An inverse problem for strongly degenerate heat equation

Mykola Ivanchov, Nataliya Saldina

Department of Mechanics and Mathematics, Lviv National University, 79002 Lviv, Ukraine E-mail:

ivanchov@franko.lviv.ua

AMS Mathematics Subject Classification 35R30, 35K65

Abstract. In this paper we consider an inverse problem for determining time - dependent heat conduction coefficient which vanishes at initial moment as a power t^β. The case of strong degeneration ($\beta \geq 1$) is studied. To prove the existence of solution we employ the Schauder fixed point theorem. The uniqueness of the solution is established too.

1. Introduction

Degenerate parabolic problems arise in a lot of fields of natural and social sciences and technology (see, e.g., [1-5]). These problems may be divided on different classes accordingly to the way of degeneration with respect either to spatial variables or to time variable, weak or strong degeneration. Direct problems for degenerate parabolic equations are sufficiently well studied. As examples, we can mention the works [6-12]. On the other hand, inverse problems for non-degenerate parabolic equations are no less investigated [13-17]. However, inverse problems for degenerate partial differential equations are almost not considered. There are some works [18-20] dedicated to inverse problems for partial differential equations degenerating with respect to a spatial variable.

In this paper we consider an inverse problem for the heat equation with unknown heat conduction coefficient depending on time variable t. It is supposed that the unknown coefficient vanishes at the initial moment as a power t^β. The case of weak degeneration ($\beta < 1$) was studied in [21]. Here we investigate the case of strong degeneration ($\beta \geq 1$).

In a domain $Q_T \equiv \{(x,t) : 0 < x < h, 0 < t < T\}$ we consider the following heat equation

$$u_t = a(t)u_{xx} + f(x,t) \tag{1}$$

with unknown coefficient $a(t) > 0, t \in (0,T]$, initial condition

$$u(x,0) = \varphi(x), \quad x \in [0,h], \tag{2}$$

boundary conditions

$$u(0,t) = \mu_1(t), \quad u(h,t) = \mu_2(t), \quad t \in [0,T], \tag{3}$$

and overdetermination condition

$$a(t)u_x(0,t) = \mu_3(t), \quad t \in [0,T]. \tag{4}$$

In this problem both $u(x,t)$ and the coefficient $a(t)$ are unknown, are to be determined from data $f(x,t), \varphi(x), \mu_1(t), \mu_2(t), \mu_3(t)$. As a solution of the problem (1)-(4) we mean a classical solution which is defined as follows.

Definition. *The pair of functions $a(t)$ and $u(x,t)$ is a solution of (1)-(4) if the following conditions are fulfilled:*

(a) $(a,u) \in C[0,T] \times C^{2,1}(Q_T) \cap C(\overline{Q}_T), u_x(0,t) \in C(0,T]$;

(b) *$a(t)$ is positive for $t \in (0,T]$;*

(c) *there exists the limit* $\lim\limits_{t \to +0} \dfrac{a(t)}{t^\beta} > 0$, $\beta \geq 1$ *- a given number;*

(d) *(1)-(4) are satisfied.*

Note that the analogous problem for a non-degenerate heat equation was for the first time studied in [22].

We establish the conditions of existence and uniqueness of solution for the problem (1)-(4) which are formulated in the following theorem.

Theorem. *Suppose that the following conditions hold:*

1) $\varphi \in C^2[0,h]; \mu_i \in C^1[0,T], i=1,2; \mu_3 \in C[0,T]; f \in C^{2,0}(\overline{Q}_T);$

2) $\varphi'(x) \geq 0, x \in [0,h]; f(0,t)-\mu_1'(t) > 0, \mu_2'(t)-f(h,t) \geq 0, t \in [0,T]; \mu_3(t) > 0, t \in (0,T]$, *the limit* $\lim\limits_{t\to+0} \frac{\mu_3(t)}{t^{\frac{\beta+1}{2}}} > 0$ *exists;* $f_x(x,t) \geq 0, (x,t) \in \overline{Q}_T;$

3) $\varphi(0)=\mu_1(0), \varphi(h)=\mu_2(0).$

Then there exists an unique solution of problem (1)-(4).

To prove the existence of solution, the Schauder fixed point theorem is applied. The proof of the uniqueness of solution is divided in two parts: first we establish it for a small time interval and after this we prove a global (in time) uniqueness of solution of the problem (1)-(4).

