Three and Two-Point One-Loop Integrals in Heavy Particle Effective Theories

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Abstract

We give a complete analytical computation of three and two-point loop integrals occurring in heavy-particle theories, involving a velocity change, for arbitrary real values of the external masses and residual momenta.

1 Introduction

The study of the dynamics and spectroscopy of hadrons containing a heavy quark has been greatly simplified and systematized with the introduction of heavy quark effective theory (HQET) [1]. Heavy-particle theories along similar lines have also been succesfully applied in other, related contexts. Thus, in those cases where a chiral approach to the strong interactions of heavy hadrons with light mesons is applicable, a combination of chiral and heavy-quark symmetries leads to heavy hadron chiral perturbation theory (HHChPT) [2]. A heavy-particle expansion has also been developed in the chiral-perturbative framework for nucleon-meson interactions, which constitutes the so-called heavy baryon chiral perturbation theory (HBChPT) [3].

In the heavy-quark limit the interaction of a heavy quark, or hadron, with the light degrees of freedom cannot change its four-velocity v^μ. In consequence, v^μ becomes a good quantum number and, therefore, heavy-particle effective theories of the strong interactions are expressed in terms of velocity-dependent fields. Weak interactions, or other external sources, however, can change the velocity and/or flavor of a heavy quark or hadron. Strong-interaction corrections to velocity-changing interaction vertices then involve loop integrals with two different velocities.

In this paper we report on a complete analytic computation of three-point loop integrals involving a velocity change, and two-point loop integrals. We consider a class of one-loop integrals occurring in heavy-particle theories, with arbitrary real values for the external masses and residual momenta. Since our aim here is mainly methodological, we will not discuss specific phenomenological applications. For definiteness, however, we adopt the language of HHChPT in the sequel.

In the next section, we define the integrals to be studied, establish our notations and conventions, and discuss the method we use, which involves a combination of the HQET technique and of standard methods for computing loop integrals [4, 5, 6]. In section 3 we give technical details about

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the computation of the scalar three-point integral, state our results and discuss several important limits and particular cases and cross-checks. In section 4 we briefly consider the two-point integral, which has already been given in the previous literature. In section 5 the vector and second-rank tensor integrals are given in terms of form factors. Finally, in section 6 we give some final remarks.

2 Method. Notation and Conventions

The loop integrals we consider are of the form,

$$\mathcal{I}_3{}^{\alpha_1\cdots\alpha_n} = \frac{i\mu^{4-d}}{(2\pi)^d}\int d^dq \frac{q^{\alpha_1}\cdots q^{\alpha_n}}{(2v_1\cdot(q+k_1)-\delta M_1+i\varepsilon)\,(2v_2\cdot(q+k_2)-\delta M_2+i\varepsilon)\,(q^2-m^2+i\varepsilon)} \quad (1)$$

$$\mathcal{I}_2{}^{\alpha_1\cdots\alpha_n} = \frac{i\mu^{4-d}}{(2\pi)^d}\int d^dq \frac{q^{\alpha_1}\cdots q^{\alpha_n}}{(2v\cdot(q+k)-\delta M+i\varepsilon)\,(q^2-m^2+i\varepsilon)}. \quad (2)$$

Here v_i^μ, $i=1,2$, are the velocities of the external heavy legs, k_i^μ their residual momenta, and δM_i their mass splittings relative to the common heavy mass of the corresponding heavy-flavor/spin multiplet. m is the mass of the light particle, which in HHChPT corresponds to a light pseudoscalar meson. These integrals are defined in $d=4-\epsilon$ dimensions, μ being the mass scale of dimensional regularization. Their degrees of divergence are $n+d-4$ for $\mathcal{I}_3{}^{\alpha_1\cdots\alpha_n}$ and $n+d-3$ for $\mathcal{I}_2{}^{\alpha_1\cdots\alpha_n}$. The factor of 2 in front of v_i^μ corresponds to our normalization of the heavy-particle propagators.

In this section we will restrict ourselves to the scalar case $n=0$. The cases $n=1,2$ will be considered in detail in section 5. Together with $\mathcal{I}_{2,3}$ we consider also the auxiliary integrals,

$$\widetilde{\mathcal{I}}_3 = \frac{i\mu^{4-d}}{(2\pi)^d}\int d^dq \frac{1}{((q+p_1)^2-M_1^2+i\varepsilon)\,((q+p_2)^2-M_2^2+i\varepsilon)\,(q^2-m^2+i\varepsilon)} \quad (3)$$

$$\widetilde{\mathcal{I}}_2 = \frac{i\mu^{4-d}}{(2\pi)^d}\int d^dq \frac{1}{((q+p)^2-M^2+i\varepsilon)\,(q^2-m^2+i\varepsilon)}. \quad (4)$$

$\widetilde{\mathcal{I}}_3$ is convergent in four dimensions, with degree of divergence $d-6$. $\widetilde{\mathcal{I}}_2$ has degree of divergence $d-4$, diverging logarithmically at $d=4$. The relations among external momenta and masses in $\mathcal{I}_3$ and $\widetilde{\mathcal{I}}_3$ are, $(i=1,2)$

$$p_i^\mu = Mv_i^\mu + k_i^\mu\ ,\quad p_i^\mu p_{i\mu} > 0\ ;\qquad M_i = M + \frac{1}{2}\delta M_i\ ,\quad M_i>0, \quad (5)$$

and similarly for $\mathcal{I}_2$ and $\widetilde{\mathcal{I}}_2$. We remark at this point that $\widetilde{\mathcal{I}}_{2,3}$ need not be related to Feynman diagrams in any existing physical theory. The similarity of the limit $M\to\infty$ studied below with the heavy-quark limit is purely formal. $\widetilde{\mathcal{I}}_{2,3}$ are just intermediate steps in our calculation of $\mathcal{I}_{2,3}$, as we discuss next.

In the limit $M\to\infty$ we have, $(k=1,2)$

$$\frac{1}{(q+p_k)^2-M_k^2+i\varepsilon} = \frac{1}{M}\frac{1}{2v_k\cdot(q+k_k)-\delta M_k+i\varepsilon} + \mathcal{O}\left(\frac{1}{M^2}\right). \quad (6)$$

Since $\partial\mathcal{I}_{2,3}/\partial m^2$ are convergent for $d=4$, equation (6) leads to,

$$\frac{\partial\mathcal{I}_3}{\partial m^2} = M^2\frac{\partial\widetilde{\mathcal{I}}_3}{\partial m^2} + \mathcal{O}\left(\frac{1}{M}\right),\qquad \frac{\partial\mathcal{I}_2}{\partial m^2} = M\frac{\partial\widetilde{\mathcal{I}}_2}{\partial m^2} + \mathcal{O}\left(\frac{1}{M}\right)\ . \quad (7)$$

Therefore, at $d = 4$ we must have,

$$\mathcal{I}_3 = \mathcal{I}_3|_{m=0} + M^2 \left(\widetilde{\mathcal{I}}_3 - \widetilde{\mathcal{I}}_3|_{m=0}\right) + \mathcal{O}\left(\frac{1}{M}\right) \tag{8}$$

$$\mathcal{I}_2 = \mathcal{I}_2|_{m=0} + M \left(\widetilde{\mathcal{I}}_2 - \widetilde{\mathcal{I}}_2|_{m=0}\right) + \mathcal{O}\left(\frac{1}{M}\right). \tag{9}$$

Moreover, using equations (5) and defining $\Delta_j \equiv \delta M_j - 2v_j{\cdot}k_j$, we can write $\mathcal{I}_3$, $\widetilde{\mathcal{I}}_3$ in terms of Δ_i. Differentiating we obtain, to leading order in $1/M$,

$$\frac{\partial \mathcal{I}_3}{\partial \Delta_j} = M^2 \frac{\partial \widetilde{\mathcal{I}}_3}{\partial \Delta_j} + \mathcal{O}\left(\frac{1}{M}\right), \qquad\qquad j = 1, 2, \tag{10}$$

or, equivalently,

$$\mathcal{I}_3 = M^2 \widetilde{\mathcal{I}}_3 + C_1(\Delta_1) + \mathcal{O}\left(\frac{1}{M}\right) = M^2 \widetilde{\mathcal{I}}_3 + C_2(\Delta_2) + \mathcal{O}\left(\frac{1}{M}\right) . \tag{11}$$

Here, the dependence of $C_{1,2}$ on d, M, μ, m and $v_1 \cdot v_2$ is understood, but it is shown explicitly that C_1 can depend on Δ_1 but not on Δ_2, and the oposite is true for C_2. Subtracting the two equations (11) term by term, we conclude that $C_1 = C_2 = C(d, M, \mu, m, v_1 \cdot v_2)$ do not depend on $\Delta_{1,2}$. (Furthermore, (7) together with (10) imply that C does not depend on m either.) Thus,

$$\mathcal{I}_3 = \mathcal{I}_3|_{\Delta_1=0=\Delta_2} + M^2 \left(\widetilde{\mathcal{I}}_3 - \widetilde{\mathcal{I}}_3|_{\Delta_1=0=\Delta_2}\right) + \mathcal{O}\left(\frac{1}{M}\right) . \tag{12}$$

We notice that $\mathcal{I}_3$ is straightforward to compute for $\Delta_j = 0$ by using the HQET method for combining denominators (see, *e.g.,* [7]). On the other hand, $\widetilde{\mathcal{I}}_3$ is needed in (12) only at $d = 4$, and to leading order in M, including logarithmic corrections. Eq. (12) will then be the starting point for our computation of $\mathcal{I}_3$. Equations for $\mathcal{I}_2$ analogous to (11) and (12) can also be obtained, involving two derivatives. We will find it more convenient to use eq. (9) in order to compute $\mathcal{I}_2$.

