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Cohomology operations and algebraic geometry

SIMONE BORGHESI

The manuscript is an overview of the motivations and foundations lying behind Voevodsky's ideas of constructing categories similar to the ordinary topological homotopy categories. The objects of these categories are strictly related to algebraic varieties and preserve some of their algebraic invariants.

14F42, 18D10; 14F20, 18F10

1 Introduction

This manuscript is based on a ten hours series of seminars I delivered in August of 2003 at the Nagoya Institute of Technology as part of the workshop on homotopy theory organized by Norihiko Minami and following the Kinosaki conference in honor of Goro Nishida. One of the most striking applications of homotopy theory in "exotic" contexes is Voevodsky's proof of the Milnor Conjecture. This conjecture can be reduced to statements about algebraic varieties and "cohomology theories" of algebraic varieties. These contravariant functors are called *motivic cohomology* with coefficients in abelian groups A. Since they share several properties with singular cohomology in classical homotopy theory, it is reasonable to expect "motivic cohomology operations" acting naturally on these cohomology theories. By assuming the existence of certain motivic Steenrod operations and guessing their right degrees, Voevodsky was able to prove the Milnor Conjecture. This strategy reduced the complete proof of the conjecture to the construction of these operations and to an appropriate category in whcih motivic cohomology is a "representable". In homotopy theory there are several ways of doing this. We now know two ways of obtaining such operations on motivic cohomology: one is due to P Brosnan [7] and the other to V Voevodsky [22]. The latter approach follows a systematic developement of homotopy categories containing algebraic information of the underlying objects and it is the one we will discuss in this manuscript. It turns out that, if we think of an algebraic variety as something like a topological space with an algebraic structure attached to it, it makes sense to try to construct homotopy categories in which objects are algebraic varieties, as opposed to just topological spaces, and, at the same time motivic cohomology representable. In the classical homotopy

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categories such representability always holds for any cohomology theory because of Brown Representability Theorem. If such exotic homotopy categories existed in our algebraic setting, they would tautologically contain all the algebraic invariants detected by the cohomology theories represented.

This manuscript begins with a motivational part constituted by an introduction to Voevodsky's reduction steps of the Milnor Conjecture. By the end, the conjecture is reduced to the existence of motivic Steenrod operations and of an (unstable) homotopy category of schemes. In the second section we will discuss some of the main issues involved on the construction of the homotopy category of schemes and in the proof of the representability of motivic cohomology, assuming perfectness of the base field. Another reference for representability of motivic cohomology are the lecture notes by Voevodsky and Deligne [19]. We will focus particularly on the identifications between objects in the localized categories.

For the beautiful memories of the period spent in Japan, which included the workshop at the Nagoya Institute of Technology, I am greatly indebted to Akito Futaki and to Norihiko Minami.

2 Motivic Steenrod operations in the Milnor Conjecture

Throughout this section, the word scheme will refer to a separable scheme of finite type over a field k. Alternatively, we may consider a scheme to be an algebraic variety over a field k. By Milnor Conjecture we mean the statement known as Bloch–Kato Conjecture at the prime $p = 2$ (for more information about the origin of this conjecture, see the introduction of the paper [20] by Voevodsky). This conjecture asserts:

Conjecture (Bloch–Kato) *Let k be a field of characteristic different from a prime number p. Then the norm residue homomorphism*

$$N\colon\ K_n^M(k)/(p) \to \mathbb{H}^n_{\mathrm{et}}(\operatorname{Spec} k, \mu_p^{\otimes n}) \tag{1}$$

is an isomorphism for any nonnegative integer n.

We assume the reader to have mainly a homotopy theoretic background, therefore we will occasionally include some descriptions of objects used in algebraic geometry. The *Milnor K–theory* is defined as the graded ring

$$\{K_0^M(k), K_1^M(k), K_2^K(k), \dots\}$$

where $K_n^M(k)$ is the quotient of the group $k^* \otimes_{\mathbb{Z}} \overset{n}{\cdots} \otimes_{\mathbb{Z}} k^*$ by the subgroup generated by the elements $a_1 \otimes \cdots \otimes a_n$ where $a_i + a_{i+1} = 1$ for some i. It is useful to mention that, in the literature, when dealing with Milnor K–theory, the multiplicative group k^* of the field k is written additively: for instance the element in $K_n^M(k)$ represented by $a_1 \otimes \cdots \otimes a_m^c \otimes \cdots a_n$ belongs to the subgroup $c \cdot K_n^M(k)$, being equal to $a_1 \otimes \cdots c \cdot a_m \otimes \cdots a_n$ where $c \cdot a_m$ is in $K_1^M(k) = (k^*, +)$. The group lying as target of the norm residue homomorphism is a (hyper)cohomology group of the algebraic variety $\operatorname{Spec} k$. As first approximation we may think of it as being a sort of ordinary cohomology functor $H^*(-, A)$ on algebraic varieties in which, instead of having the abelian group A as the only input, we have two inputs: the *Grothendieck topology* (*et*=étale in this case), and a *complex of sheaves of abelian groups* for the topology considered as "coefficients" of the cohomology (the étale complex of sheaves $\mu_p^{\otimes n}$ in the statement of the conjecture). The complex of sheaves μ_p is zero at any degree except in degree zero where it is the sheaf that associates the elements $f \in \mathcal{O}(X)$ such that $f^p = 1$ to any smooth scheme of finite type X over a field k. $\mu_p^{\otimes n}$ is the n–fold tensor product of μ_p in the derived category (one can construct this monoidal structure in a similiar way as in the derived category of complexes of abelian groups). As we would expect, the cohomology theory $\mathbb{H}^*_{\text{et}}(-, \mu_p^{\otimes *})$ is endowed of a commutative ring structure given by the *cup product*. The norm residue homomorphism is defined as

$$N(\overline{a_1 \otimes \cdots \otimes a_n}) = \delta(a_1) \cup \cdots \cup \delta(a_n) \in \mathbb{H}^n_{\text{et}}(\operatorname{Spec} k, \mu_p^{\otimes n}),$$

where δ is the coboundary operator $\mathbb{H}^0_{\text{et}}(\operatorname{Spec} k, \mathbb{G}_m) \to \mathbb{H}^1_{\text{et}}(\operatorname{Spec} k, \mu_p)$ associated to the short exact (for the étale topology) sequence of sheaves

$$0 \longrightarrow \mu_p \longrightarrow \mathbb{G}_m \xrightarrow{\widehat{p}} \mathbb{G}_m \longrightarrow 0 \tag{2}$$

We recall that *the multiplicative group* sheaf $\mathbb{G}_m$ is defined as $\mathbb{G}_m(X) = \mathcal{O}(X)^*$ for any smooth scheme X and that $H^0(X, \mathcal{F}) = \mathcal{F}(X)$ for any sheaf $\mathcal{F}$. N defines indeed an homomorphism from $K_*^M(k)$ because $\delta(a) \cup \delta(1-a) = 0$ in $\mathbb{H}^2_{\text{et}}(\operatorname{Spec} k, \mu^{\otimes 2})$ as a consequence of a result of Bass and Tate in [1].

2.1 Reduction steps

There are several reduction steps in Voevodsky's program to prove the general Bloch–Kato Conjecture before motivic cohomology operations are used. Firstly, the conjecture follows from another one:

Conjecture (Beilinson–Lichtenbaum, p–local version) *Let k be a field and $w \geq 0$. Then*

$$\mathbb{H}^{w+1}_{\text{et}}(\operatorname{Spec} k, \mathbb{Z}_{(p)}(w)) = 0 \tag{3}$$

In the proof that this statement implies the Bloch–Kato conjecture we begin to use the so called *motivic cohomology theory* $H^{i,j}(-, A)$ with coefficients in an abelian group A.

Proof (that the Beilinson–Licthembaum Conjecture implies the Bloch–Kato Conjecture) Let $H^{i,j}(X, \mathbb{Z}/p)$ be the (i,j)th motivic cohomology group of X with coefficients in $\mathbb{Z}/p$ (see Definition 4.1). Assuming that the Beilinson–Lichtenbaum conjecture holds in degrees less or equal than n, Voevodsky proved in [20, Corollary 6.9(2)] that $H^{i,j}(X, \mathbb{Z}/p) \cong \mathbb{H}^i_{\text{et}}(X, \mathbb{Z}/p(j))$ for all $i \leq j \leq n$. On the other hand, in Theorem 6.1 of the same paper, he shows that $\mathbb{H}^i_{\text{et}}(X, \mathbb{Z}/p(j)) \cong \mathbb{H}^i_{\text{et}}(X, \mu_p^{\otimes j})$. We conclude by recalling that, if $i = j$ and $X = \operatorname{Spec} k$, the natural homomorphism $K^M_i(k)/(p) \to H^{i,i}(\operatorname{Spec} k, \mathbb{Z}/p)$ is an isomorphism (see Suslin–Voevodsky [17, Theorem 3.4]). □

The proof of the Beilinson–Lichtenbaum Conjecture is by induction on the index w. For $w = 0$ we have that

$$\mathbb{H}^1_{\text{et}}(\operatorname{Spec} k, \mathbb{Z}(0)) = \mathbb{H}^1_{\text{et}}(\operatorname{Spec} k, \mathbb{Z}) = H^{1,0}(\operatorname{Spec} k, \mathbb{Z}) = 0$$

and in the case $w = 1$, we know that $\mathbb{Z}(1) = \mathbb{G}_m[-1]$, thus

$$\mathbb{H}^2_{\text{et}}(\operatorname{Spec} k, \mathbb{Z}(1)) = H^1_{\text{et}}(\operatorname{Spec} k, \mathbb{G}_m) = \operatorname{Pic}(\operatorname{Spec} k) = 0$$

The first reduction step is that we can assume that k has no finite extensions of degree prime to p (for the time being such field will be called p–special). Indeed, $\mathbb{H}^{w+1}_{\text{et}}(\operatorname{Spec} k, \mathbb{Z}_{(p)}(w))$ injects in $\mathbb{H}^{w+1}_{\text{et}}(\operatorname{Spec} L, \mathbb{Z}_{(p)}(w))$ for any prime to p degree field extension L/k, because the composition

$$\mathbb{H}^*_{\text{et}}(\operatorname{Spec} k, \mathbb{Z}_{(p)}(*)) \xrightarrow{\text{transfer}} \mathbb{H}^*_{\text{et}}(\operatorname{Spec} L, \mathbb{Z}_{(p)}(*)) \xrightarrow{i^*} \mathbb{H}^*_{\text{et}}(\operatorname{Spec} k, \mathbb{Z}_{(p)}(*))$$

is multiplication by $[L:k]$, hence it is an isomorphism. Therefore, letting F to be the colimit over all the prime to p field extensions of k, to conclude it suffices to show the vanishing statement for F.

Secondly, if k is p–special and $K^M_w(k)/(p) = 0$, it is possible to prove directly that $\mathbb{H}^{w+1}_{\text{et}}(\operatorname{Spec} k, \mathbb{Z}_{(p)}(w)) = 0$. Hence, to prove the Bloch–Kato Conjecture it suffices to prove the following statement.

Theorem 2.1 *For any* $0 \neq \{a_1, a_2, \ldots, a_w\} = \underline{a} \in K^M_w(k)/(p)$*, there exists a field extension* $k_{\underline{a}}/k$ *with* k *being* p*–special such that:*

(1) $\underline{a} \in \ker i_*$*, where* $i_* \colon K^M_w(k)/(p) \to K^M_w(k_{\underline{a}})/(p)$ *is the induced homomorphism;*

(2) $i^* \colon \mathbb{H}^{w+1}_{\text{et}}(\operatorname{Spec} k, \mathbb{Z}_{(p)}(w)) \hookrightarrow \mathbb{H}^{w+1}_{\text{et}}(k_{\underline{a}}, \mathbb{Z}_{(p)}(w))$ *is an injection.*

Surprisingly, the best candidates for such fields, we have knowledge of, are function fields of appropriate algebraic varieties. Indeed, $k(a_i^{1/p})$ are all fields satisfying condition (1) for any $1 \leq i \leq w$, and condition (2) is precisely the complicated one. The approach to handle (2) is to give a sort of underlying "algebraic variety" structure to the field $k_{\underline{a}}$. Voevodsky proved the Milnor Conjecture by showing that, if $p = 2$, we can take $k_{\underline{a}}$ to be the function field of the projective quadric $Q_{\underline{a}}$ given by the equation

$$\langle 1, -a_1\rangle \otimes \langle 1, -a_2\rangle \otimes \cdots \otimes \langle 1, -a_{w-1}\rangle \oplus \langle -a_w\rangle = 0 \tag{4}$$

with the convention that

$$\langle a, b\rangle \otimes \langle c, d\rangle = act_1^2 + adt_2^2 + bct_3^2 + bdt_4^2$$

and

$$\langle k_1, k_2, \ldots, k_m\rangle \oplus \langle h\rangle = k_1t_1^2 + k_2t_2^2 + \cdots + k_mt_m^2 + ht_{m+1}^2$$

Such variety is known as *Pfister neighborhood* of the quadric $\langle 1, -a_1\rangle \otimes \langle 1, -a_2\rangle \otimes \cdots \otimes \langle 1, -a_{w-1}\rangle$. In order to prove this, Voevodsky used two results of Markus Rost about such quadrics translated in a context in which it makes sense to use homotopy theoretical tools on algebraic varieties and an argument involving motivic cohomology operations. We are now going to examine more carefully these techniques.

Remark 2.2 Recently Rost worked on finding substitute varieties for Pfister neighborhoods at odd primes. His candidates are called *norm varieties* and he uses in his program certain formulae called *(higher) degree formulae* (see Rost [13, 15] and Borghesi [5]) to prove the relevant properties in order to fit in Voevodsky's framework.

At this stage, we will take for granted the existence of a localized category $\mathcal{H}_\bullet(k)$ whose objects are pointed simplicial algebraic varieties (or, more generally, pointed simplicial sheaves for the Nisnevich topology on the site of smooth schemes over k). The localizing structure is determined by $\mathcal{H}_\bullet(k)$ being the homotopy category associated to an $(\mathbb{A}^1_k)$ model structure on the category of simplicial sheaves. These categories will be discussed more extensively later in the manuscript (see Section 4). At this point all we need to know is that

(1) the objects of $\mathcal{H}_\bullet(k)$ include pointed simplicial algebraic varieties;

(2) any morphism $f\colon \mathcal{X} \to \mathcal{Y}$ in $\mathcal{H}_\bullet(k)$ can be completed to a sequence

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow \mathcal{X} \wedge S^1_s \xrightarrow{f\wedge \mathrm{id}} \mathcal{Y} \wedge S^1_s \longrightarrow \cdots \tag{5}$$

inducing long exact sequences of sets, or abelian groups when appropriate, by applying the functor $\mathrm{Hom}_{\mathcal{H}_\bullet(k)}(-,\mathcal{W})$ for any $\mathcal{W} \in \mathcal{H}_\bullet(k)$;

(3) for all integers i, j and abelian group A, there exist objects $K(A(j), i) \in \mathcal{H}_\bullet(k)$ such that $\mathrm{Hom}_{\mathcal{H}_\bullet(k)}(X_+, K(A(j), i)) = H^{i,j}(X, A)$ for any smooth algebraic variety X (X_+ is the pointed object associated to X, that is $X \amalg \mathrm{Spec}\, k$). Moreover, defining $H^{i,j}(\mathcal{X}, A) = \mathrm{Hom}_{\mathcal{H}_\bullet(k)}(\mathcal{X}, K(A(j), i))$ for any pointed simplicial algebraic variety $\mathcal{X}$, we have that, if k is a perfect field, the following equalities hold (see Voevodsky [22, Theorem 2.4]):

$$H^{i+1,j}(X \wedge S^1_s, A) \cong H^{i,j}(X, A) \tag{6}$$

$$H^{i+2,j+1}(X \wedge \mathbb{P}^1_k, A) \cong H^{i,j}(X, A) \tag{7}$$

where $\wedge$ is the usual categorical smash product defined as $\mathcal{X} \times \mathcal{Y} / \mathcal{X} \vee \mathcal{Y}$.

Perfectness of the base field is not restrictive for the purpose of the Bloch–Kato Conjecture since such conjecture holding on characteristic zero fields implies the same result for fields of characteristic different from p. Let X be a variety; denote by $\check{C}(X)$ the simplicial variety given by $\check{C}(X)_n = X \times \overset{n+1}{\cdots} \times X$ with projections and diagonals as structure maps. The main feature of such simplicial variety is that it becomes simplicially equivalent to a point (ie $\mathrm{Spec}\, k$) if X has a rational point $x\colon\ \mathrm{Spec}\, k \to X$, a contracting homotopy being

$$\mathrm{id} \times x\colon\ X \times \overset{i}{\cdots} \times X \to X \times \overset{i+1}{\cdots} \times X \tag{8}$$

By (2), the canonical map $\check{C}(X)_+ \to \mathrm{Spec}\, k_+$ can be completed to a sequence

$$\check{C}(X)_+ \to \mathrm{Spec}\, k_+ \to \widetilde{C}(X) \to \check{C}(X)_+ \wedge S^1_s \to \cdots \tag{9}$$

for some object $\widetilde{C}(X)$ of $\mathcal{H}_\bullet(k)$.