The part of the paper that follows is composed of four sections. In Section 2 the inverse problem (1)-(4) is reduced to an integral equation with respect to unknown coefficient $a(t)$. In Section 3 we study the behavior of $u_x(0,t)$ as $t \to 0$ and we show that under the assumptions of the theorem the solution of the integral equation is bounded from below and above by a power t^β with coefficients which depend on given data. In Section 4 we apply the Schauder fixed point theorem to the integral equation and we complete the proof of the existence of solution of the problem (1)-(4). In the first part of Section 5 we show that the integral equation with respect to unknown coefficient $a(t)$ admits at most one solution on the interval $[0,\tilde{t}]$ where $\tilde{t} > 0$ is, in general, a small number defined by given data. Then using this statement, we prove the uniqueness of solution of the problem (1)-(4) in whole.

2. Reduction of the problem (1)-(4) to an integral equation

Apply the overdetermination condition (4) to obtain an equation for the function $a(t)$. Denote by $G_k(x,t,\xi,\tau), k=1,2$, the Green functions of the first ($k=1$) and the second ($k=2$) boundary value problems for equation (1)

$$G_k(x,t,\xi,\tau) = \frac{1}{2\sqrt{\pi(\theta(t)-\theta(\tau))}} \sum_{n=-\infty}^{\infty} \left(\exp\left(-\frac{(x-\xi+2nh)^2}{4(\theta(t)-\theta(\tau))} \right) + (-1)^k \exp\left(-\frac{(x+\xi+2nh)^2}{4(\theta(t)-\theta(\tau))} \right) \right), \tag{5}$$

where $\theta(t) = \int\limits_0^t a(\tau)d\tau$. Temporarily assuming that the function $a(t)$ is known, we can write the solution of direct problem (1)-(3) with the aid of the Green function

$$u(x,t) = \int\limits_0^h G_1(x,t,\xi,0)\varphi(\xi)d\xi + \int\limits_0^t G_{1\xi}(x,t,0,\tau)a(\tau)\mu_1(\tau)d\tau - \int\limits_0^t G_{1\xi}(x,t,h,\tau)a(\tau)\mu_2(\tau)d\tau + \int\limits_0^t \int\limits_0^h G_1(x,t,\xi,\tau)f(\xi,\tau)d\xi d\tau. \tag{6}$$

Evaluate the first derivative $u_x(x,t)$, taking into account the relationships

$$G_{1x}(x,t,\xi,\tau) = -G_{2\xi}(x,t,\xi,\tau) \quad \text{and} \quad G_{2\xi\xi} = -\frac{G_{2\tau}(x,t,\xi,\tau)}{a(\tau)}, \tag{7}$$

and integrating by parts with using compatibility condition. We obtain

$$u_x(x,t)=\int\limits_0^h G_2(x,t,\xi,0)\varphi'(\xi)d\xi+\int\limits_0^t G_2(x,t,0,\tau)(f(0,\tau)-\mu_1'(\tau))d\tau+$$
$$+\int\limits_0^t G_2(x,t,h,\tau)(\mu_2'(\tau)-f(h,\tau))d\tau+\int\limits_0^t\int\limits_0^h G_2(x,t,\xi,\tau)f_\xi(\xi,\tau)d\xi d\tau. \tag{8}$$

We substitute this expression into overdetermination condition (4) and we come to the equation for $a(t)$:

$$a(t)=\mu_3(t)\Big(\int\limits_0^h G_2(0,t,\xi,0)\varphi'(\xi)d\xi+\int\limits_0^t G_2(0,t,0,\tau)(f(0,\tau)-\mu_1'(\tau))d\tau$$
$$+\int\limits_0^t G_2(0,t,h,\tau)(\mu_2'(\tau)-f(h,\tau))d\tau+\int\limits_0^t\int\limits_0^h G_2(0,t,\xi,\tau)f_\xi(\xi,\tau)d\xi d\tau\Big)^{-1}, t\in[0,T]. \tag{9}$$

Taking into account the conditions of the theorem it is easy to verify that the function $a(t)$ is positive on $(0,T]$ and belongs to $C(0,T]$.