Scalar integrals can depend on v_1^μ, v_2^μ only through $\omega = v_1 \cdot v_2$. If we denote by Ω the magnitude of the three-velocity associated to v_1^μ or v_2^μ in the rest frame of $v_1^\mu + v_2^\mu$ then [8, §11.5],

$$\omega = v_1 \cdot v_2 = \frac{1 + \Omega^2}{1 - \Omega^2} ; \qquad \Omega = \sqrt{-\frac{(v_1^\mu - v_2^\mu)^2}{(v_1^\mu + v_2^\mu)^2}} = \sqrt{\frac{\omega - 1}{\omega + 1}} . \tag{13}$$

Together with Ω, the roots of $(v_1^\mu - \alpha v_2^\mu)^2 = 0$, given by

$$\alpha_\pm = \omega \pm \sqrt{\omega^2 - 1} = \frac{1 \pm \Omega}{1 \mp \Omega} , \tag{14}$$

will appear frequently below. For physical values of $v_{1,2}^\mu$, such that $(v_{1,2}^\mu)^2 = 1$, we have $\omega > 1$, $0 < \Omega < 1$, $0 < \alpha_- < 1 < \alpha_+$. We will always assume these inequalities to hold in what follows.

Logarithms have a cut along the negative real axis. The log of a product can be split as $\log(ab) = \log(a) + \log(b)$ if $\mathrm{Im}(a)$ and $\mathrm{Im}(b)$ have opposite sign, or if $a > 0$. Similarly, $\log(ab) = \log(a) - \log(b)$ if $\mathrm{Im}(a)$ and $\mathrm{Im}(b)$ have the same sign, or if $a > 0$ [4]. Given a complex number z, we use that determination of the argument such that $-\pi < \arg(z) < \pi$. In particular, $\log(1/z) = -\log(z)$. We use the same definition and conventions as [4] for the dilogarithm, which we denote by Li_2.

3 The Scalar Three-Point Integral

We will now consider in detail the calculation of $\mathcal{I}_3$. Our first step is to compute $\widetilde{\mathcal{I}}_3$ to leading order in M. As mentioned above, we only need to evaluate $\widetilde{\mathcal{I}}_3$ at $d = 4$. We introduce a standard Feynman parametrization of the integrand in (3). Integrating over d^4q and over the Feynman parameter associated with the third propagator in (3), we obtain,

$$\widetilde{\mathcal{I}}_3 = \frac{1}{(4\pi)^2}\int_0^1 dx \int_0^{1-x} dy \frac{1}{(yp_1 + xp_2)^2 + y(M_1^2 - p_1^2 - m^2) + x(M_2^2 - p_2^2 - m^2) + m^2 - i\varepsilon}. \quad (15)$$

In the limit $M \to \infty$ the polynomial in the denominator is a sum of terms of the form $M^p x^q y^{p-q}$, with $p = 0, 1, 2$ and $0 \leq q \leq p$, the term with $p = 0$ being $m^2 - i\varepsilon$. These are properties we want to maintain, in order to be able to take the limit $M \to \infty$ later, retaining only the leading terms in M in each coefficient. Thus, we will not make a change of variable $y \to 1 - y$ at this stage, as the limits of integration suggest. Following [4], we shift variables according to $y \to y - \alpha x$. This shift is homogeneous in x, y, so it does not change the order of each term as $M \to \infty$. The parameter α is taken to be one of the roots $\alpha_\pm$ of $(p_2 - \alpha p_1)^2 = 0$. For M large, we have $\alpha_+ > 1 > \alpha_- > 0$. Choosing $\alpha = \alpha_+$ and exchanging the order of integration, we are led to,

$$\begin{aligned} \widetilde{\mathcal{I}}_3 &= \frac{1}{(4\pi)^2}\left\{\int_0^1 dy \int_0^{y/\alpha_+} dx + \int_1^{\alpha_+} dy \int_{\frac{y-1}{\alpha_+-1}}^{y/\alpha_+} dx\right\}\frac{1}{D} \\ D &= \Big(2y(p_1\cdot p_2 - \alpha_+ p_1^2) + (M_2^2 - p_2^2 - m^2) - \alpha_+(M_1^2 - p_1^2 - m^2)\Big)\, x \\ &\quad + p_1^2 y^2 + y(M_1^2 - p_1^2 - m^2) + m^2 - i\varepsilon \ . \end{aligned} \quad (16)$$

Using equations (5) and retaining only leading powers of M in each coefficient, the previous expression simplifies considerably. In the limit $M \to \infty$, $\alpha_\pm$ are given by (14). Performing the integration over x we obtain,

$$\begin{aligned} \widetilde{\mathcal{I}}_3 &= -\frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\frac{1}{M^2}\int_0^1 dy \frac{1}{G}\log\left(\frac{K}{H}\right) - \frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\frac{1}{M^2}\int_1^{\alpha_+} dy \frac{1}{G}\log\left(\frac{K}{L}\right) \\ L &= H - \frac{4\Omega}{1-\Omega^2}\frac{y-1}{\alpha_+ - 1}G\ ; \qquad K = H - \frac{4\Omega}{1-\Omega^2}\frac{y}{\alpha_+}G \\ H &= y^2 + \frac{\Delta_1}{M}y + \frac{m^2}{M^2} - i\varepsilon\ ; \qquad G = y - \frac{y_0}{M}\ , \end{aligned} \quad (17)$$

with

$$y_0 = -\frac{1+\Omega}{2\Omega}(\Omega\Delta + \delta)\ ; \qquad \Delta = \frac{1}{2}(\Delta_1 + \Delta_2)\ ; \qquad \delta = \frac{1}{2}(\Delta_1 - \Delta_2)\ . \quad (18)$$

In these last two equations we have introduced several notations that will be needed later. In the second integral in (17) the variable y is $\mathcal{O}(1)$ over the entire domain of integration. Therefore, $My = \mathcal{O}(M)$ and the integral is given, to leading order in M, by

$$-\frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\frac{1}{M^2}\int_1^{\alpha_+} dy\, \frac{1}{y}\, \log\left(\alpha_- \frac{y - i\varepsilon}{-y + \alpha_+ + 1 - i\varepsilon}\right)\ . \quad (19)$$

This expression does not depend on $\Delta_{1,2}$. Therefore, it will cancel when we subtract $\widetilde{\mathcal{I}}_3|_{\Delta_1=0=\Delta_2}$ from $\widetilde{\mathcal{I}}_3$, and will not contribute to $\mathcal{I}_3$ as given in (12). We shall then drop this term from $\widetilde{\mathcal{I}}_3$ from now on. The remaining integral can be re-written as,

$$\widetilde{\mathcal{I}}_3 = \frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\frac{1}{M^2}\int_0^1 dy\,\frac{1}{G}\left\{\log(H)-\log(K)\right\}+\cdots\ . \tag{20}$$

We notice that there is no singularity at the zero of the denominator, since the numerator vanishes there. We will denote $y_{1\pm}/M$ and $y_{2\pm}/M$ the roots of H and K, respectively. They are given by,

$$y_{1\pm} = \frac{1}{2}\left(-\Delta_1 \pm \sqrt{\Delta_1^2-4m^2+i\varepsilon}\right)\ ; \qquad y_{2\pm} = \frac{\alpha_+}{2}\left(-\Delta_2 \pm \sqrt{\Delta_2^2-4m^2+i\varepsilon}\right)\ . \tag{21}$$

From their definition, (17), it is clear that H and K are equal at the zero of G. Defining,

$$z_{k\sigma} = y_{k\sigma} - y_0\ , \qquad k=1,2, \quad \sigma=\pm\ , \tag{22}$$

the equality of H and K at $y = y_0/M$ can be expressed as,

$$z_{1+}z_{1-} = \alpha_-^2 z_{2+}z_{2-}\ , \tag{23}$$

an identity that will be important below.