We now assume $p = 2$ as in that case things are settled and we let $\mathcal{X}_{\underline{a}}$ to be the simplicial smooth variety $\check{C}(R_{\underline{a}})$, $R_{\underline{a}}$ being the Pfister neighborhood associated to the symbol $\underline{a}$. Part (1) of Theorem 2.1 for $k_{\underline{a}} = k(Q_{\underline{a}})$ is consequence of a standard property of Pfister quadrics: if $R_{\underline{a}}$ has a rational point on a field extension L over k, then $\underline{a}$ is in the kernel of the map $i^*\colon\ K^M_*(k)/(2) \to K^M_*(L)/(2)$ (see Voevodsky [20, Proposition 4.1]). Part (2) follows from two statements:

(i) there exists a surjective map $H^{w+1,w}(\mathcal{X}_{\underline{a}}, \mathbb{Z}_{(2)}) \to \ker i^*$, and

(ii) $H^{w+1,w}(\mathcal{X}_{\underline{a}}, \mathbb{Z}_{(2)}) = 0$.

The first statement is already nontrivial and uses various exact triangles in certain triangulated categories, one of which is derived by a result of Rost on the Chow motive of $R_{\underline{a}}$, and a very technical argument (not mentioned by Voevodsky [20]) involving commutativity of the functor $\mathbb{H}^*_{\text{et}}(-,\mathbb{Z}_{(2)}(*))$ with limits. The reason for introducing the simplicial algebraic variety $\mathcal{X}_{\underline{a}}$ is to have an object sufficiently similar to $\operatorname{Spec} k$, and at the same time sufficiently different to carry homotopy theoretic information. The similarity to $\operatorname{Spec} k$ is used to prove (i), whereas the homotopy information plays a fundamental role in showing that

Theorem 2.3 $H^{w+1,w}(\mathcal{X}_{\underline{a}},\mathbb{Z}_{(2)}) = 0$

Proof Let $\widetilde{\mathcal{X}}_{\underline{a}}$ be $\widetilde{C}(R_{\underline{a}})$. Since $R_{\underline{a}}$ has points of degree two $\operatorname{Spec} E \to R_{\underline{a}}$ over k, we have that $(\widetilde{\mathcal{X}}_{\underline{a}})_E \cong \operatorname{Spec} E$, because f_E (the base change of the structure map $f\colon \widetilde{\mathcal{X}}_{\underline{a}} \to \operatorname{Spec} k$ over $\operatorname{Spec} E$) is a simplicial weak equivalence. By a transfer argument, we see that $2H^{*,*}(\widetilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}_{(2)}) = 0$. Moreover, property (3) of $\mathcal{H}_\bullet(k)$ implies that $H^{w+1,w}(\mathcal{X}_{\underline{a}},\mathbb{Z}_{(2)}) \cong \widetilde{H}^{w+2,w}(\widetilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}_{(2)})$, because $H^{i,j}(\operatorname{Spec} k,\mathbb{Z}) = 0$ if $i > j$. Thus, it suffices to show that the image of the reduction modulo 2 map

$$\widetilde{H}^{w+2,w}(\widetilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}_{(2)}) \to \widetilde{H}^{w+2,w}(\widetilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}/(2)) \tag{10}$$

is zero. The groups $H^{i,j}(\widetilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}/2)$ are known for $i \le j \le w-1$, because of the following comparison result (cf Voevodsky [20, Corollary 6.9]):

Theorem 2.4 *Assume that the Beilinson–Lichtenbaum Conjecture holds in degree n. Then for any field k and any smooth simplicial scheme $\mathcal{X}$ over k,*

(1) *the homomorphisms*

$$H^{i,j}(\mathcal{X},\mathbb{Z}_{(p)}) \to \mathbb{H}^i_{\text{et}}(\mathcal{X},\mathbb{Z}_{(p)}(j))$$

are isomorphisms for $i-1 \le j \le n$ and monomorphisms for $i = j+2$ and $j \le n$; and

(2) *the homomorphisms*

$$H^{i,j}(\mathcal{X},\mathbb{Z}/p^m) \to \mathbb{H}^i_{\text{et}}(\mathcal{X},\mathbb{Z}/p^m(j))$$

are isomorphisms for $i \le j \le n$ and monomorphisms for $i = j+1$ and $j \le n$ and for any nonnegative integer m.

In our case we have

$$\widetilde{H}^{i,j}(\widetilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}/2) \cong \widetilde{\mathbb{H}}^i_{\text{et}},(\widetilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}/2(j)) \cong \widetilde{\mathbb{H}}^i_{\text{et}},(\operatorname{Spec} k,\mathbb{Z}/2(j)) = 0$$

because of the inductive assumption on the Beilinson–Lichtenbaum Conjecture holding through degree $w-1$ and [20, Lemma 7.3]. In other degrees almost nothing is known, except that $H^{2^w-1,2^{w-1}}(\mathcal{X}_{\underline{a}},\mathbb{Z}) = \widetilde{H}^{2^w,2^{w-1}}(\widetilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}) = 0$ by Theorem 4.9, which uses the second result of Rost [14]. Let u be a nonzero class in the image of (10), by means of some hypothetical motivic cohomology operation θ acting on $H^{*,*}(-,\mathbb{Z}/2)$, we can first try to move u up to the degree $(2^w,2^{w-1})$ and then compare it with the datum $\widetilde{H}^{2^w,2^{w-1}}(\widetilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}) = 0$. What we are about to write now is strictly related to Section 5 of this manuscript. Let Q_i^{top} be the topological Steenrod operations defined by Milnor [9]. Let us assume that there exist operations which we still denote by Q_i that act on $H^{*,*}(-,\mathbb{Z}/p)$ and that satisfy similar properties as Q_i^{top}. In particular, we should expect that $Q_i^2 = 0$ and we could compute the bidegrees of Q_i from the equality

$$Q_i = Q_0(0,\dots,1) + (0,\dots,1)Q_0 \tag{11}$$

where $(0,\dots,1)$, with the 1 in the ith place, is the hypothetical motivical cohomological operations defined as the dual to the canonical class ξ_i of the dual of the motivic Steenrod algebra (cf Voevodsky [22] and Milnor [9]). Indeed, the bidegree of $(0,\dots,1)$ is $(2p^i-2, p^i-1)$ and Q_0 should be the Bockstein, hence bidegree $(0,1)$. This shows that $|Q_i| = (2p^i-1, p^i-1)$ or $(2^{i+1}-1, 2^i-1)$ if $p=2$. Section 5 will be devoted to the construction of such cohomology operations. Thus, $Q_{w-2}Q_{w-3}\cdots Q_1 u$ belongs to $\widetilde{H}^{2^w,2^{w-1}}(\widetilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}/2)$. By assumption, the class u is the reduction of an integral cohomology class and, by the equality (11), so does the class $Q_{w-2}Q_{w-3}\cdots Q_1 u$, that therefore must be zero. To finish the proof it suffices to show that multiplication by Q_i on the relevant motivic cohomology group $\widetilde{H}^{w-i+2^{i+1}-1,w-i+2^1-1}(\widetilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}/2)$ is injective. Since

$$Q_i \cdot \widetilde{H}^{w-i,w-1}(\widetilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}/2) \subset \widetilde{H}^{w-i+2^{i+1}-1,w-i+2^1-1}(\widetilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}/2) \tag{12}$$

and the former group is zero as mentioned above, the obstruction to injectivity of left multiplication by Q_i in degree $(w-i+2^{i+1}-1, w-i+2^1-1)$ is given by the *ith Margolis homology* of the module $\widetilde{H}^{*,*}(\widetilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}/2)$ in degree $(w-i+2^{i+1}-1, w-i+2^1-1)$. Recall that given a graded left module over the Steenrod algebra M, the Margolis homology of M is defined as H_1 of the chain complex

$$\{M \xrightarrow{Q_i\cdot} M \xrightarrow{Q_i\cdot} M\}$$

and is usually denoted with $HM(M,Q_i)$.

Remark 2.5 This argument, along with the successful use of Q_i to prove the vanishing of Margolis homology of $\widetilde{\mathcal{X}}_{\underline{a}}$, constituted the strongest motivation to construct cohomological operations analogous to the Steenrod operations. Voevodsky did this in [22] and it turned out that the Hopf algebra (actually Hopf algebroid) structure of these operations may depend on the base field. More precisely, at odd primes or if $\sqrt{-1} \in k$, then the multiplication and comultiplication between the motivic Steenrod operations are as expected, but at the prime 2 and in the case $\sqrt{-1} \notin k$ both the product *and* the coproduct are different. For more details see Theorem 5.5, Theorem 5.7, Proposition 5.10 of this text, [22] and [4]. From these formulae we see that the formula (11) barely holds even if $\sqrt{-1} \notin k$.

It suffices to prove that $HM(H^{*,*}(\widetilde{\mathcal{X}}_{\underline{a}}, \mathbb{Z}/2), Q_n) = 0$ for all $n \geq 0$. If X is a smooth, projective variety of pure dimension i, then $s_i(X)$ is the zero cycle in X represented by $-\operatorname{Newt}_i(c_1(T_X), c_2(T_X), \dots, c_i(T_X))$, that is the ith Newton polynomial in the Chern classes of the tangent bundle of X. Such polynomials depend only on the index i, thus the notation $s_i(X)$ for them may seem strange. However, it is motivated by the property of these polynomials being dual to certain monomials in the motivic homology of the *classifying sheaf* BU. Indeed, in analogy to the topological case, the set $\{s_\alpha(\gamma)\}_{\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n,\dots)}$ forms a basis for the motivic cohomology of BU, where α_i are nonnegative integers and $\gamma \to BU$ is the colimit of universal n–plane bundles over the infinite grassmanians $Gr_n(\mathbb{A}^\infty)$ (see the author's article [4]).

Theorem 2.6 *Let Y be a smooth, projective variety over k with a map $X \to Y$ such that X is smooth, projective variety of dimension $d = p^t - 1$ with $\deg(s_d(X)) \not\equiv 0 \mod p^2$. Then $HM(H^{*,*}(\widetilde{C}(Y), \mathbb{Z}/p), Q_t) = 0$ for all $n \geq 0$.*

To prove Theorem 2.3 we apply the proposition to $X = X_i$ for smooth subquadrics X_i of dimension $2^i - 1$ for all $1 \leq i \leq w-2$. The fact that the condition on the characteristic number of the X_i is satisfied is readily checked (see Voevodsky [20, Proposition 3.4]).

Proof The pointed object $\widetilde{C}(Y)$, which in topology would just be equivalent to a point, it looks very much like $\operatorname{Spec} k$, but it is different enough to contain homotopy theoretic invariants of algebraic varieties mapping to Y. To prove the vanishing of the Margolis homology, we analyze the two possible cases: in the first the scheme Y has points of degree prime with p, in the second the prime p divides all the degrees of points of Y. The former case implies that $\widetilde{H}^{*,*}(\widetilde{C}(Y), \mathbb{Z}/p) = 0$ by a transfer argument. Thus, we can assume the latter property holds for Y. We want to present a contracting homotopy

$$\phi\colon\ \widetilde{H}^{*,*}(\widetilde{C}(Y), \mathbb{Z}/p) \to \widetilde{H}^{*-2d-1,*-d}(\widetilde{C}(Y), \mathbb{Z}/p) \tag{13}$$

satisfying the relation

$$\phi Q_t - Q_t \phi = c \tag{14}$$

for some nonzero $c \in \mathbb{Z}/p$. In topology, the assumption on the characteristic number of X and the fact that all the caracteristic numbers of Y (a subset of degrees of points of Y) are divisible by p are equivalent to $Q_t\tau \neq 0$ in the cohomology of the cone of $f_X \colon S^{d+n} \to \mathrm{Th}(\nu_X)$ obtained via the Thom–Pontryagin construction, where ν is the normal bundle to an embedding $X \hookrightarrow \mathbb{R}^{d+n}$ such that ν has a complex structure and $\tau \in H^n(\mathrm{Th}(\nu_X), \mathbb{Z}/p)$ is the Thom class. Existence of an "algebraic" Thom–Pontryagin construction (see Voevodsky [20, Theorem 2.11] or Borghesi [5, Section 2]) allows to conclude the same statement in the category $\mathcal{H}_\bullet(k)$. We will now transform the equality $Q_t\tau \neq 0$ in the equality (14). Consider the sequence

$$(\mathbb{P}^1_k)^{\wedge d+n} \xrightarrow{f_X} \mathrm{Th}(\nu_X) \to \mathrm{Cof}(f_X) \xrightarrow{\delta} (\mathbb{P}^1_k)^{\wedge d+n} \wedge S^1_s \to \cdots \tag{15}$$

and let $Q_t\tau = c\gamma \neq 0$, where $\gamma = \delta^*\iota$ and ι is the canonical generator of

$$H^{2(d+n)+1,d+n}((\mathbb{P}^1_k)^{\wedge d+n} \wedge S^1_s, \mathbb{Z}/p).$$

Consider now the chain of morphisms

$$\begin{array}{ccc} \widetilde{H}^{*+2n,*+n}(\mathrm{Cof}(f_X), \mathbb{Z}/p) & \xleftarrow{\delta^* \wedge \mathrm{id}} & \widetilde{H}^{*+2n,*+n}((\mathbb{P}^1_k)^{n+d} \wedge S^1_s \wedge \widetilde{C}(Y), \mathbb{Z}/p) \\ {\scriptstyle \tau\wedge -}\uparrow & & \downarrow{\scriptstyle \cong} \\ \widetilde{H}^{*,*}(\widetilde{C}(Y), \mathbb{Z}/p) & \xrightarrow[\phi]{} & \widetilde{H}^{*-2d-1,*-d}(\widetilde{C}(Y), \mathbb{Z}/p) \end{array} \tag{16}$$

Notice that $\delta^* \wedge \mathrm{id}$ is an isomorphism. Indeed, one thing that makes $\widetilde{C}(Y)$ similar to $\mathrm{Spec}\, k$ is that $\mathrm{Th}(\nu_X) \wedge \widetilde{C}(Y)$ is simplicially weak equivalent to $\mathrm{Spec}\, k$. We can see this by smashing the sequence (9) with ν_+ and using that the projection $(\check{C}(Y) \times \nu)_+ \to \nu_+$ is a simplicial weak equivalence because of the assumption on existence of a morphism $X \to Y$ and [20, Lemma 9.2]. Let now $y \in \widetilde{H}^{*,*}(\widetilde{C}(Y), \mathbb{Z}/p)$ and consider the equality

$$c\gamma \wedge y = Q_t\tau \wedge y = \tau \wedge Q_t(y) - Q_t(\tau \wedge y) \tag{17}$$

with the last equality following from the motivic Steenrod algebra coproduct structure (cf Proposition 5.10, Voevodsky [22, Proposition 13.4] or Borghesi [4, Corollary 5]) and that $H^{i,j}(X, \mathbb{Z}/p) = 0$ if X is a smooth variety and $i > 2j$, hence $Q_i\tau = 0$ for $i < t$ because of the motivic Thom isomorphism. Equality (17) is equality (14) if we let ϕ as in diagram (16) and we identify isomorphisms. □

This finishes the proof of Theorem 2.3 and the motivational part of this manuscript. □

3 Foundations and the Dold–Kan theorem

We will proceed now with the creation of the environment in which algebraic varieties preserve various algebraic invariants and at the same time we can employ the homotopy theoretic techniques which have been used by Voevodsky to prove Theorem 2.3.

Definition 3.1 A functor $F\colon \mathcal{E} \to \mathcal{F}$, with every object of $\mathcal{F}$ being a set, is *representable* in the category $\mathcal{E}$ if there exists an object $K_F \in \mathcal{E}$ such that the bijection of sets

$$F(X) \cong \mathrm{Hom}_{\mathcal{E}}(X, K_F) \tag{18}$$

holds and it is natural in X for any $X \in \mathcal{E}$.