3. A priori estimates

In order to prove the existence of the solution of equation (9) we apply the Schauder fixed point theorem. First of all, we estimate the solution of equation (9). As a consequence of the second condition of the theorem and the explicit representation of the function $G_2(x,t,\xi,\tau)$ we obtain the inequality

$$u_x(0,t)\ge\int\limits_0^t\frac{f(0,\tau)-\mu_1'(\tau)}{\sqrt{\pi(\theta(t)-\theta(\tau))}}d\tau. \tag{10}$$

This allows us to write the following inequality for the function $a(t)$

$$a(t)\le\frac{\mu_3(t)}{\int\limits_0^t\frac{f(0,\tau)-\mu_1'(\tau)}{\sqrt{\pi(\theta(t)-\theta(\tau))}}d\tau}. \tag{11}$$

Let

$$a_0(t)\equiv\frac{a(t)}{t^\beta},\quad a_{\max}(t)\equiv\max_{0\le\tau\le t}a_0(\tau). \tag{12}$$

Then from (11) we find

$$a_0(t)\le\frac{\sqrt{\pi}\mu_3(t)}{\sqrt{\beta+1}t^\beta\int\limits_0^t\frac{f(0,\tau)-\mu_1'(\tau)}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}}d\tau}\sqrt{a_{\max}(t)}.$$

Denote

$$H(t)\equiv\frac{\sqrt{\pi}\mu_3(t)}{\sqrt{\beta+1}t^\beta\int\limits_0^t\frac{f(0,\tau)-\mu_1'(\tau)}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}}d\tau}. \tag{13}$$

It follows from the conditions of the theorem that the function $H(t)$ is positive on $(0,T]$ and belongs to $C(0,T]$. Establish the existence of the limit $\lim\limits_{t\to+0} H(t)$. To this end, we apply the theorem on average and change of variables $z=\dfrac{\tau}{t}$:

$$\lim_{t\to+0} H(t) = \lim_{t\to+0} \frac{\sqrt{\pi}\mu_3(t)}{\sqrt{\beta+1}t^{\beta}(f(0,\bar{t})-\mu_1'(\bar{t}))\int\limits_0^t \dfrac{d\tau}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}}} =$$

$$= \sqrt{\frac{\pi}{\beta+1}} \lim_{t\to+0} \frac{\mu_3(t)}{t^{(\beta+1)/2}(f(0,\bar{t})-\mu_1'(\bar{t}))\int\limits_0^1 \dfrac{dz}{\sqrt{1-z^{\beta+1}}}}, \quad \bar{t}\in[0,t].$$

Denote

$$\int\limits_0^1 \frac{dz}{\sqrt{1-z^{\beta+1}}} = I_1, \tag{14}$$

then we obtain $\lim\limits_{t\to+0} H(t) = \dfrac{\sqrt{\pi}M}{\sqrt{\beta+1}(f(0,0)-\mu_1'(0))I_1} > 0$, where $M = \lim\limits_{t\to+0} \dfrac{\mu_3(t)}{t^{\frac{\beta+1}{2}}}$. Applying (13) we come to the inequality $a_0(t) \le H(t)\sqrt{a_{\max}(t)}$, which leads to an estimation of $a_{\max}(t)$ from above

$$a_{\max}(t) \le H_{\max}^2(t) < \infty, \quad t\in[0,T], \tag{15}$$

where $H_{\max}(t) \equiv \max\limits_{0\le\tau\le t} H(\tau)$. Taking into account the existence of the limit $\lim\limits_{t\to+0} \dfrac{\mu_3(t)}{t^{\frac{\beta+1}{2}}}$ and notation (14), we estimate $H(t)$

$$H(t) \le \frac{\sqrt{\pi}M_1}{\sqrt{\beta+1}\min\limits_{[0,T]}(f(0,t)-\mu_1'(t))I_1} \equiv H_1, \quad \text{where} \quad M_1 = \max_{[0,T]} \frac{\mu_3(t)}{t^{\frac{\beta+1}{2}}}. \tag{16}$$

Then the estimate of $a(t)$ from above follows:

$$a(t) \le H_{\max}^2(t)t^{\beta} \le H_1^2 t^{\beta}, \quad t\in[0,T]. \tag{17}$$

