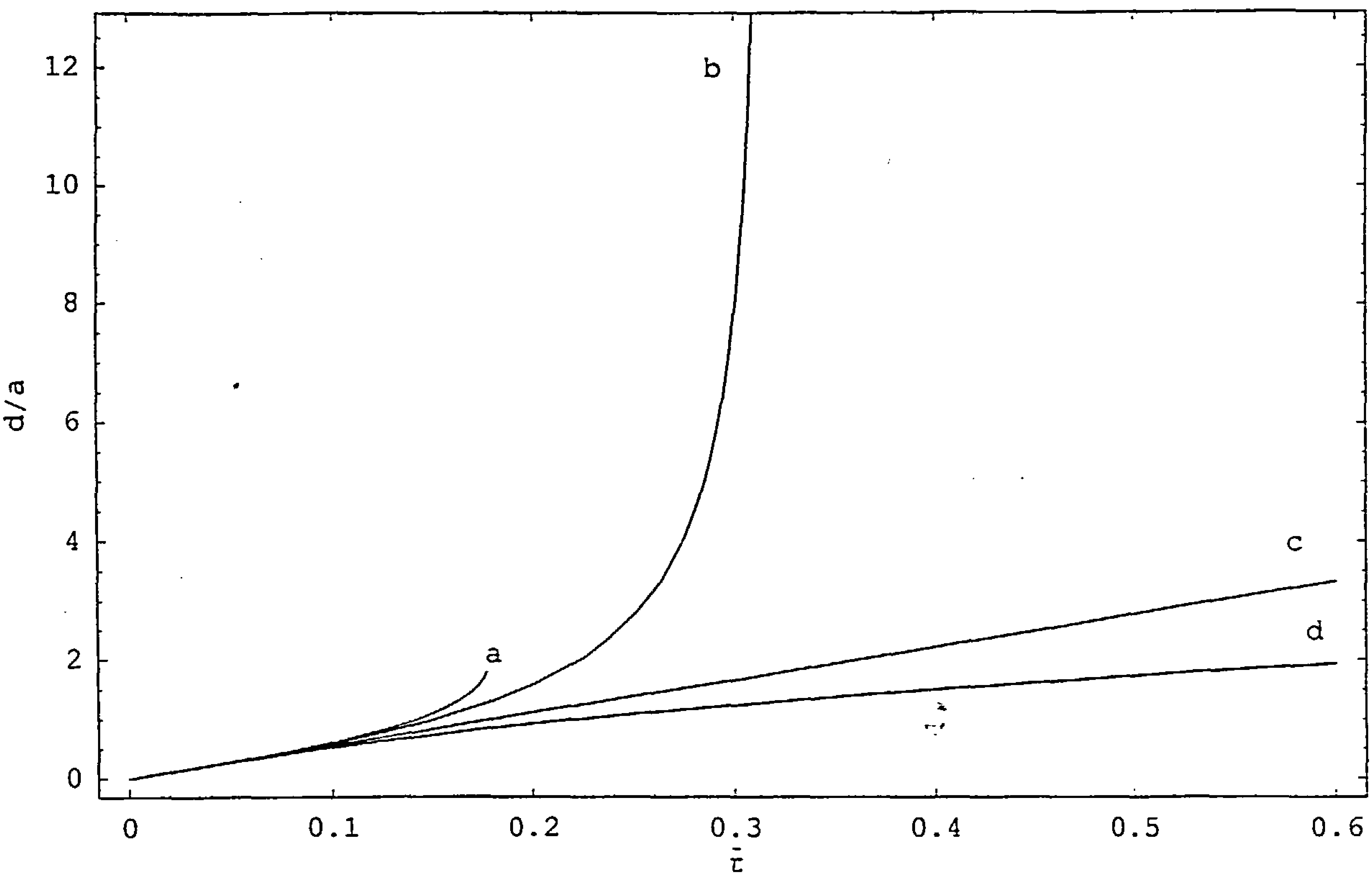


Figure 5.1: Plot of the dimensionless axial displacement d/a against the dimensionless axial shear stress $\bar{\tau}$ for strain-energy functions in the class (5.4.25) with (5.4.28) for $\bar{\eta} = 2$: (a) $k = 1/2$; (b) $k = 1/4$; (c) $k = -1/2$; (d) $k = 0$.

5.5 The incompressible problem

For an incompressible material the axial shear deformation is necessarily *pure* axial shear. Let the strain energy of an incompressible (isotropic) elastic material, regarded as a function of I_1 and I_2 , be written $\bar{\omega}(I_1, I_2)$. This may be identified with the strain energy of a compressible material, specialized for isochoric deformations, by writing

$$\bar{\omega}(I_1, I_2) \equiv \tilde{W}(I_1, I_2, 1). \tag{5.5.1}$$

Let

$$\hat{\omega}'(\gamma) = \bar{\omega}(\gamma^2 + 3, \gamma^2 + 3). \tag{5.5.2}$$

Then, the equilibrium equation (5.1.18) yields

$$\hat{\omega}'(\gamma) = \frac{\tau b}{r}, \tag{5.5.3}$$

which is equivalent to (5.2.5).

The radial equilibrium equation (5.1.19) serves to determine the hydrostatic stress p arising from the incompressibility constraint. In the present proble this entails the replacement of (5.1.12) by

$$t_1^{(1)} = \frac{\partial W}{\partial \lambda_1} - p\lambda_1^{-1}, \quad t_2^{(1)} = \frac{\partial W}{\partial \lambda_2} - p, \quad t_3^{(1)} = \frac{\partial W}{\partial \lambda_3} - p\lambda_3^{-1}, \tag{5.5.4}$$

evaluated for $\lambda_1\lambda_3 = 1, \lambda_2 = 1$, in (5.1.14) and (5.1.15).

Solutions for the incompressible proble may be obtained from those for the compressible problem disscussed in Section 5.4 (and for others obtained similarly) by identifying $\bar{\omega}(I_1, I_2)$, or its equivalent in terms of other variables, with the appropriate specific (compressible) strain-energy functions specialized for isochoric deformations. For example, in respect of (5.4.25) with (5.4.28), we set $\bar{\omega}(I_1, I_2) = g(I_2) - g(3)$, which is independent of I_1, and we recall that for the problem discussed here $I_1 = I_2$. The stress distribution in the materials is determined by integration of (5.2.6) with (5.5.4) duly specialized in order to obtain p. This procedure was illustrated for azimuthal shear in Chapter 3 and we do not give further details here.

Chapter 6

Azimuthal shear of an eccentric annulus

In Chapter 2 we discussed the plane strain character of the finite azimuthal shear of a circular cylindrical annulus of compressible isotropic elastic material and used this to express the strain energy as a function of two independent strain invariants. Some closed-form solutions for particular strain-energy functions were found. In this chapter we continue to consider the (plane strain) azimuthal shear of a circular tube but here we consider the centre circle to be displaced (in the considered plane) by a small displacement ϵ while the outer circle is fixed. The objective is to use the theory of small elastic deformations superimposed on the initial finite strain (which will be known) to determine the change in this initial strain due to the boundary displacement.

We use the results of Case (ii) in Section 3.4 as the basis for our calculations and consider that the solutions have the form of $u_r = A(r)\cos\theta + B(r)\sin\theta$ and

$u_\theta = C(r)\cos\theta + D(r)\sin\theta$. The governing incremental equations and boundary conditions are described and then numerical results are obtained to study the effect of the superimposed deformation on the displacement and the incremental nominal stresses.

6.1 The basic azimuthal shear deformation

As in Section 3.1 we let (R, Θ, Z) and (r, θ, z) be cylindrical polar coordinates in the reference and deformed configurations respectively. The compressible nonlinear elastic annulus (cross section of the tube) is defined by

$$0 < A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi. \tag{6.1.1}$$

We consider an azimuthal shear deformation of the form

$$r = r(R), \quad \theta = \Theta + g(R), \quad z = Z, \tag{6.1.2}$$

where $r(R)$ and $g(R)$ are functions to be found – in general they will depend on the properties of the material.

We write $\mathbf{x} = r\mathbf{e}_r$, where $\mathbf{e}_r$, $\mathbf{e}_\theta$ are polar coordinate axes associated with (r, θ) and, similarly, $\mathbf{X} = R\mathbf{e}_R$, with $\mathbf{e}_R$, $\mathbf{e}_\Theta$ the polar coordinate axes associated with (R, Θ). Here, $\mathbf{x}$ and $\mathbf{X}$ are (two-dimensional) position vectors in the plane of the annular cross-section, with the origin taken at the centre of the cross-section.

Henceforth, attention is restricted to the plane of the annular cross-section.

The *deformation gradient* given by equation (3.1.5) may be rewritten as

$$\begin{aligned}\mathbf{A} &= \operatorname{Grad}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathrm{R}} \otimes \mathbf{e}_{\mathrm{R}} + \frac{1}{\mathrm{R}}\frac{\partial \mathbf{x}}{\partial \Theta} \otimes \mathbf{e}_{\Theta} \\ &= r'\mathbf{e}_r \otimes \mathbf{e}_R + rg'\mathbf{e}_\theta \otimes \mathbf{e}_R + \frac{r}{R}\mathbf{e}_\theta \otimes \mathbf{e}_\Theta, \end{aligned} \tag{6.1.3}$$

from which we calculate the *left Cauchy-Green deformation tensor*

$$\mathbf{A}\mathbf{A}^T = r'^2\mathbf{e}_r \otimes \mathbf{e}_r + rr'g'\left(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r\right) + \left(r^2g'^2 + \frac{r^2}{R^2}\right)\mathbf{e}_\theta \otimes \mathbf{e}_\theta. \tag{6.1.4}$$

With respect to the polar coordinate axes the matrix of components of $\mathbf{A}\mathbf{A}^T$ is therefore

$$\mathbf{A}\mathbf{A}^T = \begin{bmatrix} r'^2 & rr'g' \\ rr'g' & r^2g'^2 + r^2/R^2 \end{bmatrix}. \tag{6.1.5}$$

Let $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ be the *Eulerian principal axes* associated with this deformation, that is the principal axes of $\mathbf{A}\mathbf{A}^T$. Then we express $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ in terms of the basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ in the form

$$\begin{aligned}\mathbf{v}^{(1)} &= \cos\phi\,\mathbf{e}_r + \sin\phi\,\mathbf{e}_\theta, \\ \mathbf{v}^{(2)} &= -\sin\phi\,\mathbf{e}_r + \cos\phi\,\mathbf{e}_\theta,\end{aligned} \tag{6.1.6}$$

where ϕ defines the orientation of the axis $\mathbf{v}^{(1)}$ relative to $\mathbf{e}_r$. We may also write $\mathbf{A}\mathbf{A}^T$ in the spectral form

$$\mathbf{A}\mathbf{A}^T = \lambda_1^2\mathbf{v}^{(1)} \otimes \mathbf{v}^{(1)} + \lambda_2^2\mathbf{v}^{(2)} \otimes \mathbf{v}^{(2)}, \tag{6.1.7}$$

where λ_1 and λ_2 are the principal stretches. Substituting equation (6.1.6) in equation (6.1.7) and then comparing with equation (6.1.5), we deduce that the eigenvalues λ_1^2 and λ_2^2 of $\mathbf{A}\mathbf{A}^T$ are such that

$$\begin{aligned}\lambda_1^2 + \lambda_2^2 &= r'^2 + r^2g'^2 + \frac{r^2}{R^2}, \\ \lambda_1^2\lambda_2^2 &= \frac{r^2r'^2}{R^2},\end{aligned} \tag{6.1.8}$$

and the angle ϕ is given by

$$\tan\phi = \frac{\lambda_1^2 - r'^2}{rr'g'}. \tag{6.1.9}$$

Note that λ_1^2 and λ_2^2 are solutions of the quadratic equation

$$\lambda^4 - \left(r'^2 + r^2 g'^2 + \frac{r^2}{R^2}\right)\lambda^2 + \frac{r^2 r'^2}{R^2} = 0 \tag{6.1.10}$$

for λ^2.

6.2 Superimposed incremental deformation and equilibrium equations

6.2.1 Superimposed incremental deformation

Let the small displacement $\mathbf{u}$ be superimposed on the basic deformation $\mathbf{x}$ and let

$$\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta, \tag{6.2.1}$$

where u_r and u_θ are functions of r and θ. The displacement gradient $\mathbf{\Gamma} \equiv \text{grad}\,\mathbf{u}$ is

$$\begin{aligned} &\frac{\partial}{\partial r}(u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta)\otimes\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}(u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta)\otimes\mathbf{e}_\theta \\ = &\frac{\partial u_r}{\partial r}\mathbf{e}_r\otimes\mathbf{e}_r + \frac{1}{r}(\frac{\partial u_r}{\partial\theta} - u_\theta)\mathbf{e}_r\otimes\mathbf{e}_\theta + \frac{\partial u_\theta}{\partial r}\mathbf{e}_\theta\otimes\mathbf{e}_r + \frac{1}{r}(\frac{\partial u_\theta}{\partial\theta} + u_r)\mathbf{e}_\theta\otimes\mathbf{e}_\theta, \end{aligned} \tag{6.2.2}$$

and the components of $\mathbf{\Gamma}$ referred to polar coordinates are therefore

$$\begin{aligned} \Gamma_{rr} &= \frac{\partial u_r}{\partial r}, \qquad \Gamma_{r\theta} = \frac{1}{r}(\frac{\partial u_r}{\partial\theta} - u_\theta), \\ \Gamma_{\theta r} &= \frac{\partial u_\theta}{\partial r}, \qquad \Gamma_{\theta\theta} = \frac{1}{r}(\frac{\partial u_\theta}{\partial\theta} + u_r). \end{aligned} \tag{6.2.3}$$

We take the incremental boundary conditions in the form

$$\begin{aligned} \mathbf{u} &= \mathbf{0} \qquad \text{on} \quad r = b, \\ \mathbf{u} &= \epsilon \qquad \text{on} \quad r = a, \end{aligned} \tag{6.2.4}$$

where $\boldsymbol{\epsilon}$ is a small rigid displacement (constant). In Cartesians we may write

$$\boldsymbol{\epsilon} = \epsilon(\cos\psi, \sin\psi), \tag{6.2.5}$$

where ψ is the angle between $\boldsymbol{\epsilon}$ and the x-axis, or, in polars

$$\boldsymbol{\epsilon} = \epsilon\cos(\theta-\psi)\mathbf{e}_r - \epsilon\sin(\theta-\psi)\mathbf{e}_\theta. \tag{6.2.6}$$

Without loss of generality, we take $\psi = 0$, so that

$$\boldsymbol{\epsilon} = \epsilon\cos\theta\ \mathbf{e}_r - \epsilon\sin\theta\ \mathbf{e}_\theta, \tag{6.2.7}$$

and, hence, the boundary conditions may be written

$$\begin{aligned} &u_r = u_\theta = 0 && \text{on} \quad r = b, \\ &u_r = \epsilon\cos\theta, \quad u_\theta = -\epsilon\sin\theta && \text{on} \quad r = a, \end{aligned} \tag{6.2.8}$$

or, equivalently, as

$$u_r + i\, u_\theta = \begin{cases} 0 & \text{on } r = b \\ \epsilon e^{-i\theta} & \text{on } r = a \end{cases} \tag{6.2.9}$$

with $i = \sqrt{-1}$.

6.2.2 Incremental constitutive law

The linearized form of the incremental stress-deformation relation, referred to the finitely deformed configuration, may be written

$$\dot{\mathbf{S}}_0 = \mathcal{A}_0\boldsymbol{\Gamma}, \tag{6.2.10}$$

where $\dot{\mathbf{S}}_0$ is the *increment* of the nominal stress tensor evaluated in the current configuration, and $\mathcal{A}_0$ is the tensor of *first-order instantaneous elastic moduli* associated with the conjugate pair $(\mathbf{S}, \mathbf{A})$ in the current configuration. The components of (6.2.10) referred to polar coordinates are then written as

$$\dot{S}_{0rr} = \mathcal{A}_{0rrrr}\Gamma_{rr} + \mathcal{A}_{0rrr\theta}\Gamma_{\theta r} + \mathcal{A}_{0rr\theta r}\Gamma_{r\theta} + \mathcal{A}_{0rr\theta\theta}\Gamma_{\theta\theta},$$

$$\dot{S}_{0r\theta} = \mathcal{A}_{0r\theta rr}\Gamma_{rr} + \mathcal{A}_{0r\theta r\theta}\Gamma_{\theta r} + \mathcal{A}_{0r\theta\theta r}\Gamma_{r\theta} + \mathcal{A}_{0r\theta\theta\theta}\Gamma_{\theta\theta},$$

$$\dot{S}_{0\theta r} = \mathcal{A}_{0\theta rrr}\Gamma_{rr} + \mathcal{A}_{0\theta rr\theta}\Gamma_{\theta r} + \mathcal{A}_{0\theta r\theta r}\Gamma_{r\theta} + \mathcal{A}_{0\theta r\theta\theta}\Gamma_{\theta\theta},$$

$$\dot{S}_{0\theta\theta} = \mathcal{A}_{0\theta\theta rr}\Gamma_{rr} + \mathcal{A}_{0\theta\theta r\theta}\Gamma_{\theta r} + \mathcal{A}_{0\theta\theta\theta r}\Gamma_{r\theta} + \mathcal{A}_{0\theta\theta\theta\theta}\Gamma_{\theta\theta}. \qquad (6.2.11)$$

Let $\bar{\mathcal{A}}_{0ijkl}$ be the components of $\mathcal{A}_0$ referred to the Eulerian principal axes $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$. Then, the only non-zero components are given by

$$J\bar{\mathcal{A}}_{01111} = \lambda_1^2 \frac{\partial^2 W}{\partial {\lambda_1}^2}, \quad J\bar{\mathcal{A}}_{01122} = \lambda_1\lambda_2 \frac{\partial^2 W}{\partial\lambda_1\partial\lambda_2}, \quad J\bar{\mathcal{A}}_{02222} = \lambda_2^2 \frac{\partial^2 W}{\partial {\lambda_2}^2},$$
$$J\bar{\mathcal{A}}_{01212} = \frac{\left(\lambda_1 \frac{\partial W}{\partial\lambda_1} - \lambda_2 \frac{\partial W}{\partial\lambda_2}\right)\lambda_1^2}{\lambda_1^2 - \lambda_2^2}, \quad J\bar{\mathcal{A}}_{02121} = \frac{\left(\lambda_1 \frac{\partial W}{\partial\lambda_1} - \lambda_2 \frac{\partial W}{\partial\lambda_2}\right)\lambda_2^2}{\lambda_1^2 - \lambda_2^2},$$
$$J\bar{\mathcal{A}}_{01221} = J\bar{\mathcal{A}}_{02112} = J\bar{\mathcal{A}}_{01212} - \lambda_1 \frac{\partial W}{\partial\lambda_1}. \qquad (6.2.12)$$

It is convenient to write

$$\alpha_{11} = \bar{\mathcal{A}}_{01111}, \qquad \alpha_{12} = \bar{\mathcal{A}}_{01122}, \qquad \alpha_{22} = \bar{\mathcal{A}}_{02222},$$

$$\gamma_{12} = \bar{\mathcal{A}}_{01212}, \qquad \gamma_{21} = \bar{\mathcal{A}}_{02121}, \qquad \delta_{12} = \alpha_{12} + \bar{\mathcal{A}}_{01221}. \qquad (6.2.13)$$

The connection between the componts $\bar{\mathcal{A}}_{0ijkl}$ and $\mathcal{A}_{0ijkl}$ (in polars, with indices r, θ corresponding to 1, 2) is

$$\mathcal{A}_{0ijkl} = l_{ip}l_{jq}l_{kr}l_{ls}\bar{\mathcal{A}}_{0pqrs}, \qquad (6.2.14)$$

where l_{ij} $(i,\ j = 1,\ 2)$ are the components of $\mathbf{L}$ given by

$$\mathbf{L} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \qquad (6.2.15)$$

and ϕ is determined by equation (6.1.9). Making use of equations (6.2.12) and (6.2.13), this leads to

$$\mathcal{A}_{0rrrr} = \alpha_{11}\cos^4\phi + (\gamma_{12}+\gamma_{21}+2\delta_{12})\sin^2\phi\cos^2\phi + \alpha_{22}\sin^4\phi,$$

$$\mathcal{A}_{0rrr\theta} = \sin\phi\cos\phi[(\alpha_{11}-\gamma_{12}-\delta_{12})\cos^2\phi - (\alpha_{22}-\gamma_{21}-\delta_{12})\sin^2\phi],$$

$$\mathcal{A}_{0rr\theta r} = \sin\phi\cos\phi[(\alpha_{11}-\gamma_{21}-\delta_{12})\cos^2\phi - (\alpha_{22}-\gamma_{12}-\delta_{12})\sin^2\phi],$$

$$\mathcal{A}_{0rr\theta\theta} = \sin^2\phi\cos^2\phi(\alpha_{11}+\alpha_{22}-\gamma_{12}-\gamma_{21}-2\delta_{12}) + \alpha_{12},$$

$$\mathcal{A}_{0r\theta rr} = \mathcal{A}_{0rrr\theta},$$

$$\mathcal{A}_{0r\theta r\theta} = (\alpha_{11}+\alpha_{22}-2\delta_{12})\sin^2\phi\cos^2\phi + \gamma_{21}\sin^4\phi + \gamma_{12}\cos^4\phi,$$

$$\mathcal{A}_{0r\theta\theta r} = \sin^2\phi\cos^2\phi(\alpha_{11}+\alpha_{22}-\gamma_{12}-\gamma_{21}-2\delta_{12}) + \delta_{12} - \alpha_{12},$$

$$\mathcal{A}_{0r\theta\theta\theta} = \sin\phi\cos\phi[(\alpha_{11}-\gamma_{21}-\delta_{12})\sin^2\phi - (\alpha_{22}-\gamma_{12}-\delta_{12})\cos^2\phi],$$

$$\mathcal{A}_{0\theta rrr} = \mathcal{A}_{0rr\theta r}, \tag{6.2.16}$$

$$\mathcal{A}_{0\theta r\theta r} = (\alpha_{11}+\alpha_{22}-2\delta_{12})\sin^2\phi\cos^2\phi + \gamma_{12}\sin^4\phi + \gamma_{21}\cos^4\phi,$$

$$\mathcal{A}_{0\theta rr\theta} = \mathcal{A}_{0r\theta\theta r},$$

$$\mathcal{A}_{0\theta r\theta\theta} = \sin\phi\cos\phi[(\alpha_{11}-\gamma_{12}-\delta_{12})\sin^2\phi - (\alpha_{22}-\gamma_{21}-\delta_{12})\cos^2\phi],$$

$$\mathcal{A}_{0\theta\theta rr} = \mathcal{A}_{0rr\theta\theta},$$

$$\mathcal{A}_{0\theta\theta r\theta} = \mathcal{A}_{0r\theta\theta\theta},$$

$$\mathcal{A}_{0\theta\theta\theta r} = \mathcal{A}_{0\theta r\theta\theta},$$

$$\mathcal{A}_{0\theta\theta\theta\theta} = \alpha_{11}\sin^4\phi + (\gamma_{12}+\gamma_{21}+2\delta_{12})\sin^2\phi\cos^2\phi + \alpha_{22}\cos^4\phi.$$

6.2.3 Incremental equilibrium

When there are no body force, the incremental equilibrium equations are

$$\operatorname{div}\dot{\mathbf{S}}_0 = \mathbf{0}. \tag{6.2.17}$$

In polar coordinates, with restriction to plane strain, this leads to the two equations

$$\frac{\partial \dot{S}_{0rr}}{\partial r}+\frac{1}{r}\frac{\partial \dot{S}_{0\theta r}}{\partial \theta}+\frac{1}{r}\left(\dot{S}_{0rr}-\dot{S}_{0\theta\theta}\right)=0,$$
$$\frac{\partial \dot{S}_{0r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \dot{S}_{0\theta\theta}}{\partial \theta}+\frac{1}{r}\left(\dot{S}_{0r\theta}+\dot{S}_{0\theta r}\right)=0. \qquad (6.2.18)$$