In our situation, $\mathcal{F}$ will be the category of abelian groups, and the congruence (18) is to be understood as abelian groups. In algebraic topology there are several ways to prove that singular cohomology is a representable functor in the unstable homotopy category $\mathcal{H}$. This is the ordinary homotopy category associated to the model structure on topological spaces in which fibrations are the *Serre* fibrations, as opposed to *Hurewicz* fibrations. Here we will recall one way to show such representability which serves as a model for proving the same result in the algebraic context. Originally, singular cohomology has been defined as the homology of a (cochain) complex of abelian groups, hence it is resonable to find some connection between the category of complexes of abelian groups and the underlying objects of $\mathcal{H}$. The latter maybe seen as a localized category of the category of simplicial sets, thus we may set as our starting point the

Theorem 3.2 (Dold–Kan) *Let $\mathcal{A}$ be an abelian category with enough injective objects. Then there exists a pair of adjoint functors N and K (N left adjoint to K) which induce and equivalence of categories*

$$\mathrm{Ch}_{\geq 0}(\mathcal{A}) \underset{N}{\overset{K}{\rightleftarrows}} \Delta^{\mathrm{op}}(\mathcal{A}) \tag{19}$$

in which simplicially homotopic morphisms correspond to chain homotopic maps and viceversa.

Consider the case of $\mathcal{A}$ to be the category of abelian groups. Then we have a chain of adjunctions

$$\mathrm{Ch}_{\geq 0}(\mathcal{A}) \underset{N}{\overset{K}{\rightleftarrows}} \Delta^{\mathrm{op}}(\mathcal{A}) \underset{\mathbb{Z}[-]}{\overset{\text{forget}}{\rightleftarrows}} \Delta^{\mathrm{op}}(\mathrm{Sets}) \underset{\mathrm{Sing}(-)}{\overset{|-|}{\rightleftarrows}} \mathrm{Top} \tag{20}$$

where $\mathbb{Z}[\mathcal{X}]$ is the free simplicial abelian group generated by the simplicial set $\mathcal{X}$, $|\mathcal{X}|$ is the topological realization of $\mathcal{X}$, $\mathrm{Sing}(X)$ is the simplicial set $\mathrm{Hom}_{\mathrm{Top}}(\Delta^*_{\mathrm{top}}, X)$ and Δ^*_{top} is the cosimplicial topological space

$$\Big\{(t_0, t_1, \dots, t_n)/0 \le t_i \le 1, \sum t_i = 1\Big\}$$

Each of the categories appearing in the diagram admits localizing structures which are preserved by the functors. In $\mathrm{Ch}_{\ge 0}(\mathcal{A})$ the class of morphisms that is being inverted is the quasi isomorphisms, in the simplicial homotopy categories are the simplicial homotopy equivalences. This lead us to the first result:

Proposition 3.3 *The chain of adjunctions of diagram (20) induce bijection of sets in the relevant localized categories*

$$\mathrm{Hom}_{\mathcal{D}_{\ge 0}(\mathrm{Ab})}(N\mathbb{Z}[\mathcal{X}], D_*) \cong \mathrm{Hom}_{\mathcal{H}(\Delta^{\mathrm{op}}(\mathrm{Sets}))}(\mathcal{X}, K(D_*)) \tag{21}$$

for any simplicial set $\mathcal{X}$ and complex of abelian groups D_*.

Since we will need to take "shifted" complexes we have to enlarge the category $\mathrm{Ch}_{\ge 0}(\mathrm{Ab})$: let $\mathrm{Ch}_+(\mathrm{Ab})$ be the category of chain (ie with differential of degree -1) complexes of abelian groups which are bounded below, that is for any $C_* \in \mathrm{Ch}_+(\mathrm{Ab})$, there exists an integer $n_C \in \mathbb{Z}$ such that $C_i = 0$ for all $i < n_C$. We relate these two categories of chain complexes by means of a pair of adjoint functors

$$\mathrm{Ch}_{\ge 0}(\mathrm{Ab}) \underset{\text{truncation}}{\overset{\text{forget}}{\rightleftarrows}} \mathrm{Ch}_+(\mathrm{Ab}) \tag{22}$$

where the truncation is the functor which sends a complex

$$\{\cdots \to C_i \xrightarrow{d_i} C_{i-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} \cdots\} \tag{23}$$

to the complex

$$\{\cdots \to C_i \xrightarrow{d_i} C_{i-1} \to \cdots \to C_1 \xrightarrow{d_1} \ker d_0\} \tag{24}$$

Notice that this functor factors through the derived categories. Given a complex D_* and an integer n, let $D[n]_*$ be the complex such that $D[n]_i = D_{i-n}$. Proposition 3.3 implies

Proposition 3.4 *The adjoint functors of diagram (20) induce a bijection of sets*

$$\mathrm{Hom}_{\mathcal{D}_+(\mathrm{Ab})}(N\mathbb{Z}[\mathcal{X}], D[n]_*) \cong \mathrm{Hom}_{\mathcal{H}(\Delta^{\mathrm{op}}(\mathrm{Sets}))}(\mathcal{X}, K(\mathrm{trunc}(D[n]_*))) \tag{25}$$

for any integer n, simplicial set $\mathcal{X}$ and bounded below complex of abelian groups D_*.

We wish to understand better the morphisms in the derived category in order to use the equality (25). Recall that we want to show that the functor *simplicial cohomology* of topological spaces is representable in the unstable homotopy category $\mathcal{H}$. Quillen showed in [12] that this category is equivalent to $\mathcal{H}(\Delta^{\text{op}}(\text{Sets}))$, thus representability of singular cohomology in $\mathcal{H}$ amounts to the group isomorphism

$$H^n_{\text{sing}}(X,A) \cong \text{Hom}_{\mathcal{D}_+(\text{Ab})}(N\mathbb{Z}[\text{Sing}(X)], D_A[n]_*) \tag{26}$$

for some bounded below complex of abelian groups D_A and any CW-complex X.

Definition 3.5 Given a topological space X and an abelian group A , we define:

(1) the nth singular homology group $H_n^{\text{sing}}(X,A)$ to be the nth homology of the chain complex $N\mathbb{Z}[\text{Sing}(X)]$;

(2) the nth singular cohomology group $H^n_{\text{sing}}(X,A)$, to be the nth homology of the cochain complex $\text{Hom}_{\text{Ab}}(N\mathbb{Z}[\text{Sing}(X)], A)$.

In conclusion, we are reduced to compare the homology of the cochain complex $\{\text{Hom}_{\text{Ab}}(N\mathbb{Z}[\text{Sing}(X)], A)\}$ with the group $\text{Hom}_{\mathcal{D}_+(\text{Ab})}(N\mathbb{Z}[\text{Sing}(X)], D_A[n]_*)$ for some appropriate complex $D_A[n]_*$. Morphisms in the derived category are described by this important result

Theorem 3.6 *Let $\mathcal{A}$ be an abelian category with enough projective (respectively injective) objects. Moreover, let C_* and D_* be chain complexes, $P_* \to C_*$ and $D_* \to I_*$ quasi isomorphisms, P_* being projective objects and I_* injective objects for all $*$. Then*

$$\begin{aligned} \text{Hom}_{\mathcal{D}_+(\mathcal{A})}(C, D[n]) &= H^n(\text{Tot}^*(\text{Hom}_{\mathcal{A}}(P, D))) \\ &(\textit{resp.} = H^{-n}(\text{Tot}^*(\text{Hom}_{\mathcal{A}}(C, I)))) \end{aligned} \tag{27}$$

where Tot^* is the total complex of the cochain bicomplex $\text{Hom}_{\mathcal{A}}(P_*, D_*)$ (resp. $\text{Hom}_{\mathcal{A}}(C_*, I^{-*})$) and H^* denotes the homology of the cochain complex.

Theorem 3.6 is rather classical and we refer to Weibel [25, Section 10.7] for its proof. It is based on the fact that, via calculus of fractions, one shows that

$$\text{Hom}_{\mathcal{D}_+(\mathcal{A})}(C_*, D[n]_*) = \lim_{B_* \overset{\text{q.iso}}{\to} C_*} \text{Hom}_{\mathcal{K}_+(\mathcal{A})}(B_*, D[n]_*)$$

where $\mathcal{K}_+(\mathcal{A})$ is the localization of $\text{Ch}_+(\mathcal{A})$ with respect to the chain equivalences of complexes and $\text{Hom}_{\mathcal{K}_+(\mathcal{A})}(B_*, D[n]_*)$ is the quotient set of

$$\text{Hom}_{\text{Ch}_+(\mathcal{A})}(B_*, D[n]_*)$$

modulo chain equivalences. According to the definitions given, applying Theorem 3.6 to $C_* = N\mathbb{Z}[\mathrm{Sing}(X)]$ and $D_* = A[i]$ (here the abelian group A is seen as complex concentrated in degree zero), we conclude that

$$\mathrm{Hom}_{\mathcal{D}_+(\mathrm{Ab})}(N\mathbb{Z}[\mathrm{Sing}(X)], A[i]) = H^i_{\mathrm{sing}}(X,A) \tag{28}$$

and, in view of Proposition 3.4, we conclude

Theorem 3.7 *The homotopy class of $K(\mathrm{trunc}\, A[i])$ represents singular cohomology in the sense that, for any topological space X and abelian group A,*

$$\mathrm{Hom}_{\mathcal{H}(\Delta^{\mathrm{op}}\,\mathrm{Sets})}(\mathrm{Sing}(X), K(\mathrm{trunc}\, A[i])) = H^i_{\mathrm{sing}}(X,A) \tag{29}$$

or equivalently,

$$\mathrm{Hom}_{\mathcal{H}}(X, |K(\mathrm{trunc}\, A[i])|) = H^i_{\mathrm{sing}}(X,A) \tag{30}$$

This represents the reasoning we wish to reproduce in the algebraic setting.

4 Motivic cohomology and its representability

4.1 Sites and sheaves

There are two main issues we should consider: the first is finding the category playing the role of $\mathcal{D}_+(\mathrm{Ab})$ and the second is the category replacing $\mathcal{H}$. The former question is dictated by the definition we wish to give to *motivic cohomology groups* $H^{i,j}(X,A)$ with coefficients in an abelian group A of a smooth variety over a field k. If we set them to be right hyperderived functors of $\mathrm{Hom}_{\mathcal{A}}(-, D_*)$ for some object $D_* \in \mathrm{Ch}_+(\mathcal{A})$ and some abelian category $\mathcal{A}$ with enough projective or injective objects, then Theorem 3.6 automatically tells us that motivic cohomology is representable in the derived category $\mathcal{D}_+(\mathcal{A})$ and we are well poised for attempting to repeat the topological argument of the previous section, with $\mathcal{D}_+(\mathcal{A})$ in place of $\mathcal{D}_+(\mathrm{Ab})$. Up to a decade or so ago the best approximation of what we wished to be motivic cohomology was given by the Bloch's *higher Chow groups* [2] defined as the homology of a certain complex. Most of Voevodsky's work in [24] is devoted to prove that, if k admits resolution of singularities, for a smooth variety X, these groups are canonically isomorphic to $\mathbb{H}^i_{\mathrm{Nis}}(X, A(j))$: the Nisnevich (or even Zariski) hypercohomology of X with coefficients in a certain complex of sheaves of abelian groups $A(j)$. This makes possible to apply the previous remark since hypercohomology with coefficients in a complex of sheaves D_* is defined as right derived functors of the global sections and $\mathrm{Hom}_{\mathrm{Ab-Shv(Sm}/k)}(\mathbb{Z}\,\mathrm{Hom}_{\mathrm{Sm}/k}(-,X),D_*)$ are precisely the global sections of D_*, by the Yoneda Lemma. Thus, we can take the abelian category $\mathcal{A}$ to be $\mathcal{N}_k$, the category of sheaves of abelian groups for the Nisnevich topology over the site of smooth algebraic varieties over k, and D_* to be $A(j)$. Therefore, we define

Definition 4.1 The (i,j)th motivic cohomology group of a smooth algebraic variety X is $\mathbb{H}^i_{\mathrm{Nis}}(X,A(j))$, where the complex of sheaves of abelian groups $A(j)$ will be defined below. Such groups will be denoted in short by $H^{i,j}(X,A)$.

The quest for the algebraic counterpart of the category $\mathcal{H}$ begins with the question on what are the "topological spaces" or more precisely what should we take as the category $\Delta^{\mathrm{op}}(\mathrm{Sets})$ in the algebraic setting. The first candidate is clearly Sch/k, the category of schemes of finite type over a field k. This category has all (finite) limits but, the finiteness type condition on the objects of Sch/k prevents this category from being closed under (finite) colimits and this is too strong of a limitation on a category for trying to do homotopy theory on. The less painful way to solve this problem it seems to be to embed Sch/k in $\mathrm{Funct}((\mathrm{Sch}/k)^{\mathrm{op}},\mathrm{Sets})$, the category of contravariant functors from Sch/k to Sets, via the Yoneda embedding $\mathbf{Y}\colon X \to \mathrm{Hom}_{\mathrm{Sch}/k}(-,X)$. The category $\mathrm{Funct}((\mathrm{Sch}/k)^{\mathrm{op}},\mathrm{Sets})$ has all limits and colimits induced by the ones of Sets and the functor $\mathbf{Y}$ has some good properties like being faithfully full, because of Yoneda Lemma. However, although it preserves limits, it does not preserve existing colimits in Sch/k. Thus, the entire category $\mathrm{Funct}((\mathrm{Sch}/k)^{\mathrm{op}},\mathrm{Sets})$ does not reflect enough existing structures of the original category Sch/k. It turns out that smaller categories are more suited for this purpose and the question of which colimits we wish to be preserved by $\mathbf{Y}$ is related to *sheaf theory*. Let us recall some basic definitions of this theory.

Definition 4.2 Let $\mathcal{C}$ be a category such that for each object U there exists a set of maps $\{U_i \to U\}_{i\in I}$, called a *covering*, satisfying the following axioms:

(1) for any $U \in \mathcal{C}$, $\{U \xrightarrow{\mathrm{id}} U\}$ is a covering of U;

(2) for any covering $\{U_i \to U\}_{i\in I}$ and any morphism $V \to U$ in $\mathcal{C}$, the fibre products $U_i \times_U V$ exist and $\{U_i \times_U V \to V\}_{i\in I}$ is a covering of V;

(3) if $\{U_i \to U\}_{i\in I}$ is a covering of U, and if for each i, $\{V_{ij} \to U_i\}_{j\in J_i}$ is a covering of U_i, then the family $\{V_{ij} \to U\}_{i,j}$ is a covering of U.

Then the datum T of the covering is called a *Grothendieck topology* and the pair $(\mathcal{C},T)$ is a *site*.

Examples 4.3 (1) If X is a topological space, $\mathcal{C}$ is the category whose objects are the open subsets of X, and morphisms are the inclusions, then the family $\{U_i \to U\}_{i \in I}$ for which $\amalg_i U_i \to U$ is surjective and U has the quotient topology is an open covering of U. This is a Grothendieck topology on $\mathcal{C}$. In particular, if the topology of X is the one of Zariski, we denote by X_{Zar} the site associated to it.

(2) Let X be a scheme and $\mathcal{C}$ be the category whose objects are étale (ie locally of finite type, flat and unramified) morphisms $U \to X$ and as arrows the morphisms over X. If we let the covering be the surjective families of étale morphisms $\{U_i \to U\}_i$, then we obtain a site denoted with X_{et}.

(3) If X is a scheme over a field k, and in the previous example we replace étale coverings with étale coverings $\{U_i \to U\}_{i \in I-finite}$ with the property that for each (not necessarily closed point) $u \in U$ there is an i and $u_i \in U_i$ such that the induced map on the residue fields $k(u_i) \to k(u)$ is an isomorphism, then we get a Grothendieck topology called *completely decomposed* or *Nisnevich*.

(4) Let $\mathcal{C}$ be a subcategory of the schemes over a base scheme S (eg Sm/k: locally of finite type, smooth schemes over a field k). Then Grothendieck topologies are obtained by considering coverings of objects of $\mathcal{C}$ given by surjective families of open embeddings, locally of finite type étale morphisms, locally of finite type étale morphisms with the condition of the previous example, locally of finite type flat morphisms, et cetera. The corresponding sites are denoted by $\mathcal{C}_{\mathrm{Zar}}$, $\mathcal{C}_{\mathrm{et}}$, $\mathcal{C}_{\mathrm{Nis}}$, $\mathcal{C}_{\mathrm{fl}}$. In the remaining part of this manuscript we will mainly deal with the site $(\mathrm{Sm}/k)_{\mathrm{Nis}}$.