Estimate $a(t)$ from below. To this end, provide some estimates for expression in the denominator of (8). From the equality $\int\limits_0^h G_2(x,t,\xi,\tau)d\xi = 1$ the first and the fourth summands are estimated

$$\int\limits_0^h G_2(0,t,\xi,\tau)\varphi'(\xi)d\xi \le C_1, \quad \int\limits_0^t\int\limits_0^h G_2(0,t,\xi,\tau)f_{\xi}(\xi,\tau)d\xi d\tau \le C_2.$$

It can be easily verified that

$$\int\limits_0^t G_2(0,t,h,\tau)(\mu_2'(\tau)-f(h,\tau))d\tau \le C_3,$$

where $C_1, C_2, C_3 > 0$ — the constants determined by the problem data. Transform the second summand, separating out of the series the term that corresponds to $n=0$:

$$\int\limits_0^t G_2(0,t,0,\tau)(f(0,\tau)-\mu_1'(\tau))d\tau = \frac{1}{\sqrt{\pi}}\int\limits_0^t \frac{f(0,\tau)-\mu_1'(\tau)}{\sqrt{\theta(t)-\theta(\tau)}}d\tau$$

$$+\frac{2}{\sqrt{\pi}}\int\limits_0^t \frac{f(0,\tau)-\mu_1'(\tau)}{\sqrt{\theta(t)-\theta(\tau)}}\sum_{n=1}^{\infty}\exp\left(-\frac{n^2h^2}{\theta(t)-\theta(\tau)}\right)d\tau.$$

Since the integral function of the last summand has no singularities and is bounded, we estimate it by a constant. Denote $a_{\min}(t) \equiv \min\limits_{0\le\tau\le t} a_0(\tau)$. By formula (8) and previous estimates we obtain

$$a_0(t) \ge \frac{\mu_3(t)}{t^\beta\left(C_4 + \dfrac{\sqrt{\beta+1}}{\sqrt{\pi a_{\min}(t)}}\displaystyle\int\limits_0^t \frac{f(0,\tau)-\mu_1'(\tau)}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}}d\tau\right)}$$

$$\ge \frac{\sqrt{a_{\min}(t)}}{\dfrac{C_5 t^\beta}{\mu_3(t)} + \dfrac{t^\beta\sqrt{\beta+1}}{\sqrt{\pi}\mu_3(t)}\displaystyle\int\limits_0^t \frac{f(0,\tau)-\mu_1'(\tau)}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}}d\tau}.$$

Taking into account (13), reduce the inequality to the form

$$a_0(t) \ge \frac{\sqrt{a_{\min}(t)}}{\dfrac{C_5 t^\beta}{\mu_3(t)} + \dfrac{1}{H(t)}} = \frac{\sqrt{a_{\min}(t)}H(t)}{\dfrac{C_5 t^\beta H(t)}{\mu_3(t)} + 1}.$$

From (16) it follows

$$\frac{C_5 t^\beta H(t)}{\mu_3(t)} \le C_6 t^{\frac{\beta-1}{2}}.$$

Hence, we obtain

$$a_0(t) \ge \frac{\sqrt{a_{\min}(t)}H(t)}{C_6 t^{\frac{\beta-1}{2}} + 1}.$$

From here we establish the estimate

$$a_{\min}(t) \ge \frac{H^2_{\min}(t)}{(C_6 t^{\frac{\beta-1}{2}} + 1)^2}, \tag{18}$$

where $H_{\min}(t) \equiv \min\limits_{0\le\tau\le t} H(\tau)$. Finally, we have for $a(t)$:

$$0 < A_0 \le \frac{H^2_{\min}(t)}{(C_6 t^{\frac{\beta-1}{2}} + 1)^2} \le \frac{a(t)}{t^\beta} \le H^2_{\max}(t) \le A_1 < \infty, \quad t\in[0,T]. \tag{19}$$

Therefore, we have established a priori estimates for the solution (9). Having a priori estimates of solution of equation (9), we can apply to it the Schauder fixed point theorem.