We now substitute equations (6.2.11) into equations (6.2.18) and make use of the expressions (6.2.3). This yields

$$\begin{aligned}
&\mathcal{A}_{0rrrr}u_{r,rr}+\frac{2}{r}\mathcal{A}_{0rr\theta r}u_{r,r\theta}+\frac{1}{r^2}\mathcal{A}_{0\theta r\theta r}u_{r,\theta\theta}+\mathcal{A}_{0rrr\theta}u_{\theta,rr}\\
&+\frac{1}{r}\left(\mathcal{A}_{0rr\theta\theta}+\mathcal{A}_{0\theta rr\theta}\right)u_{\theta,r\theta}+\frac{1}{r^2}\mathcal{A}_{0\theta r\theta\theta}u_{\theta,\theta\theta}+\left(\mathcal{A}'_{0rrrr}+\frac{1}{r}\mathcal{A}_{0rrrr}\right)u_{r,r}\\
&+\frac{1}{r}\mathcal{A}'_{0rr\theta r}u_{r,\theta}+\left(\mathcal{A}'_{0rrr\theta}-\frac{1}{r}\mathcal{A}_{0rr\theta r}+\frac{1}{r}\mathcal{A}_{0rrr\theta}-\frac{1}{r}\mathcal{A}_{0\theta\theta r\theta}\right)u_{\theta,r}\\
&+\frac{1}{r}\left(\mathcal{A}'_{0rr\theta\theta}-\frac{1}{r}\mathcal{A}_{0\theta\theta\theta\theta}-\frac{1}{r}\mathcal{A}_{0\theta r\theta r}\right)u_{\theta,\theta}\\
&+\frac{1}{r}\left(\mathcal{A}'_{0rr\theta\theta}-\frac{1}{r}\mathcal{A}_{0\theta\theta\theta\theta}\right)u_{r}-\frac{1}{r}\left(\mathcal{A}'_{0rr\theta r}-\frac{1}{r}\mathcal{A}_{0\theta\theta\theta r}\right)u_{\theta}=0,
\end{aligned} \qquad (6.2.19)$$

$$\begin{aligned}
&\mathcal{A}_{0r\theta rr}u_{r,rr}+\frac{1}{r}\left(\mathcal{A}_{0rr\theta\theta}+\mathcal{A}_{0r\theta\theta r}\right)u_{r,r\theta}+\frac{1}{r^2}\mathcal{A}_{0\theta\theta\theta r}u_{r,\theta\theta}\\
&+\mathcal{A}_{0r\theta r\theta}u_{\theta,rr}+\frac{2}{r}\mathcal{A}_{0\theta\theta r\theta}u_{\theta,r\theta}+\frac{1}{r^2}\mathcal{A}_{0\theta\theta\theta\theta}u_{\theta,\theta\theta}\\
&+\left(\mathcal{A}'_{0r\theta rr}+\frac{1}{r}\mathcal{A}_{0rr\theta r}+\frac{1}{r}\mathcal{A}_{0rrr\theta}+\frac{1}{r}\mathcal{A}_{0r\theta\theta\theta}\right)u_{r,r}\\
&+\frac{1}{r}\left(\mathcal{A}'_{0r\theta\theta r}+\frac{1}{r}\mathcal{A}_{0\theta r\theta r}+\frac{1}{r}\mathcal{A}_{0\theta\theta\theta\theta}\right)u_{r,\theta}+\left(\mathcal{A}'_{0r\theta r\theta}+\frac{1}{r}\mathcal{A}_{0r\theta r\theta}\right)u_{\theta,r}\\
&+\frac{1}{r}\mathcal{A}'_{0r\theta\theta\theta}u_{\theta,\theta}+\frac{1}{r}\left(\mathcal{A}'_{0r\theta\theta\theta}+\frac{1}{r}\mathcal{A}_{0\theta r\theta\theta}\right)u_{r}-\frac{1}{r}\left(\mathcal{A}'_{0r\theta\theta r}+\frac{1}{r}\mathcal{A}_{0\theta r\theta r}\right)u_{\theta}=0.
\end{aligned} \qquad (6.2.20)$$

In these equations, $u_{\alpha,\beta}$ and $u_{\alpha,\beta\gamma}$ denote $\partial u_\alpha/\partial\beta$ and $\partial^2 u_\alpha/\partial\beta\partial\gamma$ $(\alpha,\ \beta,\ \gamma=r,\ \theta)$ respectively, and a prime denotes differentiation with respect to r.

In view of the boundary conditions (6.2.8) we consider solutions of the form

$$u_r=A(r)\cos\theta+B(r)\sin\theta,$$
$$u_\theta=C(r)\cos\theta+D(r)\sin\theta, \qquad (6.2.21)$$

and, hence, the boundary conditions (6.2.8) become

$$A(a) = \epsilon, \quad B(a) = 0, \quad C(a) = 0, \quad D(a) = -\epsilon,$$
$$A(b) = B(b) = C(b) = D(b) = 0. \tag{6.2.22}$$

Substitution of equations (6.2.21) into equations (6.2.19) and (6.2.20) enables the equilibrium problem to be expressed in the form of the four ordinary differential equations, namely

$$\begin{aligned}
&\mathcal{A}_{0rrrr}A'' + \mathcal{A}_{0rrr\theta}C'' + \left(\mathcal{A}'_{0rrrr} + \frac{1}{r}\mathcal{A}_{0rrrr}\right)A' + \frac{2}{r}\mathcal{A}_{0rr\theta r}B' \\
&+\left(\mathcal{A}'_{0rrr\theta} - \frac{1}{r}\mathcal{A}_{0rr\theta r} + \frac{1}{r}\mathcal{A}_{0rrr\theta} - \frac{1}{r}\mathcal{A}_{0\theta\theta r\theta}\right)C' + \frac{1}{r}\left(\mathcal{A}_{0\theta\theta rr} + \mathcal{A}_{0\theta rr\theta}\right)D' \\
&+\frac{1}{r}\left(\mathcal{A}'_{0rr\theta\theta} - \frac{1}{r}\mathcal{A}_{0\theta\theta\theta\theta} - \frac{1}{r}\mathcal{A}_{0\theta r\theta r}\right)A + \frac{1}{r}\mathcal{A}'_{0rr\theta r}B - \frac{1}{r}\mathcal{A}_{0rr\theta r}C \\
&+\frac{1}{r}\left(\mathcal{A}'_{0rr\theta\theta} - \frac{1}{r}\mathcal{A}_{0\theta\theta\theta\theta} - \frac{1}{r}\mathcal{A}_{0\theta r\theta r}\right)D = 0,
\end{aligned} \tag{6.2.23}$$

$$\begin{aligned}
&\mathcal{A}_{0rrrr}B'' + \mathcal{A}_{0rrr\theta}D'' + \left(\mathcal{A}'_{0rrrr} + \frac{1}{r}\mathcal{A}_{0rrrr}\right)B' - \frac{2}{r}\mathcal{A}_{0rr\theta r}A' \\
&-\frac{1}{r}\left(\mathcal{A}_{0\theta\theta rr} + \mathcal{A}_{0\theta rr\theta}\right)C' + \left(\mathcal{A}'_{0rrr\theta} - \frac{1}{r}\mathcal{A}_{0rr\theta r} + \frac{1}{r}\mathcal{A}_{0rrr\theta} - \frac{1}{r}\mathcal{A}_{0\theta\theta r\theta}\right)D' \\
&+\frac{1}{r}\left(\mathcal{A}'_{0rr\theta\theta} - \frac{1}{r}\mathcal{A}_{0\theta\theta\theta\theta} - \frac{1}{r}\mathcal{A}_{0\theta r\theta r}\right)B - \frac{1}{r}\mathcal{A}'_{0rr\theta r}A \\
&-\frac{1}{r}\left(\mathcal{A}'_{0rr\theta\theta} - \frac{1}{r}\mathcal{A}_{0\theta\theta\theta\theta} - \frac{1}{r}\mathcal{A}_{0\theta r\theta r}\right)C - \frac{1}{r}\mathcal{A}_{0rr\theta r}D = 0,
\end{aligned} \tag{6.2.24}$$

$$\begin{aligned}
&\mathcal{A}_{0r\theta rr}A'' + \mathcal{A}_{0r\theta r\theta}C'' + \left(\mathcal{A}'_{0r\theta rr} + \frac{1}{r}\mathcal{A}_{0rr\theta r} + \frac{1}{r}\mathcal{A}_{0rrr\theta} + \frac{1}{r}\mathcal{A}_{0r\theta\theta\theta}\right)A' \\
&+\frac{1}{r}\left(\mathcal{A}_{0rr\theta\theta} + \mathcal{A}_{0r\theta\theta r}\right)B' + \left(\mathcal{A}'_{0r\theta r\theta} + \frac{1}{r}\mathcal{A}_{0r\theta r\theta}\right)C' + \frac{2}{r}\mathcal{A}_{0\theta\theta r\theta}D' + \frac{1}{r}\mathcal{A}'_{0r\theta\theta\theta}A \\
&+\frac{1}{r}\left(\mathcal{A}'_{0r\theta\theta r} + \frac{1}{r}\mathcal{A}_{0\theta r\theta r} + \frac{1}{r}\mathcal{A}_{0\theta\theta\theta\theta}\right)B - \frac{1}{r}\left(\mathcal{A}'_{0r\theta\theta r} + \frac{1}{r}\mathcal{A}_{0\theta r\theta r} + \frac{1}{r}\mathcal{A}_{0\theta\theta\theta\theta}\right)C \\
&+\frac{1}{r}\mathcal{A}_{0r\theta\theta\theta}D = 0,
\end{aligned} \tag{6.2.25}$$

$$\begin{aligned}
&\mathcal{A}_{0r\theta r\theta}B'' + \mathcal{A}_{0r\theta r\theta}D'' - \frac{1}{r}\left(\mathcal{A}_{0rr\theta\theta} + \mathcal{A}_{0r\theta\theta r}\right)A' \\
&+\left(\mathcal{A}'_{0r\theta rr} + \frac{1}{r}\mathcal{A}_{0rr\theta r} + \frac{1}{r}\mathcal{A}_{0rrr\theta} + \frac{1}{r}\mathcal{A}_{0r\theta\theta\theta}\right)B' - \frac{2}{r}\mathcal{A}_{0\theta\theta r\theta}C' + \frac{1}{r}\mathcal{A}'_{0r\theta\theta\theta}B
\end{aligned}$$

$$+\left(\mathcal{A}'_{0r\theta r\theta}+\frac{1}{r}\mathcal{A}_{0r\theta r\theta}\right)D'-\frac{1}{r}\left(\mathcal{A}'_{0r\theta\theta r}+\frac{1}{r}\mathcal{A}_{0\theta r\theta r}+\frac{1}{r}\mathcal{A}_{0\theta\theta\theta\theta}\right)A$$

$$-\frac{1}{r}\mathcal{A}_{0r\theta\theta\theta}C-\frac{1}{r}\left(\mathcal{A}'_{0r\theta\theta r}+\frac{1}{r}\mathcal{A}_{0\theta r\theta r}+\frac{1}{r}\mathcal{A}_{0\theta\theta\theta\theta}\right)D=0, \quad (6.2.26)$$

for $A(r)$, $B(r)$, $C(r)$ and $D(r)$.

6.3 The pure azimuthal shear problem

Pure azimuthal shear is the isochoric specialization of the deformation (6.1.2) corresponding to $r = R$. Then, equations (6.1.8) reduce to

$$\lambda_1^2+\lambda_2^2 = 2+r^2g'^2,$$

$$\lambda_1\lambda_2 = 1, \quad (6.3.1)$$

as discussed in Section 3.3. We therefore write

$$\lambda_1 = \lambda, \qquad \lambda_2 = \lambda^{-1}, \quad (6.3.2)$$

so that

$$\lambda-\lambda^{-1} = rg', \qquad \tan\phi = \lambda, \quad (6.3.3)$$

and, locally, the deformation corresponds to a simple shear.

For this case $g(r)$ can be found for special choices of strain-energy function, and the strain-energy function depends only on λ or, equivalently, on the amount of shear $rg'(r)$.

Examples of these solutions are given in Chapter 3 and can be used as the basis for considering superimposed incremental deformations. Corresponding boundary conditions for the basic problem are as discussed in Section 3.1.

Making use of equation (6.3.2) we note that

$$\sin\phi = \frac{\lambda}{\sqrt{\lambda^2+1}}, \qquad \cos\phi = \frac{1}{\sqrt{\lambda^2+1}}, \tag{6.3.4}$$

and equations (6.2.16) may be rewritten as

$$\begin{aligned}
\mathcal{A}_{0rrrr} &= \frac{1}{(\lambda^2+1)^2}[\alpha_{11} + (\gamma_{12}+\gamma_{21}+2\delta_{12})\lambda^2 + \alpha_{22}\lambda^4],\\
\mathcal{A}_{0rrr\theta} &= \frac{\lambda}{(\lambda^2+1)^2}[\alpha_{11} - \gamma_{12} - \delta_{12} - (\alpha_{22}-\gamma_{21}-\delta_{12})\lambda^2],\\
\mathcal{A}_{0rr\theta r} &= \frac{\lambda}{(\lambda^2+1)^2}[\alpha_{11} - \gamma_{21} - \delta_{12} - (\alpha_{22}-\gamma_{12}-\delta_{12})\lambda^2],\\
\mathcal{A}_{0rr\theta\theta} &= \frac{\lambda^2}{(\lambda^2+1)^2}(\alpha_{11}+\alpha_{22}-\gamma_{12}-\gamma_{21}-2\delta_{12}) + \alpha_{12},\\
\mathcal{A}_{0r\theta rr} &= \mathcal{A}_{0rrr\theta},\\
\mathcal{A}_{0r\theta r\theta} &= \frac{1}{(\lambda^2+1)^2}[(\alpha_{11}+\alpha_{22}-2\delta_{12})\lambda^2 + \gamma_{21}\lambda^4 + \gamma_{12}],\\
\mathcal{A}_{0r\theta\theta r} &= \frac{\lambda^2}{(\lambda^2+1)^2}(\alpha_{11}+\alpha_{22}-\gamma_{12}-\gamma_{21}-2\delta_{12}) + \delta_{12} - \alpha_{12},\\
\mathcal{A}_{0r\theta\theta\theta} &= \frac{\lambda}{(\lambda^2+1)^2}[(\alpha_{11}-\gamma_{21}-\delta_{12})\lambda^2 - \alpha_{22} - \gamma_{12} - \delta_{12}],\\
\\
\mathcal{A}_{0\theta rrr} &= \mathcal{A}_{0rr\theta r},\\
\mathcal{A}_{0\theta r\theta r} &= \frac{1}{(\lambda^2+1)^2}[(\alpha_{11}+\alpha_{22}-2\delta_{12})\lambda^2 + \gamma_{12}\lambda^4 + \gamma_{21}],\\
\mathcal{A}_{0\theta rr\theta} &= \mathcal{A}_{0rr\theta\theta},\\
\mathcal{A}_{0\theta r\theta\theta} &= \frac{\lambda}{(\lambda^2+1)^2}[(\alpha_{11}-\gamma_{12}-\delta_{12})\lambda^2\phi - \alpha_{22} - \gamma_{21} - \delta_{12}],\\
\mathcal{A}_{0\theta\theta rr} &= \mathcal{A}_{0rr\theta\theta},\\
\mathcal{A}_{0\theta\theta r\theta} &= \mathcal{A}_{0r\theta\theta\theta},\\
\mathcal{A}_{0\theta\theta\theta rr} &= \mathcal{A}_{0\theta r\theta\theta},\\
\mathcal{A}_{0\theta\theta\theta\theta} &= \frac{1}{(\lambda^2+1)^2}[\alpha_{11}\lambda^4 + (\gamma_{12}+\gamma_{21}+2\delta_{12})\lambda^2 + \alpha_{22}].
\end{aligned} \tag{6.3.5}$$

6.4 A special form of strain-energy function

Now we consider Case (ii) in Section 3.2 with the strain-energy function given by

$$W(I_1, I_3) = \frac{1}{2}\mu I_1 h_1(I_3) + h_2(I_3), \tag{6.4.1}$$

where

$$I_1 = \lambda_1^2 + \lambda_2^2, \qquad I_3 = \lambda_1\lambda_2, \tag{6.4.2}$$

with

$$h_1(1) = 1, \quad h_1'(1) = -\frac{1}{4}, \quad h_2(1) = -\mu, \quad h_2'(1) = -\frac{1}{4}\mu, \tag{6.4.3}$$

and

$$\mu h_1''(1) + h_2''(1) = \frac{1}{4}\kappa + \frac{7\mu}{12}. \tag{6.4.4}$$

For this form of strain-energy function the finite underlying deformation is a pure azimuthal shear, and the solution of the problem is

$$\mu\gamma = \frac{\tau b^2}{r^2}, \tag{6.4.5}$$

with

$$rg'(r) \equiv \gamma = \lambda - \lambda^{-1}. \tag{6.4.6}$$

Making use of equation (6.3.2), we calculate that

$$\begin{aligned}
\frac{\partial W}{\partial \lambda_1} &= \frac{3}{4}\mu\lambda - \frac{1}{2}\mu\lambda^{-1} - \frac{1}{4}\mu\lambda^{-3},\\
\frac{\partial W}{\partial \lambda_2} &= \frac{3}{4}\mu\lambda^{-1} - \frac{1}{2}\mu\lambda - \frac{1}{4}\mu\lambda^{3},\\
\frac{\partial^2 W}{\partial {\lambda_1}^2} &= 2\mu(1+\lambda^{-4})h_1''(1) + 4\lambda^{-2}h_2''(1) - \frac{1}{4}\mu - \frac{1}{4}\mu\lambda^{-4} - \frac{1}{2}\mu\lambda^{-2},\\
\frac{\partial^2 W}{\partial {\lambda_2}^2} &= 2\mu(1+\lambda^{4})h_1''(1) + 4\lambda^{2}h_2''(1) - \frac{1}{4}\mu - \frac{1}{4}\mu\lambda^{4} - \frac{1}{2}\mu\lambda^{2},\\
\frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} &= 2\mu(\lambda^2+\lambda^{-2})h_1''(1) + 4h_2''(1) - \mu - \mu\lambda^2 - \mu\lambda^{-2}.
\end{aligned}$$

Substituting equations (6.4.6) and (6.4.7) into equations (6.2.12), it follows that

$$
\begin{aligned}
J\bar{\mathcal{A}}_{01111} &= 2\mu(\lambda^2 + \lambda^{-4})h_1''(1) + 4h_2''(1) - \frac{1}{4}\mu\lambda^2 - \frac{1}{4}\mu\lambda^{-2} - \frac{1}{2}\mu, \\
J\bar{\mathcal{A}}_{01122} &= 2\mu(\lambda^2 + \lambda^{-2})h_1''(1) + 4h_2''(1) - \mu - \mu\lambda^2 - \mu\lambda^{-2}, \\
J\bar{\mathcal{A}}_{02222} &= J\bar{\mathcal{A}}_{01111}, \\
J\bar{\mathcal{A}}_{01212} &= \mu\lambda^2, \\
J\bar{\mathcal{A}}_{02121} &= \mu\lambda^{-2}, \\
J\bar{\mathcal{A}}_{01221} &= J\bar{\mathcal{A}}_{02112} = \frac{1}{4}\mu(\lambda^2 + \lambda^{-2}) + \frac{1}{2}\mu.
\end{aligned} \tag{6.4.7}
$$

For the reason of simplification we set $\delta_{12} =$ constant, by setting

$$
h_1''(1) = \frac{3}{8}, \qquad h_2''(1) = \frac{1}{4}\left(\kappa + \frac{5}{6}\mu\right), \tag{6.4.8}
$$

with equation (6.4.4) holding identically. The material parameters in equations (6.2.13) are then given by

$$
\begin{aligned}
\alpha_{11} &= \alpha_{22} = \frac{1}{2}\mu(\lambda^2 + \lambda^{-2}) + \delta, \\
\alpha_{12} &= -\frac{1}{4}\mu(\lambda^2 + \lambda^{-2}) + \delta - \frac{1}{2}\mu, \\
\gamma_{12} &= \mu\lambda^2, \quad \gamma_{21} = \mu\lambda^{-2}, \quad \delta_{12} = \delta = \kappa + \frac{1}{3}\mu.
\end{aligned} \tag{6.4.9}
$$

It follows from equation (6.3.5) with the help of equation (6.4.6), that

$$
\begin{aligned}
\mathcal{A}_{0rrrr} &= \mathcal{A}_{0\theta\theta\theta\theta} = \frac{1}{2}\mu\gamma^2 + \delta + \mu, \\
\mathcal{A}_{0rrr\theta} &= \mathcal{A}_{0r\theta rr} = \mathcal{A}_{0\theta r\theta\theta} = \mathcal{A}_{0\theta\theta\theta r} = -\frac{1}{2}\mu\gamma, \\
\mathcal{A}_{0rr\theta r} &= \mathcal{A}_{0r\theta\theta\theta} = \mathcal{A}_{0\theta rrr} = \mathcal{A}_{0\theta\theta r\theta} = \frac{1}{2}\mu\gamma, \\
\mathcal{A}_{0rr\theta\theta} &= \mathcal{A}_{0\theta\theta rr} = -\frac{1}{4}\mu\gamma^2 + \delta - \mu, \\
\mathcal{A}_{0r\theta r\theta} &= \mu
\end{aligned} \tag{6.4.10}
$$

$$\mathcal{A}_{0r\theta\theta r} = \mathcal{A}_{0r\theta\theta r} = \frac{1}{4}\mu\gamma^2 + \mu,$$

$$\mathcal{A}_{0\theta r\theta r} = \mu\gamma^2 + \mu,$$

Let $T = \tau b^2/\mu$. Then equation (6.4.5) provides

$$\gamma = Tr^{-2}. \tag{6.4.11}$$

Substituting equations (6.4.10) and (6.4.11) into equations (6.2.25) - (6.2.28) and using the notation of $\bar{\delta} = \delta/\mu$, we obtain

$$\begin{aligned}
&\left(\frac{1}{2}T^2r^{-4} + \bar{\delta} + 1\right)A'' - \frac{1}{2}Tr^{-2}C'' - \left[\frac{3}{2}T^2r^{-5} - (\bar{\delta}+1)r^{-1}\right]A' + \frac{1}{2}Tr^{-3}B' \\
&-\frac{1}{2}Tr^{-3}C' + \bar{\delta}r^{-1}D' - \left[\frac{1}{2}T^2r^{-6} + (\bar{\delta}+2)r^{-2}\right]A - Tr^{-4}B + Tr^{-4}C \\
&-\left[\frac{1}{2}T^2r^{-6} + (\bar{\delta}+2)r^{-2}\right]D = 0,
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{1}{2}T^2r^{-4} + \bar{\delta} + 1\right)B'' - \frac{1}{2}Tr^{-2}D'' - \left[\frac{3}{2}T^2r^{-5} - (\bar{\delta}+1)r^{-1}\right]B' - \frac{1}{2}nTr^{-3}A' \\
&-\frac{1}{2}Tr^{-3}D' - \bar{\delta}r^{-1}C' - \left[\frac{1}{2}T^2r^{-6} + (\bar{\delta}+2)r^{-2}\right]B + Tr^{-4}A + Tr^{-4}D \\
&+\left[\frac{1}{2}T^2r^{-6} + (\bar{\delta}+2)r^{-2}\right]C = 0,
\end{aligned} \tag{6.4.12}$$

$$\begin{aligned}
&-\frac{1}{2}Tr^{-2}A'' + C'' + \frac{3}{2}Tr^{-3}A' + \bar{\delta}r^{-1}B' + r^{-1}C' + Tr^{-3}D' - Tr^{-4}A \\
&+\left[\frac{1}{2}T^2r^{-6} + (\bar{\delta}+2)r^{-2}\right]B - \left[\frac{1}{2}T^2r^{-6} + (\bar{\delta}+1)r^{-2}\right]C - Tr^{-4}D = 0,
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{2}Tr^{-2}B'' + D'' + \frac{3}{2}Tr^{-3}B' - \bar{\delta}r^{-1}A' + r^{-1}D' - Tr^{-3}C' - Tr^{-4}B \\
&-\left[\frac{1}{2}T^2r^{-6} + (\bar{\delta}+2)r^{-2}\right]A + Tr^{-4}C - \left[\frac{1}{2}T^2r^{-6} + (\bar{\delta}+1)r^{-2}\right]D = 0.
\end{aligned}$$