After factorizing H and K and splitting the logs in (20), we find,

$$\begin{aligned}\widetilde{\mathcal{I}}_3 \;=\; & \frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\frac{1}{M^2}\int_0^1 dy\,\frac{1}{y-y_0/M}\left\{\log\left(y-\frac{y_{1+}}{M}\right)+\log\left(y-\frac{y_{1-}}{M}\right)\right.\\ & \left.-\log\left[\alpha_-\left(y-\frac{y_{2+}}{M}\right)\right]-\log\left[\alpha_-\left(y-\frac{y_{2-}}{M}\right)\right]\right\}\ . \end{aligned}\tag{24}$$

In order to be able to distribute the integral inside the braces without introducing spurious singularities, we use (23) to add and subtract the value of each log at the pole. In this way we obtain,

$$\widetilde{\mathcal{I}}_3 = \frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\frac{1}{M^2}\sum_{k,\sigma}(-1)^{k+1}\int_0^1 dy\,\frac{1}{y-y_0/M}\left\{\log\left(y-\frac{y_{k\sigma}}{M}\right)-\log\left(\frac{y_0}{M}-\frac{y_{k\sigma}}{M}\right)\right\}\ , \tag{25}$$

where the sum runs over $k = 1, 2$ and $\sigma = \pm$. These integrals are already in standard form. Evaluating them to leading order in M, we arrive at,

$$\begin{aligned}\widetilde{\mathcal{I}}_3 \;=\; & \frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\frac{1}{M^2}\sum_{k,\sigma}(-1)^{k+1}\left\{\frac{1}{2}\log^2\left(\frac{M}{\mu}\right)-\frac{\pi^2}{6}-\log\left(\frac{M}{\mu}\right)\log\left(-\frac{z_{k\sigma}}{\mu}\right)\right.\\ & \left.-\frac{1}{2}\log^2\left(\frac{z_{k\sigma}}{\mu}\right)+\log\left(\frac{z_{k\sigma}}{\mu}\right)\log\left(-\frac{z_{k\sigma}}{\mu}\right)-\log\left(-\frac{y_0}{z_{k\sigma}}\right)\log\left(\frac{y_{k\sigma}}{z_{k\sigma}}\right)-\mathrm{Li}_2\left(\frac{y_{k\sigma}}{z_{k\sigma}}\right)\right\}\ . \end{aligned}\tag{26}$$

In order to simplify this result we have explicitly used the relation $\log(\mu/z_{k\sigma}) = -\log(z_{k\sigma}/\mu)$ as explained in section 2. Notice that we have introduced a mass scale μ that makes the arguments of the logs dimensionless. The first two terms in (26) are independent of k and therefore they cancel out in the sum. The third term can be simplified by making use of (23), which results in a term of the form $(1-\Omega^2)/(64\pi^2M^2\Omega)\log(\alpha_+^2)\log(M/\mu)$.

The integral $\widetilde{\mathcal{I}}_3$ at $\Delta_j = 0$ is straightforward to compute. Its logarithmic dependence on M cancels that of $\widetilde{\mathcal{I}}_3$, so that we obtain,

$$\begin{aligned}\widetilde{\mathcal{I}}_3 - \widetilde{\mathcal{I}}_3|_{\Delta_j=0} \;=\; & \frac{1}{32\pi^2M^2}\frac{1-\Omega^2}{2\Omega}\left\{\log^2(\alpha_+)+\log\left(\alpha_+^2\right)\log\left(\frac{m}{\mu}\right)+\sum_{k,\sigma}(-1)^k\left[\frac{1}{2}\log^2\left(\frac{z_{k\sigma}}{\mu}\right)\right.\right.\\ & \left.\left.-\log\left(\frac{z_{k\sigma}}{\mu}\right)\log\left(-\frac{z_{k\sigma}}{\mu}\right)+\log\left(-\frac{y_0}{z_{k\sigma}}\right)\log\left(\frac{y_{k\sigma}}{z_{k\sigma}}\right)+\mathrm{Li}_2\left(\frac{y_{k\sigma}}{z_{k\sigma}}\right)\right]\right\}\ . \end{aligned}\tag{27}$$

This is essentially the final result, except for the dimensional regularization pole term, which is supplied by $\mathcal{I}_3|_{\Delta_j=0}$ (see eq. (12)). Setting $n=0$, $\delta M_j = 0 = k_j$, $j=1,2$, in (1), we obtain, after using the HQET method for combining denominators and integrating over d^dq,

$$\mathcal{I}_3|_{\Delta_j=0} = \frac{\mu^{4-d}}{(4\pi)^{d/2}} \Gamma\Big(3-\frac{d}{2}\Big) \int_0^1 dx \int_0^\infty d\lambda \frac{\lambda}{\left[\lambda^2\left(x^2+(1-x)^2+2x(1-x)\omega\right)+m^2-i\varepsilon\right]^{3-d/2}} \,. \quad (28)$$

The innermost integral can be evaluated by changing variable to $u=\lambda^2$. We get,

$$\begin{aligned}
\mathcal{I}_3|_{\Delta_j=0} &= \frac{1}{2(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{\mu}{m}\right)^{4-d}\int_0^1 dx \frac{1}{x^2+(1-x)^2+2x(1-x)\omega} \\
&= \frac{1}{2(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{\mu}{m}\right)^{4-d}\frac{1-\Omega^2}{4\Omega}\log\left(\alpha_+^2\right) \\
&= \frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\log(\alpha_+)\left(\frac{2}{\epsilon}-\gamma_{\mathrm{E}}+\log(4\pi)+\log\left(\frac{\mu^2}{m^2}\right)\right)+\mathcal{O}(\epsilon) \,, \qquad (29)
\end{aligned}$$

where γ_{E} is Euler's gamma.

Thus, finally, using (12), (27) and (29), we obtain,

$$\begin{aligned}
\mathcal{I}_3 = \frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\Bigg\{&\frac{2}{\epsilon}\log(\alpha_+)+\log^2(\alpha_+)+\sum_{k,\sigma}(-1)^k\left[\frac{1}{2}\log^2\left(\frac{z_{k\sigma}}{\overline{\mu}}\right)\right. \\
&\left.-\log\left(\frac{z_{k\sigma}}{\overline{\mu}}\right)\log\left(-\frac{z_{k\sigma}}{\overline{\mu}}\right)+\log\left(-\frac{y_0}{z_{k\sigma}}\right)\log\left(\frac{y_{k\sigma}}{z_{k\sigma}}\right)+\mathrm{Li}_2\left(\frac{y_{k\sigma}}{z_{k\sigma}}\right)\right]\Bigg\} \,, \qquad (30)
\end{aligned}$$

where we used (23) to rewrite $\log(\mu)$ in (27) as $\log(\overline{\mu}) = \log(\mu\sqrt{4\pi}) - \gamma_{\mathrm{E}}$. Equation (30) is our main result. It gives the analytical expression for $\mathcal{I}_3$ for general real values of the external masses and residual momenta. Below we consider some particular values of the parameters which are important in practice, and in which the expression for $\mathcal{I}_3$ takes on a simplified form. They can also serve as cross-checks of (30).

We note that $\mathcal{I}_3$, given by (1) with $n=0$, is symmetric under exchange of heavy particles $1 \leftrightarrow 2$. The general result (30) is not manifestly invariant under $\Delta_1 \leftrightarrow \Delta_2$, but its symmetry has been thoroughly checked numerically.

3.1 The case $y_0 < 0$ and the zero-recoil limit

An important particular case to consider is the case $v_1^\mu = v_2^\mu$ or, equivalently, $\Omega = 0$ ($\omega = 1$). $\mathcal{I}_3$ at $\Omega = 0$ can be computed more directly by differentiating $\mathcal{I}_2$ (see section 4). Therefore, its calculation from (30) constitutes a cross-check.

The limit $\Omega \to 0$, however, is difficult to take in (30). As indicated by the factor $1/\Omega$ in that equation, the limit results from complicated cancellations among different terms in the sum in (30). Individually, some of those terms may be large as $\Omega \to 0$. This situation arises mainly from the addition and subtraction of terms of the form $\log(y_0/M - y_{k\sigma}/M)$ in (25), which is necessary when $y_0 > 0$, since in that case there is a singularity in the integration domain. If, however, we restrict ourselves to the case $y_0 < 0$, then the general expression (30) takes a simpler form, which makes the zero-recoil limit transparent.