For the time being all the maps of a covering will be locally of finite type. We have the following inclusions:

$$\{\text{Zariski covers}\} \subset \{\text{Nisnevich covers}\} \subset \{\text{étale covers}\} \subset \{\text{flat covers}\}$$

We now return to the colimit preserving properties of the Yoneda embedding functor $\mathbf{Y}\colon \mathrm{Sch}/k \hookrightarrow \mathrm{Funct}((\mathrm{Sch}/k)^{\mathrm{op}}, \mathrm{Sets})$. Taking for granted the notion of limits and colimits of diagrams in the category Sets, we recall that an object A of a small category $\mathcal{C}$ is a colimit of a diagram D in $\mathcal{C}$ if $\mathrm{Hom}_{\mathcal{C}}(A, X) = \lim_{\mathrm{Sets}} \mathrm{Hom}_{\mathcal{C}}(D, X)$ for all $X \in \mathcal{X}$. The diagrams in Sch/k whose colimits we are mostly interested in being preserved by

$\mathbf{Y}$ are

$$\begin{array}{ccc} V \times_X V & \longrightarrow & V \\ \downarrow & & \\ V & & \end{array} \tag{31}$$

where $V \to X$ is a flat covering for a Grothendieck topology. Such diagrams admit colimits on Sch/k: they are each isomorphic to X. This can be rephrased by saying that the square obtained by adding X and the canonical maps $V \to X$ on the lower right corner of diagram (31) is cocartesian. In order for $\mathbf{Y}$ to preserve such colimits we are going to consider the largest full subcategories of $\mathrm{Funct}((\mathrm{Sch}/k)^{\mathrm{op}}, \mathrm{Sets})$ in which $\mathbf{Y}(X)$ is the colimit of

$$\begin{array}{ccc} \mathbf{Y}(V \times_X V) & \longrightarrow & \mathbf{Y}(V) \\ \downarrow & & \\ \mathbf{Y}(V) & & \end{array} \tag{32}$$

Definition 4.4 Let $(\mathrm{Sch}/k, T)$ be a site. Then F is a *sheaf* on the site $(\mathrm{Sch}/k, T)$ if

$$\mathrm{Hom}_{\mathrm{Funct}((\mathrm{Sch}/k)^{\mathrm{op}},\mathrm{Sets})}\left(\begin{array}{ccc} \mathbf{Y}(V \times_X V) & \longrightarrow & \mathbf{Y}(V) \\ \downarrow{\scriptstyle \mathbf{Y}(T)} & & \downarrow \\ \mathbf{Y}(V) & \xrightarrow{\mathbf{Y}(T)} & \mathbf{Y}(X) \end{array}, F \right) \tag{33}$$

is a cartesian square in Sets. This property will be refered to as F making the diagram

$$\begin{array}{ccc} \mathbf{Y}(V \times_X V) & \longrightarrow & \mathbf{Y}(V) \\ \downarrow{\scriptstyle \mathbf{Y}(T)} & & \downarrow \\ \mathbf{Y}(V) & \xrightarrow{\mathbf{Y}(T)} & \mathbf{Y}(X) \end{array}$$

cocartesian in $\mathrm{Funct}((\mathrm{Sch}/k)^{\mathrm{op}}, \mathrm{Sets})$.

The full subcategory of $\mathrm{Funct}((\mathrm{Sch}/k)^{\mathrm{op}}, \mathrm{Sets})$ of sheaves for the T topology will be denoted as $\mathrm{Shv}(\mathrm{Sch}/k)_T$.

Remark 4.5 (1) This definition is a rephrasing of the classical one found in algebraic geometry texts.

(2) The Yoneda embedding $\mathbf{Y}$ embeds Sch/k as a full subcategory of $\mathrm{Shv}(\mathrm{Sch}/k)_T$ for any topology T coarser than the flat.

Strange it may seem, the category $\mathrm{Shv}(\mathrm{Sch}/k)_T$ is going to be the replacement for the category Sets in topology. This time, though, the category depends upon one parameter: the Grothendieck topology T. The role played by such parameter can be deduced by the results below. For the time being, we will just consider sites with Sm/k as underlying category of objects

Proposition 4.6 (Morel–Voevodsky [11, Proposition 1.4, p96]) *F is a sheaf for the Nisnevich topology if and only if F makes cocartesian $\mathbf{Y}(D)$ where D is the following diagram:*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \overset{i}{\hookrightarrow} & X \end{array} \tag{34}$$

where all the schemes are smooth, p is étale, i is an open immersion and

$$p\colon\ p^{-1}(X - i(U))_{\mathrm{red}} \to (X - i(U))_{\mathrm{red}}$$

(ie reduced structure on the closed subschemes) is an isomorphism. Such squares are called elementary squares.

Remark 4.7 There are analogous results for the Zariski and étale topologies according to whether we require p to be an open embedding or i to be just étale, respectively.

We now drop the $\mathbf{Y}$ from the notation and therefore think of a scheme X as being $\mathbf{Y}(X)$, the sheaf represented by X, as an object in $\mathrm{Shv}(\mathrm{Sch}/k)_T$. Denote with $\mathrm{Cof}(f)$ the colimit (of course in the category $\mathrm{Shv}(\mathrm{Sch}/k)_T$) of the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & X \\ \downarrow & & \\ \mathrm{Spec}\,k & & \end{array}$$

and call it the *cofiber* of the map f. If f is injective, we will write X/U for such cofiber. By general nonsense, in a cocartesian square we can identify cofibers of parallel maps. In particular, in the case of the cocartesian square (34) we get a canonical isomorphism $V/(U \times_X V) \to X/U$. This implies that we are able to identify special sheaves in the isomorphism classes of such objects.

Examples 4.8 (1) Let Z_1 and Z_2 be two disjoint closed subvarieties of a smooth variety X. Then,

$$
\begin{array}{ccc}
X - Z_1 - Z_2 & \longrightarrow & X - Z_1 \\
\downarrow & & \downarrow \\
X - Z_2 & \longrightarrow & X
\end{array}
\tag{35}
$$

is an elementary square and the canonical morphism

$$X - Z_1/(X - Z_1 - Z_2) \to X/(X - Z_2)$$

is, not surprisingly, an isomorphism.

(2) Let L/k be a finite, separable field extension $y\colon \operatorname{Spec} L \to X$ be an L–point, X a smooth variety over k and X_L the base change of X over L. Then there exist points $x_1, x_2 \cdots, x_n, x_L$ of X_L with x_L rational (if L/k is a Galois extension then all x_i are rational), such that

$$
\begin{array}{ccc}
X_L - \{x_1, \ldots, x_n, x_L\} & \longrightarrow & X_L - \{x_1, \ldots, x_n\} \\
\downarrow & & \downarrow{\scriptstyle p} \\
X - y & \xrightarrow{\quad i \quad} & X
\end{array}
\tag{36}
$$

is an elementary square. For instance,

$$
\begin{array}{ccc}
\mathbb{A}^1_{\mathbb{C}} - \{i, -i\} & \longrightarrow & \mathbb{A}^1_{\mathbb{C}} - \{i\} \\
\downarrow & & \downarrow \\
\mathbb{A}^1_{\mathbb{R}} - \operatorname{Spec} \mathbb{C} & \longrightarrow & \mathbb{A}^1_{\mathbb{R}}
\end{array}
\tag{37}
$$

is an elementary square and $\mathbb{A}^1_{\mathbb{C}} - \{i\}/(\mathbb{A}^1_{\mathbb{C}} - \{i, -i\}) \to \mathbb{A}^1_{\mathbb{R}}/(\mathbb{A}^1_{\mathbb{R}} - \operatorname{Spec} \mathbb{C})$ is an isomorphism.

Combining the two previous examples we prove:

Proposition 4.9 *Let L/k be a finite, separable field extension. Assume that a smooth variety has an L–point $y\colon \operatorname{Spec} L \to X$. Then, the canonical morphism $X/(X - y) \to X_L/(X_L - y)$ is an isomorphism of sheaves for a topology finer or equivalent to the Nisnevich one.*

The next results require the introduction of some notation most of the homotopy theorists are familiar with.

Definition 4.10 (1) A *pointed sheaf* is a pair (F,x) of a sheaf F and an element $x \in F(\operatorname{Spec} k)$. The symbol F_+ will denote the sheaf $F \amalg \operatorname{Spec} k$. The cofiber of the unique morphism $\emptyset \to X$ is set by convention to be X_+. Notice that, unlike in topology, there exist sheaves F with the property that $F(\operatorname{Spec} k)$ is the empty set, thus such F cannot be pointed (unless by adding a separate basepoint). This fact is the source of the most interesting phenomena in motivic homotopy theory such as the various degree formulae.

(2) Let (A,a) and (B,b) two pointed sheaves. Then $A \vee B$ is the colimit of

(38)
$$\begin{array}{ccc} \operatorname{Spec} k & \xrightarrow{a} & A \\ \big\downarrow{\scriptstyle b} & & \\ B & & \end{array}$$

pointed by $a = b$.

(3) The pointed sheaf $A \wedge B$ is defined to be $A \times B / A \vee B$ pointed by the image of $A \vee B$.

(4) Let $V \to X$ be a vector bundle (ie the scheme associated with a locally free sheaf of $\mathcal{O}_X$ modules). Then the *Thom sheaf* of V is $V/(V - i(X))$, where $i\colon X \to V$ is the zero section of V. The Thom sheaf will be denoted by $\mathrm{Th}_X(V)$.

(5) The sheaf T will always stand for $\mathbb{A}^1_k/(\mathbb{A}^1_k - 0)$.

The Thom sheaves enjoy of properties similar to the topological counterparts.

Proposition 4.11 *Let $V_1 \to X_1$ and $V_2 \to X_2$ be two vector bundles. Then there is a canonical isomorphism of pointed sheaves*

$$\mathrm{Th}_{X_1 \times X_2}(V_1 \times V_2) \xrightarrow{\cong} \mathrm{Th}_{X_1}(V_1) \wedge Th_{X_2}(V_2)$$

Corollary 4.12 *Let $\mathbb{A}^n_X := \mathbb{A}^n_k \times_k X$ be the trivial vector bundle of rank n over X. Then there is a canonical isomorphism of pointed sheaves*

$$\mathrm{Th}(\mathbb{A}^n_X) \xrightarrow{\cong} T^n \wedge X_+$$

Proof Proposition 4.11 implies that $\mathrm{Th}(\mathbb{A}^n_k) \cong T^{\wedge n}$ and

$$\mathrm{Th}_X(\mathbb{A}^n_X) = Th_{\operatorname{Spec} k \times_k X}(\mathbb{A}^n_k \times_k \mathbb{A}^0_X) \cong T^{\wedge n} \wedge \mathrm{Th}_X(\mathbb{A}^0_X) = T^{\wedge n} \wedge X_+ \qquad \square$$

Corollary 4.13 *Let L/k be a finite and separable field extension and $y\colon \operatorname{Spec} L \to \mathbb{A}^n_k$ be an L point. For any positive integer n there is an isomorphism of sheaves*

$$\mathbb{A}^n_k/(\mathbb{A}^n_k - y) \cong T^{\wedge n} \wedge (\operatorname{Spec} L)_+$$

if the topology T is finer or equivalent to the Nisnevich one.

Proof Proposition 4.9 gives an isomorphism $\mathbb{A}^n_k/(\mathbb{A}^n_k - y) \cong \mathbb{A}^n_L/(\mathbb{A}^n_L - y)$. Since $\mathbb{A}^n_L$ is the trivial rank n vector bundle over $\operatorname{Spec} L$, Corollary 4.12 gives

$$\mathbb{A}^n_L/(\mathbb{A}^n_L - y) \cong \operatorname{Th}(\mathbb{A}^n_L) \cong T^{\wedge n} \wedge (\operatorname{Spec} L)_+. \qquad \square$$

This last isomorphism can be generalized to any closed embedding of smooth schemes, if we allow to work in a suitable localized category of $\Delta^{\mathrm{op}} \operatorname{Shv}(\operatorname{Sm}/k)_{\mathrm{Nis}}$. In that situation the equivalence will be $M/(M - i(X)) \cong \operatorname{Th}_X(\nu_i)$ if $i\colon X \hookrightarrow M$ is a closed embedding of smooth schemes. We will go back to this in the next section.

4.2 Homotopy categories of schemes

In Section 4.1 we have mentioned that we will be taking the category $\Delta^{\mathrm{op}} \operatorname{Shv}(\operatorname{Sm}/k)_{\mathrm{Nis}}$ as replacement for $\Delta^{\mathrm{op}}(\mathrm{Sets})$ in the program of Section 3. The chain of adjunctions of diagram (20) has this algebraic counterpart:

$$\operatorname{Ch}_+(\mathcal{N}_k) \underset{\text{forg}}{\overset{\text{trunc}}{\rightleftarrows}} \operatorname{Ch}_{\geq 0}(\mathcal{N}_k) \underset{N}{\overset{K}{\rightleftarrows}} \Delta^{\mathrm{op}}(\mathcal{N}_k) \underset{\mathbb{Z}[-]}{\overset{\text{forget}}{\rightleftarrows}} \Delta^{\mathrm{op}} \operatorname{Shv}(\operatorname{Sm}/k)_{\mathrm{Nis}} \tag{39}$$

where $\mathcal{N}_k$ denotes the category of sheaves of abelian groups for the Nisnevich topology (the definition is the same as for $\operatorname{Shv}(\operatorname{Sm}/k)_{\mathrm{Nis}}$ except that it is a full subcategory of $\operatorname{Funct}((\operatorname{Sm}/k)^{\mathrm{op}}, \mathrm{Ab})$). Notice that the diagram (39) is a generalization of (20) since there is a full embedding $\mathrm{Sets} \hookrightarrow \operatorname{Shv}(\operatorname{Sm}/k)_T$ sending a set S to the sheaf associated to the presheaf such that $F(U) = S$ for any $U \in \operatorname{Sm}/k$ and $F(f) = \mathrm{id}$ for any f morphism in Sm/k. In Section 3 we passed to localized categories associated with the categories appearing in the adjunction. The functors respected the localizing structures and induced adjunctions on the localized categories. Here we will do the same. Since $\mathcal{N}_k$ is an abelian category, it admits a derived category $\mathcal{D}(\mathcal{N}_k)$. In general, localizing a category may be a very complicated operation to do on a category, being unclear even what morphisms look like in the localized category, assuming it exists. D Quillen in [12] developed a systematic approach to this issue, under the assumption that the category in question had a *model structure*. This is a set of axioms which three classes of morphisms (weak equivalences, cofibrations and fibrations) have to satisfy. In a sense, this makes the category similar to the category of topological spaces and the localization with repect of weak equivalences becomes treatable much in the same way as the unstable homotopy category of topological spaces $\mathcal{H}$. The category $\Delta^{\mathrm{op}}(\mathrm{Shv}(\mathrm{Sm}_k)_T$ can be given a structure of model category as follows:

Definition 4.14 (1) A *point* in $\Delta^{\mathrm{op}}\,\mathrm{Shv}(\mathcal{C})_T$ is a functor $\Delta^{\mathrm{op}}\,\mathrm{Shv}(\mathcal{C})_T \to \mathrm{Sets}$ commuting with finite limits and all colimits.

(2) Let a square

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle g}\downarrow & \exists h \nearrow & \downarrow{\scriptstyle f} \\ C & \longrightarrow & D \end{array}$$

be given in a category. We say that f has the *right lifting property* with respect of g if there exists a lifting h making the diagram commute. In the same instance, we say that g has the *left lifting property* with respect of f.

Definition 4.15 Let $\mathcal{C}_T$ be a site and $f\colon\ \mathcal{X} \to \mathcal{Y}$ a morphism in the site $\Delta^{\mathrm{op}}\,\mathrm{Shv}(\mathcal{C}_T)$. Then f is called

(1) a *simplicial weak equivalence* if for any "point" x of $\Delta^{\mathrm{op}}\,\mathrm{Shv}(\mathcal{C})_T$ the morphism of simplicial sets $x(f)\colon\ x(\mathcal{X}) \to x(\mathcal{Y})$ is a simplicial weak equivalence;

(2) a *cofibration* if it is a monomorphism;

(3) a *fibration* if it has the right lifting property with respect to any cofibration which is a weak equivalence.

Examples 4.16 Consider the site $\mathcal{C} = (\mathrm{Sch}\,/k)_{\mathrm{Zar}}$ or $(\mathrm{Sm}\,/k)_{\mathrm{Zar}}$ and $x\colon\ \mathrm{Spec}\,L \to X$ be a point of a scheme X over k. Let I_x be the cofiltered category of Zariski open subschemes of X containing x. Then the functor $F \to \mathrm{colim}_{U\in I_x} F(U)$ is a point for $\mathrm{Shv}(\mathcal{C})_{\mathrm{Zar}}$. If we change the topology to Nisnevich or étale, to get points for the respective categories we can take

$$I_x^{\mathrm{Nis}} = \{p\colon\ U \to X \text{ étale such that } \exists y \in p^{-1}x \text{ with } p\colon\ k(y) \xrightarrow{\cong} k(x)\}$$

and

$$I_x^{\mathrm{et}} = \left\{ \text{diagrams} \quad \begin{array}{ccc} & & U \\ & & \downarrow{\scriptstyle p} \\ \mathrm{Spec}\,K & \xrightarrow{\bar{x}} & X \end{array} \quad \text{with } p \text{ étale} \right\}$$

where $\bar{x}\colon\ \operatorname{Spec} K \to \operatorname{Spec} k(x) \xrightarrow{x} X$ and K is a separably algebraically closed field. Notice that the latter is a K dependent category.