These can be also rearranged as

$$\begin{aligned}
A'' = {} & \frac{\frac{3}{4}T^2r^{-5} - (\bar{\delta}+1)r^{-1}}{\frac{1}{4}T^2r^{-4} + \bar{\delta} + 1}A' - \frac{\frac{1}{2}(\bar{\delta}+2)Tr^{-3}}{\frac{1}{4}T^2r^{-4} + \bar{\delta} + 1}B' - \frac{\frac{1}{2}T^2r^{-5} + \bar{\delta}r^{-1}}{\frac{1}{4}T^2r^{-4} + \bar{\delta} + 1}D' \\
&+\frac{T^2r^{-6} + (\bar{\delta}+2)r^{-2}}{\frac{1}{4}T^2r^{-4} + \bar{\delta} + 1}A - \frac{\frac{1}{4}T^3r^{-8} + \frac{1}{2}\bar{\delta}Tr^{-4}}{\frac{1}{4}T^2r^{-4} + \bar{\delta} + 1}B + \frac{\frac{1}{4}T^3r^{-8} + \frac{1}{2}\bar{\delta}Tr^{-4}}{\frac{1}{4}T^2r^{-4} + \bar{\delta} + 1}C \\
&+\frac{T^2r^{-6} + (\bar{\delta}+2)r^{-2}}{\frac{1}{4}T^2r^{-4} + \bar{\delta} + 1}D,
\end{aligned}$$

$$
\begin{aligned}
B'' = {} & \frac{\frac{1}{2}(\bar\delta+2)Tr^{-3}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}A' + \frac{\frac{3}{4}T^2r^{-5}-(\bar\delta+1)r^{-1}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}B' - \frac{\frac{1}{2}T^2r^{-5}+\bar\delta r^{-1}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}C' \\
& + \frac{\frac{1}{4}T^3r^{-8}+\frac{1}{2}\bar\delta Tr^{-4}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}A + \frac{T^2r^{-6}+(\bar\delta+2)r^{-2}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}B - \frac{T^2r^{-6}+(\bar\delta+2)r^{-2}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}C \\
& + \frac{\frac{1}{4}T^3r^{-8}+\frac{1}{2}\bar\delta Tr^{-4}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}D, \qquad (6.4.13)
\end{aligned}
$$

$$
\begin{aligned}
C'' = {} & \frac{-2(\bar\delta+1)Tr^{-3}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}A' - \frac{(\bar\delta+1)\left(\frac{1}{2}T^2r^{-5}+\bar\delta r^{-1}\right)}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}B' - \frac{\frac{1}{4}T^2r^{-4}+(\bar\delta+1)r^{-1}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}C' \\
& \frac{\frac{1}{2}T^3r^{-7}+(\frac{3}{2}\bar\delta+2)Tr^{-4}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}D' + \frac{\frac{3}{4}T^3r^{-8}+(\frac{3}{2}\bar\delta+2)Tr^{-4}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}A \\
& - \frac{\frac{1}{4}T^4r^{-10}+(\bar\delta+1)[T^2r^{-6}+(\bar\delta+2)r^{-2}]}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}B \\
& + \frac{\frac{1}{4}T^4r^{-10}+(\bar\delta+1)[T^2r^{-6}+(\bar\delta+2)r^{-2}]}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}C + \frac{\frac{3}{4}T^3r^{-8}+(\frac{3}{2}\bar\delta+2)Tr^{-4}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}D,
\end{aligned}
$$

$$
\begin{aligned}
D'' = {} & \frac{(\bar\delta+1)\left(\frac{1}{2}T^2r^{-5}+\bar\delta r^{-1}\right)}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}A' - \frac{2(\bar\delta+1)Tr^{-3}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}B' + \frac{\frac{1}{2}T^3r^{-7}+(\frac{3}{2}\bar\delta+2)Tr^{-4}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}C' \\
& - \frac{\frac{1}{4}T^2r^{-4}+(\bar\delta+1)r^{-1}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}D' + \frac{\frac{1}{4}T^4r^{-10}+(\bar\delta+1)[T^2r^{-6}+(\bar\delta+2)r^{-2}]}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}A \\
& + \frac{\frac{3}{4}T^3r^{-8}+(\frac{3}{2}\bar\delta+2)Tr^{-4}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}B - \frac{\frac{3}{4}T^3r^{-8}+(\frac{3}{2}\bar\delta+2)Tr^{-4}}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}C \\
& + \frac{\frac{1}{4}T^4r^{-10}+(\bar\delta+1)[T^2r^{-6}+(\bar\delta+2)r^{-2}]}{\frac{1}{4}T^2r^{-4}+\bar\delta+1}D.
\end{aligned}
$$

For computational purposes equations (6.4.13) are non-dimensionalized using the dimensionless variables

$$
\begin{aligned}
&\bar r = \frac{r}{a}, \\
&(\bar A,\ \bar B,\ \bar C,\ \bar D) = (A/a,\ B/a,\ C/a,\ D/a), \qquad (6.4.14)
\end{aligned}
$$

and

$$
\bar T = \frac{T}{a^2}. \qquad (6.4.15)
$$

Then, the range of values of r is given by $1 \leq \bar{r} \leq \bar{b}$, where $\bar{b} = b/a$. Making use of equation (6.2.22), the non-dimensionalized boundary conditions may be described by

$$\bar{A}(1) = \bar{\epsilon}, \quad \bar{B}(1) = 0, \quad \bar{C}(1) = 0, \quad \bar{D}(1) = -\bar{\epsilon},$$

$$\bar{A}(\bar{b}) = \bar{B}(\bar{b}) = \bar{C}(\bar{b}) = \bar{D}(\bar{b}) = 0, \tag{6.4.16}$$

where $\bar{\epsilon} = \epsilon/a$. Since ϵ is a small displacement, we may, therefore, assume that $\bar{\epsilon} << 1$.

Let the prime now indicate differentiation with respect to $\bar{r}$. We introduce the notation

$$(y_1,\ y_2,\ y_3,\ y_4) = (\bar{A},\ \bar{B},\ \bar{C},\ \bar{D})$$

$$(y_5,\ y_6,\ y_7,\ y_8) = (y_1',\ y_2',\ y_3',\ y_4')$$

and equations (6.4.14) can be rewritten as a first-order system

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \tag{6.4.17}$$

where

$$\mathbf{y}^T = (y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6,\ y_7,\ y_8), \tag{6.4.18}$$

and the matrix $\mathbf{A}$ has components $a_{ij},\ i,\ j \in \{1,\ 2,\ ...,\ 8\}$, given by

$$a_{1i} = \delta_{i5}, \quad a_{2i} = \delta_{i6}, \quad a_{3i} = \delta_{i7}, \quad a_{4i} = \delta_{i8},$$

$$a_{51} = 4\bar{r}^{-2} - \frac{(3\bar{\delta}+2)\bar{r}^2}{\frac{1}{4}\bar{T}^2 + (\bar{\delta}+1)\bar{r}^4},$$

$$a_{52} = -a_{53} = -\bar{T}\bar{r}^{-4} - \frac{(\frac{1}{2}\bar{\delta}+1)\bar{T}}{\frac{1}{4}\bar{T}^2 + (\bar{\delta}+1)\bar{r}^4},$$

$$a_{54} = 4\bar{r}^{-2} - \frac{(3\bar{\delta}+2)\bar{r}^2}{\frac{1}{4}\bar{T}^2 + (\bar{\delta}+1)\bar{r}^4},$$

$$
\begin{aligned}
a_{55} &= 3\bar{r}^{-1} - \frac{4(\bar{\delta}+1)\bar{r}^3}{\frac{1}{4}\bar{T}^2 + (\bar{\delta}+1)\bar{r}^4}, \\
a_{56} &= -\frac{\frac{1}{2}(\bar{\delta}+2)\bar{T}\bar{r}}{\frac{1}{4}\bar{T}^2 + (\bar{\delta}+1)\bar{r}^4}, \\
a_{57} &= 0, \\
a_{58} &= -2\bar{r}^{-1} - \frac{(\bar{\delta}+2)\bar{r}^3}{\frac{1}{4}\bar{T}^2 + (\bar{\delta}+1)\bar{r}^4}, \\
a_{61} &= -a_{52}, \quad a_{62} = a_{51}, \quad a_{63} = -a_{54}, \quad a_{64} = a_{53}, \\
a_{65} &= -a_{56}, \quad a_{66} = a_{55}, \quad a_{67} = -a_{58}, \quad a_{68} = 0, \\
a_{71} &= 3\bar{T}\bar{r}^{-4} - \frac{(\frac{3}{2}\bar{\delta}+1)\bar{T}}{\frac{1}{4}\bar{T}^2 + (\bar{\delta}+1)\bar{r}^4}, \\
a_{72} &= -\bar{T}^2\bar{r}^{-6} - \frac{(\bar{\delta}+1)(\bar{\delta}+2)\bar{r}^2}{\frac{1}{4}\bar{T}^2 + (\bar{\delta}+1)\bar{r}^4}, \\
a_{73} &= -\bar{T}^2\bar{r}^{-6} - \frac{(\bar{\delta}+1)(\bar{\delta}+2)\bar{r}^2}{\frac{1}{4}\bar{T}^2 + (\bar{\delta}+1)\bar{r}^4}, \\
a_{74} &= 3T\bar{r}^{-4} - \frac{(\frac{3}{2}\bar{\delta}+1)\bar{T}}{\frac{1}{4}\bar{T}^2 + (\bar{\delta}+1)\bar{r}^4}, \\
a_{75} &= -\frac{2(\bar{\delta}+1)\bar{T}\bar{r}}{\frac{1}{4}\bar{T}^2 + (\bar{\delta}+1)\bar{r}^4}, \\
a_{76} &= -2(\bar{\delta}+1)\bar{r}^{-1} - \frac{(\bar{\delta}+1)(\bar{\delta}+2)\bar{r}^3}{\frac{1}{4}\bar{T}^2 + (\bar{\delta}+1)\bar{r}^4}, \\
a_{77} &= -\bar{r}^{-1}, \\
a_{78} &= -2T\bar{r}^{-3} - \frac{(\frac{1}{2}\bar{\delta}+1)\bar{T}\bar{r}}{\frac{1}{4}\bar{T}^2 + (\bar{\delta}+1)\bar{r}^4}, \\
a_{81} &= -a_{72}, \quad a_{82} = a_{71}, \quad a_{83} = -a_{74}, \quad a_{84} = a_{73}, \\
a_{85} &= -a_{76}, \quad a_{86} = a_{75}, \quad a_{87} = -a_{78}, \quad a_{88} = -r^{-1}.
\end{aligned}
\tag{6.4.19}
$$

Equations (6.4.17) can be solved numerically for $\bar{r} \in (1, \bar{b})$ with chosen values of the parameters $\bar{b}$, $\bar{\delta}$ and $\bar{T}$, with $\bar{T}$ reflecting the dependence on the underlying finite strain. The boundary conditions are given by

$$
\begin{aligned}
&y_1(1) = \bar{\epsilon}, \quad y_2(1) = y_3(1) = 0, \quad y_4(1) = -\bar{\epsilon}, \\
&y_1(\bar{b}) = y_2(\bar{b}) = y_3(\bar{b}) = y_4(\bar{b}) = 0.
\end{aligned}
\tag{6.4.20}
$$

6.5 Numerical solutions

The solutions for the dimensionless incremental radial and axial displacements $\bar{u}_r = u_r/a$ and $\bar{u}_\theta = u_\theta/a$ as functions of $\bar{r}$ and θ were obtained numerically using the D02 GBF in NAG FORTRAN Library Routine Document. The programme of D02 GBF may be used to solve a general linear two-point boundary value problem for a system of N ordinary differential equations by using a deferred correction technique. The components of the incremental nominal stress associated with the solutions were also calculated in dimensionless form.

The variation of the radial displacement as a function of $\bar{r}$ is quite different for the different values of θ. Results obtained from the numerical method for $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, $\bar{T} = 10.0$ and $\bar{b} = 2.0$ are shown in Figure 6.1. The figures change to approximate straight lines when $\bar{T} = 2.0$ in Figure 6.2. Note that in Figure 6.3, when the dependence of $\bar{u}_r$ on $\bar{T}$ for a fixed value of $\bar{r}$ is shown with $\epsilon = 0.01$, $\bar{\delta} = 5.0$ and $\bar{b} = 2.0$, $\bar{u}_r$ seems to be sensitive to changes in $\bar{T}$ for small $\bar{T}$ but otherwise depends little on $\bar{T}$. This effect is more pronounced near $\bar{r} = 1.6$ as Figure 6.4 illustrates. The influence of $\bar{\delta}$ on the response in the radial direction is insensitive near $\bar{r} = 1.2$ but quite significant for small $\bar{\delta}$ and, then, become insensitively as $\bar{\delta}$ increase near $\bar{r} = 1.8$ as Figures 6.5 and 6.6 respectively show for $\bar{\epsilon} = 0.01$, $\bar{T} = 10.0$ and $\bar{b} = 2.0$.

The displacement in the axial direction is also strongly dependent on the value of θ, and the changes in $\bar{u}_\theta$ for $\bar{T} = 10.0$ is more pronounced than that for $\bar{T} = 2.0$. This is shown graphically in Figures 6.7 and 6.8 with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, and $\bar{b} = 2.0$. The displacement of $\bar{u}_\theta$ on $\bar{T}$ is illustrated in Figures 6.9 and 6.10. When $\bar{r} \simeq 1.2$ with $\bar{b} = 2.0$ $\bar{u}_\theta$ is seen to be depend only slightly on $\bar{T}$ for $\bar{T} < 2$ but is quite sensitive to changes in $\bar{T}$ for $\bar{T} > 2$, while $\bar{T}$ affects u_θ strongly near $\bar{r} = 1.8$. On the other hand, $\bar{\delta}$ is seen to affect $\bar{u}_\theta$ more strongly than $\bar{u}_r$, while $\bar{u}_\theta$ with different θ increase as $\bar{\delta}$ increases for $\bar{r} \simeq 1.2$, and $\bar{u}_\theta$ become quite significant near $\bar{r} = 1.8$. These are indicated in Figures 6.11 and 6.12 with $\bar{\epsilon} = 0.01$, $\bar{T} = 10.0$ and $\bar{b} = 2.0$.

The numerical results show the expected linear effect of ϵ on both the radial and axial displacements at an arbitrary point $\bar{r}$ with θ for $\bar{\delta} = 5.0$, $\bar{T} = 10.0$ and $\bar{b} = 2.0$. Figures 6.13 and 6.14 provide the evidence of this for $\bar{r} \simeq 1.2$. A similar effect is observed on the components of the incremental nominal stress in the inner and outer surfaces in Figures 6.15 and 6.16 for $\bar{\delta} = 5.0$, $\bar{T} = 10.0$ and $\bar{b} = 2.0$.

The variation of the components of the incremental nominal stress as a function of $\bar{r}$ for $\bar{T} = 10.0$ are shown graphically for the different values of θ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, and $\bar{b} = 2.0$ in Figure 6.17. The changes of these are quite different from those for $\bar{T} = 2.0$, which are displayed in Figure 6.18. Therefore, $\bar{T}$ affects the incremental nominal stresses strongly. This is also indicated graphically by the results in Figures 6.19 and 6.20 for $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$ and $\bar{b} = 2.0$, which show that the components of the incremental nominal stress are dependent on $\bar{T}$ strongly on the inner surface as well as on the outer surface. The dependence of the incremental nominal stresses on $\bar{\delta}$ on the inner and outer surface is illustrated in Figures 6.21 and 6.22, respectively.

We also consider the case of thicker annulus with $\bar{b} = 10.0$. It seems that the variation of the displacements and incremental nominal stresses as functions of $\bar{r}$ with different θ are generally more pronounced than for $\bar{b} = 2.0$. Note that for smaller $\bar{r}$ (< 1.3) $\bar{u}_r$ remains approximately the same as on the inner surface for $\bar{T} = 10.0$, while $\bar{u}_r$ changes sharply with $\bar{T} = 2.0$ but $\bar{u}_r$ is close to zero in the case of $\theta = 90°$. These results are presented respectively in Figures 6.23 and 6.24 with

$\bar{\epsilon} = 0.01$ and $\bar{\delta} = 5.0$. Note that $\bar{u_\theta}$ is also strongly dependent on the values of θ and the changes in $\bar{u_\theta}$ for $\bar{T} = 10.0$ is more pronounced than that for $\bar{T} = 2.0$. This is shown graphically in Figures 6.25 and 6.26 with $\bar{\epsilon} = 0.01$ and $\bar{\delta} = 5.0$ respectively. Similarly, the dependence of the incremental nominal stresses on $\bar{r}$ is displayed graphically for the different values of θ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$ and $\bar{T} = 2.0$ in Figure 6.27 for $\bar{T} = 2.0$, repectively. The changes are seen to be rather significant for small $\bar{r}$ but then approach zero as $\bar{r}$ increases, while $\bar{T} = 10.0$ they are seen to be more pronounced for $\bar{r} < 4$ on inner surface and $\bar{r} < 3$ on outer surface, and then change more quickly to close to zero in Figures 6.28 - 6.29 respectively.

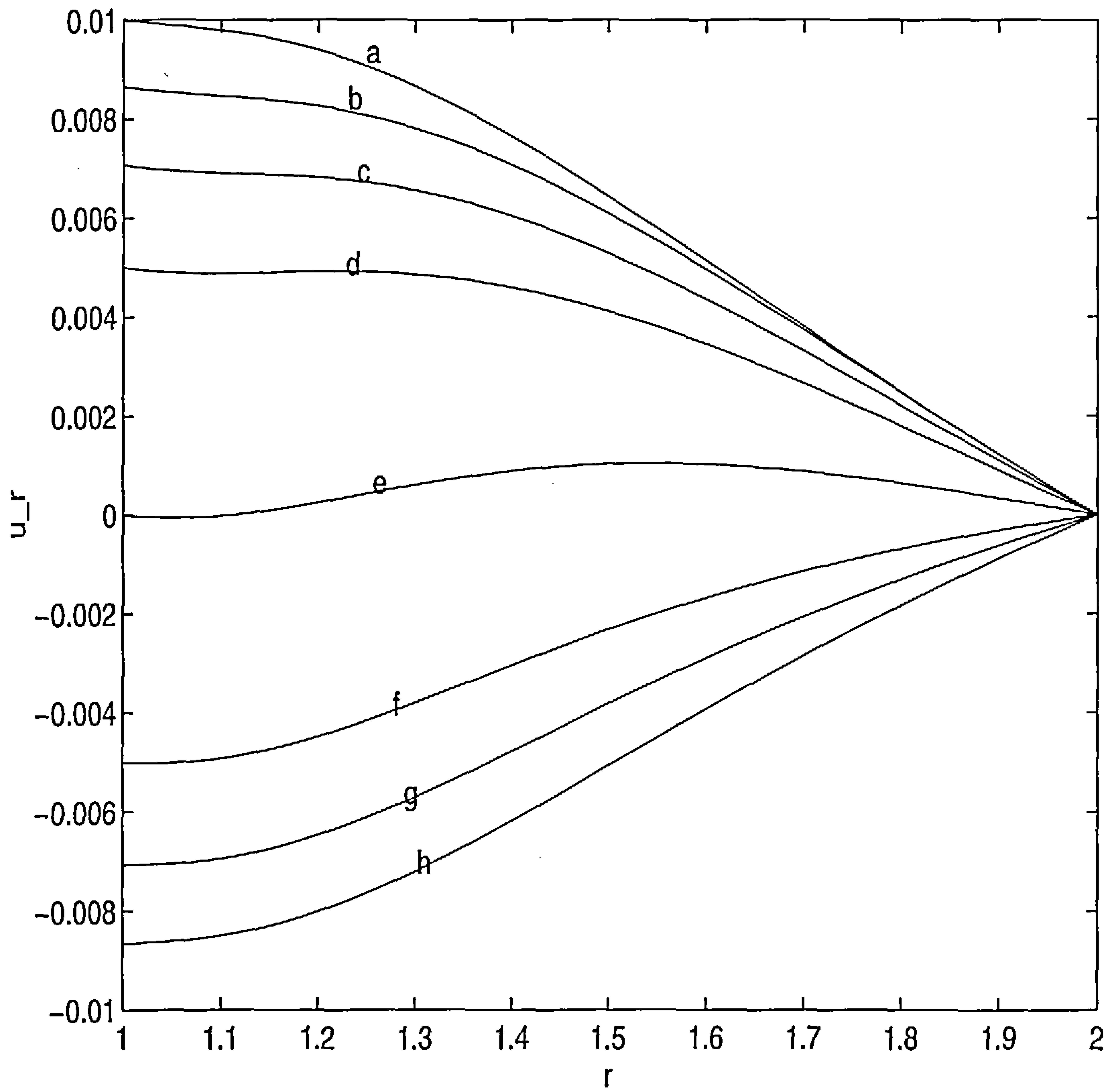


Figure 6.1: Plot of the dimensionless radial displacement $\bar{u}_r$ for $\bar{r} \in (1, 2)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

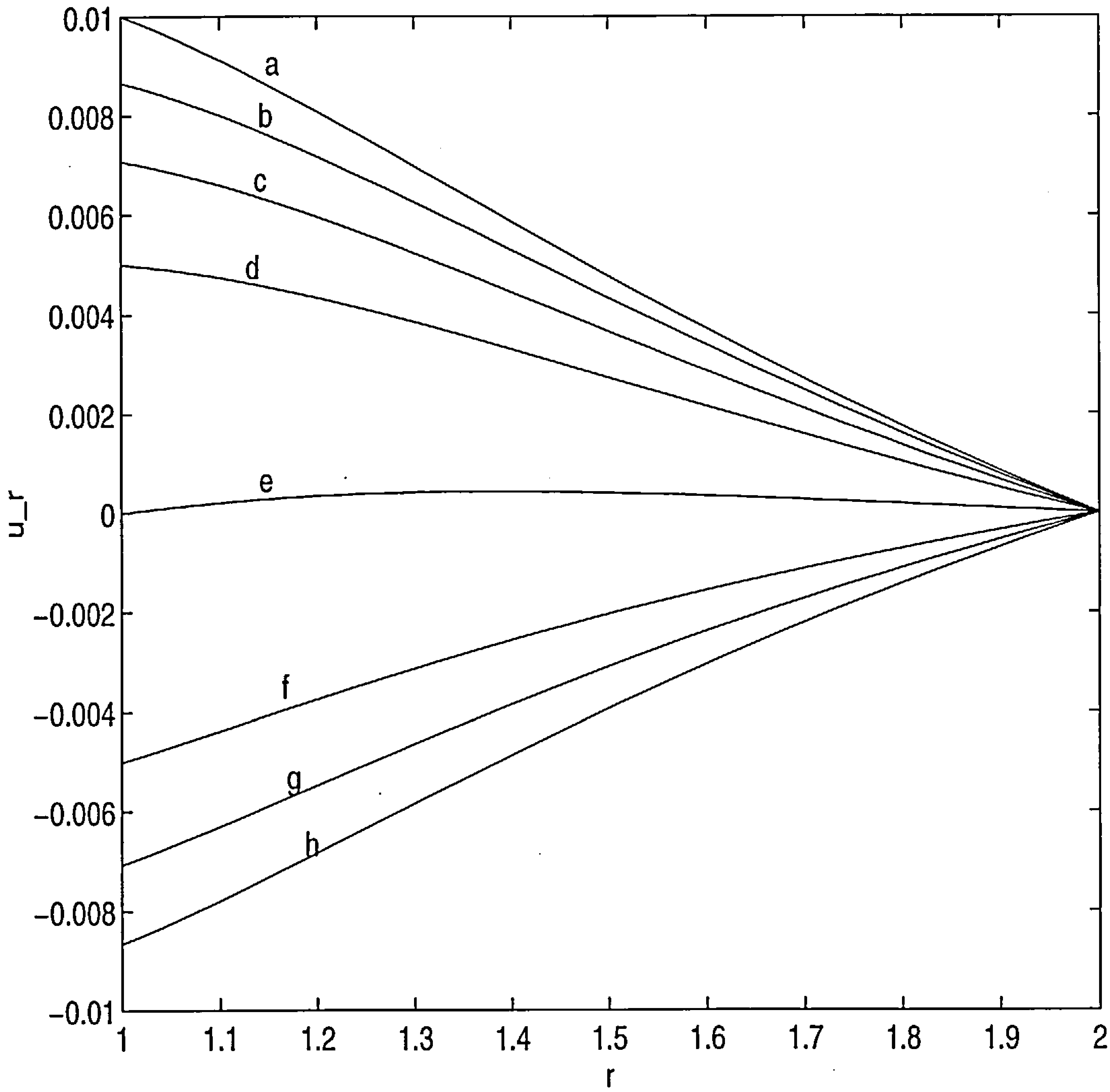


Figure 6.2: Plot of the dimensionless radial displacement $\bar{u}_r$ for $\bar{r} \in (1,2)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, and $\bar{T} = 2.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