As mentioned at the end of the previous section, $\mathcal{I}_3$ is symmetric under exchange of Δ_1 and Δ_2. Hence, without loss of generality we can assume $\delta > 0$ (see eq. (18). The case $\delta = 0$ will be considered afterwards as a limiting case). From the definition of y_0 in (18) we see that, if $\delta > 0$, then for sufficiently small Ω we will have $y_0 < 0$. This is therefore the relevant parameter region to consider when $\Omega \to 0$.

Assuming, then, $y_0 < 0$, we can go back to (24) and distribute the integral inside the braces without adding extra terms. The calculation goes through unchanged, yielding,

$$\begin{aligned} \mathcal{I}_3\Big|_{\substack{\delta>0 \\ y_0<0}} &= \frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\left\{ \log(\alpha_+)\left[\frac{2}{\epsilon} + \log(\alpha_+) + \log\left(\frac{\overline{\mu}^2}{4\Omega^2 y_0^2}\right)\right]\right. \\ &\left. + \sum_{k,\sigma}(-1)^k\left[\frac{1}{2}\log^2\left(1-\frac{y_{k\sigma}}{y_0}\right) + \log\left(1-\frac{y_{k\sigma}}{y_0}\right)\log\left(\frac{2\Omega y_0}{y_{k\sigma}}\right) + \mathrm{Li}_2\left(\frac{y_{k\sigma}}{z_{k\sigma}}\right)\right]\right\} . \end{aligned} \quad (31)$$

The zero-recoil limit of this expression can be easily obtained. Using the assumption $\delta > 0$ and the relation (valid for $\Omega = 0$)

$$\sum_{k,\sigma}(-1)^k y_{k\sigma} = 2\delta \ ,$$

we find,

$$\mathcal{I}_3\Big|_{\substack{\delta>0 \\ \Omega=0}} = \frac{1}{32\pi^2}\left(\frac{2}{\epsilon} + 2 + \sum_{k,\sigma}(-1)^{k+1}\frac{y_{k\sigma}}{\delta}\log\left(-\frac{y_{k\sigma}}{\overline{\mu}}\right)\right) . \quad (32)$$

It is understood that in this equation we must set $\Omega = 0$ in the expression (21) for $y_{k\sigma}$. In section 4 we will obtain this result from $\mathcal{I}_2$.

We give, finally, the result for $\mathcal{I}_3$ at zero-recoil when $\delta = 0$ ($\Delta_1 = \Delta_2 \equiv \Delta$). We just take the limit of (32) as $\delta \to 0$ to obtain,

$$\mathcal{I}_3\Big|_{\substack{\delta=0 \\ \Omega=0}} = \frac{1}{32\pi^2}\left\{\frac{2}{\epsilon} + \log\left(\frac{\overline{\mu}^2}{m^2}\right) + \frac{\Delta}{\sqrt{\cdot}}\left[\log\left(\frac{\Delta-\sqrt{\cdot}}{\overline{\mu}}\right) - \log\left(\frac{\Delta+\sqrt{\cdot}}{\overline{\mu}}\right)\right]\right\} , \quad (33)$$

where $\sqrt{\cdot} = \sqrt{\Delta^2 - 4m^2 + i\varepsilon}$. Notice that, in general, we cannot express the difference of logs in (33) as the log of the ratio of the arguments , because their imaginary parts have opposite sign.

3.2 The case $y_0 = 0$

When $y_0 = 0$ the general expression (30) for $\mathcal{I}_3$ is singular. The singularity is avoidable, though, so that we can take the limit $y_0 \to 0$ in (30) to get,

$$\begin{aligned} \mathcal{I}_3\Big|_{y_0=0} &= \frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\log(\alpha_+)\left(\frac{2}{\epsilon} + \log(\alpha_+)\right) \\ &+ \frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\sum_{k,\sigma}(-1)^k\left[\frac{1}{2}\log^2\left(\frac{y_{k\sigma}}{\overline{\mu}}\right) - \log\left(\frac{y_{k\sigma}}{\overline{\mu}}\right)\log\left(-\frac{y_{k\sigma}}{\overline{\mu}}\right)\right] . \end{aligned} \quad (34)$$

This expression is valid, in particular, when $\Delta_1 = 0 = \Delta_2$, which is the point in parameter space we used to "match" $\mathcal{I}_3$ and $\widetilde{\mathcal{I}}_3$. As is easily seen, in that case we recover (29).

3.3 The case $m = 0$

The case $m = 0$ is relevant to theories involving massless particles, such as gluons in HQET and Goldstone bosons in chiral theories in the limit of massless quarks . The value of $\mathcal{I}_3$ at $m = 0$

can be obtained from the general expression (30), and also by direct computation from (1) using the HQET method for combining denominators. We will consider both approaches in this section. Together with the zero-recoil case studied in sections 3.1 and 4, this is one of our main cross-checks.

We will now study the limit $m \to 0$ of $\mathcal{I}_3$ as given in (30). We notice that when $m = 0$ one of the two possible values $\sigma = \pm$ leads to $y_{k\sigma} = 0$ (see eq. (21)). In fact, $m = 0$ and $\Delta_k > 0$ (resp. $\Delta_k < 0$) implies $y_{k+} = 0$ (resp. $y_{k-} = 0$). Calling $\sigma_k = -\mathrm{sgn}(\Delta_k) = -\sigma'_k$, so that $y_{k\sigma_k} \neq 0 = y_{k\sigma'_k}$, we have $z_{k\sigma'_k} = -y_0 + i\varepsilon\sigma'_k$ and, therefore,

$$\log\left(-\frac{y_0}{z_{k\sigma'_k}}\right)\log\left(\frac{y_{k\sigma'_k}}{z_{k\sigma'_k}}\right) = 0\ ; \qquad \mathrm{Li}_2\left(\frac{y_{k\sigma'_k}}{z_{k\sigma'_k}}\right) = 0\ ,$$

so these terms drop from the sum in (30). On the other hand, since y_0 is by definition (18) a real number, we have the equalities,

$$\begin{aligned}\sum_{k=1}^{2}(-1)^k\frac{1}{2}\log^2(-y_0+i\varepsilon\sigma'_k) &= i\pi\theta(y_0)\log(y_0)\sum_{k=1}^{2}(-1)^k\sigma'_k \\ \sum_{k=1}^{2}(-1)^k\log(-y_0+i\varepsilon\sigma'_k)\log(y_0-i\varepsilon\sigma'_k) &= i\pi\mathrm{sgn}(y_0)\log(|y_0|)\sum_{k=1}^{2}(-1)^k\sigma'_k\ ,\end{aligned}$$

$\theta(x)$ being a step function. With these relations taken into account, from (30) we find for $\mathcal{I}_3$,

$$\begin{aligned}\mathcal{I}_3\Big|_{m=0} &= \frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\left\{\frac{2}{\epsilon}\log(\alpha_+) + \log^2(\alpha_+) + \sum_{k=1}^{2}(-1)^k\left[\frac{1}{2}\log^2\left(\frac{z_{k\sigma_k}}{\bar\mu}\right)\right.\right. \\ &\left.\left. -\log\left(\frac{z_{k\sigma_k}}{\bar\mu}\right)\log\left(-\frac{z_{k\sigma_k}}{\bar\mu}\right) + \log\left(-\frac{y_0}{z_{k\sigma_k}}\right)\log\left(\frac{y_{k\sigma_k}}{z_{k\sigma_k}}\right) + \mathrm{Li}_2\left(\frac{y_{k\sigma_k}}{z_{k\sigma_k}}\right) + \mathcal{E}(y_0)\right]\right\}, \quad (35) \\ \mathcal{E}(y_0) &\equiv i\pi\theta(-y_0)\log\left(\frac{|y_0|}{\bar\mu}\right)\sum_{k=1}^{2}(-1)^k\sigma'_k = -2i\pi\log\left(\frac{-y_0}{\bar\mu}\right)\theta(\Delta_1)\theta(-\Delta_2)\ . \quad (36)\end{aligned}$$

Notice that the sum in (35) runs over $k = 1, 2$ but not over σ_k, and that $-y_0 > 0$ in (36) due to the step functions. As before, $y_{k\sigma_k}$ and $z_{k\sigma_k}$ in (35) are given by (21) and (22) with m set to zero. When $\delta = 0$ ($\Delta_1 = \Delta_2 \equiv \Delta$), equation (35) becomes

$$\begin{aligned}\mathcal{I}_3\Big|_{m=0=\delta} &= \frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\left\{\frac{2}{\epsilon}\log(\alpha_+) - \frac{1}{2}\log(\alpha_+)\log\left(\frac{1-\Omega^2}{4}\right) - 2\log(\alpha_+)\log\left(\frac{\Delta-i\varepsilon}{\bar\mu}\right)\right. \\ &\left. +\mathrm{Li}_2\left(\frac{1-\Omega}{2}\right) - \mathrm{Li}_2\left(\frac{1+\Omega}{2}\right)\right\}\ , \quad (37)\end{aligned}$$

a result we will explicitly cross-check below.