Jardine [8] checked that the classes of morphisms of Definition 4.15 satisfy the axioms of a model structure. Its homotopy category, that is the localization inverting the weak equivalences, will be denoted as $\mathcal{H}_s(k)$. A generalization of the Dold–Kan Theorem implies

Proposition 4.17 *The maps in diagram (39) preserve the mentioned localizing structures. In particular, the adjoint functors in that diagram induce a bijection of sets*

$$\operatorname{Hom}_{\mathcal{D}_+(\mathcal{N}_k)}(N\mathbb{Z}[\mathcal{X}], D[i]_*) \cong \operatorname{Hom}_{\mathcal{H}_s(k)}(\mathcal{X}, K(\operatorname{trunc}(D[i]_*))) \tag{40}$$

for any $D_* \in \operatorname{Ch}_+(\mathcal{N}_k)$, $\mathcal{X} \in \Delta^{\mathrm{op}} \operatorname{Shv}(\operatorname{Sm}/k)_{\mathrm{Nis}}$ *and* $i \in \mathbb{Z}$.

Associating to any set S the constant (simplicially and as a sheaf) simplicial sheaf S, we can embed Δ^{op} Sets in $\Delta^{\mathrm{op}} \operatorname{Shv}(\operatorname{Sm}/k)_{\mathrm{Nis}}$. Since this embedding preserves the localizing structures we get a faithfully full embedding of $\mathcal{H}$ in $\mathcal{H}_s(k)$. In particular, the canonical projection of $\Delta^1 \to \operatorname{Spec} k$ becomes an isomorphism in $\mathcal{H}_s(k)$. Thus Proposition 4.17 is a generalization of Proposition 3.4. However, while in the topological case $\mathcal{H}$ is the correct category to study topological spaces with respect of homotopy invariant functors like singular cohomology, the category $\mathcal{H}_s(k)$ is not appropriate for certain applications: for example Theorem 4.23 is false in $\mathcal{H}_s(k)$. This category can be made more effective to study algebraic varieties via motivic cohomology. Indeed, we can localize further $\mathcal{H}_s(k)$ without loosing any information detected by motivic cohomology: it is known that for any algebraic variety X, the canonical projection $X \times_k \mathbb{A}^1_k \to X$ induces via pullback of cycles an isomorphism on (higher) Chow groups and the same holds for the definition of motivic cohomology of Definition 4.1, if k is a perfect field. Therefore, inverting all such projections is a lossless operation with respect to these functors. To localize a localized category we can employ the Bousfield framework [6] and define $\mathcal{X} \in \Delta^{\mathrm{op}} \operatorname{Shv}(\operatorname{Sm}/k)_{\mathrm{Nis}}$ to be a simplicial sheaf $\mathbb{A}^1_k$ *local* if for any $\mathcal{Y} \in \Delta^{\mathrm{op}} \operatorname{Shv}(\operatorname{Sm}/k)_{\mathrm{Nis}}$, the maps of sets

$$\operatorname{Hom}_{\mathcal{H}_s(k)}(\mathcal{Y}, \mathcal{X}) \xrightarrow{p^*} \operatorname{Hom}_{\mathcal{H}_s(k)}(\mathcal{Y} \times_k \mathbb{A}^1_k, \mathcal{X})$$

induced by the projection $\mathcal{Y} \times_k \mathbb{A}^1_k \to \mathcal{Y}$, is a bijection.

Definition 4.18 A map $f\colon\ \mathcal{X} \to \mathcal{Y}$ in $\Delta^{\mathrm{op}} \operatorname{Shv}(\operatorname{Sm}/k)_{\mathrm{Nis}}$ is called:

(1) an $\mathbb{A}^1_k$ *weak equivalence* if for any $\mathbb{A}^1_k$ local sheaf $\mathcal{Z}$, the map of sets

$$\mathrm{Hom}_{\mathcal{H}_s(k)}(\mathcal{Y}, \mathcal{Z}) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{H}_s(k)}(\mathcal{X}, \mathcal{Z})$$

ia a bijection;

(2) an $\mathbb{A}^1_k$ cofibration if it is a monomorphism;

(3) an $\mathbb{A}^1_k$ fibration if it has the right lifting property (Definition 4.14 (2)) with respect to monomorphisms that are $\mathbb{A}^1_k$ weak equivalence.

Morel and Voevodsky showed in [11] that this is a (proper) model structure on $\Delta^{\mathrm{op}}\,\mathrm{Shv}(\mathrm{Sm}\,/k)_{\mathrm{Nis}}$. The associated homotopy category will be denoted as $\mathcal{H}(k)$ and called the *unstable homotopy category of schemes over* k.

Remark 4.19 (see [11]) The inclusion of the full subcategory $\mathcal{H}_{s,\mathbb{A}^1_k}(k) \hookrightarrow \mathcal{H}_s(k)$ of $\mathbb{A}^1_k$ local objects has a left adjoint, denoted by $\mathrm{Sing}(-)$, that identifies $\mathcal{H}_{s,\mathbb{A}^1_k}(k)$ with $\mathcal{H}(k)$. Such functor is equipped of a natural transformation $\Theta\colon\ \mathrm{Id} \to \mathrm{Sing}$ that is an $\mathbb{A}^1_k$ weak equivalence on objects.

Working in the category $\mathcal{H}(k)$ as opposed to $\Delta^{\mathrm{op}}\,\mathrm{Shv}(\mathrm{Sm}\,/k)_{\mathrm{Nis}}$ makes schemes more "flexible" quite like working with $\mathcal{H}$ in topology. In this case, however, we have an high algebraic content, thus such flexibility should be made more specific with a few examples. The next results are stated in unpointed categories, although they could be formulated in the pointed setting just as well.

Proposition 4.20 *Let $V \to X$ be a vector bundle and $\mathbb{P}V \hookrightarrow \mathbb{P}(V \oplus \mathbb{A}^1_X)$ be the inclusion of the projectivized scheme induced by the zero section*

$$V \to V \oplus \mathbb{A}^1_X.$$

Then there exists a canonical morphism $\mathbb{P}(V \oplus \mathbb{A}^1_X)/\mathbb{P}V \to \mathrm{Th}(V)$ in $\mathrm{Shv}(\mathrm{Sm}\,/k)_{\mathrm{Nis}}$ that is an $\mathbb{A}^1_k$ equivalence.

Proof See Morel–Voevodsky [11, Proposition 2.17, p112]. □

Corollary 4.21 *The canonical morphism $\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to T^{\wedge n}$ is an $\mathbb{A}^1_k$ weak equivalence for any positive integer n.*

One of the first consequences of the algebraic data lying in the category $\mathcal{H}(k)$ is the existence of several nonisomorphic "circles" in that category. Let S^1_s denote the sheaf $\Delta^1/\partial\Delta^1$ and $\mathbb{G}_m$ be $\mathbb{A}^1_k - \{0\}$. The following result is a particularly curious one and states that the simplicial sheaf $T = \mathbb{A}^1_k/(\mathbb{A}^1_k - \{0\})$, which should embody the properties of a sphere or a circle, is made of an algebraic and a purely simplicial part.

Proposition 4.22 *There is an $\mathbb{A}^1_k$ weak equivalence $\mathbb{G}_m \wedge S^1_s \cong T$.*

Proof See Morel–Voevodsky [11, Lemma 2.15]. □

In the previous few results the Nisnevich topology did not play any role. On the contrary, the theorem below uses this assumption. Arguably, it is the most important result proved in [11] and it represents a very useful tool which is crucial in several applications such as the computation of the motivic Steenrod operations (see Voevodsky [22]) and in the proof of the Milnor Conjecture (see Voevodsky [20]). At this point, it appears to be the main reason to work in the category $\mathcal{H}(k)$ instead of $\mathcal{H}_s(k)$.

Theorem 4.23 (Morel–Voevodsky [11, Theorem 2.23, p115]) *Let $i\colon Z \hookrightarrow X$ be a closed embedding of smooth schemes and ν_i its normal bundle. Then there is a canonical isomorphism in $\mathcal{H}(k)$*

$$X/(X - i(Z)) \cong \mathrm{Th}(\nu_i)$$

Notice the analogy between this congruence of Nisnevich sheaves and the Nisnevich sheaf isomorphism of Corollary 4.13. This is even more evident in the following corollary.

Corollary 4.24 *Let L/k be a finite and separable field extension and X a smooth scheme of dimension n with an L point $x\colon\ \operatorname{Spec} L \to X$. Then there is an isomorphism*

$$X/(X - x) \cong T^{\wedge n} \wedge (\operatorname{Spec} L)_+$$

in the category $\mathcal{H}(k)$.

Let us go back to the adjunction (39) and representability of motivic cohomology. From Proposition 4.17 and Theorem 3.6, we see that for any scheme X and complex $D_* \in \mathrm{Ch}_+(\mathcal{N}_k)$

$$(41) \qquad \mathbb{H}^i(X, D_*) \cong \mathrm{Hom}_{\mathcal{D}_+(\mathcal{N}_k)}(\mathbb{Z}[X], D[i]) \cong \mathrm{Hom}_{\mathcal{H}_s(k)}(X, K(\operatorname{trunc} D[i]_*))$$

To see this, recall that the hypercohomology groups of a scheme X are the right hyperderived functors of the global sections functor (on the small site in question). Since $\mathrm{Hom}_{\mathrm{Ab-Shv}}(\mathbb{Z}\,\mathrm{Hom}_{\mathrm{Sch}/k}(-, X), F) = F(X)$, we conclude that hypercohomology groups are *Ext* functors of $\mathbb{Z}[X] := \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Sch}/k}(-, X)$. This is isomorphic to the left end side group in the isomorphism of Proposition 4.17 because $\mathbb{Z}[X]$ is a complex concentrated in dimension zero (hence $N\mathbb{Z}[X] = \mathbb{Z}[X]$) and of Theorem 3.6. In particular, for any abelian group A we get that (cfr. Definition 4.1) $H^{i,j}(X, A) \cong \mathrm{Hom}_{\mathcal{H}_s(k)}(X, K(A(j)[i]))$, that is motivic cohomology is a representable functor in $\mathcal{H}_s(k)$. We now wish to derive an adjunction allowing to conclude an identification similar to the one of Proposition 4.17 but involving the category $\mathcal{H}(k)$. Since things work fine in the case of $\mathcal{D}_+(\mathcal{N}_k)$ and $\mathcal{H}_s(k)$ we may try to alterate the former category in the same way we already did to obtain $\mathcal{H}(k)$ from $\mathcal{H}_s(k)$. This leads us to the notion of complex $D_* \in \mathrm{Ch}_+(\mathcal{N}_k)$ to be $\mathbb{A}^1_k$ *local* if the morphism $\mathbb{A}^1_k \xrightarrow{p} \operatorname{Spec} k$ induces an isomorphism of groups

$$p^*\colon\ \operatorname{Hom}_{\mathcal{D}_+(\mathcal{N}_k)}(C_*, D_*) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{D}_+(\mathcal{N}_k)}(C_* \overset{L}{\otimes} \mathbb{Z}[\mathbb{A}^1_k], D_*)$$

for any $C_* \in \mathrm{Ch}_+(\mathcal{N}_k)$. We than can endow $\mathcal{D}_+(\mathcal{N}_k)$ of a model structure given by the classes of maps described in Definition 4.18 adapted to this derived category. The resulting category $\mathcal{D}_+^{\mathbb{A}^1}(\mathcal{N}_k)$ is equivalent to the full subcategory of $\mathcal{D}_+(\mathcal{N}_k)$ of $\mathbb{A}^1_k$ local objects, and the inclusion functor i has a left adjoint denoted by $\operatorname{Sing}(-)$. One checks that the pair of functors induced by $(N \circ \mathbb{Z}[-], K \circ \mathrm{trunc})$ of diagram (39) on derived and simplicial homotopy categories preserve $\mathbb{A}^1_k$ local objects, therefore they induce an adjunction between $\mathcal{D}_+^{\mathbb{A}^1}(\mathcal{N}_k)$ and $\mathcal{H}_s(k)$:

Theorem 4.25 *The pair of adjoint functors* $(N \circ \mathbb{Z}[-], K \circ \mathrm{trunc})$ *induce a bijection of sets:*

$$\operatorname{Hom}_{\mathcal{D}_+^{\mathbb{A}^1}(\mathcal{N}_k)}(N\mathbb{Z}[\mathcal{X}], D[i]_*) \cong \operatorname{Hom}_{\mathcal{H}(k)}(\mathcal{X}, K(\mathit{trunc}\, D[i]_*)) \tag{42}$$

for any $D_* \in \mathrm{Ch}_+(\mathcal{N}_k)$, $\mathcal{X} \in \Delta^{\mathrm{op}}\operatorname{Shv}(\mathrm{Sm}/k)_{\mathrm{Nis}}$ *and* $i \in \mathbb{Z}$.

To relate the left end side of the isomorphism (42) with hypercohomology groups and hence with motivic cohomology, we employ the adjunction $(\operatorname{Sing}(-), i)$: since $\mathcal{D}_+^{\mathbb{A}^1}(\mathcal{N}_k)$ is equivalent to the full subcategory of $\mathbb{A}^1_k$ local objects of $\mathcal{D}_+(\mathcal{N}_k)$, it follows that

$$\operatorname{Hom}_{\mathcal{D}_+^{\mathbb{A}^1}(\mathcal{N}_k)}(C_*, D_*) = \operatorname{Hom}_{\mathcal{D}_+(\mathcal{N}_k)}(C_*, D_*)$$

for any $\mathbb{A}^1_k$ local complexes C_* and D_*. Let $\operatorname{Sing}\colon \mathcal{H}_s(k) \to \mathcal{H}_s^{\mathbb{A}^1_k}(k)$ denote also the functor left adjoint to the inclusion of the full subcategory generated by the $\mathbb{A}^1_k$ local simplicial sheaves in $\mathcal{H}_s(k)$. Since $N\mathbb{Z}[-]$ preserves $\mathbb{A}^1_k$ local objects and $\mathbb{A}^1_k$ weak equivalences, $N\mathbb{Z}[\operatorname{Sing}(X)]$ is an $\mathbb{A}^1_k$ local complex and the canonical morphism $N\mathbb{Z}[\Theta]\colon N\mathbb{Z}[X] \to N\mathbb{Z}[\operatorname{Sing}(X)]$ is an $\mathbb{A}^1_k$ weak equivalence. Therefore,

$$\operatorname{Hom}_{\mathcal{D}_+^{\mathbb{A}^1}(\mathcal{N}_k)}(N\mathbb{Z}[\operatorname{Sing}(X)], L_*) = \operatorname{Hom}_{\mathcal{D}_+(\mathcal{N}_k)}(N\mathbb{Z}[\operatorname{Sing}(X)], L_*) \xrightarrow{N\mathbb{Z}[\Theta]^*} \operatorname{Hom}_{\mathcal{D}_+(\mathcal{N}_k)}(N\mathbb{Z}[X], L_*) \cong \mathbb{H}^i_{\mathrm{Nis}}(X, L_*)$$

is a group isomorphism for any $\mathbb{A}^1_k$ local complex L_*. In the case of motivic cohomology, if k is a perfect field, the complex of sheaves $A(j)$ are $\mathbb{A}^1_k$ local for all $j \in \mathbb{Z}$, hence:

Corollary 4.26 *For a smooth scheme X over a perfect field k, and an abelian group A, we have*

$$\begin{aligned} H^{i,j}(X,A) &\cong \mathrm{Hom}_{\mathcal{D}_+^{\mathbb{A}^1}(\mathcal{N}_k)}(\mathbb{Z}[\mathrm{Sing}(X)], A(j)[i]) \\ &\cong \mathrm{Hom}_{\mathcal{H}(k)}(X, K(\mathrm{trunc}\, A(j)[i])) \end{aligned} \tag{43}$$

The homotopy class of the simplicial sheaf $K(\mathrm{trunc}\, A(j)[i])$ is called the (i,j) *motivic Eilenberg–MacLane* simplicial sheaf with coefficients in A, in analogy with the topological representing space of singular cohomology and will be denoted by $K(A(j),i)$.