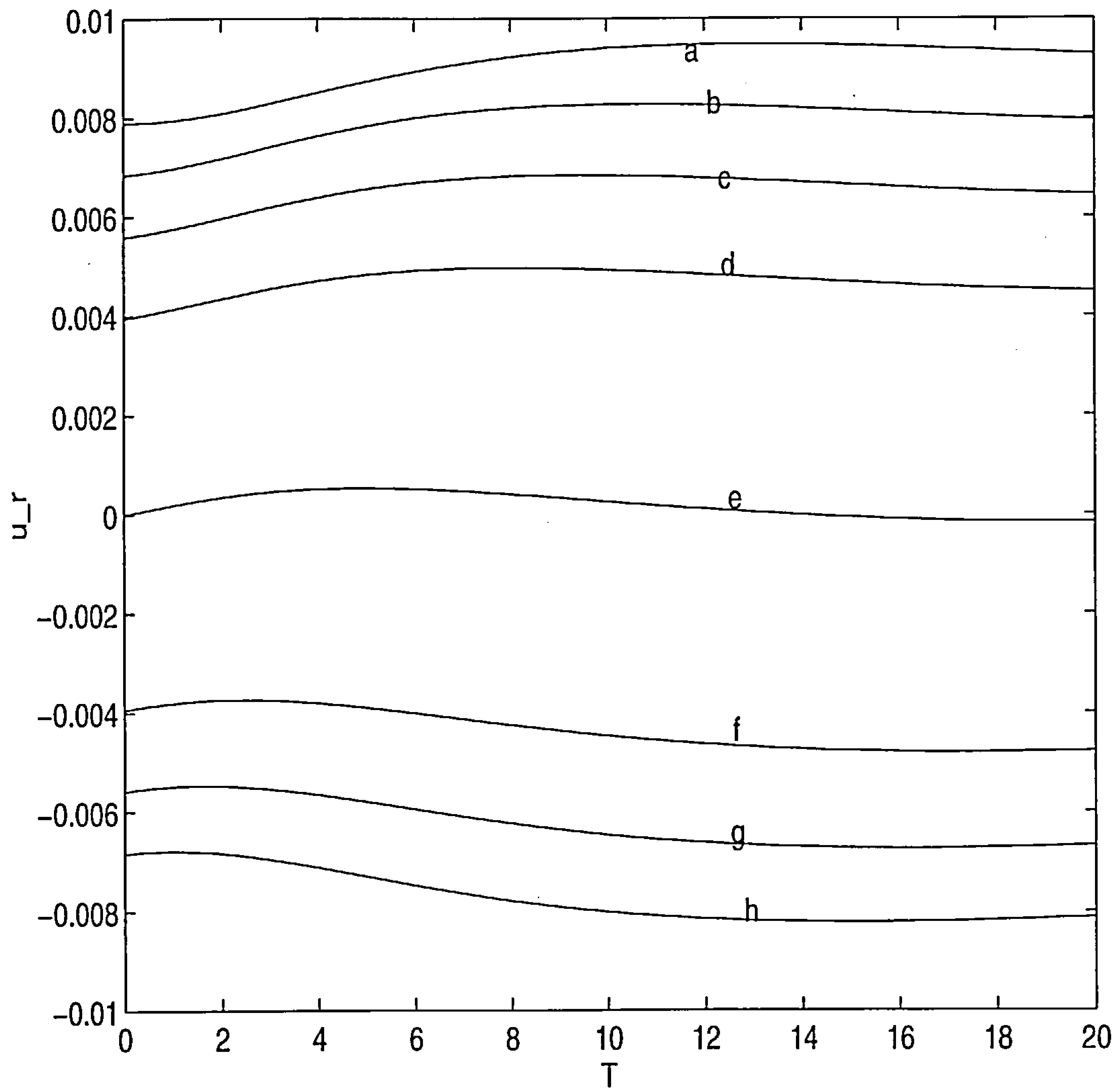


Figure 6.3: Plot of the dimensionless radial displacement $\bar{u}_r$ against the dimensionless underlying deformation parameter $\bar{T}$ at $\bar{r} \simeq 1.2$ for $\bar{r} \in (1, 2)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

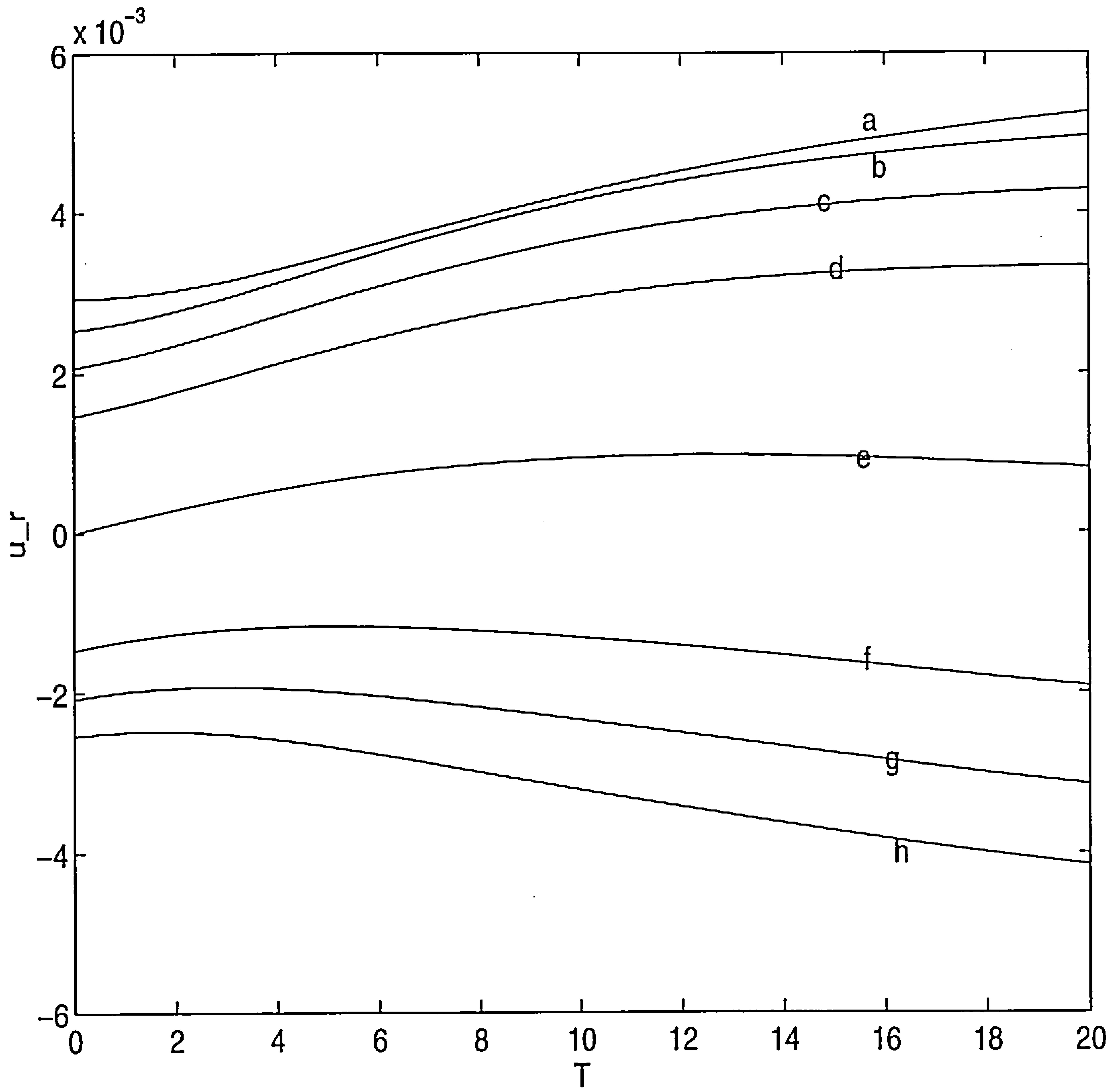


Figure 6.4: Plot of the dimensionless radial displacement $\bar{u}_r$ against the dimensionless underlying deformation parameter $\bar{T}$ at $\bar{r} \simeq 1.6$ for $\bar{r} \in (1,2)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

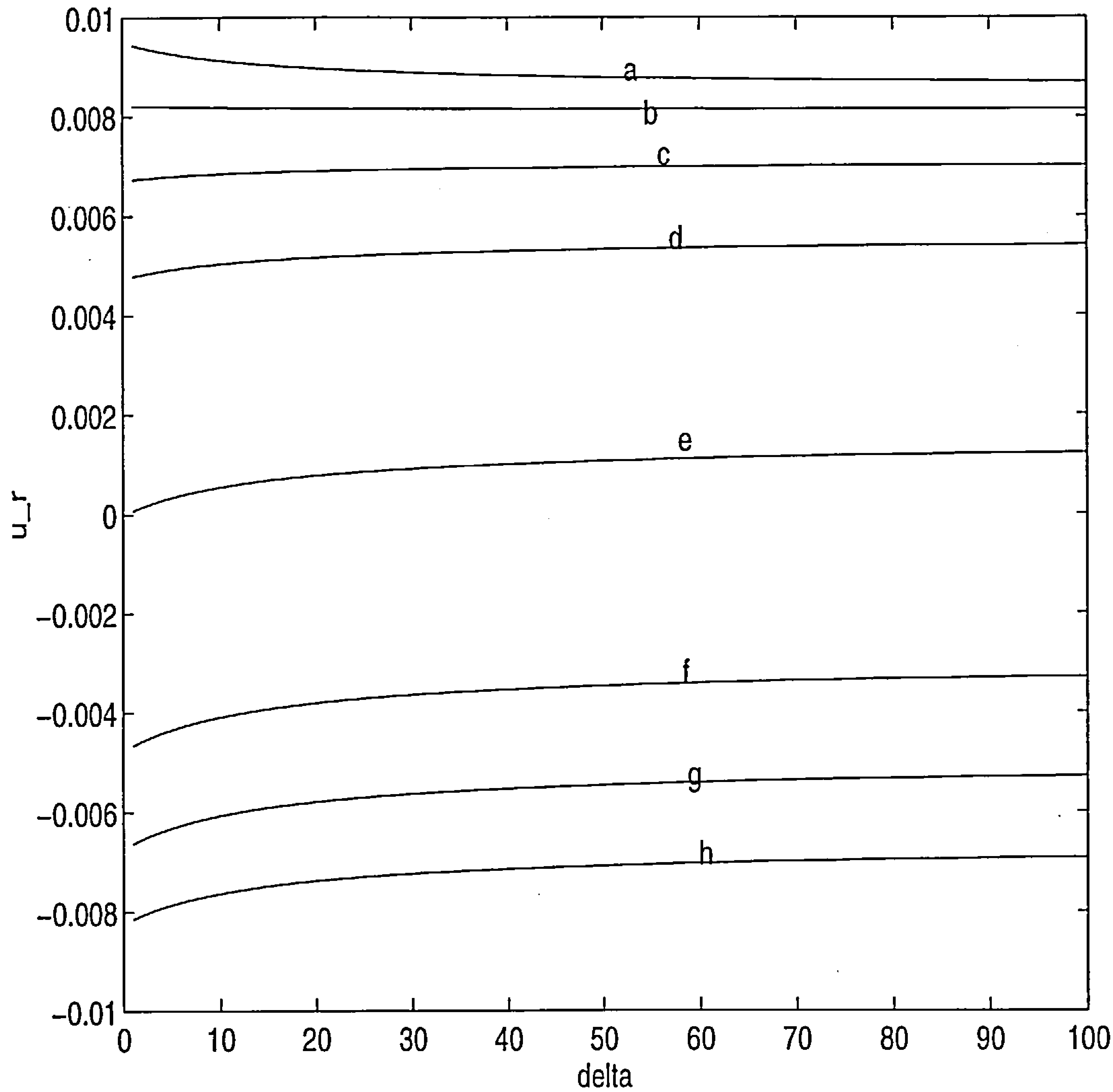


Figure 6.5: Plot of the dimensionless radial displacement $\bar{u}_r$ against the dimensionless material parameter $\bar{\delta}$ at $\bar{r} \simeq 1.2$ for $\bar{r} \in (1, 2)$ with $\bar{\epsilon} = 0.01$ and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

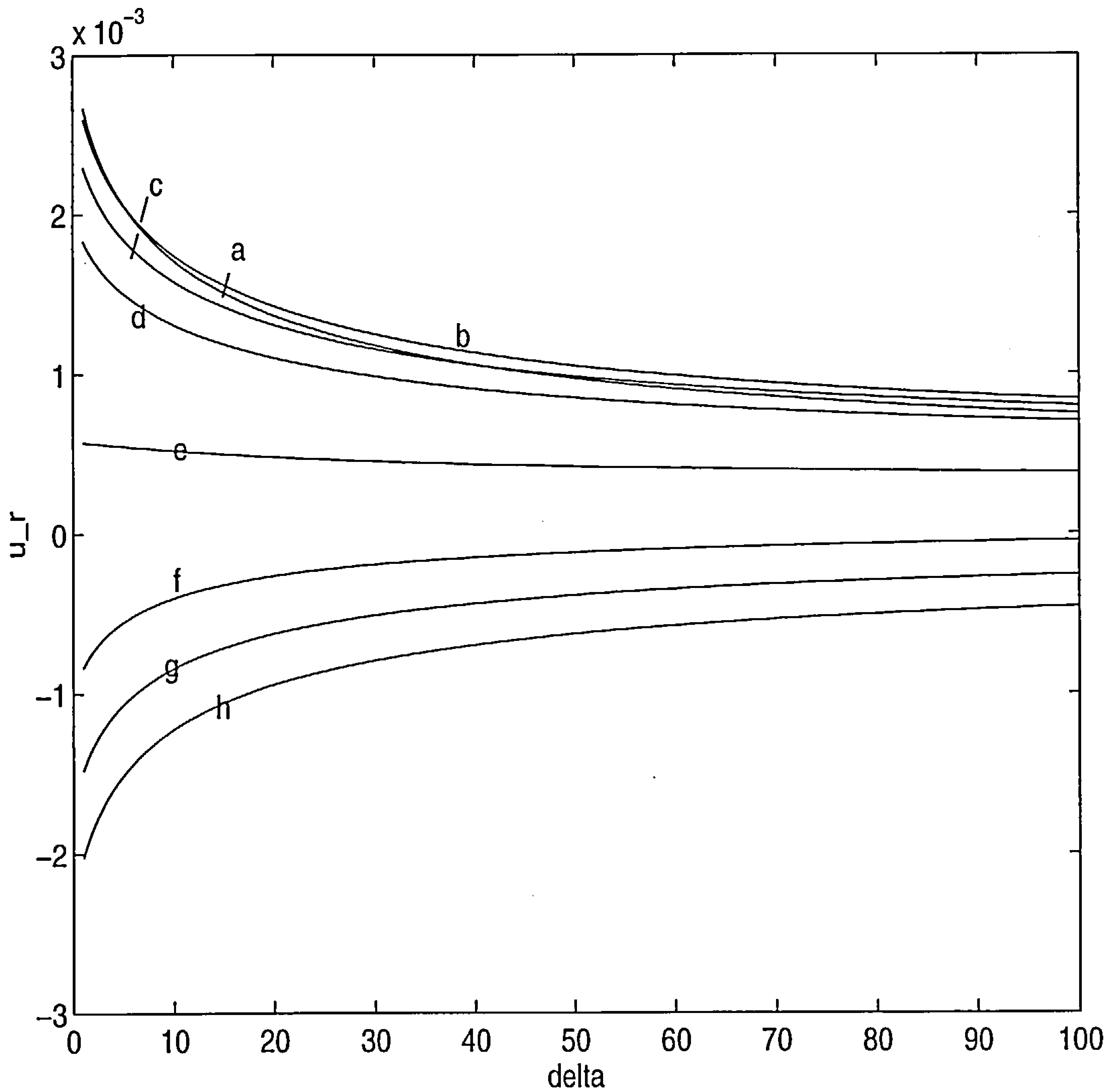


Figure 6.6: Plot of the dimensionless radial displacement $\bar{u}_r$ against the dimensionless material parameter $\bar{\delta}$ at $\bar{r} \simeq 1.83$ for $\bar{r} \in (1, 2)$ with $\bar{\epsilon} = 0.01$ and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

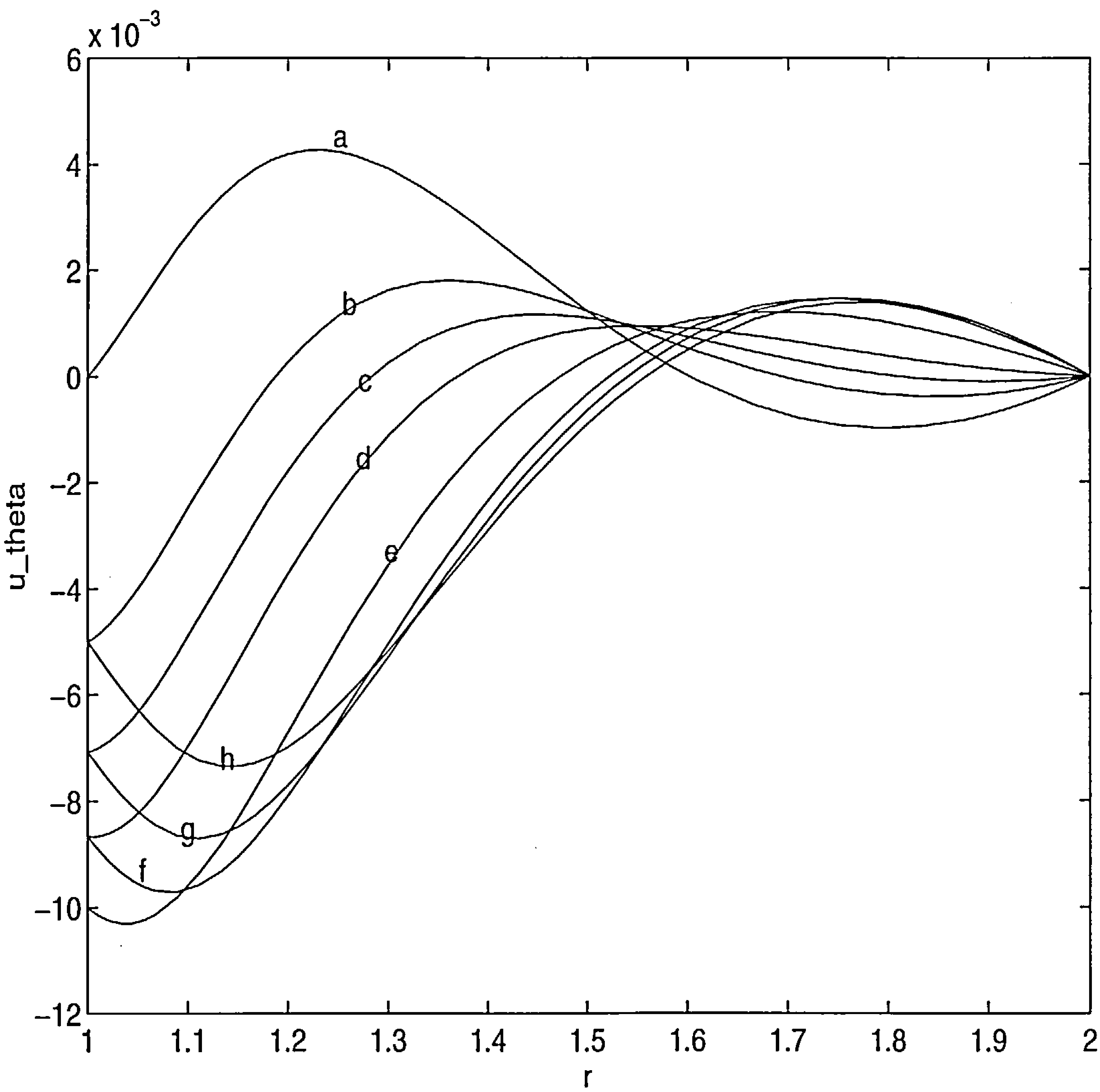


Figure 6.7: Plot of the dimensionless axial displacement $\bar{u}_\theta$ for $\bar{r} \in (1, 2)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

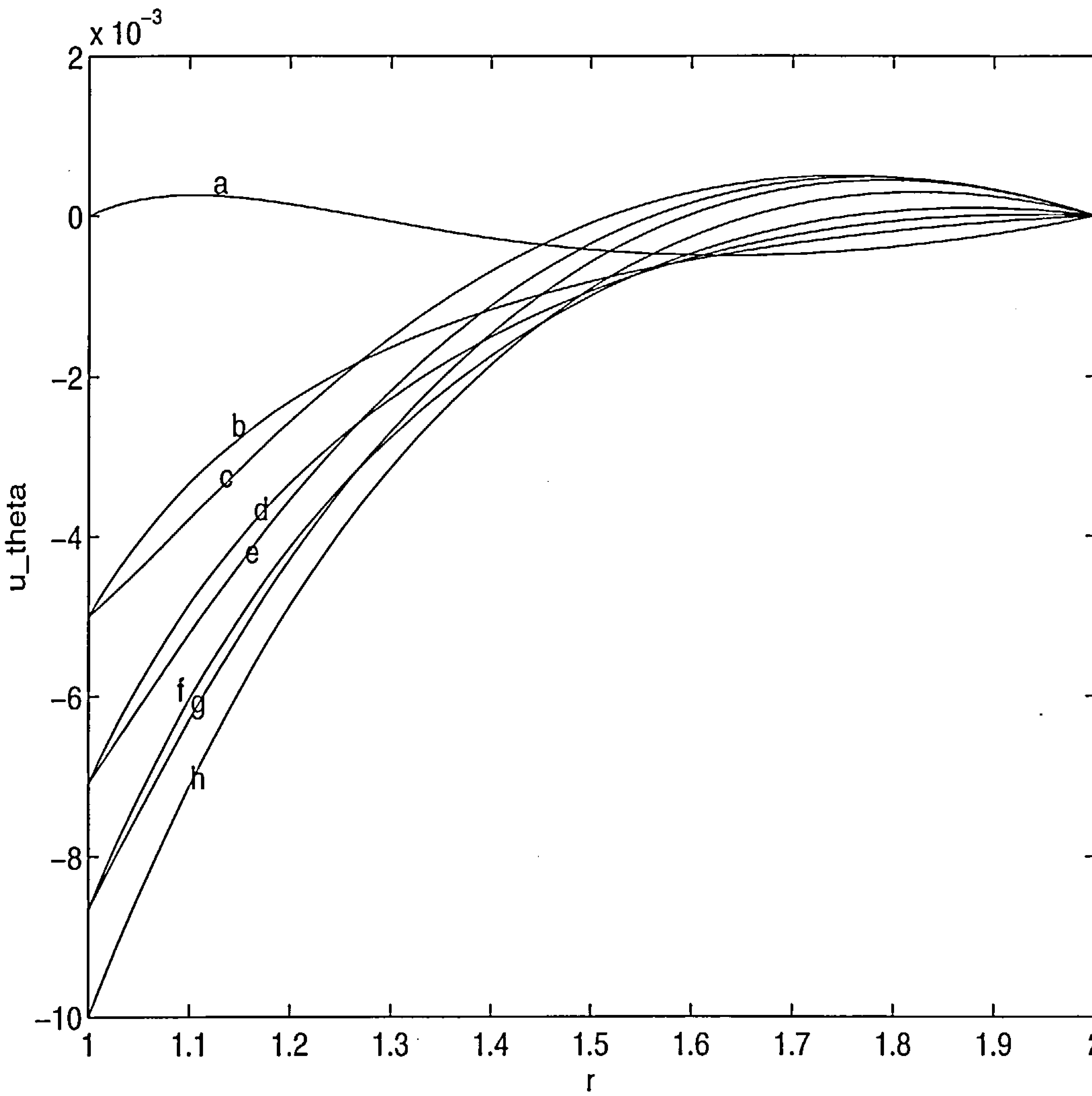


Figure 6.8: Plot of the dimensionless axial displacement $\bar{u}_\theta$ for $\bar{r} \in (1,2)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, and $\bar{T} = 2.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