We now turn to the calculation of $\mathcal{I}_3$ at $m^2 = 0$ directly from its definition (1) with the HQET method for combining denominators. For the sake of brevity, we will skip the details of the derivation and quote the final result, which can be written as,

$$\begin{aligned}\mathcal{I}_3\Big|_{m=0} &= \frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\left\{\frac{2}{\epsilon}\log(\alpha_+) - \frac{1}{2}\log(\alpha_+)\log\left(\frac{1-\Omega^2}{4}\right) + \mathrm{Li}_2\left(\frac{1-\Omega}{2}\right) - \mathrm{Li}_2\left(\frac{1+\Omega}{2}\right)\right\} \\ &+\frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\left\{-\log\left(\frac{-(1+\Omega)\delta}{\Omega\Delta-\delta-i\varepsilon}\right)\log\left(\frac{\Delta+\delta-i\varepsilon}{\bar\mu}\right) + \log\left(\frac{-(1-\Omega)\delta}{\Omega\Delta-\delta-i\varepsilon}\right)\log\left(\frac{\Delta-\delta-i\varepsilon}{\bar\mu}\right)\right. \\ &\left.-\mathrm{Li}_2\left(\frac{\Delta+\delta-i\varepsilon}{\Delta-\delta/\Omega-i\varepsilon}\right) + \mathrm{Li}_2\left(\frac{\Delta-\delta-i\varepsilon}{\Delta-\delta/\Omega-i\varepsilon}\right) + (\delta\to-\delta)\right\}\ . \quad (38)\end{aligned}$$

Here, the terms within braces in the second and third line are to be repeated with δ replaced by $-\delta$ as indicated.

When $\delta = 0$, (38) reduces to (37). Thus, in the case $m = 0 = \delta$ we have an analytic cross-check of our results. In the more general case $\delta \neq 0$, we have numerically compared general expression (30) for small values m, with equations (35) and (38). The three expressions for $\mathcal{I}_3|_{m=0}$ were found to agree over a wide range of real values for Δ_1, Δ_2, and $0 < \Omega < 1$, again providing a cross-check for (30).

4 The Scalar Two-Point Integral

The scalar two-point integral $\mathcal{I}_2$ is given by (2) with $n = 0$. $\mathcal{I}_2$ is a function of m and $\Delta = \delta M - 2v\cdot k$. The starting point for the calculation of $\mathcal{I}_2$ is (9). The computations of both $\widetilde{\mathcal{I}}_2$ and $\mathcal{I}_2|_{m=0}$ are standard. Defining,

$$x_\pm = \frac{1}{2}\left(-\Delta \pm \sqrt{\Delta^2 - 4m^2 + i\varepsilon}\right) \ , \tag{39}$$

from (9) we obtain,

$$\mathcal{I}_2 = \frac{\Delta}{32\pi^2}\left(\frac{2}{\epsilon} + 2\right) + \frac{1}{16\pi^2}\left(x_+ \log\left(-\frac{x_+}{\overline{\mu}}\right) + x_- \log\left(-\frac{x_-}{\overline{\mu}}\right)\right) \ . \tag{40}$$

This result can also be obtained by using the HQET method for combining denominators, which yields an equivalent expression in terms of hypergeometric functions.

In order to compare our result (40) for $\mathcal{I}_2$ with those in the previous literature, we rewrite it in terms of $\mathsf{x} = \Delta/(2m)$,

$$\mathcal{I}_2(\Delta, m) = \frac{\Delta}{32\pi^2}\left(\frac{2}{\epsilon} + \log\left(\frac{\overline{\mu}^2}{m^2}\right) + 2\right) + \frac{m}{16\pi^2}\mathcal{F}(\mathsf{x}) \tag{41}$$

with $\mathcal{F}(\mathsf{x}) = \sqrt{\mathsf{x}^2 - 1 + i\varepsilon}\left[\log\left(\mathsf{x} - \sqrt{\mathsf{x}^2 - 1 + i\varepsilon}\right) - \log\left(\mathsf{x} + \sqrt{\mathsf{x}^2 - 1 + i\varepsilon}\right)\right]$. The coefficient of the dimensional regularization pole vanishes when $\Delta = 0$. This is due to the fact that the real part of the integrand in (2) is parity-odd when $\Delta = 0$.

Equation (41) agrees with [9] for all real values of Δ, once their different normalization and conventions are taken into account. It agrees with the results from [10, 11] (see also the second of [2]) only in the region $\mathsf{x} > 0$, our result being different from theirs over the entire negative semiaxis.

We consider now $\mathcal{I}_3$ at zero recoil. As shown in [10], and seen from its definition (1), $\mathcal{I}_3$ at $\Omega = 0$ can be obtained from $\mathcal{I}_2$ as,

$$\mathcal{I}_3\Big|_{\Omega=0}(\Delta_1, \Delta_2) = \frac{1}{\Delta_1 - \Delta_2}\left(\mathcal{I}_2(\Delta_1, m) - \mathcal{I}_2(\Delta_2, m)\right), \quad \text{and} \quad \mathcal{I}_3\Big|_{\Omega=0}(\Delta, \Delta) = \frac{\partial}{\partial\Delta}\mathcal{I}_2(\Delta, m). \tag{42}$$

Substituting the value of $\mathcal{I}_2(\Delta, m)$ given by (41) in (42), we recover our previous results (32) and (33), as can be easily checked.

5 Vector and Tensor Integrals

In this section we give general expressions for vector and second-rank tensor integrals in terms of form factors. We also compare our results to those in the literature, when available. The form factors will be expressed in terms of scalar integrals. Of those, $\mathcal{I}_2$ and $\mathcal{I}_3$ have been given in previous sections. We will also need,

$$\mathcal{I}_1 = \frac{i\mu^{4-d}}{(2\pi)^d}\int d^dq \frac{1}{(q^2-m^2+i\varepsilon)} = -\frac{m^2}{16\pi^2}\left(\frac{2}{\epsilon}+\log\left(\frac{\overline{\mu}^2}{m^2}\right)+1\right)+\mathcal{O}(\epsilon)\ , \tag{43}$$

with $m>0$, $d=4-\epsilon$, $\log(\overline{\mu}^2)=\log(\mu^2 4\pi)-\gamma_{\mathrm{E}}$. Two other scalar integrals appear in the evaluation of tensor ones,

$$\mathcal{I}_1{}' = \frac{i\mu^{4-d}}{(2\pi)^d}\int d^dq \frac{1}{(2v\cdot q-\Delta+i\varepsilon)} \quad \text{and} \quad \mathcal{I}_2{}' = \frac{i\mu^{4-d}}{(2\pi)^d}\int d^dq \frac{1}{(2v_1\cdot q-\Delta_1+i\varepsilon)\,(2v_2\cdot q-\Delta_2+i\varepsilon)} \tag{44}$$

Both $\mathcal{I}_1{}'$, and $\mathcal{I}_2{}'$ vanish, as we will now show. The easiest way to see that $\mathcal{I}_1{}'=0$ is by applying the axioms of dimensional regularization [12, §4.1]. We consider $\mathcal{I}_1{}'(v^\mu,d)$ as a function of d and v^μ, momentarily allowing $v^\mu v_\mu>0$, not necessarily equal to 1. Then, we can always shift the integration variable so that,

$$\mathcal{I}_1{}'(v^\mu,d) = \frac{i\mu^{4-d}}{(2\pi^d)}\int d^dq \frac{1}{(2v\cdot q+i\varepsilon)}\ .$$

Let $s>0$, and consider $\mathcal{I}_1{}'(sv^\mu,d)$. By factoring s out of the integral we get, $\mathcal{I}_1{}'(sv^\mu,d) = 1/s\mathcal{I}_1{}'(v^\mu,d)$, whereas by rescaling the integration variable q^μ we find, $\mathcal{I}_1{}'(sv^\mu,d)=1/s^d\mathcal{I}_1{}'(v^\mu,d)$. Therefore, we must have $1/s\mathcal{I}_1{}'(v^\mu,d)=1/s^d\mathcal{I}_1{}'(v^\mu,d)$ for all $s>0$ and all complex d, excluding positive integer values. Since $\mathcal{I}_1{}'$ must be an analytic function of d, we conclude that it vanishes for $v^\mu v_\mu>0$.