5 Motivic cohomology operations

In this section the word *sheaf* will always mean *Nisnevich sheaf*, unless otherwise specified. The objective of this section is to give a more solid basis to key techniques used to prove Theorem 2.6, namely the use of *motivic cohomology operations* with finite coefficients. By such term we mean natural transformations

$$H^{*,*}(-,\mathbb{Z}/p) \to H^{*+n,*+m}(-,\mathbb{Z}/p)$$

commuting with suspension isomorphisms

$$\widetilde{H}^{*,*}(\mathcal{X},\mathbb{Z}/p) \xrightarrow{\sigma_s} \widetilde{H}^{*+1,*}(\mathcal{X}\wedge S^1_s,\mathbb{Z}/p)$$

and

$$\widetilde{H}^{*,*}(\mathcal{X},\mathbb{Z}/p) \xrightarrow{\sigma_t} \widetilde{H}^{*+1,*+1}(\mathcal{X}\wedge \mathbb{G}_m,\mathbb{Z}/p)$$

for any pointed sheaf $\mathcal{X}$. See Voevodsky [22, Theorem 2.4] for a proof that these canonical morphisms are isomorphisms in the case the base field is perfect. We are interested to produce motivic cohomology operations as similar as possible to the ones generating the *Steenrod algebra* $\mathcal{A}^*_{\mathrm{top}}$: they are classically denoted by P^i and by β, the *Bockstein* operation, and the collection of them over all nonnegative i generate $\mathcal{A}^*_{\mathrm{top}}$ as (graded) algebra over $\mathbb{Z}/p$. At any prime p, the operation P^0 is the identity. At the prime $p=2$, the operation P^i is usually denoted by Sq^{2i} and βP^i by Sq^{2i+1} and are sometimes called the *Steenrod squares*. This is a classical subject and there are several texts available covering it. Among them, the ones of Steenrod and Epstein [16] and of Milnor [9] contain the constructions and ideas which will serve as models to follow in the "algebraic" case. The topological Steenrod algebra is in fact an Hopf algebra and is completely determined by

(1) generators $\{\beta, P^i\}_{i\geq 0}$,

(2) the relations $P^0 = 1$ and the so called *Adem relations*,

(3) the diagonal $\psi^*\colon \mathcal{A}^*_{\mathrm{top}} \to \mathcal{A}^*_{\mathrm{top}} \otimes_{\mathbb{Z}/p} \mathcal{A}^*_{\mathrm{top}}$ described by the so called *Cartan formulae*.

The standard reference for the "algebraic" constructions is Voevodsky's paper [22]. To construct the operations P^i, we need a suitable and cohomologically rich enough simplicial sheaf $\mathcal{B}$ and an homomorphism

$$P\colon H^{2*,*}(-,\mathbb{Z}/p) \to H^{2p*,p*}(- \wedge \mathcal{B}, \mathbb{Z}/p) \tag{44}$$

called the *total power operation*. The "suitability" that the simplicial sheaf shall satisfy is that $H^{*,*}(\mathcal{X} \wedge \mathcal{B}, \mathbb{Z}/p)$ is free as left $H^{*,*}(\mathcal{X}, \mathbb{Z}/p)$ module over a basis $\{b_i\}_i$ for any pointed simplicial sheaf $\mathcal{X}$. We can then obtain individual motivic cohomology operations from P by defining classes $A_i(x)$ satisfying the equality $P(x) = \sum_i A_i(x) b_i$ for any $x \in H^{*,*}(\mathcal{X}, \mathbb{Z}/p)$. In analogy with the topological case, the pointed simplicial sheaf $\mathcal{B}$ is going to be chosen among the ones of the kind BG for a finite group (or, more generally, a group scheme) G. In particular, we will consider G to be S_p, the group of permutations on p elements. There are various models for the homotopy class of BG in $\mathcal{H}_\bullet(k)$. We are interested to one of them whose motivic cohomology can be computed more easily for the groups G considered: let $r\colon G \to Gl_d(k)$ be a faithful representation of G and U_n the open subset in $\mathbb{A}^{dn}_k$ where G acts freely with respect of r. The open subschemes $\{U_n\}_n$ fit into a direct system induced by the embeddings $\mathbb{A}^{dn} \hookrightarrow \mathbb{A}^{d(n+1)}$ given by $(x_1, x_2, \dots, x_n) \to (x_1, x_2, \dots, x_n, 0)$. For instance, let μ_p defined as the group scheme given by the kernel of the homomorphism $(-)^p\colon \mathbb{G}_m \to \mathbb{G}_m$. We can take $d = 1$ and $r\colon \mathbb{Z}/p \hookrightarrow Gl_1(k) = k^*$. It follows that $U_n = \mathbb{A}^n_k - 0$ in this case.

Definition 5.1 *BG is defined to be the pointed sheaf $\mathrm{colim}_n(U_n/G)$, where U_n/G is the quotient in the category of schemes and the colimit is taken in the category of sheaves.*

Of all the results of this section, the computation of the motivic cohomology of BG is the only one that will be given a complete proof. This is because of its importance and relevance in the structure of the motivic cohomology operations we are going to study later in this section.

Theorem 5.2 (see Voevodsky [22, Theorem 6.10]) *Let k be any field and $\mathcal{X}$ a pointed simplicial sheaf over k. Then, if p is odd,*

$$H^{*,*}(\mathcal{X} \wedge (B\mu_p)_+, \mathbb{Z}/p) = \frac{H^{*,*}(\mathcal{X}, \mathbb{Z}/p)[\![u,v]\!]}{(u^2)}. \tag{45}$$

If $p = 2$,

$$H^{*,*}(\mathcal{X} \wedge (B\mu_p)_+, \mathbb{Z}/p) = \frac{H^{*,*}(\mathcal{X}, \mathbb{Z}/p)[\![u,v]\!]}{(u^2 - \tau v + \rho u)}, \tag{46}$$

where

1. $u \in H^{1,1}(B\mu_p, \mathbb{Z}/p)$ *and* $v \in H^{2,1}(B\mu_p, \mathbb{Z}/p)$*;*
2. ρ *is the class of* -1 *in* $H^{1,1}(k, \mathbb{Z}/p) \cong k^*/(k^*)^p$*;*
3. τ *is zero if* $p \neq 2$ *or* $\mathrm{char}(k) = 2$*;*
4. τ *is the generator of* $H^{0,1}(k, \mathbb{Z}/2) = \Gamma(\mathrm{Spec}\, k, \mu_2) = \mu_2(k)$ *if* $p = 2$ *and* $\mathrm{char}(k) \neq 2$.

Proof It suffices to consider the case of $\mathcal{X} = X_+$ smooth scheme, essentially because of [11, Lemma 1.16] (see also [5, Appendix B]). To prove the statement of the theorem as left $H^{*,*}(\mathcal{X}, \mathbb{Z}/p)$ modules, we are going to use the existence of a cofibration sequence

$$(B\mu_p)_+ \to (\mathcal{O}_{\mathbb{P}^\infty_k}(-p))_+ \to \mathrm{Th}(\mathcal{O}(-p)) \to S^1_s \wedge (B\mu_p)_+ \cdots \tag{47}$$

where $\mathcal{O}_{\mathbb{P}^\infty_k}(-1)$ is the dual of the canonical line bundle over $\mathbb{P}^\infty_k$. By a vector bundle V we sometimes mean its affinization $\mathrm{Spec}(\mathrm{Symm}^*(V\check{-}))$. This sequence is a consequence of the fact that $B\mu_p = \mathcal{O}_{\mathbb{P}^\infty_k}(-p) - z(\mathbb{P}^\infty_k)$, where z is the zero section and $B\mu_p$ is constructed by means of the representation

$$\mu_p \hookrightarrow GL(\mathcal{O}) = \mathbb{G}_m$$

(cf Voevodsky [22, Lemma 6.3]). Smashing the sequence (47) with a smooth scheme X_+ we get a cofibration sequence

$$X_+ \wedge (B\mu_p)_+ \longrightarrow X_+ \wedge (\mathcal{O}_{\mathbb{P}^\infty_k}(-p))_+ \xrightarrow{f} X_+ \wedge \mathrm{Th}(\mathcal{O}(-p)) \longrightarrow S^1_s \wedge X_+ \wedge (B\mu_p)_+ \longrightarrow \cdots \tag{48}$$

which yields a long exact sequence

$$\cdots \to \widetilde{H}^{*-2,*-1}(X,A)[\![c]\!] \xrightarrow{\widehat{f^*}} H^{*,*}(X,A)[\![c]\!] \xrightarrow{\alpha} H^{*,*}(X_+ \wedge (B\mu_p)_+, A) \longrightarrow \widetilde{H}^{*-1,*-1}(X,A)[\![c]\!] \to \cdots \tag{49}$$

for any abelian group A and where $c \in H^{2,1}(\mathbb{P}^\infty, A)$ is the first Chern class of $\mathcal{O}_{\mathbb{P}^\infty}(1)$. We define $v \in H^{2,1}(B\mu_p, A)$ to be the image of c and $u \in H^{1,1}(B\mu_p, \mathbb{Z}/p)$ the class mapping to $1 \in H^{0,0}(\mathrm{Spec}\, k, \mathbb{Z}/p)[\![c]\!]$ and restricting to 0 in $H^{1,1}(\mathrm{Spec}\, k, \mathbb{Z}/p) \cong$

$\{H^{*,*}(\operatorname{Spec} k, \mathbb{Z}/p)[\![c]\!]\}^{1,1}$ via any rational point $\operatorname{Spec} k \to B\mu_p$. The long exact sequence (49) splits in short exact sequences

$$0 \to H^{*,*}(X, \mathbb{Z}/p)[\![c]\!] \to H^{*,*}(X_+ \wedge B\mu_p, \mathbb{Z}/p) \to H^{*-1,*-1}(X, \mathbb{Z}/p)[\![c]\!] \to 0 \tag{50}$$

if $A = \mathbb{Z}/p$ since $\widehat{f}^* = 0$, in that case. To see this, notice that $\widehat{f}^*$ is the composition $z^* \circ f^* \circ t$ where

$$t\colon H^{*,*}(\mathbb{P}^\infty) \to \widetilde{H}^{*+2,*+1}(X_+ \wedge \operatorname{Th}(\mathcal{O}(-p))) \cong H^{*+2,*+1}(\operatorname{Th}(X \times \mathcal{O}(-p)))$$

is the Thom isomorphism (see Borghesi [4, Corollary 1]) and

$$z\colon X_+ \wedge \mathbb{P}^\infty_+ \to X_+ \wedge \mathcal{O}_{\mathbb{P}^\infty_k}(-p)_+$$

is the zero section. This composition sends a class x to $x \cdot pc$ which is zero with $\mathbb{Z}/p$ coefficients. By means of the sequences (50) we prove the theorem as left $H^{*,*}(X, \mathbb{Z}/p)$ modules. The exotic part of the theorem lies in the multiplicative structure of the motivic cohomology at the prime 2, more specifically the relation $u^2 = \tau v + \rho u$. Since the multiplicative structure in motivic cohomology is graded commutative (see Voevodsky [22, Theorem 2.2]), at odd primes we have $u^2 = 0$. Let $p = 2$. The class u^2 belongs to the group $H^{2,2}(B\mu_p, \mathbb{Z}/p)$ which is isomorphic to

$$H^{0,1}(\operatorname{Spec} k, \mathbb{Z}/p)v \oplus H^{1,1}(\operatorname{Spec} k, \mathbb{Z}/p)u \oplus H^{2,2}(\operatorname{Spec} k, \mathbb{Z}/p) \tag{51}$$

because of what we have just proved. By definition of u, it restricts to 0 on $\operatorname{Spec} k$, thus $u^2 = xv + yu$ for coefficients $x \in H^{0,1}(\operatorname{Spec} k, \mathbb{Z}/2)$ and $y \in H^{1,1}(\operatorname{Spec} k, \mathbb{Z}/2)$. We wish to prove that $x = \tau$ and $y = \rho$ as described in the statement of the theorem.

Proof that $y = \rho$ We reduce the question to the group $H^{*,*}(\mathbb{A}^1 - 0, \mathbb{Z}/2)$ by means of the map

$$\mathbb{A}^1 - 0 \cong (\mathbb{A}^1 - 0)/\mu_2 \to \operatorname{colim}_i(\mathbb{A}^i - 0)/\mu_2 = B\mu_2 \tag{52}$$

The class $u \in H^{1,1}(B\mu_p, \mathbb{Z}/2)$ pulls back to the generator u_1 of

$$H^{1,1}(\mathbb{A}^1_k - 0, \mathbb{Z}/2) \cong \mathbb{Z}/2$$

because so does the similarly defined class $u_i \in H^{1,1}(\mathbb{A}^i_k - 0/\mu_2, \mathbb{Z}/2)$ for each i: the sequence (47) in this case reduces to

$$\mathbb{A}^1_k - 0_+ \to (\mathbb{A}^1_k)_+ \to \operatorname{Th}(\mathbb{A}^1_k) = \mathbb{A}^1_k/(\mathbb{A}^1_k - 0) \to \dots$$

and the generator of $H^{1,1}(\mathbb{A}^1_k - 0, \mathbb{Z}/2)$ is precisely u_1, ie the class coming from the Thom sheaf. The class u pulls back to each u_i because of a $\lim^1$ argument on the motivic cohomology of $B\mu_2$. We have thus reduced the question to the following rather curious lemma

Lemma 5.3 *Let* $u_1 \in \widetilde{H}^{1,1}(\mathbb{A}^1_k - 0, \mathbb{Z}/2) \cong \mathbb{Z}/2$ *be the generator. Then* $u_1^2 = \rho u_1$ *in* $H^{2,2}(\mathbb{A}^1_k - 0, \mathbb{Z}/2)$.

One further reduction involves the use of the canonical map $A \to \mathbb{A}^1_k - 0$ where A is a sheaf whose motivic cohomology is $\mathrm{colim}_{U \subset \mathbb{A}^1_k \text{ open}} H^{*,*}(U, \mathbb{Z}/2)$. Because of [18], $H^{i,j}(A, \mathbb{Z}/2)$ is thus isomorphic to the higher Chow group $CH^j(\mathrm{Spec}\, k(t), 2j - i)$ and in particular $H^{i,i}(A, \mathbb{Z}/2) \cong K^M_i(k(t))/(2)$. The sheaf A can be manufactured in many different non isomorphic ways as sheaves although there is a canonical one. In any case, their class in $\mathcal{H}(k)$ coincides. Such class is known as the *homotopy limit* of the cofiltered category $\{U\}_{U \subset \mathbb{A}^1_k \, open}$ and denoted by $\mathrm{holim}\, U$. It comes equipped with a map

$$\mathrm{holim}\, U \to U$$

for any such U and the localizing sequence for motivic cohomology implies the injectivity of the map $H^{*,*}(\mathbb{A}^1_k - 0, \mathbb{Z}/2) \to H^{*,*}(\mathrm{holim}\, U, \mathbb{Z}/2)$. Therefore, it suffices to prove the statement of the lemma in

$$H^{2,2}(\mathrm{holim}\, U, \mathbb{Z}/2) \cong CH^2(k(t), 0) \otimes \mathbb{Z}/2 \cong K^M_2(k(t))/(2) \tag{53}$$

With these identifications we may represent u_1 by $t \in K^M_1(k(t))$ and show that $t^2 = -1 \cdot t$, which is a known relation in K^M_2.