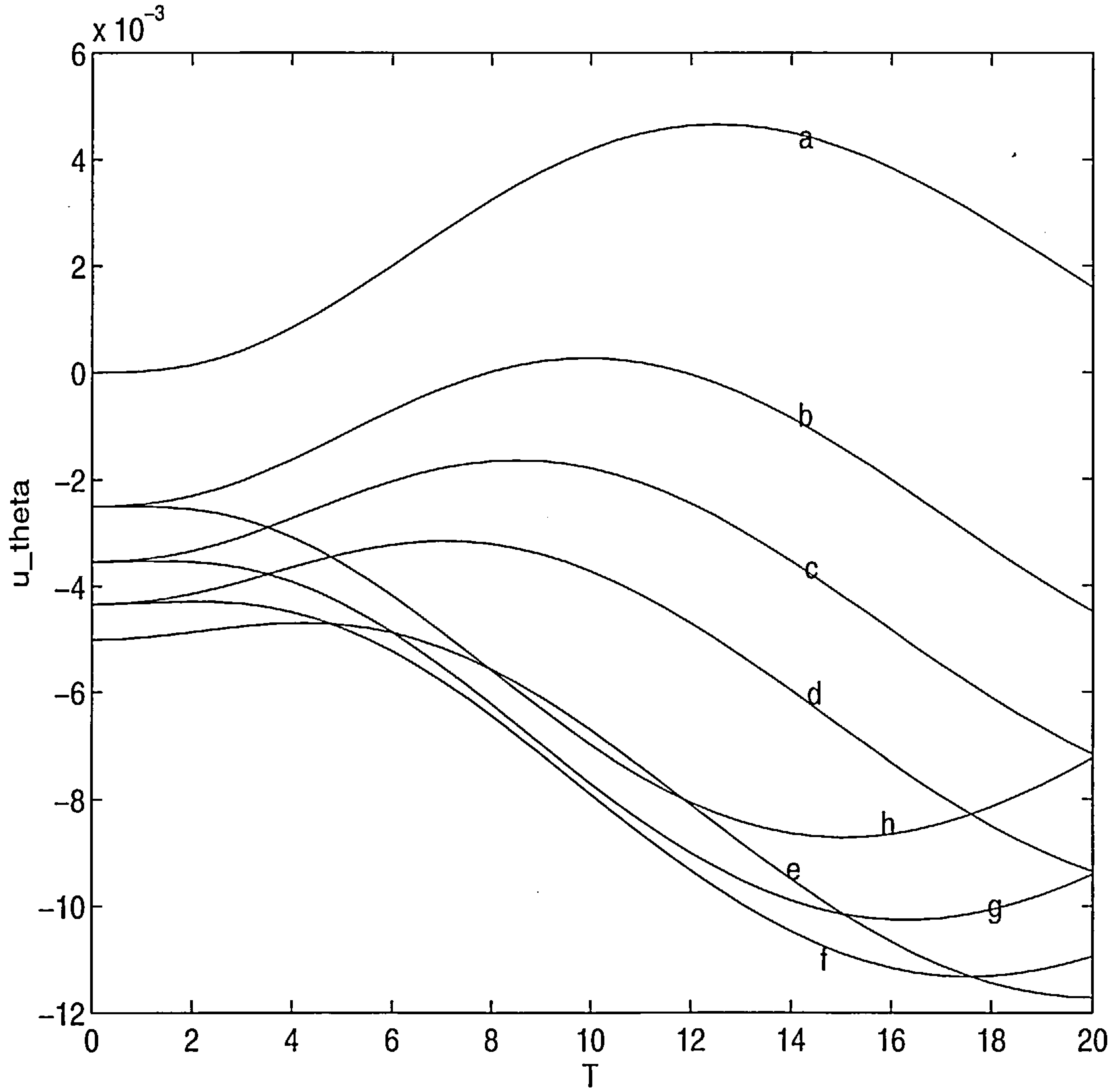


Figure 6.9: Plot of the dimensionless axial displacement $\bar{u}_\theta$ against the dimensionless underlying deformation parameter $\bar{T}$ at $\bar{r} \simeq 1.2$ for $\bar{r} \in (1,2)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

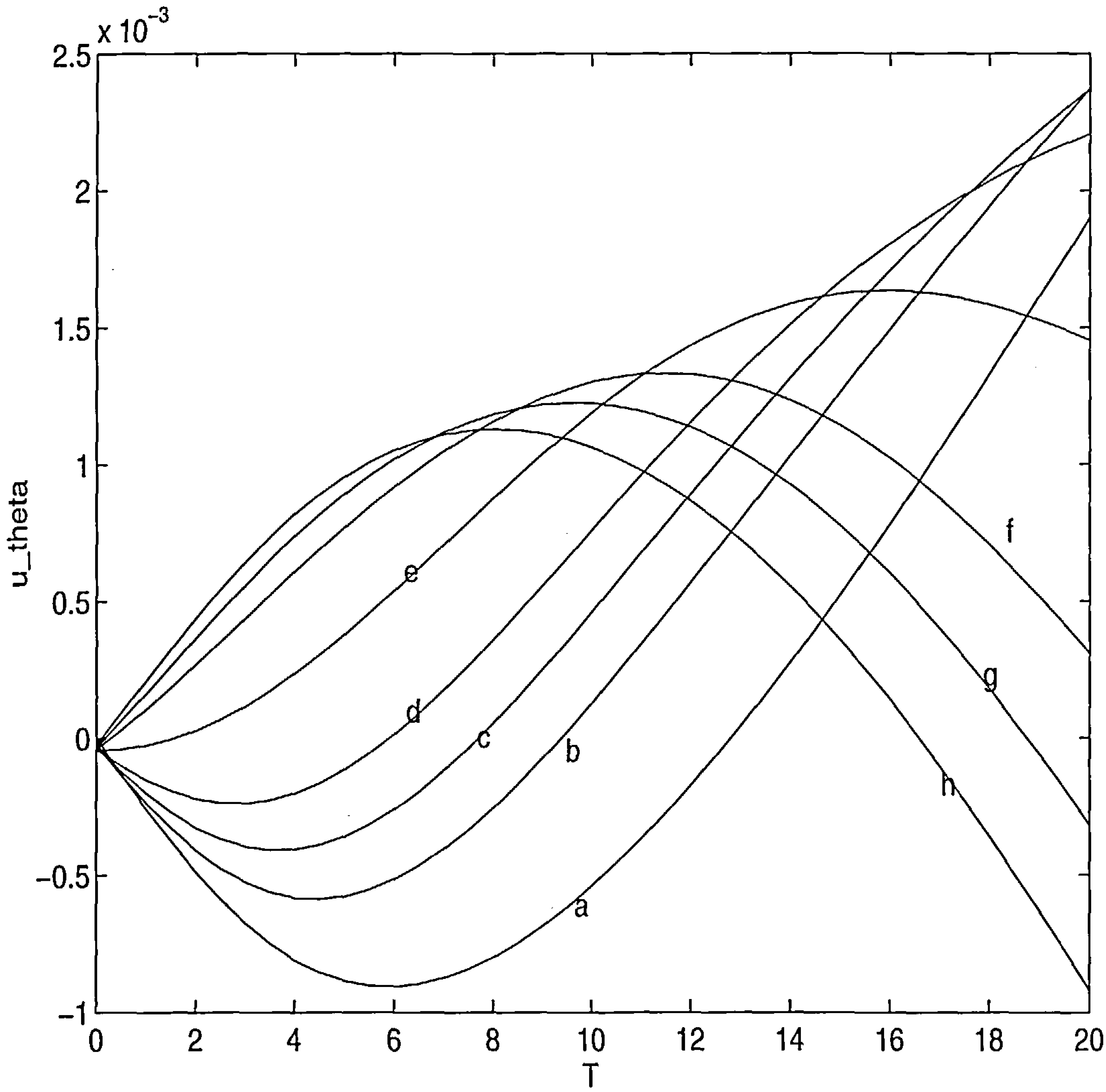


Figure 6.10: Plot of the dimensionless axial displacement $\bar{u}_\theta$ against the dimensionless underlying deformation parameter $\bar{T}$ at $\bar{r} \simeq 1.6$ for $\bar{r} \in (1, 2)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

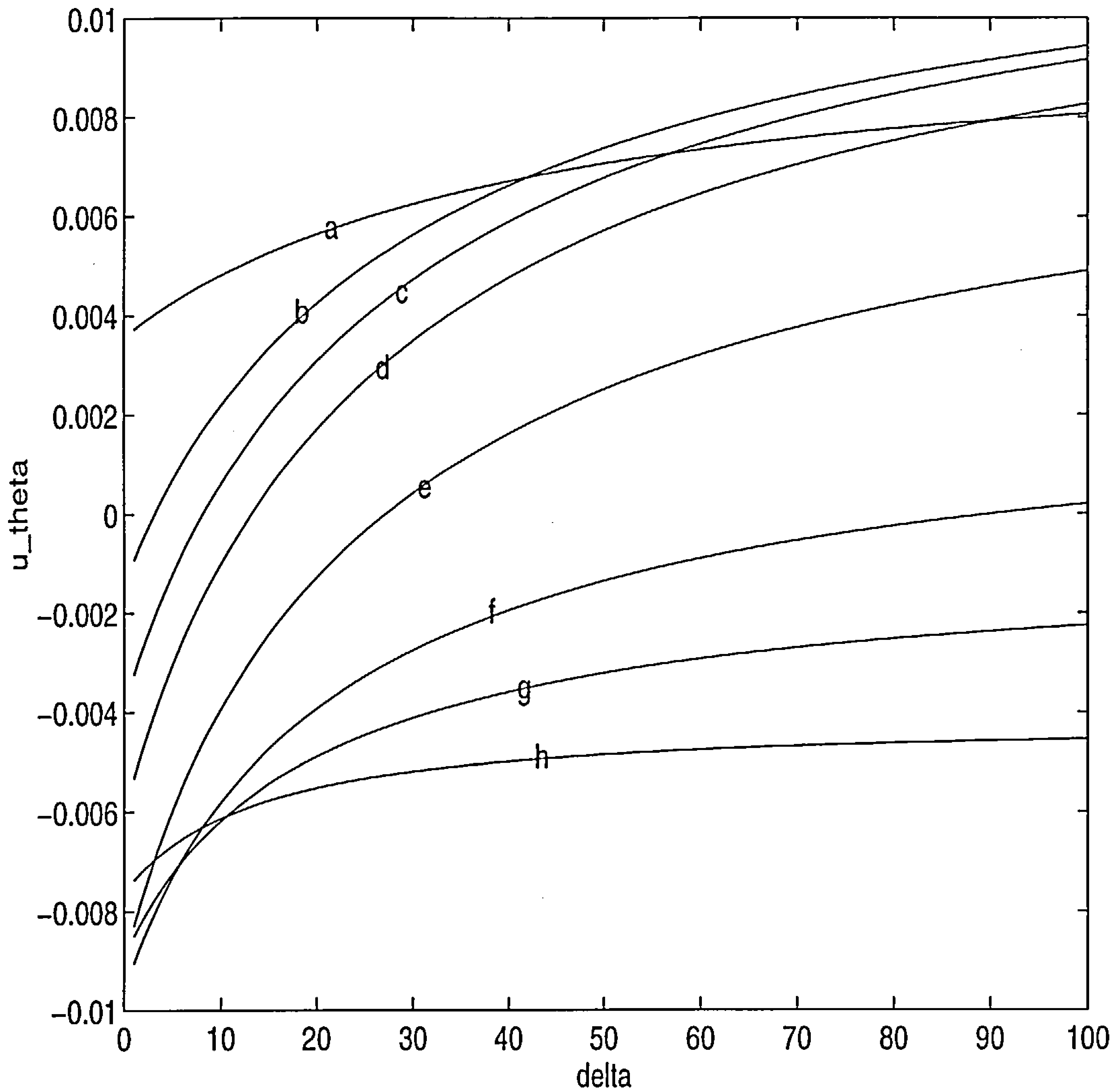


Figure 6.11: Plot of the dimensionless axial displacement $\bar{u}_\theta$ against the dimensionless material parameter $\bar{\delta}$ at $\bar{r} \simeq 1.2$ for $\bar{r} \in (1,2)$ with $\bar{\epsilon} = 0.01$ and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

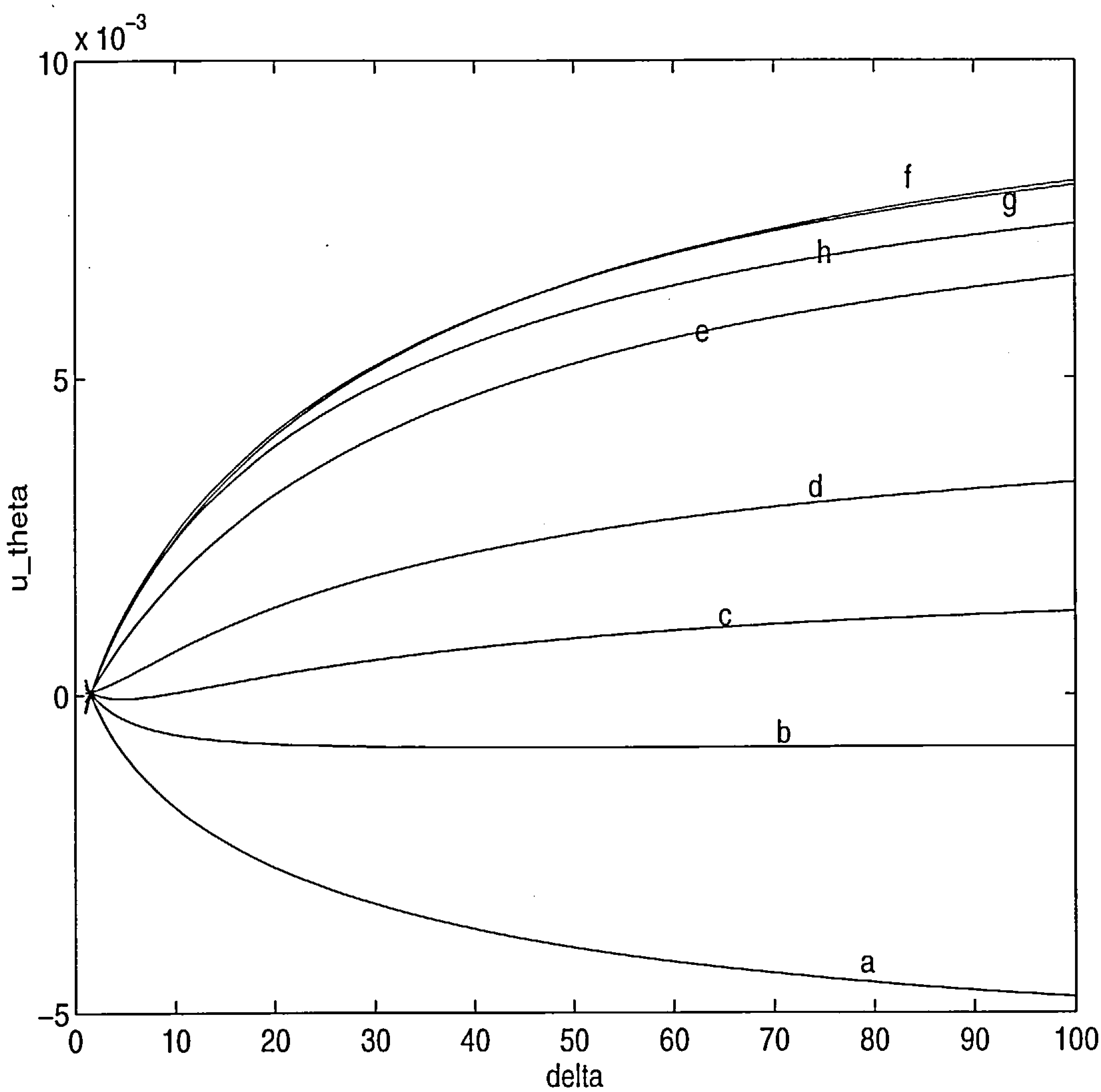


Figure 6.12: Plot of the dimensionless axial displacement $\bar{u}_\theta$ against the dimensionless material parameter $\bar{\delta}$ at $\bar{r} \simeq 1.83$ for $\bar{r} \in (1,2)$ with $\bar{\epsilon} = 0.01$ and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

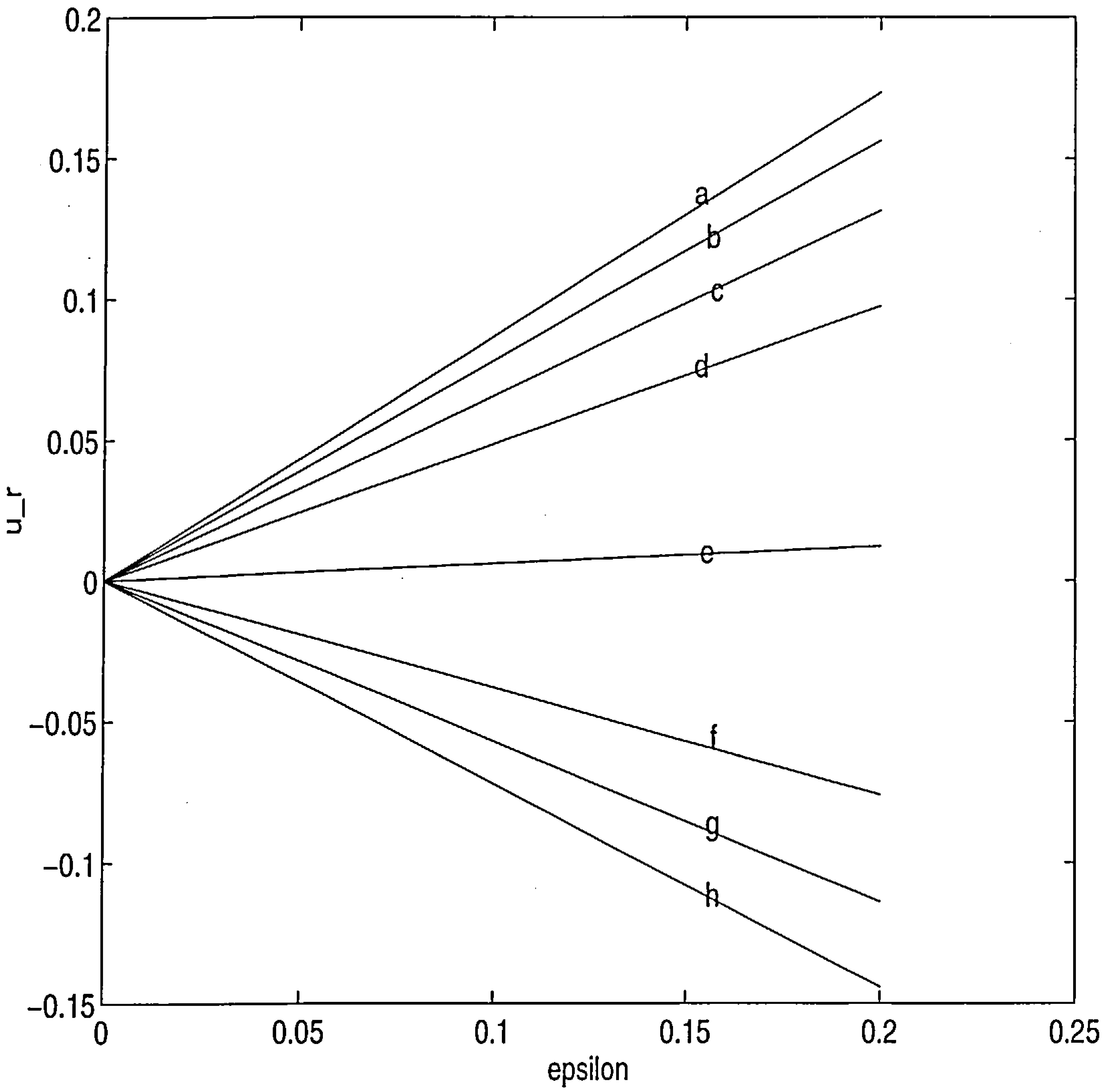


Figure 6.13: Plot of the dimensionless radial displacement $\bar{u}_r$ against the dimensionless displacement parameter $\bar{\epsilon}$ at $\bar{r} \simeq 1.3$ for $\bar{r} \in (1, 2)$ with $\bar{\delta} = 5.0$ and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