We now turn to $\mathcal{I}_2{}'$. Introducing a Feynman parameter, we can write it as,

$$\mathcal{I}_2{}' = \frac{i\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int d^dq \frac{1}{2V(x)\cdot q-\Delta(x)+i\varepsilon}\ ,$$

with $V^\mu(x)=xv_1^\mu+(1-x)v_2^\mu$ and $\Delta(x)=x\Delta_1+(1-x)\Delta_2$. For $v_1^{\mu\,2}=1=v_2^{\mu\,2}$ and $0\leq x\leq 1$ we have $V^{\mu\,2}(x)>0$. Thus, the inner integral is $\mathcal{I}_1{}'(V^\mu,d)$ and since $\mathcal{I}_1{}'=0$, $\mathcal{I}_2{}'$ also vanishes.

5.1 Vector and tensor two-point integrals

The vector two-point integral is given by,

$$\mathcal{I}_2{}^{\mu\nu}(v^\alpha,\Delta,m) = \frac{i\mu^{4-d}}{(2\pi)^d}\int d^dq \frac{q^\mu}{(2v\cdot q-\Delta+i\varepsilon)\,(q^2-m^2+i\varepsilon)}\ . \tag{45}$$

Lorentz invariance dictates that $\mathcal{I}_2$ is given in terms of only one form factor, which can be immediately evaluated by algebraic reduction [5, 6],

$$\mathcal{I}_2{}^{\mu}(v^\alpha,\Delta,m) = F(\Delta,m)v^\mu\ ,\quad \text{with}\quad F(\Delta,m)=v_\mu\mathcal{I}_2{}^\mu(v^\alpha,\Delta,m)=\frac{1}{2}\mathcal{I}_1(m)+\frac{\Delta}{2}\mathcal{I}_2(\Delta,m)\ , \tag{46}$$

where the scalar integrals $\mathcal{I}_1$ and $\mathcal{I}_2$ have been given in (43) and in section 4, respectively.

The tensor two-point integral is defined as,

$$\mathcal{I}_2{}^{\mu\nu}(v^\alpha,\Delta,m) = \frac{i\mu^{4-d}}{(2\pi)^d}\int d^dq \frac{q^\mu q^\nu}{(2v\cdot q-\Delta+i\varepsilon)\,(q^2-m^2+i\varepsilon)}. \tag{47}$$

We will introduce two sets of form factors. First, we define,

$$\mathcal{I}_2{}^{\mu\nu}(v^\alpha, \Delta, m) = I_0(\Delta, m) g^{\mu\nu} + I_1(\Delta, m) v^\mu v^\nu . \tag{48}$$

Second, we introduce F form factors which can be easily computed in terms of scalar integrals,

$$F_0(\Delta, m) \equiv g_{\mu\nu}\mathcal{I}_2{}^{\mu\nu} = dI_0 + I_1 = m^2 \mathcal{I}_2(\Delta, m) \tag{49}$$

$$F_1(\Delta, m) \equiv v_\mu v_\nu \mathcal{I}_2{}^{\mu\nu} = I_0 + I_1 = \frac{\Delta}{4}\Big(\mathcal{I}_1(m) + \Delta \mathcal{I}_2(\Delta, m)\Big) \ . \tag{50}$$

In fact, $F_0(\Delta, m) = \mathcal{I}_1{}' + m^2 \mathcal{I}_2(\Delta, m)$, so here we have used $\mathcal{I}_1{}' = 0$. Inverting the relation among F's and I's we obtain, to lowest order in $\epsilon = 4 - d$,

$$I_0(\Delta, m) = -\frac{1}{3}\left(1 + \frac{\epsilon}{3}\right)\left[\frac{\Delta}{4}\mathcal{I}_1(m) + \left(\frac{\Delta^2}{4} - m^2\right)\mathcal{I}_2(\Delta, m)\right] \tag{51}$$

$$I_1(\Delta, m) = \frac{\Delta}{3}\left(1 + \frac{\epsilon}{12}\right)\Big[\mathcal{I}_1(m) + \Delta \mathcal{I}_2(\Delta, m)\Big] - \frac{m^2}{3}\left(1 + \frac{\epsilon}{3}\right)\mathcal{I}_2(\Delta, m) \ . \tag{52}$$

Finally, we substitute the known values of $\mathcal{I}_1(m)$ and $\mathcal{I}_2(\Delta, m)$. Using the same notation as in (41),

$$I_0(\Delta, m) = -\frac{m^3}{3\cdot 16\pi^2}\left\{\left(\frac{2}{\epsilon} + \log\left(\frac{\overline{\mu}^2}{m^2}\right) + \frac{8}{3}\right)\mathsf{x}\left(\mathsf{x}^2 - \frac{3}{2}\right) + \frac{\mathsf{x}}{2} + (\mathsf{x}^2 - 1)\mathcal{F}(\mathsf{x})\right\} \tag{53}$$

$$I_1(\Delta, m) = \frac{m^3}{3\cdot 16\pi^2}\left\{\left(\frac{2}{\epsilon} + \log\left(\frac{\overline{\mu}^2}{m^2}\right) + \frac{13}{6}\right)\mathsf{x}(4\mathsf{x}^2 - 3) + \frac{3}{2}\mathsf{x} + (4\mathsf{x}^2 - 1)\mathcal{F}(\mathsf{x})\right\} \ . \tag{54}$$

This is the general form for $\mathcal{I}_2$. We do not find agreement with [11].

There are two particular cases of interest, in which $\mathcal{I}_2{}^{\mu\nu}$ can be easily computed directly by using the HQET method for combining denominators, thus providing cross-checks for our results. In the first place, we consider the case $m = 0$, $\Delta > 0$ (the case $\Delta < 0$ is analogous). A straightforward computation using the HQET method yields,

$$I_0\Big|_{\substack{m=0\\ \Delta>0}} = -\frac{\Delta^3}{3\cdot 128\pi^2}\left(\frac{2}{\epsilon} + \log\left(\frac{\overline{\mu}^2}{\Delta^2}\right) + \frac{8}{3}\right) \ ; \qquad I_1\Big|_{\substack{m=0\\ \Delta>0}} = \frac{\Delta^3}{3\cdot 32\pi^2}\left(\frac{2}{\epsilon} + \log\left(\frac{\overline{\mu}^2}{\Delta^2}\right) + \frac{13}{6}\right) \tag{55}$$

which agree with (53) and (54) evaluated at $m = 0$.

Second, in the case $\Delta = 0$ we find,

$$\mathcal{I}_2{}^{\mu\nu}(v^\alpha, \Delta = 0, m) = \frac{m^3}{3\cdot 16\pi}(g^{\mu\nu} - v^\mu v^\nu) \ , \tag{56}$$

again in agreement with the corresponding limit of (53) and (54). As remarked above, in connection with the scalar integral, there is no dimensional regularization pole in this case.

Furthermore, if we assume $0 < \mathsf{x} < 1$ ($m > \Delta/2 > 0$) and expand in powers of x, we recover the result given in [7].

5.2 Vector three-point integral

We now turn to the tensor three-point integrals, starting with the vector one,

$$\mathcal{I}_3{}^{\mu}(v_1^\alpha, v_2^\beta, \Delta_1, \Delta_2, m) = \frac{i\mu^{4-d}}{(2\pi)^d}\int d^dq \frac{q^\mu}{(2v_1\cdot q - \Delta_1 + i\varepsilon)\,(2v_2\cdot q - \Delta_2 + i\varepsilon)\,(q^2 - m^2 + i\varepsilon)}. \quad (57)$$

On the left-hand side we omitted μ and d from the argument for brevity. We define, as before, two sets of form factors,

$$\mathcal{I}_3{}^{\mu} = I_1 v_1^\mu + I_2 v_2^\mu \quad \text{and} \quad F_{1,2} = v_{1,2}\cdot \mathcal{I}_3 \ , \quad (58)$$

with $I_j(\Omega, \Delta_1, \Delta_2, m)$ and $F_j(\Omega, \Delta_1, \Delta_2, m)$ related by,

$$I_1 = \frac{1-\Omega^2}{4\Omega^2}\left[-(1-\Omega^2)F_1 + (1+\Omega^2)F_2\right] \ , \quad I_2 = \frac{1-\Omega^2}{4\Omega^2}\left[(1+\Omega^2)F_1 - (1-\Omega^2)F_2\right] \ . \quad (59)$$

The form factors F_j can be expressed in terms of scalar integrals as,

$$F_{1,2} = \frac{1}{2}\mathcal{I}_2(\Delta_{2,1}, m) + \frac{\Delta_{1,2}}{2}\mathcal{I}_3(\Omega, \Delta_1, \Delta_2, m) \ . \quad (60)$$

These equations give an explicit expression for $\mathcal{I}_3{}^{\mu}$.