4. Existence of solution

We consider the equation (9) as an operator equation $a(t) = Pa(t)$ with respect to $a(t)$ and the operator P is defined by equality $Pa(t) = \dfrac{\mu_3(t)}{u_x(0,t)}$. Denote $\mathcal{N} = \{a \in C[0,T] : A_0 \le \dfrac{a(t)}{t^\beta} \le A_1\}$. As a consequence of a priori estimates (19) the operator P maps $\mathcal{N}$ into $\mathcal{N}$. We are going to show that the set $P\mathcal{N}$ is compact or equivalently, by Arzela-Ascolli theorem, $P\mathcal{N}$ is uniformly bounded and equicontinuous. We have to establish that $\forall\varepsilon > 0\ \exists\ \delta > 0$ such that

$$|Pa(t_2) - Pa(t_1)| < \varepsilon \quad \text{for arbitrary} \quad |t_2 - t_1| < \delta, \quad a(t)\in\mathcal{N}. \tag{20}$$

As $Pa(t) = a(t)$ and $\dfrac{a(t)}{t^\beta} \le A_1$ for all $a(t)\in\mathcal{N}$ we conclude that for arbitrary $\varepsilon > 0$ there exists sufficiently small number $t^* > 0$, such that the inequality

$$|Pa(t)| < \varepsilon, \quad 0 \le t \le t^*,$$

holds.

Establish the inequality (20) in the case when $t_i > t^*, i = 1, 2$. Assume $t_2 > t_1$. Consider one of the summand which is contained in (20):

$$R_1 = \left| \int_0^{t_2} G_2(0, t_2, 0, \tau)(f(0, \tau) - \mu_1'(\tau))d\tau - \int_0^{t_1} G_2(0, t_1, 0, \tau)(f(0, \tau) - \mu_1'(\tau))d\tau \right|$$

$$\leq \left| \int_0^{t_1} (G_2(0, t_2, 0, \tau) - G_2(0, t_1, 0, \tau))(f(0, \tau) - \mu_1'(\tau))d\tau \right|$$

$$+ \left| \int_{t_1}^{t_2} G_2(0, t_2, 0, \tau)(f(0, \tau) - \mu_1'(\tau))d\tau \right| \equiv R_{1,1} + R_{1,2}.$$

The estimate of $G_2(0, t, 0, \tau)$ [17] allows us to write

$$R_{1,2} \leq \frac{\max\limits_{[0,T]}(f(0, t) - \mu_1'(t))}{\sqrt{\pi}} \int_{t_1}^{t_2} \left(\frac{1}{\sqrt{\theta(t_2) - \theta(\tau)}} + C_7 \right) d\tau$$

$$\leq C_8 \int_{t_1}^{t_2} \frac{d\tau}{\sqrt{\theta(t_2) - \theta(\tau)}} + C_9(t_2 - t_1).$$

Apply the definition of the set $\mathcal{N}$ for investigation of the first summand in $R_{1,2}$:

$$\int_{t_1}^{t_2} \frac{d\tau}{\sqrt{\theta(t_2) - \theta(\tau)}} \leq \sqrt{\frac{\beta + 1}{A_0}} \int_{t_1}^{t_2} \frac{d\tau}{\sqrt{t_2^{\beta+1} - \tau^{\beta+1}}} \leq \sqrt{\frac{\beta + 1}{A_0 t_2^{\beta}}} \int_{t_1}^{t_2} \frac{d\tau}{\sqrt{t_2 - \tau}} \leq C_{10}\sqrt{t_2 - t_1}.$$

Finally, we have

$$R_{1,2} \leq C_{11}\sqrt{t_2 - t_1} + C_9(t_2 - t_1).$$

From $R_{1,1}$, using the Green function representation and separating out of the series the term that corresponds to $n = 0$, we have

$$R_{1,1} \leq \frac{1}{\sqrt{\pi}} \max_{[0,T]}(f(0, t) - \mu_1'(t)) \left(\int_0^{t_1} \left| \frac{1}{\sqrt{\theta(t_2) - \theta(\tau)}} - \frac{1}{\sqrt{\theta(t_1) - \theta(\tau)}} \right| d\tau \right.$$

$$+ 2 \int_0^{t_1} \left| \frac{1}{\sqrt{\theta(t_2) - \theta(\tau)}} \sum_{n=1}^{\infty} \exp\left(-\frac{n^2 h^2}{\theta(t_2) - \theta(\tau)} \right) \right.$$

$$\left. \left. - \frac{1}{\sqrt{\theta(t_1) - \theta(\tau)}} \sum_{n=1}^{\infty} \exp\left(-\frac{n^2 h^2}{\theta(t_1) - \theta(\tau)} \right) \right| d\tau \right) \equiv R_{1,1,1} + R_{1,1,2}.$$