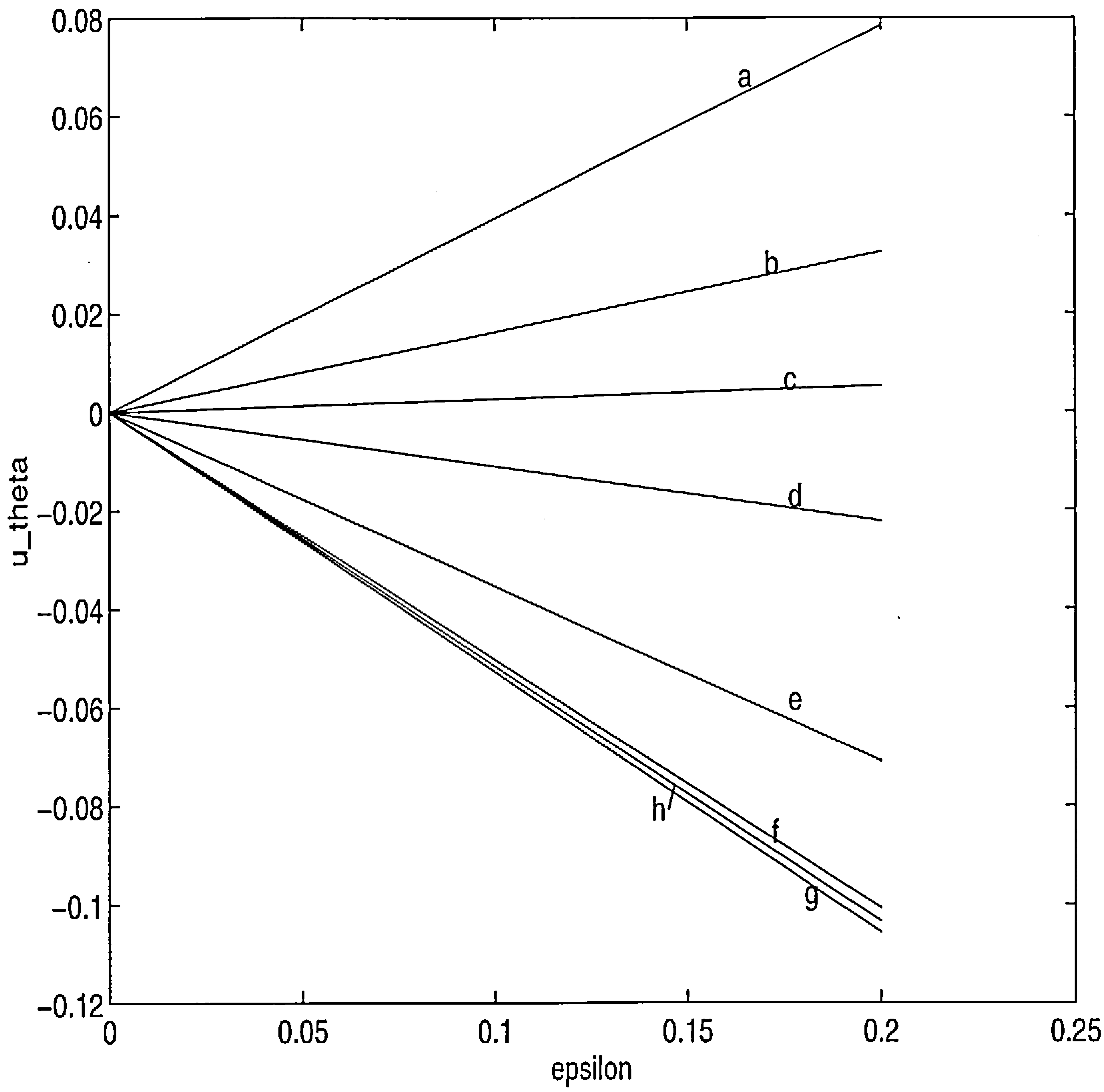


Figure 6.14: Plot of the dimensionless axial displacement $\bar{u}_\theta$ against the dimensionless displacement parameter $\bar{\epsilon}$ at $\bar{r} \simeq 1.3$ for $\bar{r} \in (1,2)$ with $\bar{\delta} = 5.0$ and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

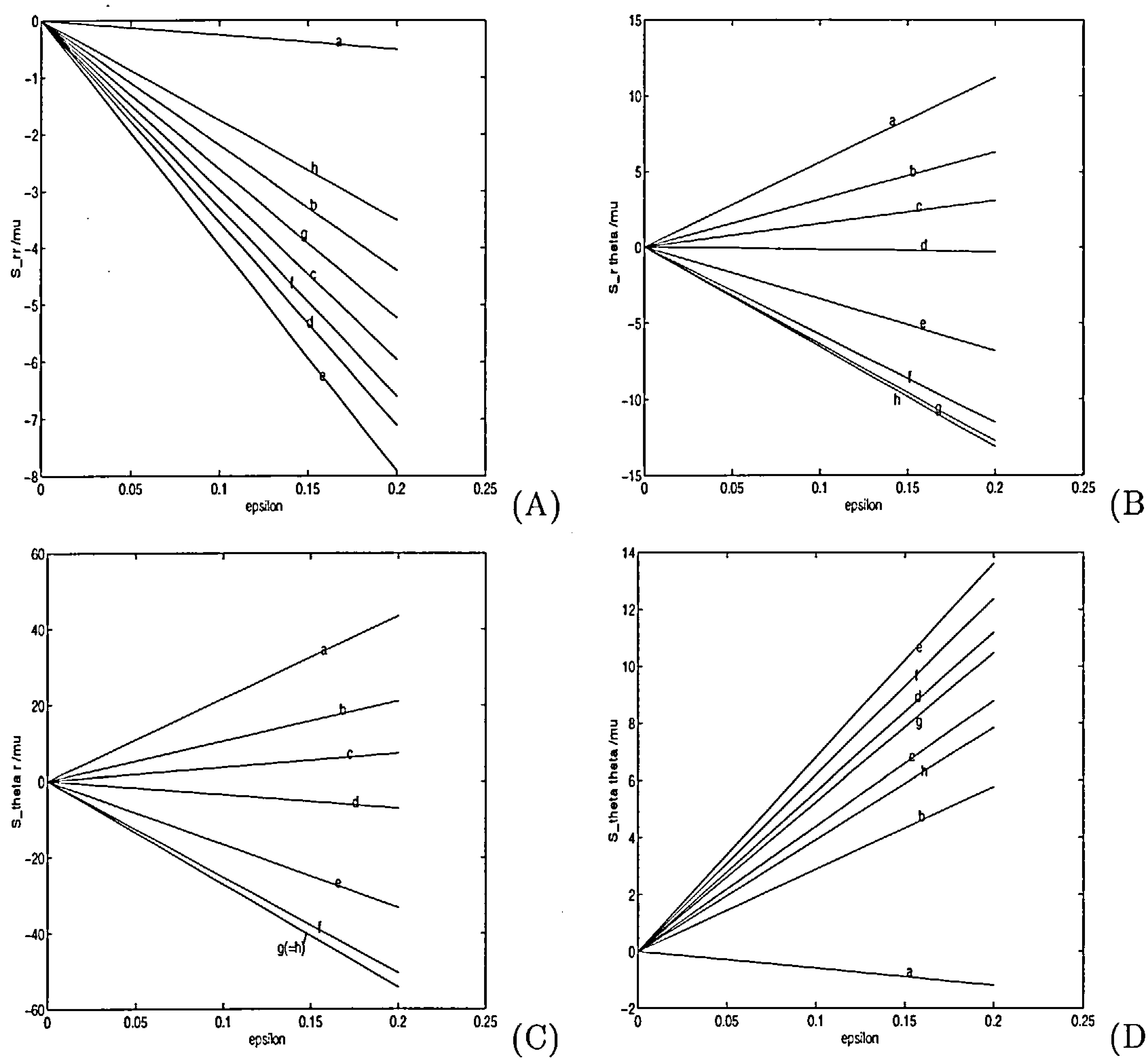


Figure 6.15: Plot of the dimensionless components (A) $\dot{S}_{rr}/\mu$, (B) $\dot{S}_{r\theta}/\mu$, (C) $\dot{S}_{\theta r}/\mu$ and (D) $\dot{S}_{\theta\theta}/\mu$ of the incremental nominal stress on the inner surface against the dimensionless displacement parameter $\bar{\epsilon}$ for $\bar{\delta} = 5.0$ and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

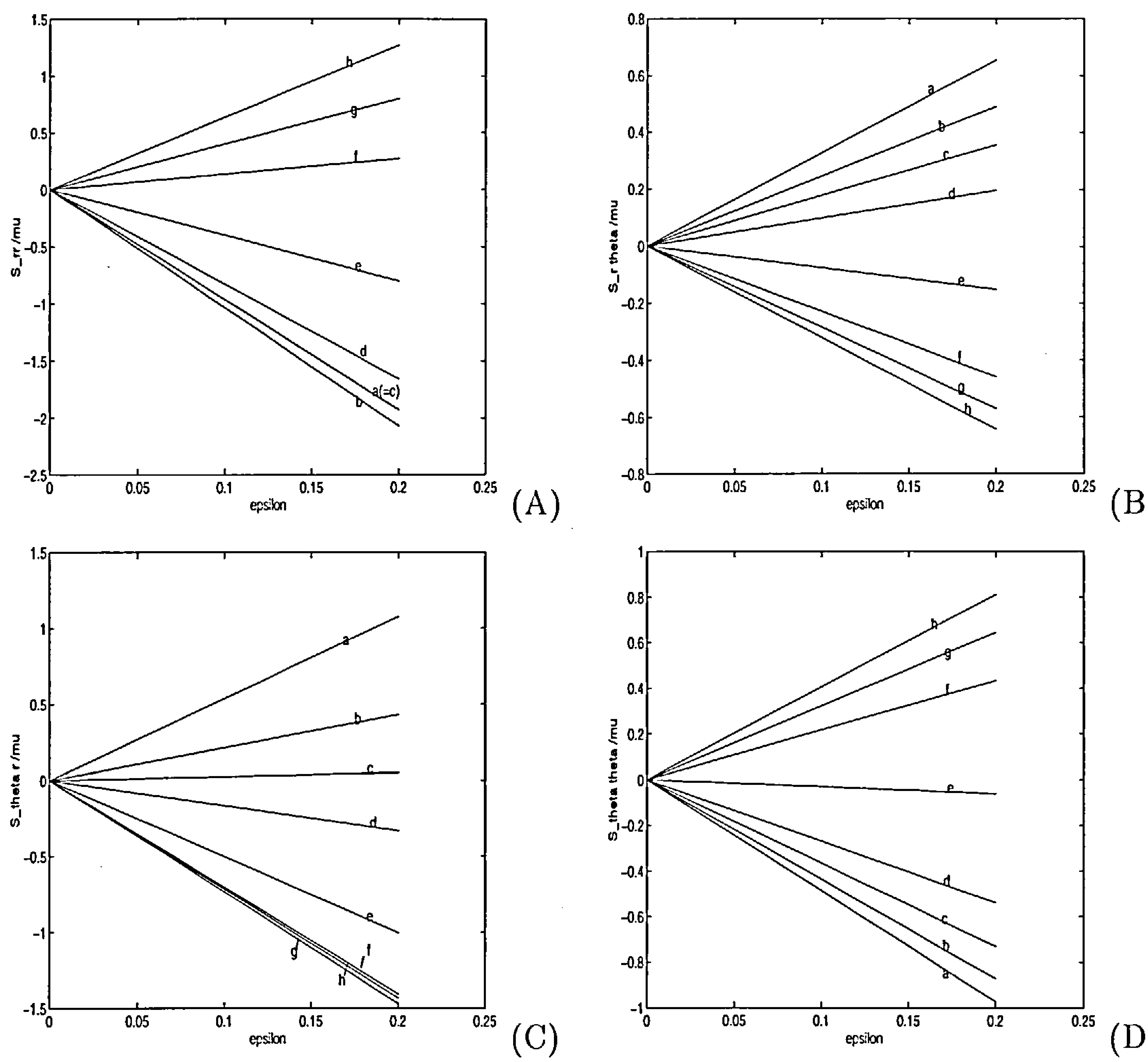
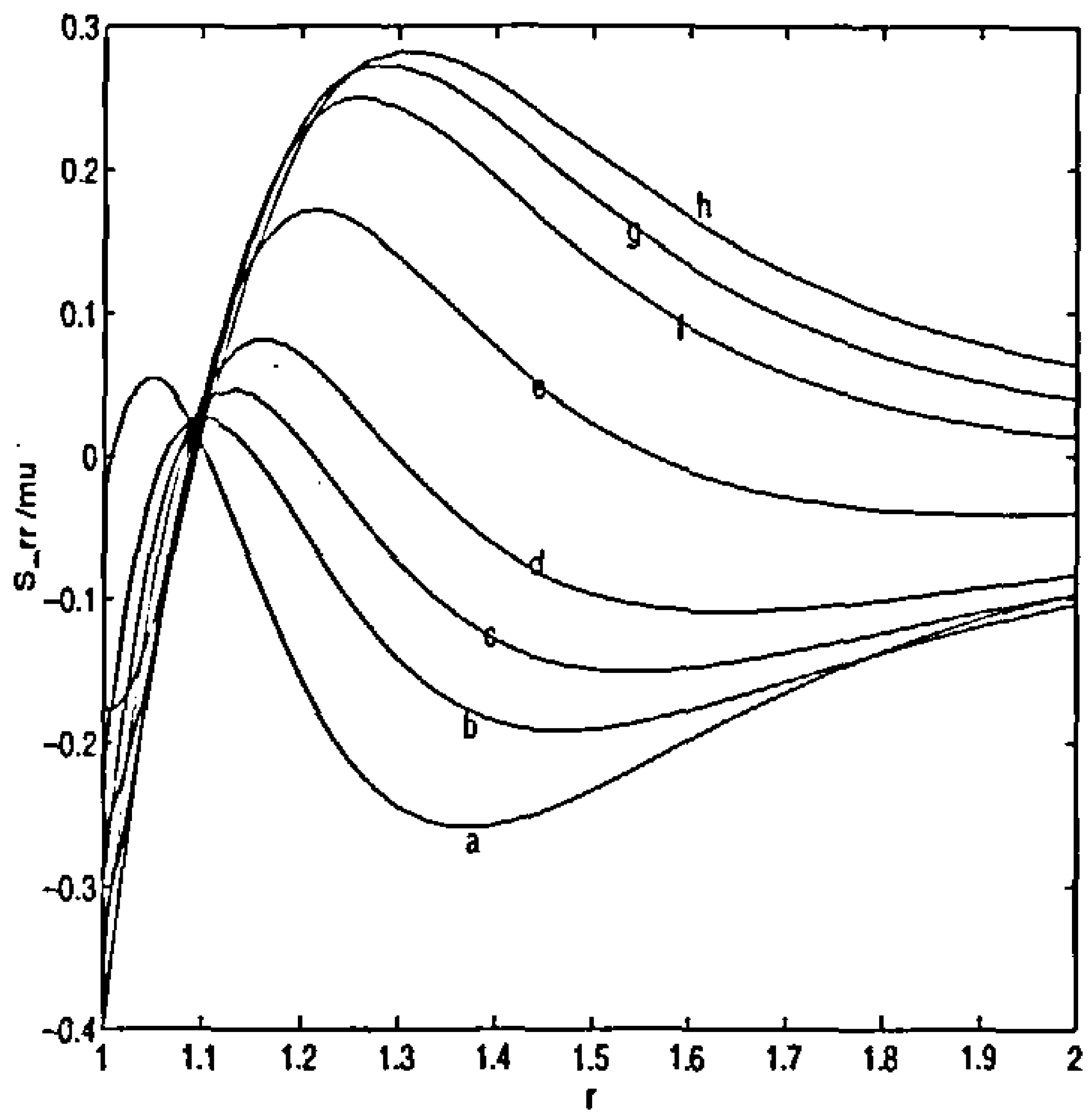
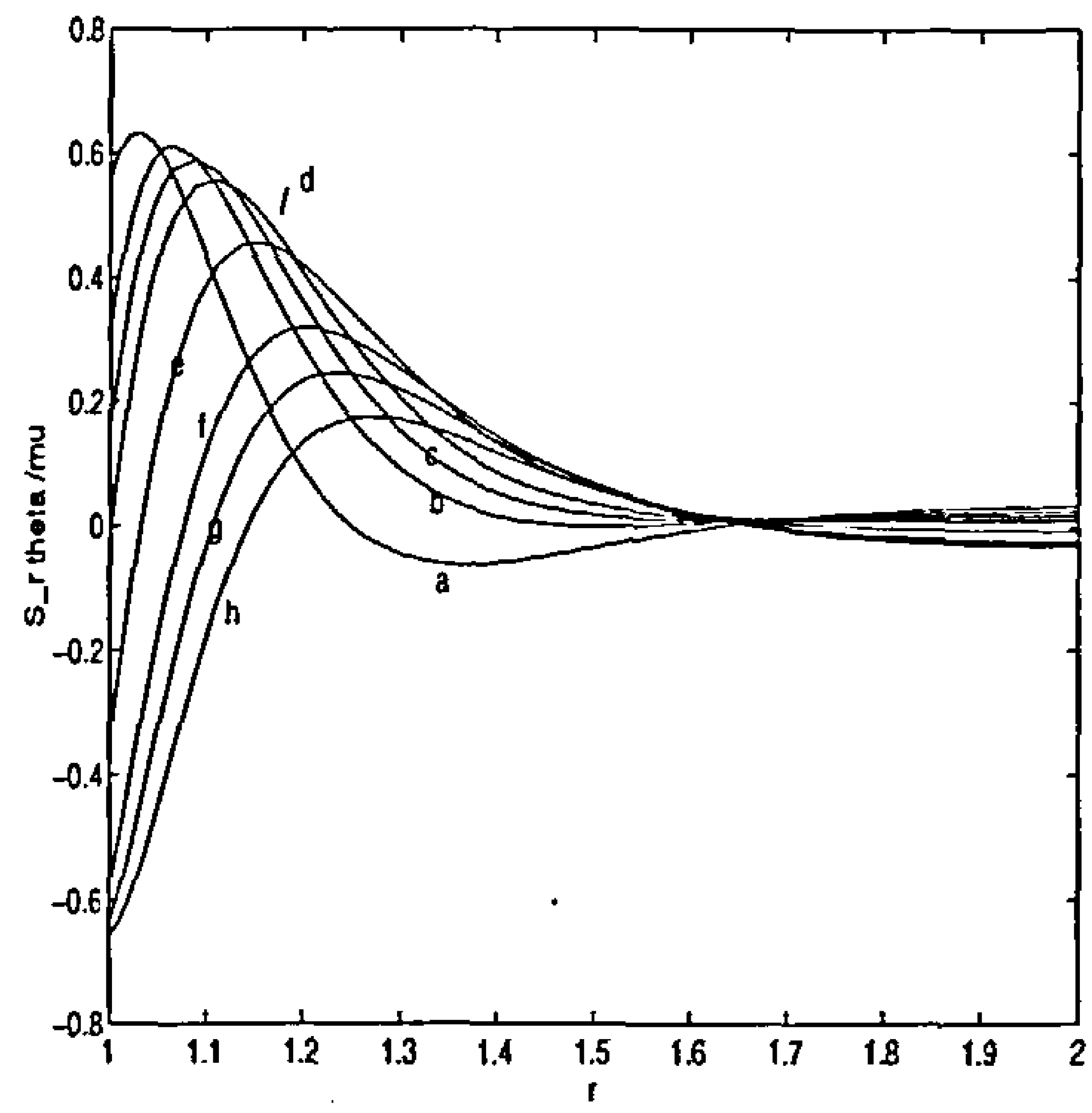


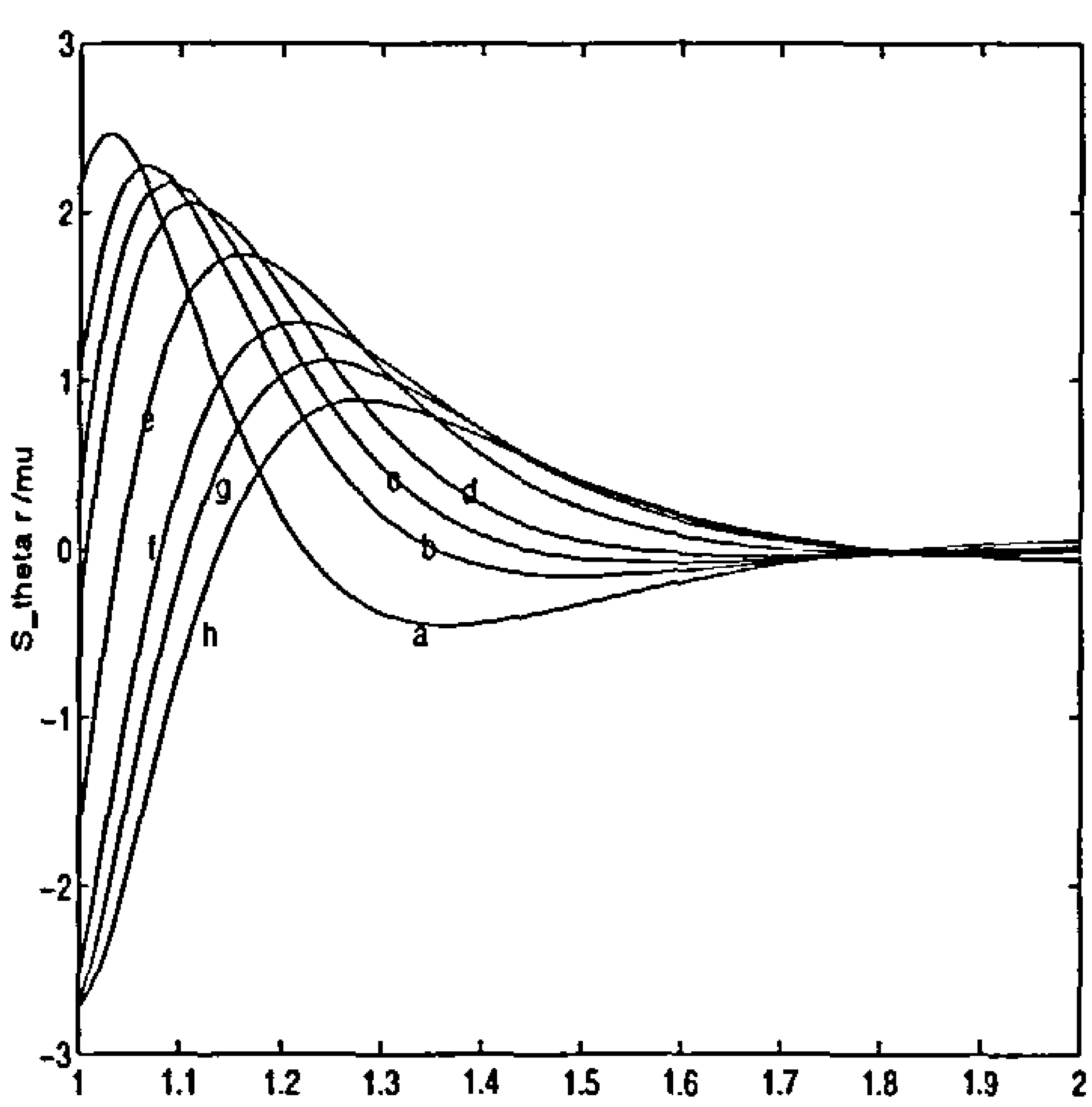
Figure 6.16: Plot of the dimensionless components (A) $\dot{S}_{rr}/\mu$, (B) $\dot{S}_{r\theta}/\mu$, (C) $\dot{S}_{\theta r}/\mu$ and (D) $\dot{S}_{\theta\theta}/\mu$ of the incremental nominal stress on the outer surface against the dimensionless displacement parameter $\bar{\epsilon}$ for $\bar{\delta} = 5.0$ and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.