At zero recoil we have $v_1^\mu = v_2^\mu = v^\mu$ and,

$$\mathcal{I}_3{}^{\mu}\Big|_{\Omega=0} = v_\nu \cdot \mathcal{I}_3{}^{\nu}\Big|_{\Omega=0} v^\mu \ , \text{ with } \ v_\nu\cdot\mathcal{I}_3{}^{\nu}\Big|_{\Omega=0} = \frac{1}{2}\mathcal{I}_2(\Delta_2, m) + \frac{\Delta_1}{2}\mathcal{I}_3(\Omega=0, \Delta_1, \Delta_2, m) \ . \quad (61)$$

Using (42) we obtain,

$$v_\nu\cdot\mathcal{I}_3{}^{\nu}\Big|_{\Omega=0} = \frac{1}{2}\frac{1}{\Delta_1-\Delta_2}\left[\Delta_1\mathcal{I}_2(\Delta_1, m) - \Delta_2\mathcal{I}_2(\Delta_2, m)\right]$$

if $\Delta_1 \neq \Delta_2$ and

$$v_\nu\cdot\mathcal{I}_3{}^{\nu}\Big|_{\Omega=0} = \frac{1}{2}\mathcal{I}_2(\Delta, m) + \frac{\Delta}{2}\frac{\partial}{\partial\Delta}\mathcal{I}_2(\Delta, m)$$

if $\Delta_1 = \Delta_2 = \Delta$. This completes our treatment of the vector integral.

5.3 The tensor three-point integral

We consider, finally, the tensor integral,

$$\mathcal{I}_3{}^{\mu\nu}(v_1^\alpha, v_2^\beta, \Delta_1, \Delta_2, m) = \frac{i\mu^{4-d}}{(2\pi)^d}\int d^dq \frac{q^\mu q^\nu}{(2v_1\cdot q - \Delta_1 + i\varepsilon)\,(2v_2\cdot q - \Delta_2 + i\varepsilon)\,(q^2 - m^2 + i\varepsilon)}. \quad (62)$$

In this case we have two sets of four form-factors each,

$$\mathcal{I}_3{}^{\mu\nu} = I_{11}v_1^\mu v_1^\nu + I_{22}v_2^\mu v_2^\nu + I_{12}v_1^{\{\mu}v_2^{\nu\}} + I_0 g^{\mu\nu} \ , \quad (63)$$

with $v_1^{\{\mu}v_2^{\nu\}} = v_1^\mu v_2^\nu + v_2^\mu v_1^\nu$, and,

$$F_{11} = v_1^\mu v_1^\nu \mathcal{I}_{3\mu\nu} \ ; \quad F_{22} = v_2^\mu v_2^\nu \mathcal{I}_{3\mu\nu} \ ; \quad F_{12} = v_1^{\{\mu}v_2^{\nu\}}\mathcal{I}_{3\mu\nu} \ ; \quad F_0 = g^{\mu\nu}\mathcal{I}_{3\mu\nu} \ . \quad (64)$$

The F's can be given in terms of the I's using (63). Inverting those relations we obtain,

$$
\begin{aligned}
I_{11} &= \frac{d-1}{d-2}\frac{(1-\Omega^2)^2}{16\Omega^4}\left\{(1-\Omega^2)^2 F_{11} + \left(\Omega^4 + 2\frac{d-3}{d-1}\Omega^2 + 1\right) F_{22} - (1-\Omega^4) F_{12}\right\} \\
&\quad + \frac{1}{d-2}\frac{(1-\Omega^2)^2}{4\Omega^2} F_0 \\
I_{22} &= \frac{d-1}{d-2}\frac{(1-\Omega^2)^2}{16\Omega^4}\left\{\left(\Omega^4 + 2\frac{d-3}{d-1}\Omega^2 + 1\right) F_{11} + (1-\Omega^2)^2 F_{22} - (1-\Omega^4) F_{12}\right\} \\
&\quad + \frac{1}{d-2}\frac{(1-\Omega^2)^2}{4\Omega^2} F_0 \\
I_{12} &= -\frac{d-1}{d-2}\frac{(1-\Omega^2)^2}{16\Omega^4}\left\{(1-\Omega^4)(F_{11}+F_{22}) - \left(\Omega^4 + \frac{2}{d-1}\Omega^2 + 1\right) F_{12}\right\} \\
&\quad - \frac{1}{d-2}\frac{1-\Omega^4}{4\Omega^2} F_0 \\
I_0 &= \frac{1}{d-2}\frac{(1-\Omega^2)^2}{4\Omega^2}(F_{11}+F_{22}) + \frac{1}{d-2} F_0 - \frac{1}{d-2}\frac{1-\Omega^4}{4\Omega^2} F_{12} \ .
\end{aligned}
\tag{65}
$$

Using the results from sections 5.1 and 5.2 we can express the F's in terms of scalar integrals as,

$$
\begin{aligned}
F_{11} &= \frac{\omega}{4}\mathcal{I}_1(m) + \frac{\Delta_1 + \omega\Delta_2}{4}\mathcal{I}_2(\Delta_2, m) + \frac{\Delta_1^2}{4}\mathcal{I}_3(\Omega, \Delta_1, \Delta_2, m) \\
F_{22} &= \frac{\omega}{4}\mathcal{I}_1(m) + \frac{\Delta_2 + \omega\Delta_1}{4}\mathcal{I}_2(\Delta_1, m) + \frac{\Delta_2^2}{4}\mathcal{I}_3(\Omega, \Delta_1, \Delta_2, m) \\
F_{12} &= \frac{1}{2}\mathcal{I}_1(m) + \frac{1}{2}\Big(\Delta_1\mathcal{I}_2(\Delta_1, m) + \Delta_2\mathcal{I}_2(\Delta_2, m)\Big) + \frac{\Delta_1\Delta_2}{2}\mathcal{I}_3(\Omega, \Delta_1, \Delta_2, m) \\
F_0 &= m^2\mathcal{I}_3(\Omega, \Delta_1, \Delta_2, m) \ .
\end{aligned}
\tag{66}
$$

Here we are using a mixed notation, in terms of both ω and Ω (see (13)), for brevity. Equations (63), (65) and (66) give an explicit analytic expression for ${\mathcal{I}_3}^{\mu\nu}$. Notice also the symmetry of (66) under exchange of Δ_1 and Δ_2. The same as with the scalar integral, there are a number of particular cases of interest which we briefly comment upon in the remainder of this section.

5.3.1 The zero recoil case

In order to study the zero recoil case, it is convenient to write ${\mathcal{I}_3}^{\mu\nu}$ in terms of vectors $v_\pm^\mu = 1/2(v_1^\mu \pm v_2^\mu)$. Instead of (63) we then have,

$$
{\mathcal{I}_3}^{\mu\nu} = I_{++}v_+^\mu v_+^\nu + I_{--}v_-^\mu v_-^\nu + I_{+-}v_+^{\{\mu}v_-^{\nu\}} + I_0 g^{\mu\nu} \ . \tag{67}
$$

In the zero recoil limit, $v_1^\mu = v_2^\mu = v_+^\mu \equiv v^\mu$. Using the results of sections 5.1 and 5.2 it is not difficult to show that when $\Omega = 0$ we have $v_-^\mu \mathcal{I}_{3\mu\nu} = 0$. Therefore, we can write,

$$
{\mathcal{I}_3}^{\mu\nu}\Big|_{\Omega=0} = I_{++}\Big|_{\Omega=0} v^\mu v^\nu + I_0\Big|_{\Omega=0} g^{\mu\nu} \ . \tag{68}
$$

These form factors can be computed as before, resulting in,

$$
I_{++}\Big|_{\Omega=0} = \frac{1}{3}\left(1 + \frac{\epsilon}{12}\right)\left\{\mathcal{I}_1(m) + \frac{\Delta_1^2 - m^2}{\Delta_1 - \Delta_2}\mathcal{I}_2(\Delta_1, m) - \frac{\Delta_2^2 - m^2}{\Delta_1 - \Delta_2}\mathcal{I}_2(\Delta_2, m))\right\} \tag{69}
$$

$$
I_0\Big|_{\Omega=0} = -\frac{1}{12}\left(1 + \frac{\epsilon}{3}\right)\left\{\mathcal{I}_1(m) + \frac{\Delta_1^2 - 4m^2}{\Delta_1 - \Delta_2}\mathcal{I}_2(\Delta_1, m) - \frac{\Delta_2^2 - 4m^2}{\Delta_1 - \Delta_2}\mathcal{I}_2(\Delta_2, m))\right\} \ . \tag{70}
$$