Note that

$$\frac{1}{\sqrt{\theta(t_2) - \theta(\tau)}} - \frac{1}{\sqrt{\theta(t_1) - \theta(\tau)}} =$$

$$= \frac{\theta(t_1) - \theta(t_2)}{\sqrt{(\theta(t_2) - \theta(\tau))(\theta(t_1) - \theta(\tau))}(\sqrt{\theta(t_1) - \theta(\tau)} + \sqrt{\theta(t_2) - \theta(\tau)})}.$$

Then we employ the definition of the set $\mathcal{N}$:

$$R_{1,1,1} \leq \frac{1}{\sqrt{\pi}} \max_{[0,T]}(f(0, t) - \mu_1'(t)) \frac{A_1}{2A_0^{3/2}} \int_0^{t_1} \left(\frac{1}{\sqrt{t_1^{\beta+1} - \tau^{\beta+1}}} - \frac{1}{\sqrt{t_2^{\beta+1} - \tau^{\beta+1}}} \right) d\tau.$$

Transform this inequality by the change of variable $z = \dfrac{\tau}{t_1}$:

$$R_{1,1,1} \le \frac{C_{12}}{t_1^{\frac{\beta-1}{2}}} \left(\int_0^1 \frac{dz}{\sqrt{1-z^{\beta+1}}} - \int_0^1 \frac{dz}{\sqrt{(t_2/t_1)^{\beta+1} - z^{\beta+1}}} \right).$$

Denote

$$I(\omega) \equiv \int_0^1 \frac{dz}{\sqrt{\omega - z^{\beta+1}}}, \quad \text{where} \quad \omega \in [1,\infty). \tag{21}$$

Obviously $\int_0^1 \frac{dz}{\sqrt{\omega - z^{\beta+1}}} \le \int_0^1 \frac{dz}{\sqrt{1-z}}$. It follows that $I(\omega)$ is continuous on $[1,\infty)$. This means that for arbitrary $\varepsilon > 0$ there exists $\delta_1 > 0$, such that $R_{1,1,1} < \varepsilon$ for $|t_2 - t_1| < \delta_1$.

We represent $R_{1,1,2}$ in the form:

$$R_{1,1,2} = \frac{2}{\sqrt{\pi}} \max_{[0,T]} (f(0,t) - \mu_1'(t)) \int_0^{t_1} \left| \int_{\theta(t_1)-\theta(\tau)}^{\theta(t_2)-\theta(\tau)} \frac{d}{dz} \left(\frac{1}{\sqrt{z}} \sum_{n=1}^{\infty} \exp\left(-\frac{n^2h^2}{z}\right)\right) dz \right| d\tau.$$

Estimating $R_{1,1,2}$ by means of inequality $x^n e^{-x^2} \le M_n < \infty, x \in [0,\infty), n \in N$, we come to the estimate

$$R_{1,1,2} \le C_{13} |\theta(t_2) - \theta(t_1)| \le C_{14} |t_2^{\beta+1} - t_1^{\beta+1}| < \varepsilon \quad \text{for} \quad |t_2 - t_1| < \delta_2,$$

where $\delta_2 > 0$ depends on ε and known data. Others summands occuring in (20) can be estimated analogously. It follows that the conditions of the Schauder theorem for equation (9) hold, and, therefore there exists the solution $a = a(t)$ of the equation (9), which belongs to $C[0,T]$.

Substituting $a(t)$ in (6), we obtain a solution $u = u(x,t)$ of direct problem (1)-(3) which possesses necessary smoothness.