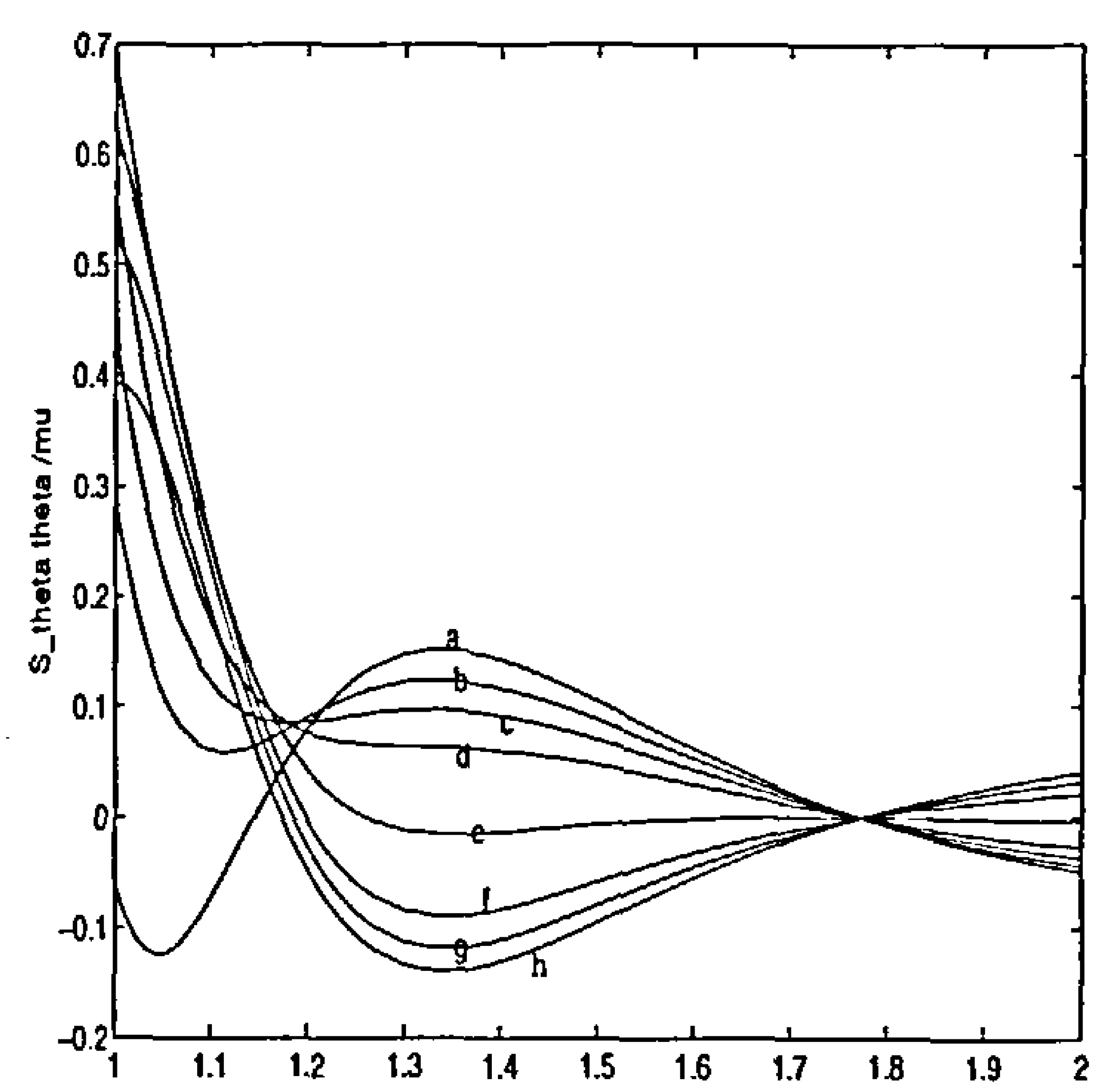
(A)



(B)



(C)



(D)

Figure 6.17: Plot of the dimensionless components (A) $\dot{S}_{rr}/\mu$, (B) $\dot{S}_{r\theta}/\mu$, (C) $\dot{S}_{\theta r}/\mu$ and (D) $\dot{S}_{\theta\theta}/\mu$ of the incremental nominal stress for $\bar{r} \in (1,2)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

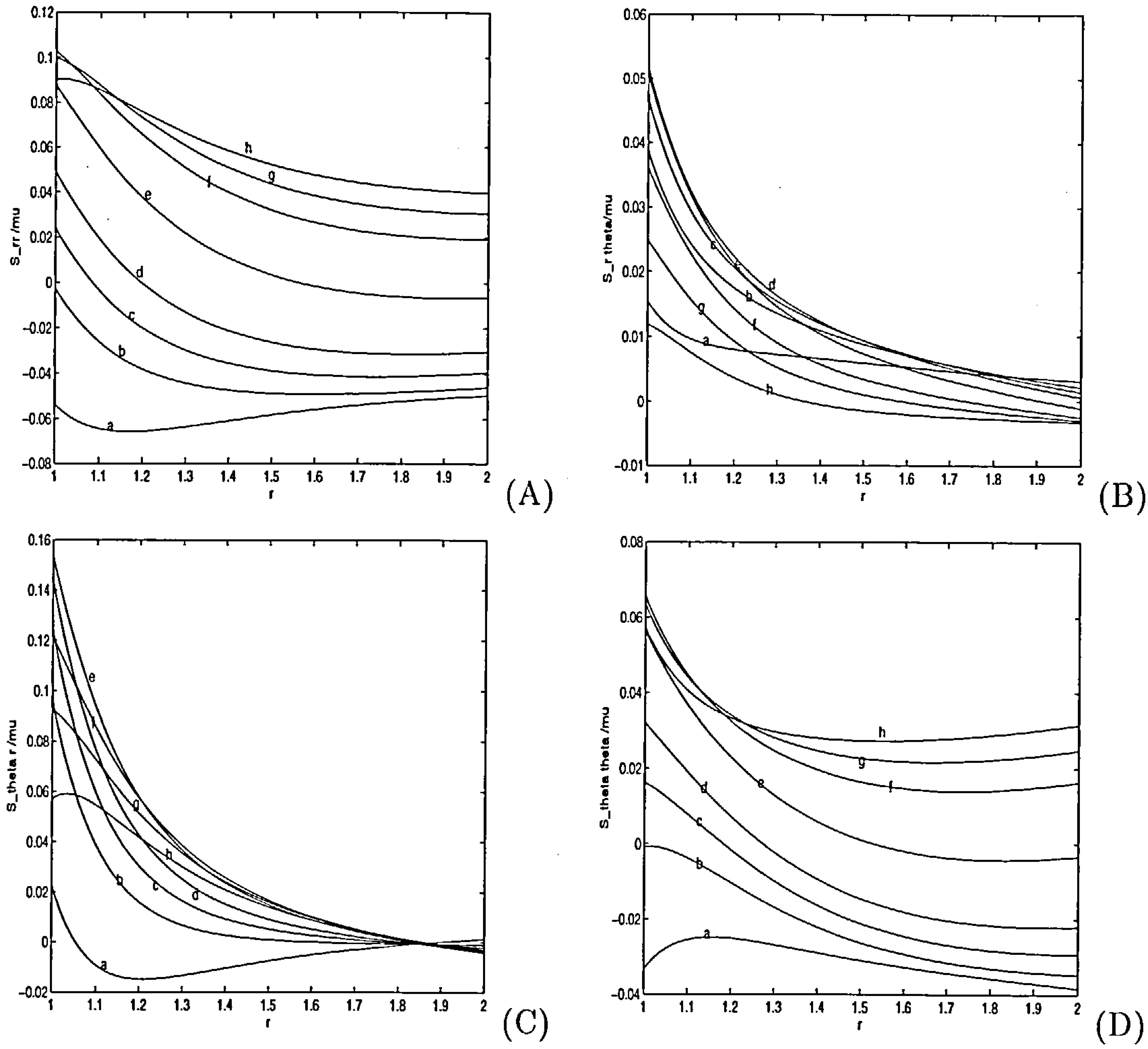


Figure 6.18: Plot of the dimensionless components (A) $\dot{S}_{rr}/\mu$, (B) $\dot{S}_{r\theta}/\mu$, (C) $\dot{S}_{\theta r}/\mu$ and (D) $\dot{S}_{\theta\theta}/\mu$ of the incremental nominal stress for $\bar{r} \in (1,2)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, and $\bar{T} = 2.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

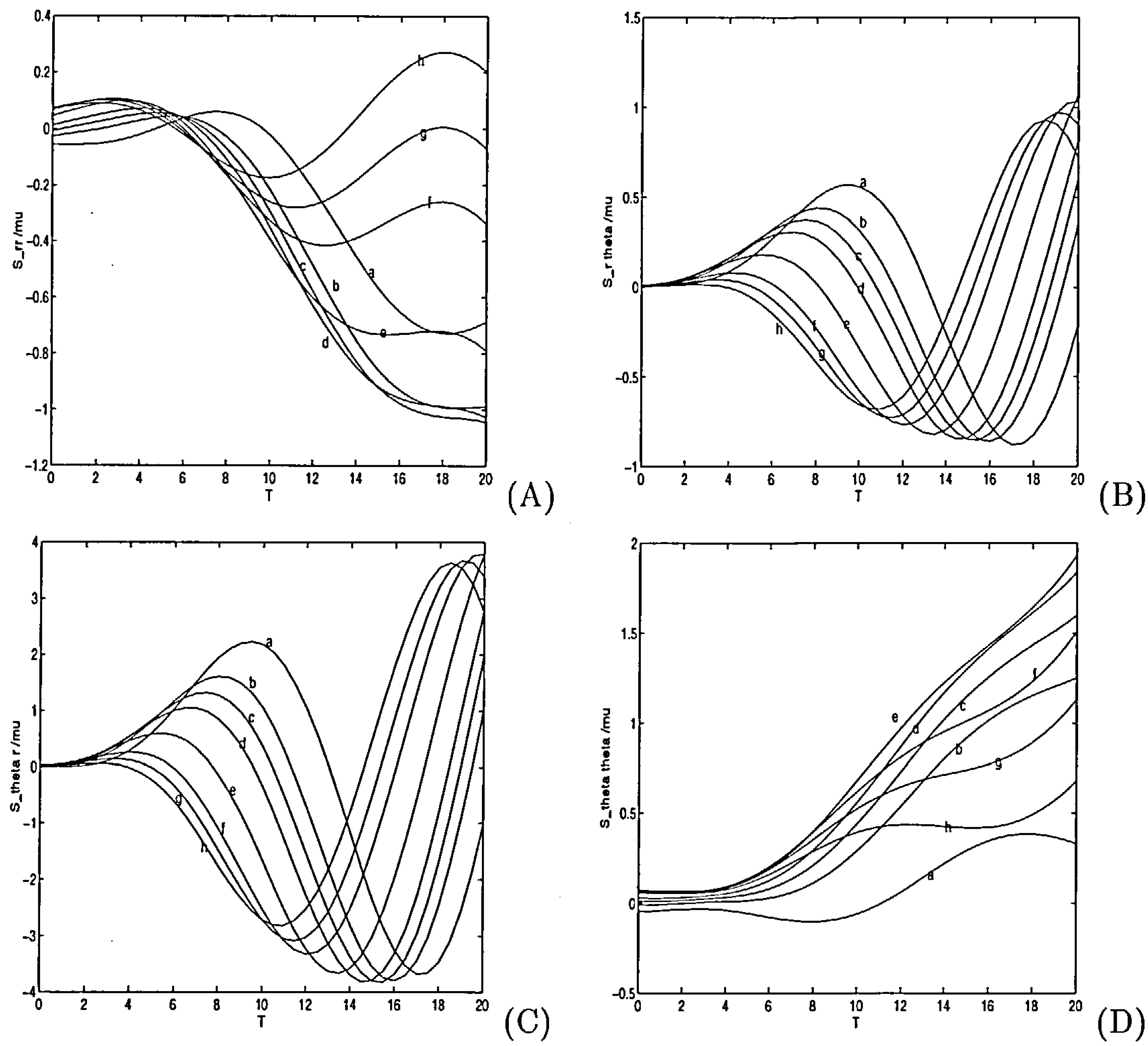


Figure 6.19: Plot of the dimensionless components (A) $\dot{S}_{rr}/\mu$, (B) $\dot{S}_{r\theta}/\mu$, (C) $\dot{S}_{\theta r}/\mu$ and (D) $\dot{S}_{\theta\theta}/\mu$ of the incremental nominal stress on the inner surface against the dimensionless underlying deformation parameter $\bar{T}$ for $\bar{r} \in (1, 2)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

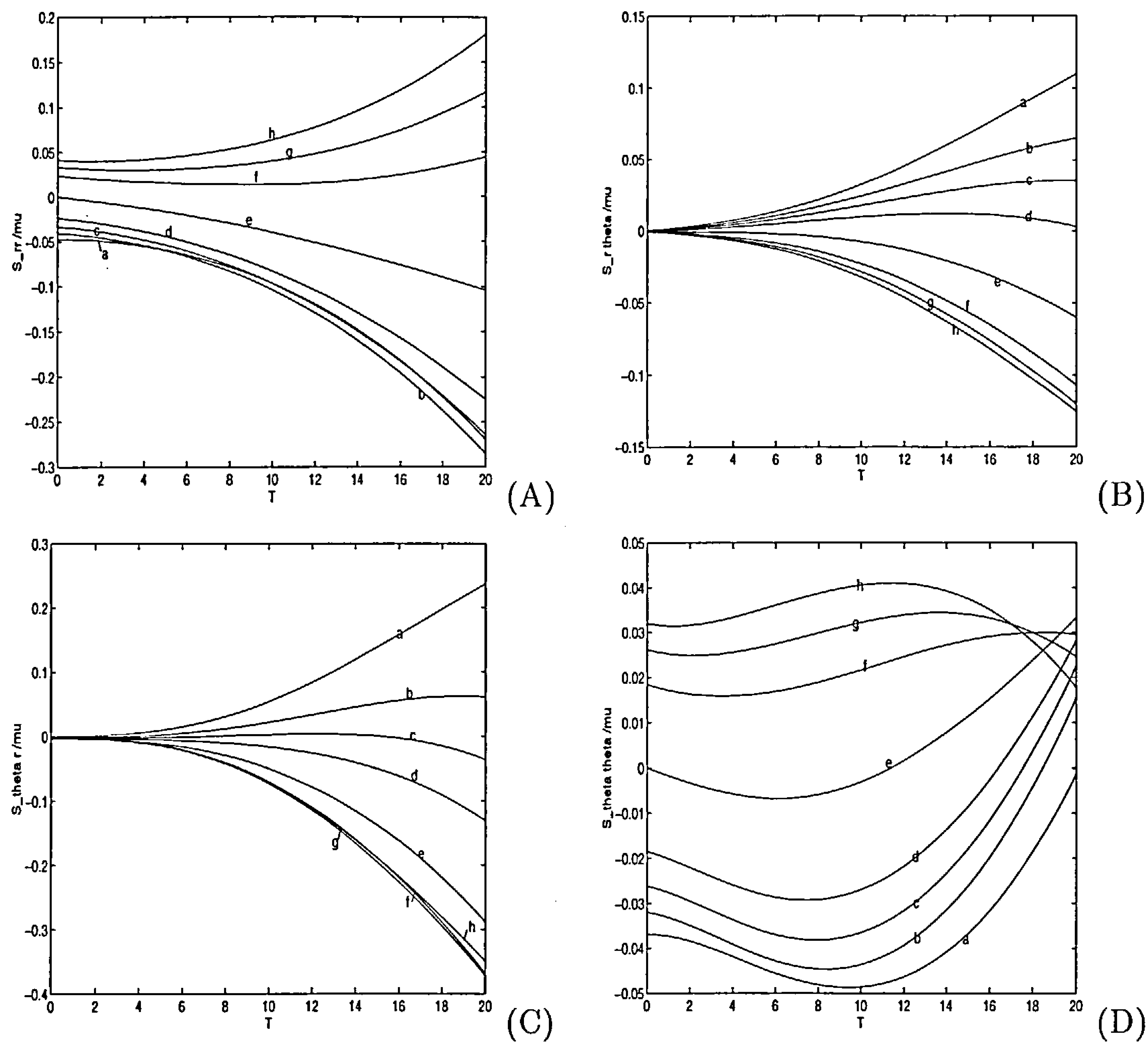


Figure 6.20: Plot of the dimensionless components (A) $\dot{S}_{rr}/\mu$, (B) $\dot{S}_{r\theta}/\mu$, (C) $\dot{S}_{\theta r}/\mu$ and (D) $\dot{S}_{\theta\theta}/\mu$ of the incremental nominal stress on the outer surface against the dimensionless underlying deformation parameter $\bar{T}$ for $\bar{r} \in (1, 2)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

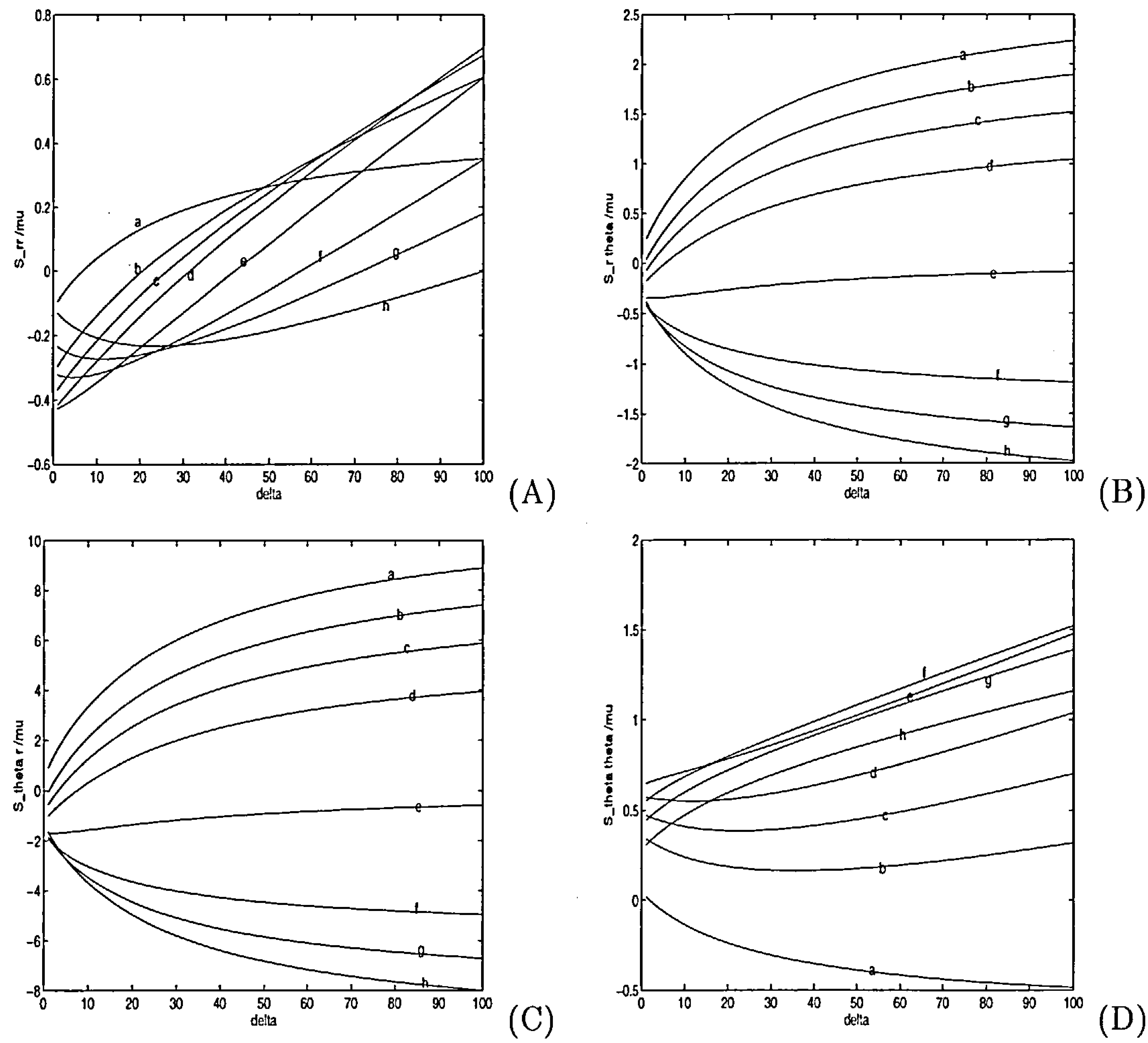


Figure 6.21: Plot of the dimensionless components (A) $\dot{S}_{rr}/\mu$, (B) $\dot{S}_{r\theta}/\mu$, (C) $\dot{S}_{\theta r}/\mu$ and (D) $\dot{S}_{\theta\theta}/\mu$ of the incremental nominal stress on the inner surface against the dimensionless material parameter $\bar{\delta}$ for $\bar{r} \in (1, 2)$ with $\bar{\epsilon} = 0.01$ and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

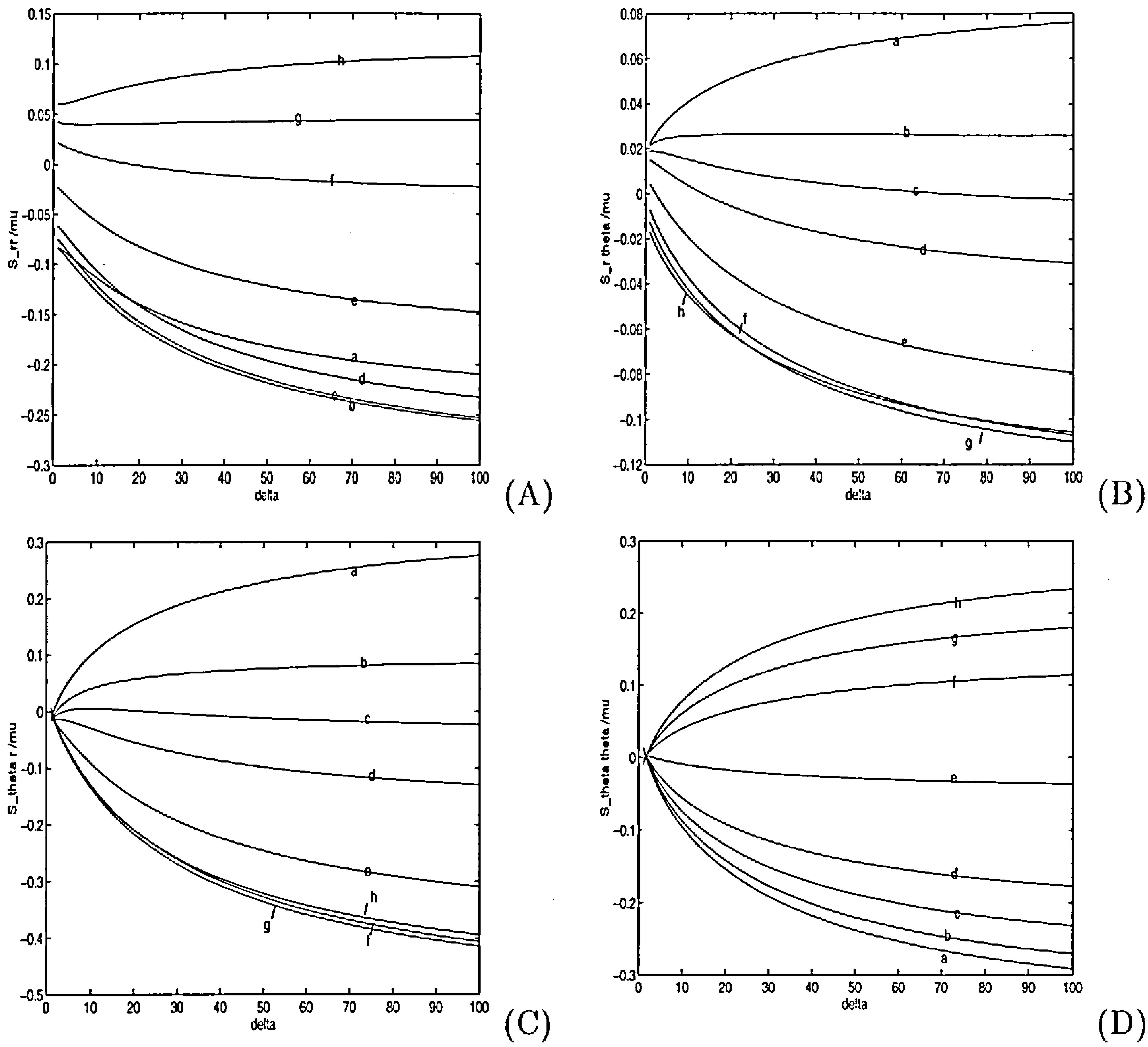


Figure 6.22: Plot of the dimensionless components (A) $\dot{S}_{rr}/\mu$, (B) $\dot{S}_{r\theta}/\mu$, (C) $\dot{S}_{\theta r}/\mu$ and (D) $\dot{S}_{\theta\theta}/\mu$ of the incremental nominal stress on the outer surface against the dimensionless material parameter $\bar{\delta}$ for $\bar{r} \in (1,2)$ with $\bar{\epsilon} = 0.01$ and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