We notice that we could have arrived at these equations by using a relation analogous to (42), namely,

$$\mathcal{I}_3{}^{\mu\nu}\Big|_{\Omega=0} = \frac{1}{\Delta_1-\Delta_2}\left(\mathcal{I}_2{}^{\mu\nu}(v^\alpha,\Delta_1,m) - \mathcal{I}_2{}^{\mu\nu}(v^\alpha,\Delta_2,m)\right) \ , \tag{71}$$

showing the consistency of our result. These expressions acquire a particularly simple form for some special values of the parameters. For instance, setting $m=0$, $\Delta_j>0$, $j=1,2$ and using either (71) and (55), or (69), we get,

$$\begin{aligned} I_0\Big|_{\substack{\Omega=0\\ m=0}} &= \frac{-1}{3\cdot 128\pi^2}\frac{1}{\Delta_1-\Delta_2}\left\{\Delta_1^3\left(\frac{2}{\epsilon}+\log\left(\frac{\overline{\mu}^2}{\Delta_1^2}\right)+\frac{8}{3}\right) - \Delta_2^3\left(\frac{2}{\epsilon}+\log\left(\frac{\overline{\mu}^2}{\Delta_2^2}\right)+\frac{8}{3}\right)\right\} &(72)\\ I_{++}\Big|_{\substack{\Omega=0\\ m=0}} &= \frac{1}{3\cdot 32\pi^2}\frac{1}{\Delta_1-\Delta_2}\left\{\Delta_1^3\left(\frac{2}{\epsilon}+\log\left(\frac{\overline{\mu}^2}{\Delta_1^2}\right)+\frac{13}{6}\right) - \Delta_2^3\left(\frac{2}{\epsilon}+\log\left(\frac{\overline{\mu}^2}{\Delta_2^2}\right)+\frac{13}{6}\right)\right\}, &(73)\end{aligned}$$

which provides another cross-check of our previous equations.

5.3.2 The case $\Delta_1 = 0 = \Delta_2$

Another case where the form factors for $\mathcal{I}_3{}^{\mu\nu}$ take a very simple form is when $\Delta_1 = 0 = \Delta_2$. In this case equations (63), (65) and (66), together with our previous results for the scalar integrals, give,

$$\begin{aligned} I_0\Big|_{\Delta_j=0} &= \frac{m^2}{128\pi^2}\frac{1-\Omega^2}{\Omega}\log(\alpha_+)\left(\frac{2}{\epsilon}+\log\left(\frac{\overline{\mu}^2}{m^2}\right)+1\right)\\ I_{11}\Big|_{\Delta_j=0} &= \frac{m^2}{128\pi^2}\left[-\frac{1-\Omega^4}{2\Omega^2}+\frac{(1-\Omega^2)^3}{4\Omega^3}\log(\alpha_+)\right]\left(\frac{2}{\epsilon}+\log\left(\frac{\overline{\mu}^2}{m^2}\right)+1\right)\\ I_{22}\Big|_{\Delta_j=0} &= I_{11}\Big|_{\Delta_j=0} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (74)\\ I_{12}\Big|_{\Delta_j=0} &= \frac{m^2}{128\pi^2}\left[\frac{(1-\Omega^2)^2}{2\Omega^2}-\frac{(1-\Omega^2)(1-\Omega^4)}{4\Omega^3}\log(\alpha_+)\right]\left(\frac{2}{\epsilon}+\log\left(\frac{\overline{\mu}^2}{m^2}\right)+1\right) \ .\end{aligned}$$

We have also computed $\mathcal{I}_3{}^{\mu\nu}$ for $\Delta_j = 0$ directly from its definition (62) by using the HQET method for combining denominators. Full agreement with (74) was found.

5.3.3 The case $m = 0$

A direct calculation of $\mathcal{I}_3{}^{\mu\nu}$ at $m=0$ with the HQET method is considerably more involved than in the previous case. The results are also much less compact. As an illustration, we will quote the result for the form factor I_0 when $m=0$ and $\Delta_1=\Delta_2\equiv\Delta>0$,

$$\begin{aligned} I_0 = -\frac{\Delta^2}{256\pi^2}(1-\Omega^2)\Bigg\{&\left(1+\frac{1-\Omega^2}{2\Omega}\log(\alpha_+)\right)\left(\frac{2}{\epsilon}+\log\left(\frac{\overline{\mu}^2}{\Delta^2}\right)\right)+3+\frac{(1-\Omega^2)}{2\Omega}\log(\alpha_+)\\ &-\frac{(1-\Omega^2)}{4\Omega}\log\left(\frac{1-\Omega^2}{4}\right)\log(\alpha_+)+\frac{(1-\Omega^2)}{2\Omega}\left(\mathrm{Li}_2\left(\frac{1-\Omega}{2}\right)-\mathrm{Li}_2\left(\frac{1+\Omega}{2}\right)\right)\Bigg\} \ . \end{aligned} \tag{75}$$

This equation agrees with the general result given by (65) and (66), evaluated at $m=0$, $\Delta_1=\Delta_2$, as it should.

5.3.4 The chiral logs

In this section we focus on the case $\Delta_1 = \Delta_2 \equiv \delta m$, with $0 < \delta m/2 < m$. We expand in powers of $\delta m/m$, retaining only those terms proportional to $\log(m)$, with coefficients computed to lowest order in $\delta m/m$. In this way, we obtain the chiral logs in ${\mathcal{I}_3}^{\mu\nu}$.

From equations (41) and (43), we get,

$$\mathcal{I}_1(m) = -\frac{m^2}{16\pi^2}\log\left(\frac{\overline{\mu}^2}{m^2}\right) + \cdots \; ; \quad \mathcal{I}_2(\delta m, m) = \frac{\delta m}{32\pi^2}\log\left(\frac{\overline{\mu}^2}{m^2}\right) + \cdots \; , \tag{76}$$

where the ellipsis denotes terms not containing $\log(m)$, or containing higher powers of δm. On the other hand, $\mathcal{I}_3$ is needed only to zeroth order in δm because it enters the form factors with δm^2 as a coefficient. From (29),

$$\mathcal{I}_3(\Omega, 0, 0, m) = \frac{1}{64\pi^2}\frac{1-\Omega^2}{\Omega}\log(\alpha_+)\log\left(\frac{\overline{\mu}^2}{m^2}\right) + \cdots \; . \tag{77}$$

With these approximations, we obtain the form factors as,

$$F_{11} = F_{22} = \frac{1}{64\pi^2}\log\left(\frac{\overline{\mu}^2}{m^2}\right)\left\{-\frac{1+\Omega^2}{1-\Omega^2}m^2 + \delta m^2\left(\frac{1}{1-\Omega^2} + \frac{1-\Omega^2}{4\Omega}\log(\alpha_+)\right)\right\} + \cdots \tag{78}$$

$$F_{12} = \frac{1}{64\pi^2}\log\left(\frac{\overline{\mu}^2}{m^2}\right)\left\{-2m^2 + \delta m^2\left(2 + \frac{1-\Omega^2}{2\Omega}\log(\alpha_+)\right)\right\} + \cdots \tag{79}$$

$$F_0 = \frac{1}{64\pi^2}\log\left(\frac{\overline{\mu}^2}{m^2}\right)m^2\frac{1-\Omega^2}{\Omega}\log(\alpha_+) + \cdots \; . \tag{80}$$

These results agree exactly with those of [7], once we take into account the differences in normalization and conventions.

6 Final Remarks

In phenomenological applications, the exact functional dependence of Feynman integrals on masses and residual momenta is usually not needed. Often, the first few terms in a series expansion in some of the parameters provides the required accuracy. We believe, however, that the exact analytic computation presented here does not require more calculational effort than approximate schemes. It has the added advantage of being valid over the entire physical region for internal and external masses.

Our result (30) for the scalar three-point integral involves four dilogarithms. This is to be compared with the analogous vertex integrals in renormalizable theories, which are generally expressed in terms of twelve dilogarithms and a collection of logarithms [4, 6]. This simplification is afforded, of course, by the effective theory formalism, which focuses only on the relevant degrees of freedom. Equation (30) is quite compact. Once the values for internal and external masses are given, so that the appropriate branches of square roots, logs and dilogs are determined with the aid of the "$i\varepsilon$" prescription, the expression for $\mathcal{I}_3$ given by (30) is easily translated into computer code.

Another possible approach to the computation of three-point integrals is to consider them strictly within the context of the effective theory, without introducing auxiliary integrals such as $\widetilde{\mathcal{I}}_3$, eq. (3). In that case, one can parametrize the integrand with the HQET method. The resulting expressions are, however, difficult to handle and, in general, they seem to lead to hypergeometric functions of two variables or, more likely, to series of hypergeometric functions. The procedure adopted in this paper avoids those difficulties.

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