Proof that $x = \tau$ We have that $x \in H^{0,1}(\mathrm{Spec}\, k, \mathbb{Z}/2) = \mu_2(k)$ and this group is zero if the characteristic of k is 2. If the characteristic is different from 2 then this group is $\mathbb{Z}/2$ and thus it suffices to show that $x \neq 0$. By contravariance of motivic cohomology, it is enough to prove this statement up to replacing the base field k with a finite field extension. In particular, by assuming that $\sqrt{-1} \in k$, we reduce the question to showing that $u^2 \neq 0$, since $\rho = 0$ in this case. Passing to étale cohomology with $\mathbb{Z}/2$ coefficients via the natural transformation from motivic cohomology with $\mathbb{Z}/2$ coefficients to étale cohomology with the same coefficients, we are reduced to show that $u^2 \neq 0$ in $H^2_{\mathrm{et}}(B\mu_2, \mathbb{Z}/2)$. All the long exact sequences involving motivic cohomology we have used in this section are valid for étale cohomology as well. Since $u \in H^1_{\mathrm{et}}$, $\beta u = u^2$. Consider the long exact sequence

$$\begin{aligned}\cdots \longrightarrow H^1_{\mathrm{et}}(B\mu_2, \mathbb{Z}) &\xrightarrow{\mathrm{mod}\, 2} H^1_{\mathrm{et}}(B\mu_2, \mathbb{Z}/2) \\ &\xrightarrow{\delta} H^2_{\mathrm{et}}(B\mu_2, \mathbb{Z}/2) \xrightarrow{2\cdot} H^2_{\mathrm{et}}(B\mu_2, \mathbb{Z}/2) \longrightarrow \cdots\end{aligned} \tag{54}$$

we see that $\delta u \neq 0$. To see this we observe the following diagram

(55)
$$
\begin{array}{ccccc}
 & & & & H^1_{\text{et}}(\operatorname{Spec} k, \mathbb{Z}/2) \\
 & & & \nearrow^{\text{mod } 2} & \downarrow \alpha \\
H^1_{\text{et}}(\operatorname{Spec} k, \mathbb{Z}) & \xrightarrow[\cong]{\alpha} & H^1_{\text{et}}(B\mu_2, \mathbb{Z}) & \xrightarrow{\text{mod } 2} & H^1_{\text{et}}(B\mu_2, \mathbb{Z}/2) \\
 & & & & \downarrow \\
 & & & & H^0_{\text{et}}(\operatorname{Spec} k, \mathbb{Z}/2)
\end{array}
$$

where the vertical sequence of maps is (49) with motivic cohomology replaced by étale cohomology and is a short exact sequence. The class $u \in H^1_{\text{et}}(B\mu_2, \mathbb{Z}/2)$ is defined as the only class which maps to the generator of $H^0_{\text{et}}(\operatorname{Spec} k, \mathbb{Z}/2)$ and is zero on the image of $H^1_{\text{et}}(\operatorname{Spec} k, \mathbb{Z}/2)$, thus it is not in the image of the mod 2 morphism. $\delta u \neq 0$ implies that $\beta u \neq 0$ since $\delta u = v \in H^2_{\text{et}}(B\mu_2, \mathbb{Z})$ which projects to the class v with $\mathbb{Z}/2$ coefficients. This finishes the proof of Theorem 5.2. □

5.1 The dual algebra $\mathcal{A}_{*,*}$

Rather than considering motivic cohomology operations, we are going to concentrate on their duals. Denote by $\mathcal{A}_m^{*,*}$ the $\mathbb{Z}/p$ algebra of all the motivic cohomology operations and let $\mathcal{A}^{*,*}$ be a locally finite and free $H^{*,*}(\operatorname{Spec} k, \mathbb{Z}/p)$ (simply written as $H^{*,*}$ from now on) submodule of it. Its dual $\mathcal{A}_{*,*}$ is the set of left $H^{*,*}$ graded module maps from $\mathcal{A}^{*,*}$ to $H^{*,*}$. We are interested in the action of $\mathcal{A}^{*,*}$ on the motivic cohomology of $B\mu_p$. Let $\theta \in \mathcal{A}^{*,*}$; its action on $H^{*,*}(B\mu_p, \mathbb{Z}/p)$ is completely determined by $\theta(u^e v^i)$ for all i and $e \in \{0, 1\}$, because of Theorem 5.2. The module $\mathcal{A}^{*,*}$ we are going to consider is in fact a $\mathbb{Z}/p$ algebra and includes certain monomial operations denoted by M_k for all nonnegative integers k satisfying:

Proposition 5.4 (see Voevodsky [22, Lemma 12.3]) (1) *$M_k(v) = M_k\beta(u) = v^{p^k}$, for all $k \geq 0$;*

(2) *if $\mathcal{A}^{*,*} \ni \theta \notin \{M_k, M_k\beta,\ k \geq 0\}$ is a monomial, then*

$$\theta \cdot H^{*,*}(B\mu_p, \mathbb{Z}/p) = 0$$

This enables us to define canonical classes $\xi_i \in \mathcal{A}_{2(p^i-1),p^i-1}$ for $i \geq 0$ and $\tau_j \in \mathcal{A}_{2p^j-1,p^j-1}$ for $j \geq 0$ as the duals of M_k and $M_k\beta$, respectively. Monomials in the classes ξ_i and τ_j are generators of $\mathcal{A}_{*,*}$ as a left $H^{*,*}$ module. More precisely, one first proves that $\omega(I) := \tau_0^{\epsilon_0}\xi_1^{r_1}\tau_1^{\epsilon_1}\xi_2^{r}\cdots$ form a free $H^{*,*}$ module basis of $\mathcal{A}_{*,*}$, where

$I = (\epsilon_0, r_1, \epsilon_1, r_2, \dots)$ ranges over all the infinite sequences of nonnegative integers r_1 and $\epsilon_i \in \{0, 1\}$ (cf [22, Theorem 12.4]). Then one derives the complete description of $\mathcal{A}_{*,*}$:

Theorem 5.5 (cf Voevodsky [22, Theorem 12.6]) *The graded left $H^{*,*}$ algebra $\mathcal{A}_{*,*}$ is (graded) commutative with respect to the first grading and is presented by generators $\xi_i \in \mathcal{A}_{2(p^i-1),p^i-1}$ and $\tau_i \in \mathcal{A}_{2p^i-1,p^i-1}$ and with relations*

(1) $\xi_0 = 1$;

(2) $$\tau_i^2 = \begin{cases} 0 & \text{for } p \neq 2 \\ \tau\xi_{i+1} + \rho\tau_{i+1} + \rho\tau_0\xi_{i+1} & \text{for } p = 2 \end{cases}$$

Remark 5.6 The degree of the product $a\gamma$ between an element $a \in H^{*,*}$ and $\gamma \in \mathcal{A}_{*,*}$ is $(\gamma_1 - a_1, \gamma_2 - a_2)$.

5.2 The algebra $\mathcal{A}^{*,*}$

To get this kind of information on $\mathcal{A}_{*,*}$ we really have to be more specific on the algebra $\mathcal{A}^{*,*}$ we are considering. As mentioned earlier, Voevodsky [22, Section 5] first defined a total power operation

$$P\colon\ H^{2*,*}(-, \mathbb{Z}/p) \to H^{2p*,p*}(- \wedge \mathcal{B}, \mathbb{Z}/p)$$

Taking $\mathcal{B}$ to be BS_p, this becomes a morphism [22, Theorem 6.16]

(56) $$P\colon\ H^{2*,*}(X, \mathbb{Z}/p) \to \begin{cases} \{H^{*,*}(X, \mathbb{Z}/p)[\![c,d]\!]/(c^2{=}\tau d{+}\rho c)\}^{4*,2*}, & p = 2 \\ \{H^{*,*}(X, \mathbb{Z}/p)[\![c,d]\!]/(c^2)\}^{2p*,p*}, & p \neq 2 \end{cases}$$

for classes $c \in H^{2p-3,p-1}(BS_p, \mathbb{Z}/p)$ and $d \in H^{2p-2,p-1}(BS_p, \mathbb{Z}/p)$ and any simplicial sheaf X. The unusual relation in the motivic cohomology of BS_p at the prime $p = 2$ is consequence of the one in $H^{*,*}(B\mu_2, \mathbb{Z}/2)$ described in Theorem 5.2. Thus, in analogy to the topological case, we can define operations P^i and B^i by the equality

(57) $$P(w) = \sum_{i \geq 0} P^i(w)d^{n-i} + B^i(w)cd^{n-i-1}$$

for $w \in H^{2n,n}(X, \mathbb{Z}/p)$. Then one proves [22, Lemma 9.6] that $B^i = P^i\beta$. For an arbitrary class $x \in H^{*,*}(X, \mathbb{Z}/p)$, we define

$$P^i(x) := \sigma_s^{-m_1}\sigma_t^{-m_2}(P^i(\sigma_s^{m_1}\sigma_t^{m_2}(x)))$$

where $\sigma_s^{m_1}\sigma_t^{m_2}(x) \in H^{2*,*}(X,\mathbb{Z}/p)$ for some $*$ and σ are the suspension isomorphisms we introduced at the beginning of this section. As left $H^{*,*}$ module we let $\mathcal{A}^{*,*}$ to be $H^{*,*} \otimes_{\mathbb{Z}/p} \mathbb{Z}/p\,\langle\beta, P^i\rangle$, where $\mathbb{Z}/p\,\langle\beta, P^i\rangle$ is the $\mathbb{Z}/p$ subalgebra (of all the motivic cohomology operations $\mathcal{A}_m^{*,*}$) generated by β and P^i. It turns out that $\mathcal{A}^{*,*}$ is a free left $H^{*,*}$ module (cf [22, Section 11]). Notice that there is canonical embedding $i\colon H^{*,*} \hookrightarrow \mathcal{A}^{*,*}$ sending $a \in H^{*,*}$ to the operation $ax = \pi^* a \cup x$, where $\pi\colon X \to \operatorname{Spec} k$ is the structure morphism. The image $i(H^{*,*})$ does not belong to the center of $\mathcal{A}^{*,*}$: the multiplication in $\mathcal{A}^{*,*}$ is the composition of the cohomology operations, therefore $(\theta a)x = \theta(ax) = \theta(a \cup x) = \sum_k \theta'_k \overset{\mathcal{A}}{\cdot} a \otimes \theta''_k(x)$ where $\sum_k \theta'_k \otimes \theta''_k$ is the image of θ through the morphism $\widehat{\psi}^*\colon \mathcal{A}_m^{*,*} \to \mathcal{A}_m^{*,*} \otimes_{H^{*,*}} \mathcal{A}_m^{*,*}$ induced by the multiplication (cf [22, Section 2])

$$\widehat{\psi}\colon K(A(j_1), i_1) \wedge K(A(j_2), i_2) \to K(A(j_1 + j_2), i_1 + i_2)$$

of the motivic Eilenberg–MacLane simplicial sheaf defined at the end of Section 4. Here $\theta \overset{\mathcal{A}}{\cdot} a$ means the motivic cohomology operation θ acting on a seen as a cohomology class of a scheme. Although, the $\mathbb{Z}/p$ vector space $H^{*,*}$ does not lie in the center of $\mathcal{A}^{*,*}$, the commutators are sums of terms of monomials of the kind aP^I. Notice that $\mathcal{A}^{*,*}$ is very much not (even graded) commutative: the relations between the products of P^i are called *Adem relations* and are very complicated already in the topological case. For the expression in the algebraic situation see [22, Theorems 10.2,10.3]. It turns out that the classes M_k are $P^{p^{k-1}}P^{p^{k-2}}\cdots P^pP^1$. This fact is needed to prove that $\mathcal{A}_{*,*}$ is a free left $H^{*,*}$ module on the classes $\omega(I)$.

To understand the product of elements in the dual, we need to endow $\mathcal{A}^{*,*}$ of an extra structure: a left $H^{*,*}$ module map $\psi^*\colon \mathcal{A}^{*,*} \to \mathcal{A}^{*,*} \otimes_{H^{*,*}} \mathcal{A}^{*,*}$ called *diagonal* or *comultiplication*. We wish to define ψ^* to be the restriction of $\widehat{\psi}^*$ to $\mathcal{A}^{*,*}$. In order to do this we have to show that the image of $\widehat{\psi}^*$ is contained in $\mathcal{A}^{*,*}$ when the domain is restricted to $\mathcal{A}^{*,*} \otimes_{H^{*,*}} \mathcal{A}^{*,*}$. This can be checked by explicitely computing the value of the the total power operation P on the exterior product of motivic cohomology classes x and y of a simplicial sheaf X and the property that $P(x \wedge y) = \Delta^*(P(x) \wedge P(y))$ (Voevodsky [22, Lemma 5.9]), where $\Delta\colon X \to X \times_k X$ is the diagonal. We include here the complete expression of ψ^* since the one in [22, Proposition 9.7] has a sign error at the prime 2:

Theorem 5.7 *Let u and v be motivic cohomology classes. Then for p odd we have*

$$P^i(u \wedge v) = \sum_{r=0}^{i} P^r(u) \wedge P^{i-r}(v) \tag{58}$$

$$\beta(u \wedge v) = \beta(u) \wedge v + (-1)^{\text{first}\deg(u)} u \wedge \beta v.$$

If $p = 2$ *then*

$$\mathrm{Sq}^{2i}(u\wedge v) = \sum_{r=0}^{i} \mathrm{Sq}^{2r}(u)\wedge \mathrm{Sq}^{2i-2r}(v) + \tau \sum_{s=0}^{i-1} \mathrm{Sq}^{2s+1}(u)\wedge \mathrm{Sq}^{2i-2s-1}(v) \tag{59}$$

$$\begin{aligned}\mathrm{Sq}^{2i+1}(u\wedge v) = \sum_{r=0}^{i}\big(\mathrm{Sq}^{2r+1}(u)\wedge \mathrm{Sq}^{2i-2r}(v) + \mathrm{Sq}^{2r}(u)\wedge \mathrm{Sq}^{2i-2r+1}(v)\big)\\ + \rho \sum_{s=0}^{i-1} \mathrm{Sq}^{2s+1}(u)\wedge \mathrm{Sq}^{2i-2s-1}(v).\end{aligned} \tag{60}$$

As usual, the strange behaviour of the motivic cohomology of $B\mu_2$ at the prime 2 results in more complicated description of ψ^* at the same prime. However, even in this case, ψ^* is associative and commutative. In such a situation we can define a product structure on $\mathcal{A}_{*,*}$ by means of the transposed map of ψ^*: let ψ_* be the map $\mathcal{A}_{*,*} \otimes_{H^{*,*}} \mathcal{A}_{*,*} \to \mathcal{A}_{*,*}$ such that

$$\langle \psi_*(\eta_1 \otimes \eta_2), \theta\rangle = \langle \eta_1 \otimes \eta_2, \psi^*(\theta)\rangle = \sum \eta_1(\theta')\eta_2(\theta'') \tag{61}$$

for any $\theta \in \mathcal{A}^{*,*}$. This makes $\mathcal{A}_{*,*}$ in an associative and graded commutative (with respect to the first grading) $H^{*,*}$ algebra.

Proof of Theorem 5.5 (sketch) Since τ_i's first degrees are odd, we have $\tau_i^2 = 0$ at odd primes because of graded commutativity. At the prime 2 the situation is more complicated. The key trick goes back to Milnor's original paper [9]: we define a morphism $\lambda\colon H^{*,*}(X, \mathbb{Z}/p) \to H^{*,*}(X, \mathbb{Z}/p) \otimes_{H^{*,*}} \mathcal{A}_{*,*}$ as

$$\lambda(x) = \sum_i e_i(x) \otimes e^i \tag{62}$$

where $\{e_i\}$ are a basis of $\mathcal{A}^{*,*}$ over $H^{*,*}$ and e^i the dual basis. The morphism λ does not depend on the choice of e_i, thus we can take $\{e_i\} = \{\beta, P^I\}$ for all indices $I = (i_1, i_2, \ldots)$. Because of Proposition 5.4, we can explicitely compute the morphism λ in the case of $X = B\mu_p$, at least on the classes u and v: $\lambda(u) = u \otimes 1 + \sum_i v^{p^i} \otimes \tau_i$ and $\lambda(v) = v \otimes 1 + \sum_i v^{p^i} \otimes \xi_i$. We now need some equality in which τ_i^2 will appear. We can take such expression to be $\lambda(u)^2$. We are interested to compare it with $\lambda(u^2) = \lambda(\tau v + \rho u)$. The crucial property of λ we use here is multiplicativity with respect to the cup product; than we derive the relation τ_i^2 by setting the homogeneous components of the expressions to be equal. Just like for the case of $\mathcal{A}^{*,*}$, the dual algebra $\mathcal{A}_{*,*}$ has two actions of $H^{*,*}$: the left one $a \otimes \xi \to a \overset{l}{\cdot} \xi$ where $a\xi$ is defined by

$$\langle a \overset{l}{\cdot} \xi, \theta\rangle = \langle \xi, a\theta\rangle = a\langle \xi, \theta\rangle$$

The right action is $\xi \otimes a \to \xi \stackrel{r}{\cdot} a$, with

$$\langle \xi \stackrel{r}{\cdot} a, \theta \rangle = \langle \xi, \theta a \rangle = \langle \xi, \sum_k (\theta'_k \stackrel{\mathcal{A}}{\cdot} a) \otimes \theta''_k \rangle = \sum_k \theta'_k \stackrel{\mathcal{A}}{\cdot} a \langle \xi, \theta''_k \rangle \tag{63}$$