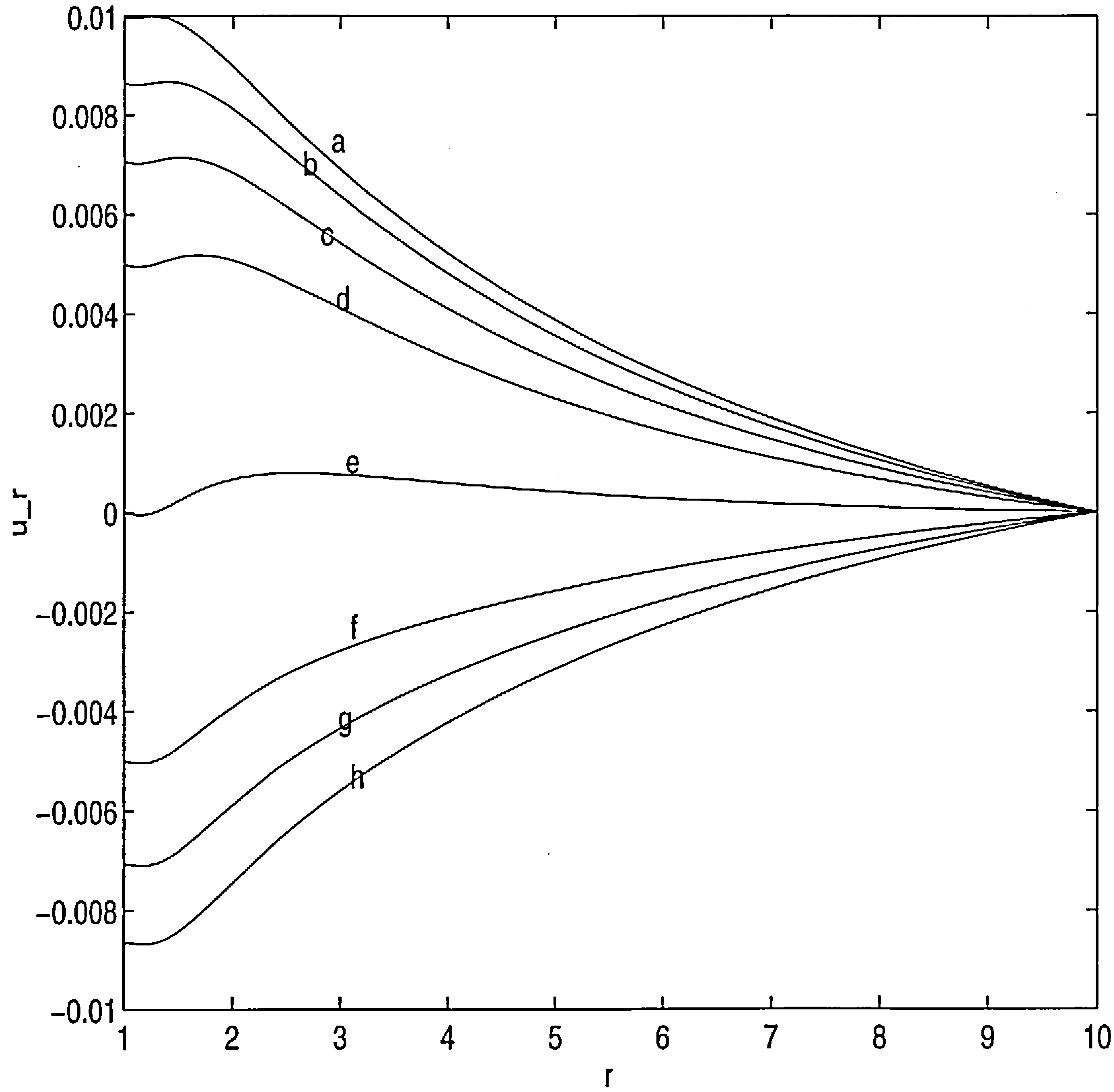


Figure 6.23: Plot of the dimensionless radial displacement $\bar{u}_r$ for $\bar{r} \in (1, 10)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

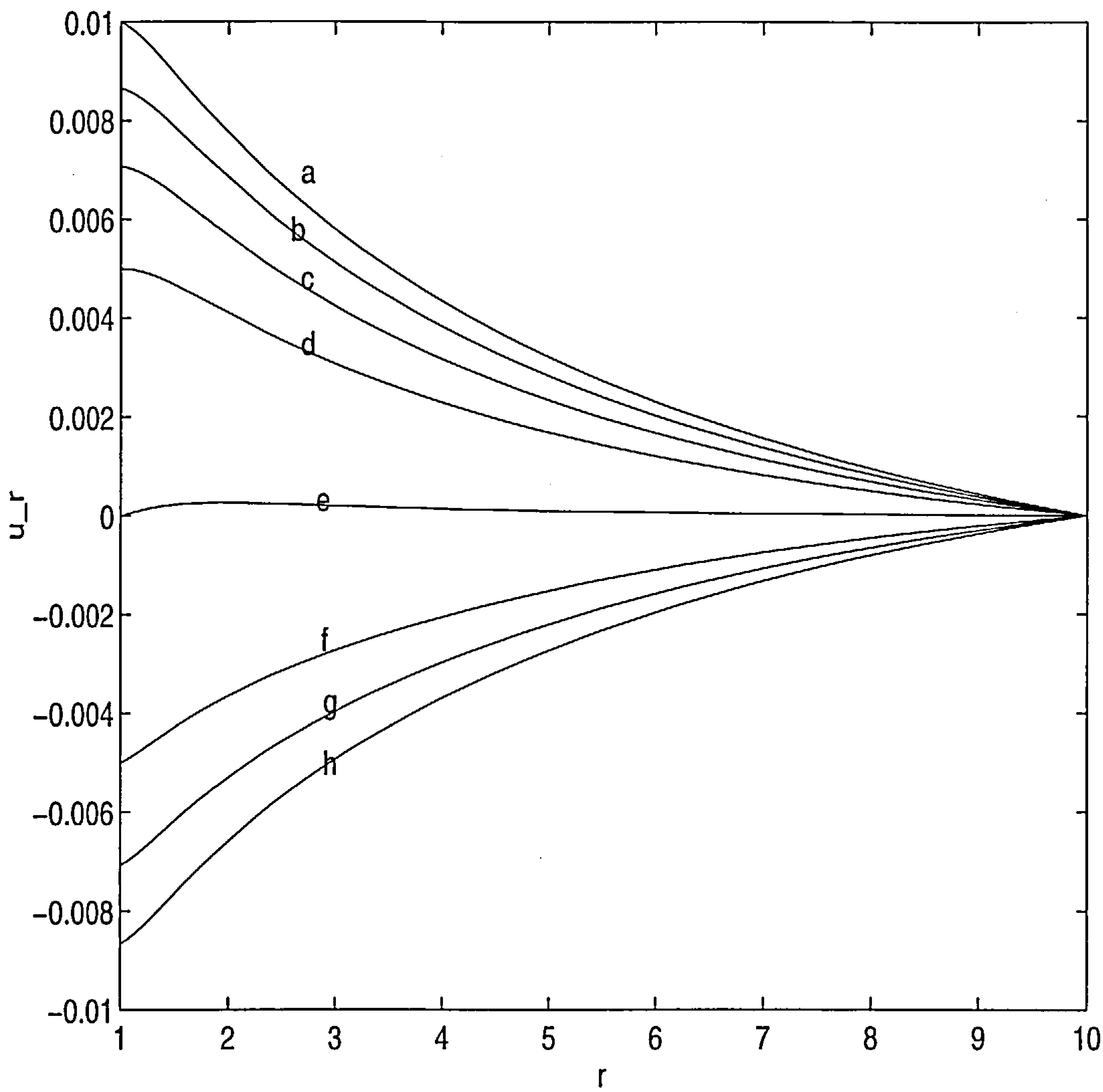


Figure 6.24: Plot of the dimensionless radial displacement $\bar{u}_r$ for $\bar{r} \in (1, 10)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, and $\bar{T} = 2.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

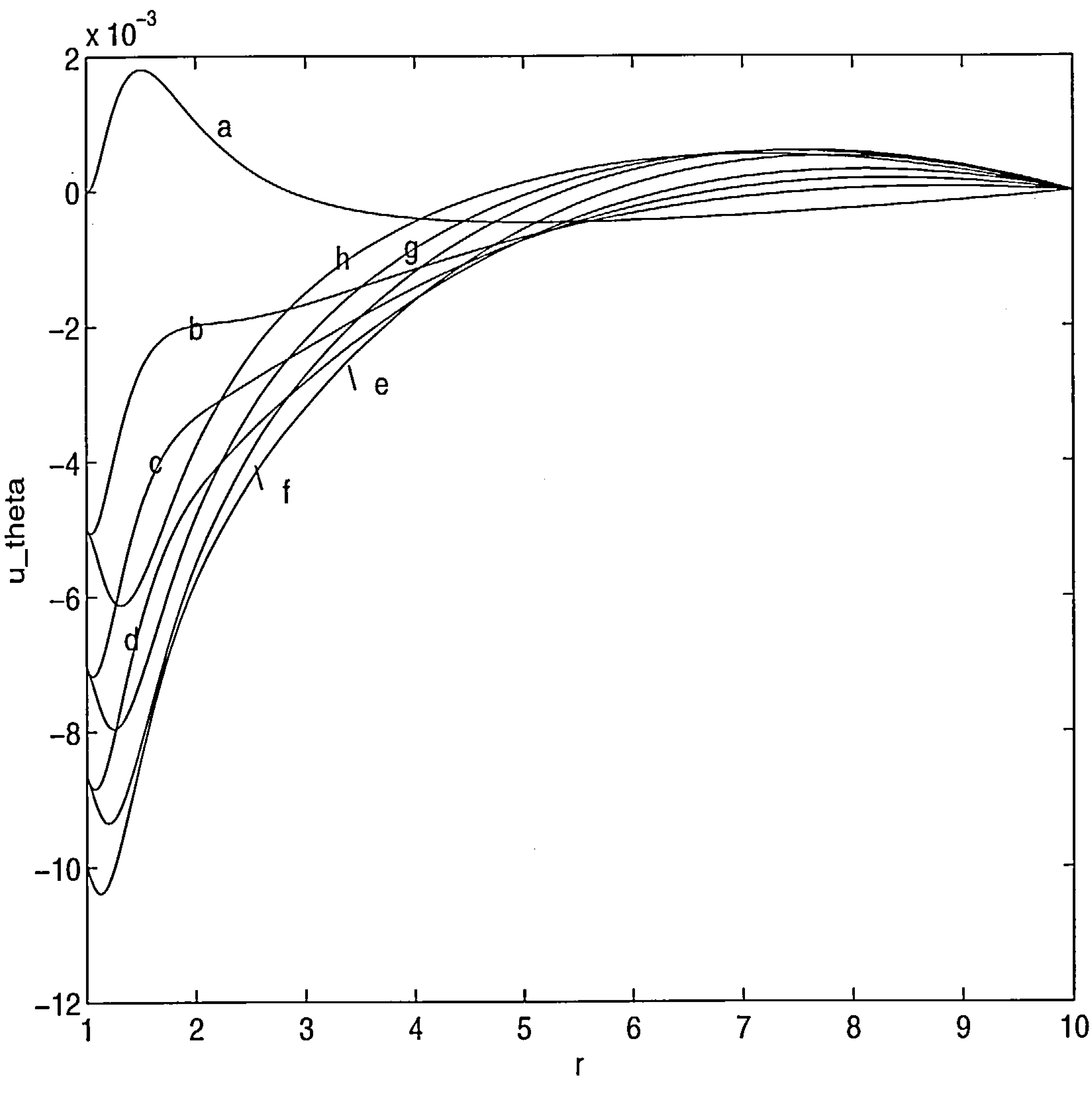


Figure 6.25: Plot of the dimensionless axial displacement $\bar{u}_\theta$ for $\bar{r} \in (1, 10)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, and $\bar{T} = 10.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

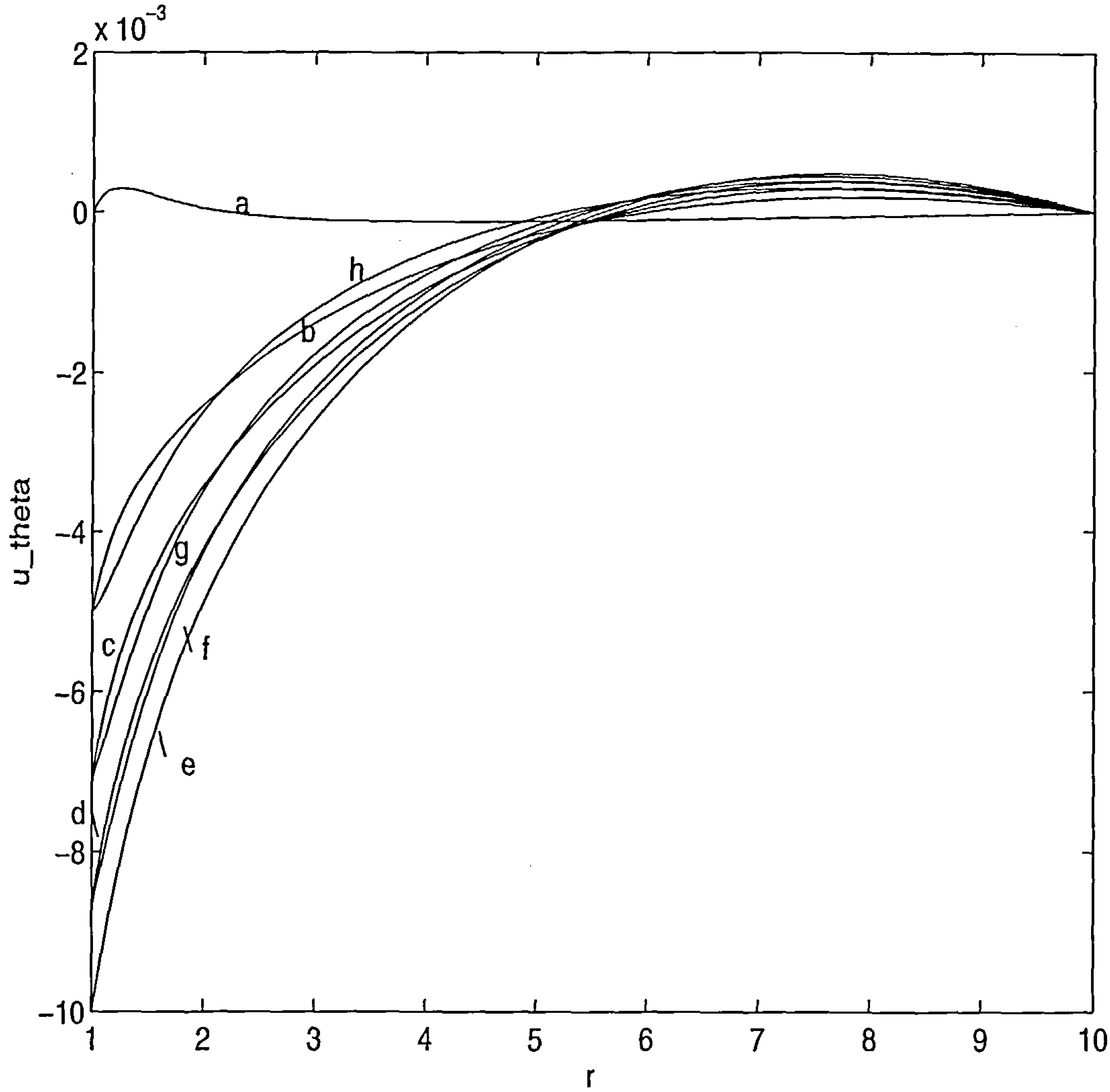


Figure 6.26: Plot of the dimensionless axial displacement $\bar{u}_\theta$ for $\bar{r} \in (1, 10)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, and $\bar{T} = 2.0$: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 45°$; (d) $\theta = 60°$; (e) $\theta = 90°$; (f) $\theta = 120°$; (g) $\theta = 135°$; (h) $\theta = 150°$.

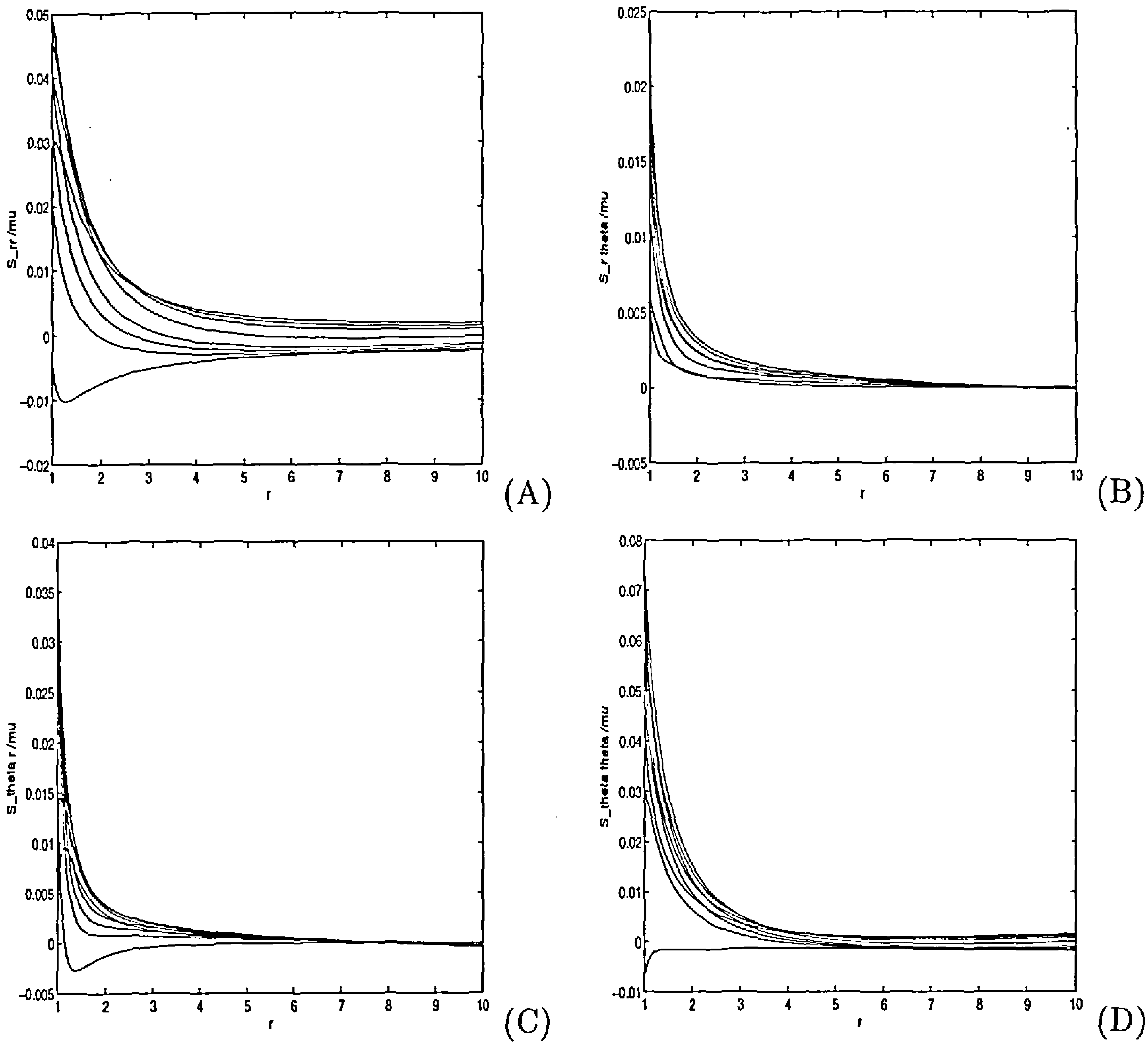


Figure 6.27: Plot of the dimensionless components (A) $\dot{S}_{rr}/\mu$, (B) $\dot{S}_{r\theta}/\mu$, (C) $\dot{S}_{\theta r}/\mu$ and (D) $\dot{S}_{\theta\theta}/\mu$ of the incremental nominal stress for $\bar{r} \in (1, 10)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, and $\bar{T} = 2.0$.

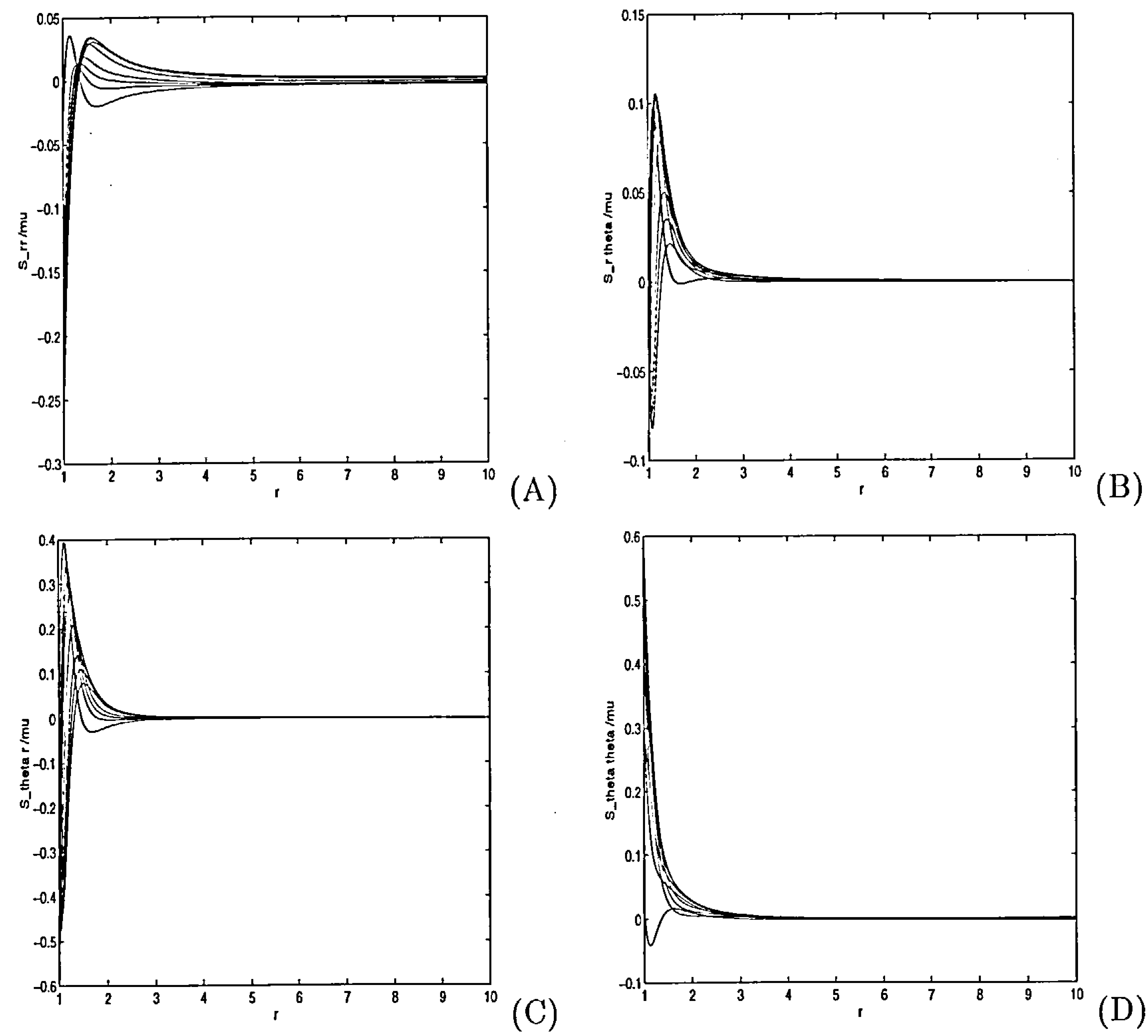


Figure 6.28: Plot of the dimensionless components (A) $\dot{S}_{rr}/\mu$, (B) $\dot{S}_{r\theta}/\mu$, (C) $\dot{S}_{\theta r}/\mu$ and (D) $\dot{S}_{\theta\theta}/\mu$ of the incremental nominal stress for $\bar{r} \in (1, 10)$ with $\bar{\epsilon} = 0.01$, $\bar{\delta} = 5.0$, and $\bar{T} = 10.0$.

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