We form the tensor product $\mathcal{A}_{*,*} \otimes_{H^{*,*}} \mathcal{A}_{*,*}$ according with these actions. To complete the picture, we shall compute the diagonal $\phi_* \colon \mathcal{A}_{*,*} \to \mathcal{A}_{*,*} \otimes_{H^{*,*}} \mathcal{A}_{*,*}$ (the left factor is understood to be endowed with the right action of $H^{*,*}$ and the right factor with the left action), defined as the transposed of the multiplication in $\mathcal{A}^{*,*}$: if $\gamma \in \mathcal{A}_{*,*}$ then its action on a product of operations α', α'' is expressed by

$$\langle \gamma, \alpha'\alpha'' \rangle = \sum_i \langle \gamma'_i, \alpha' \rangle \langle \gamma''_i, \alpha'' \rangle$$

for some classes γ' and γ''. We then define $\phi^*(\gamma) = \sum_i \gamma'_i \otimes \gamma''_i$. To compute what $\phi^*(\tau_i)$ and $\phi^*(\xi_i)$ are, we use a strategy already employed: find some equality involving the motivic cohomology classes u and v of $B\mu_p$ containing expressions like $\langle \tau_i, \alpha'\alpha'' \rangle$ and $\langle \gamma'_i, \alpha' \rangle \langle \gamma''_i, \alpha'' \rangle$ as coefficients of certain monomials and then get the result by setting equal the homogeneous components of the expressions. We can write the action of a motivic cohomology operation θ on a class $x \in H^{*,*}(X, \mathbb{Z}/p)$ in such a way to have elements of $\mathcal{A}^{*,*}$ appearing:

$$\theta(x) = \sum_i \langle \xi_i, \theta \rangle M_i(x) + \langle \tau_i, \theta \rangle M_i \beta(x) \tag{64}$$

Once again we take X to be $B\mu_p$ since we know everything about it and prove the equalities

$$\begin{aligned} \theta(u^{p^n}) &= \langle \xi_0, \theta \rangle u^{p^n} + \sum_i \langle \tau_i^{p^n}, \theta \rangle v^{p^{i+n}} \\ \theta(v^{p^n}) &= \sum_i \langle \xi_i^{p^n}, \theta \rangle v^{p^{i+n}} \end{aligned} \tag{65}$$

by means of Proposition 5.4. The expressions we are looking for are $\gamma\theta(u)$ and $\gamma\theta(v)$ for γ, $\theta \in \mathcal{A}^{*,*}$. Each of them can be written in two ways: as $(\gamma\theta)(u)$ and as $\gamma(\theta(u))$. Using equations (65) we obtain two equalities between polynomials in u and v. Equality between their coefficients give the following formulae:

Proposition 5.8

$$\begin{aligned} \phi_*(\tau_k) &= \tau_k \otimes 1 + \sum_i \xi_{k-i}^{p^i} \otimes \tau_i \\ \phi_*(\xi_k) &= \xi_{k-i}^{p^i} \otimes \xi_i \end{aligned} \tag{66}$$

The computation of the diagonal of the dual motivic Steenrod algebra is crucial to find relations between certain special classes Q_i in $\mathcal{A}^{*,*}$ we are now going to define. Let $E = (\epsilon_0, \epsilon_1, \ldots)$ and $R = (r_1, r_2, \ldots)$. and $\tau(E)\xi(R) = \prod_{i\geq 0} \tau_i^{\epsilon_i} \prod_{j\geq 1} \xi_j^{r_j}$.

Definition 5.9 The following notation will be used:

(1) $(r_1, r_2, \ldots, r_n) \in \mathcal{A}^{*,*}$ will denote the dual class to $\xi_1^{r_1}\xi_2^{r_2}\cdots\xi_n^{r_n}$;

(2) if $E = (\epsilon_0, \epsilon_1 \cdots, \epsilon_m)$, where $\epsilon_i \in \{0,1\}$, then Q_E will denote the dual to $\tau_0^{\epsilon_0}\tau_1^{\epsilon_1}\cdots\tau_m^{\epsilon_m}$;

(3) Q_i will be the dual to τ_i.

The most important properties of Q_t^{top}, the topological cohomology operations defined exactly as Q_t, are:

(1) $(Q_t^{\mathrm{top}})^2 = 0$,

(2) $\psi^*(Q_t^{\mathrm{top}}) = Q_t^{\mathrm{top}} \otimes 1 + 1 \otimes Q_t^{\mathrm{top}}$, ie Q_t^{top} is primitive and

(3) If M is a complex manifold with tangent bundle T_M is such that all the characteristic numbers are divisible by p and the integer $\deg(s_{p^n-1}(T_M))$ (see the definition of it given just after Remark 2.5) is not divisible by p^2, then $Q_t^{\mathrm{top}} t_\nu \neq 0$ in the cone of the map $S^{2t} \to \mathrm{Th}(\nu)$, coming from the Thom–Pontryagin construction, where ν is the normal bundle, with complex structure, of some embedding $M \hookrightarrow \mathbb{R}^N$ for N large enough and t_ν is its Thom class in $H^{2m,m}(\mathrm{Th}(\nu), \mathbb{Z}/p)$.

We are interested in the operations $\{Q_t\}_t$ because they appear in the proof of Theorem 2.6, that is crucial for the Voevodsky's program to the Bloch–Kato conjecture, in which property (3) is used. It turns out that the operation Q_t satisfies property (1) (use the coproduct of $\mathcal{A}_{*,*}$), but fails to satisfy property (2) at the prime 2, if $\sqrt{-1} \notin k$ for $t > 0$. Nonetheless, property (3) holds at any prime, and this is what we need for the application in Theorem 2.6. To compute the coproduct $\psi^* Q_t$ we use its adjointness with the multiplication of the dual algebra $\mathcal{A}_{*,*}$. This results in:

Proposition 5.10 (1) *if p is odd, Q_t are primitive;*

(2) *if $p = 2$*

$$\psi^*(Q_t) = Q_t \otimes 1 + 1 \otimes Q_t + \sum_{h=1}^{t} \rho^h \Bigg(\sum_{\substack{I,J \\ I\cup J = \{t-h, t-h+1, \ldots, t-1\} \\ I \cap J = \{t-h\}}} Q_I \otimes Q_J \Bigg) \tag{67}$$

To prove property (3) we use a result of Voevodsky [22, Corollary 14.3]:

Proposition 5.11 *Let X be a scheme and V a rank m vector bundle over X. If t_V is the Thom class of V in $H^{2m,m}(\mathrm{Th}(V), \mathbb{Z}/p)$ then*

$$(0,\ldots,0,\overset{n}{1})(t_V) = s_{p^n-1}(V) \in H^{2m+2(p^n-1),m+p^n-1}(\mathrm{Th}(V), \mathbb{Z}/p)$$

Given this result, to prove property (3) we use the equality

$$Q_t = [Q_0, (0,\ldots,0,\overset{t}{1})] = Q_0(0,\ldots,0,\overset{t}{1}) - (0,\ldots,0,\overset{t}{1})Q_0 \tag{68}$$

which happens to hold at any prime p. The general formulae for the commutators can be found in [4, Corollary 4] and, at the prime 2, differ from their topological counterparts. The degree of Q_0 is $(1,0)$, so $Q_0 t_V = 0$ if X is smooth by the Thom isomorphism and degree considerations. Thus, we are reduced to prove that $Q_t(t_\nu) = Q_0(0,\ldots,0,1)(t_\nu) \neq 0$, where in this algebraic case ν is a "normal" bundle suitably defined for this purpose (see Voevodsky [20] or Borghesi [5]). By Proposition 5.11 and because we know that Q_0 is the Bockstein we are reduced to show that $s_{p^n-1}(\nu)$ is nonzero, (we know that because it is the opposite of the same characteristic number of T_X which has nonzero degree by assumption) and that it is not the reduction modulo p of a class in $H^{2*,*}(\mathrm{Th}(\nu), \mathbb{Z}/p^2)$. This requires a short argument using the assumption on the degree of $s_{p^n-1}(T_X)$.

A natural question to ask is whether $\mathcal{A}_m^{*,*} = \mathcal{A}^{*,*}$, that is if all the bistable motivic cohomology operations are those of the $\mathbb{Z}/p$ vector space $\mathcal{A}^{*,*}$. In the case the characteristic of the base field is zero, this result has been claimed few times, the latest of which is in [21, Lemma 2.2]. In the general case, the canonical inclusion $\mathcal{A}^{*,*} \hookrightarrow \mathcal{A}_m^{*,*}$ makes the latter a graded left $\mathcal{A}^{*,*}$ module. Moreover, when motivic cohomology is representable in $\mathcal{H}(k)$, eg if k is a perfect field, we know that such inclusion is split in the category of graded left $\mathcal{A}^{*,*}$ modules (combine [4, Remark 5.2] with [3]). In fact, $\mathcal{A}_m^{*,*}$ is a graded free left $\mathcal{A}^{*,*}$ module because of [10, Theorem 4.4].

5.3 Final remarks

Corollary 4.26 asserts the representability of motivic cohomology groups in the category $\mathcal{H}(k)$. In this last section we will mention to two more interesting aspects, which we have not planned to cover thoroughly in this manuscript. We have been pretty vague about motivic cohomology since we were just interested in representing it in a suitable category simply as functor with values in abelian groups. However, the reader should know that such cohomology theory is expected to have several properties encoded by the *Beilinson Conjectures*. In particular, one of them states that the motivic cohomology of a smooth scheme X should be isomorphic to the *Zariski* hypercohomology of X with coefficients in some complex of sheaves. In Definition 4.1 motivic cohomology has been defined to the *Nisnevich* hypercohomology of X with coefficients in a complex of sheaves $A(j)$. In general, these groups differ from the ones obtained by taking the Zariski hypercohomology. An important result of Voevodsky in [24] states that, if X is a smooth scheme over a perfect field, and D_* is a complex of Nisnevich sheaves *with transfers* and with homotopy invariant homology sheaves then $\mathbb{H}^*_{\mathrm{Nis}}(X,D) \cong H^*_{\mathrm{Zar}}(X,D)$. The homotopy invariance of the homology sheaves of a complex of Nisnevich sheaves with transfers is strictly related to the concept of the complex being $\mathbb{A}^1_k$ local. In fact, if the base field is perfect, for such complexes the two notions are equivalent. More tricky is the condition for a sheaf to have transfers. These supplementary data on sheaves relates motivic cohomology to algebraic cycles and to the classical theory of motives (see Voevodsky's use of Rost's results on the motif of a Pfister quadric in [20]). These considerations suggest that the sheaves of the complexes $A(j)$ should have transfers and the complexes have homotopy invariant homology sheaves. Now, congruence (41) in particular implies "representability" of motivic cohomology in the category $\mathcal{D}_+(\mathcal{N}_k)$ and Corollary 4.26 in the category $\mathcal{D}_+^{\mathbb{A}^1_k}(\mathcal{N}_k)$. To preserve this property even when switching to the definition of motivic cohomology as Zariski hypercohomology (as opposed to Nisnevich hypercohomology), Voevodsky's theorem indicates we should work just with sheaves with transfers and complexes with homotopy invariant homology sheaves. The category of Nisnevich sheaves of abelian groups with transfers $\mathcal{N}_k^{\mathrm{tr}}$ is an abelian subcategory of $\mathcal{N}_k$ with enough injectives. Thus, we can consider its derived category of bounded below chain complexes. It is nontrivial to show that motivic cohomology is representable in $\mathcal{D}_+(\mathcal{N}_k^{\mathrm{tr}})$. In this case the functor $\mathbb{Z}[X]$ is replaced by an appropriate "free" sheaf with transfers $\mathbb{Z}_{\mathrm{tr}}[X]$. The further restriction to complexes with homotopy invariant homology sheaves is more straightforward as it may be encoded in a localizing functor, making the procedure very similar to the $\mathbb{A}^1_k$ localization of $\mathcal{H}_s(k)$. The outcome is a category denoted by $DM_+(k)$. For the details of these constructions, see Suslin–Voevodsky [17, Theorem 1.5]. Using the terminology introduced in that paper, the sheaf $A(j)$ is then defined to be

$$\mathbb{Z}(1) \overset{L}{\underset{\mathrm{tr}}{\otimes}} \overset{j}{\cdots} \overset{L}{\underset{\mathrm{tr}}{\otimes}} \mathbb{Z}(1) \overset{L}{\underset{\mathrm{tr}}{\otimes}} A$$

where $\mathbb{Z}(1) := C_*(\mathbb{Z}_{\mathrm{tr}}[\mathbb{G}_m])[-1]$ and $C_*(-)$ is the "homotopy invariant homology sheaves" localizing functor. An advantage of the category $DM_+(k)$ over $\mathcal{H}(k)$ is that it is triangulated, whereas the latter it is not although the latter has fibration and cofibration sequences. To overcome this disadvantage there are some procedures to *stabilize* $\mathcal{H}(k)$ and make it triangulated in a way that cofibration sequences become exact triangules. Once again it is possible to prove that representability of motivic cohomology is preserved, always under the assumption of perfectness of the base field (see [5]). This makes the *stable homotopy category of schemes* an effective working place for using homotopy theory on algebraic varieties and consistently exploited in [4] as well as [5].

References

[1] **H Bass**, **J Tate**, *The Milnor ring of a global field*, from: "Algebraic K–theory II: 'Classical' algebraic K–theory and connections with arithmetic (Proc. Conf., Seattle, Wash., Battelle Memorial Inst., 1972)", Lecture Notes in Math. 342, Springer, Berlin (1973) 349–446 MR0442061

[2] **S Bloch**, *Algebraic cycles and higher K–theory*, Adv. in Math. 61 (1986) 267–304 MR852815

[3] **S Borghesi**, *Algebraic Morava K–theory spectra over perfect fields*, preprint (2002) Available at `http://geometria.sns.it/`

[4] **S Borghesi**, *Algebraic Morava K–theories*, Invent. Math. 151 (2003) 381–413 MR1953263

[5] **S Borghesi**, *The degree formulae*, preprint (2004) Available at `http://geometria.sns.it/`

[6] **A K Bousfield**, *Constructions of factorization systems in categories*, J. Pure Appl. Algebra 9 (1976/77) 207–220 MR0478159

[7] **P Brosnan**, *Steenrod operations in Chow theory*, Trans. Amer. Math. Soc. 355 (2003) 1869–1903 MR1953530

[8] **J F Jardine**, *Simplicial presheaves*, J. Pure Appl. Algebra 47 (1987) 35–87 MR906403

[9] **J Milnor**, *The Steenrod algebra and its dual*, Ann. of Math. (2) 67 (1958) 150–171 MR0099653

[10] **J W Milnor**, **J C Moore**, *On the structure of Hopf algebras*, Ann. of Math. (2) 81 (1965) 211–264 MR0174052

[11] **F Morel**, **V Voevodsky**, $\mathbf{A}^1$*–homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. (1999) 45–143 (2001) MR1813224

[12] **D G Quillen**, *Homotopical algebra*, Lecture Notes in Math. 43, Springer, Berlin (1967) MR0223432

[13] **M Rost**, *Construction of splitting varieties*, preprint (1998) Available at `http://www.mathematik.uni-bielefeld.de/~rost/chain-lemma.html`

[14] **M Rost**, *On the spinor norm and $A_0(X, K_1)$ for quadrics*, preprint (1998) Available at `http://www.math.ohio-state.edu/~rost/spinor.html`

[15] **M Rost**, *Norm varieties and algebraic cobordism*, from: "Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)", Higher Ed. Press, Beijing (2002) 77–85 MR1957022

[16] **N E Steenrod**, **D Epstein**, *Cohomology operations*, Annals of Mathematics Studies 50, Princeton University Press, Princeton, N.J. (1962) MR0145525

[17] **A Suslin**, **V Voevodsky**, *Bloch-Kato conjecture and motivic cohomology with finite coefficients*, from: "The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)", NATO Sci. Ser. C Math. Phys. Sci. 548, Kluwer Acad. Publ., Dordrecht (2000) 117–189 MR1744945

[18] **V Voevodsky**, *Motivic cohomology are isomorphic to higher Chow groups*, preprint (1999) Available at `http://www.math.uiuc.edu/K-theory/378/`

[19] **V Voevodsky**, *Lectures on motivic cohomology*, notes by Pierre Deligne (2000–2001) Available at `http://www.math.uiuc.edu/K-theory/0527/`

[20] **V Voevodsky**, *Motivic cohomology with $\mathbb{Z}/2$ coefficients*, Publ. Math. Inst. Hautes Études Sci. (2003) 59–104 MR2031199

[21] **V Voevodsky**, *On motivic cohomology with $\mathbb{Z}/l$ coefficients* (2003)

[22] **V Voevodsky**, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. (2003) 1–57 MR2031198

[23] **V Voevodsky**, **C Mazza**, **C Weibel**, *Lectures on motivic cohomology* Available at `http://www.math.uiuc.edu/K-theory/0486`

[24] **V Voevodsky**, **A Suslin**, **E M Friedlander**, *Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies 143, Princeton University Press, Princeton, NJ (2000) MR1764197

[25] **C A Weibel**, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics 38, Cambridge University Press, Cambridge (1994) MR1269324

Universitá degli Studi di Milano-Bicocca, Dipartimento di Matematica e Applicazioni
via Cozzi 53, 20125 Milano, Italy

`mandu2@libero.it`

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