5. Uniqueness of solution

To prove the uniqueness of the solution of the problem (1)-(4), suppose that $(a_i(t), u_i(x,t)), i = 1,2$ are two solutions of the problem (1)-(4). Denote $a(t) \equiv a_1(t) - a_2(t)$, $u(x,t) \equiv u_1(x,t) - u_2(x,t)$. From the overdetermination condition (4) we obtain

$$a(t) = \frac{\mu_3(t)(u_{2x}(0,t) - u_{1x}(0,t))}{u_{2x}(0,t) u_{1x}(0,t)}. \tag{22}$$

or, after using the notation (12),

$$a_0(t) = \frac{\mu_3(t)}{t^\beta u_{1x}(0,t)} \frac{\mu_3(t)}{t^\beta u_{2x}(0,t)} \frac{t^\beta (u_{2x}(0,t) - u_{1x}(0,t))}{\mu_3(t)}.$$

Next, we apply the equality $\dfrac{\mu_3(t)}{t^\beta u_{ix}(0,t)} = \dfrac{a_i(t)}{t^\beta}$, $\quad i = 1,2$, and the estimate (15):

$$|a_0(t)| \le H_{\max}^4(t) \frac{t^\beta |u_{2x}(0,t) - u_{1x}(0,t)|}{\mu_3(t)}. \tag{23}$$

Estimate one of the summands which is contained in the expression $|u_{2x}(0,t) - u_{1x}(0,t)|$. Denote

$$I_1 \equiv \frac{1}{\sqrt{\pi}} \int_0^t (f(0,\tau) - \mu_1'(\tau)) \left(\frac{1}{\sqrt{\theta_2(t) - \theta_2(\tau)}} - \frac{1}{\sqrt{\theta_1(t) - \theta_1(\tau)}} \right) d\tau$$
$$+ \frac{2}{\sqrt{\pi}} \int_0^t (f(0,\tau) - \mu_1'(\tau)) \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{\theta_2(t) - \theta_2(\tau)}} \exp\left(-\frac{n^2h^2}{\theta_2(t) - \theta_2(\tau)}\right) \right.$$
$$\left. - \frac{1}{\sqrt{\theta_1(t) - \theta_1(\tau)}} \exp\left(-\frac{n^2h^2}{\theta_1(t) - \theta_1(\tau)}\right)\right) d\tau \equiv I_{1,1} + I_{1,2}.$$

Represent the second summand in the form:

$$|I_{1,2}| \le \frac{2}{\sqrt{\pi}} \max_{[0,T]}(f(0,t) - \mu_1'(t)) \left| \int\limits_0^t d\tau \int\limits_{\theta_1(t)-\theta_1(\tau)}^{\theta_2(t)-\theta_2(\tau)} \left| \frac{d}{dz} \left(\frac{1}{\sqrt{z}} \sum_{n=1}^{\infty} \exp\left(-\frac{n^2h^2}{z}\right)\right) \right| dz \right|.$$

From boundedness of the integrand we conclude

$$|I_{1,2}| \le C_{15} \int\limits_0^t |\theta_2(t) - \theta_2(\tau) - \theta_1(t) + \theta_1(\tau)| d\tau = C_{15} \int\limits_0^t d\tau \int\limits_\tau^t |a_2(\sigma) - a_1(\sigma)| d\sigma =$$

$$= C_{15} \int\limits_0^t d\tau \int\limits_\tau^t |a_0(\sigma)| \sigma^\beta d\sigma \le C_{15} t^{\beta+2} \tilde{a}_{\max}(t),$$

where $\tilde{a}_{\max}(t) \equiv \max\limits_{0 \le \tau \le t} |a_0(\tau)|$.

Transform $I_{1,1}$ to the form

$$I_{1,1} = \frac{1}{\sqrt{\pi}} \int\limits_0^t \frac{(f(0,\tau) - \mu_1'(\tau))(\theta_1(t) - \theta_1(\tau) - \theta_2(t) + \theta_2(\tau))}{\sqrt{(\theta_2(t) - \theta_2(\tau))(\theta_1(t) - \theta_1(\tau))}(\sqrt{\theta_1(t) - \theta_1(\tau)} + \sqrt{\theta_2(t) - \theta_2(\tau)})} d\tau.$$

Taking into account (19), we obtain

$$|\theta_1(t) - \theta_1(\tau) - \theta_2(t) + \theta_2(\tau)| \le \int\limits_\tau^t |a_0(\sigma)| \sigma^\beta d\sigma \le \frac{t^{\beta+1} - \tau^{\beta+1}}{\beta+1} \tilde{a}_{\max}(t),$$

$$\theta_i(t) - \theta_i(\tau) = \int\limits_\tau^t a_{i0}(\sigma) \sigma^\beta d\sigma \ge \frac{H^2_{\min}(t)}{(C_6 t^{\frac{\beta-1}{2}} + 1)^2} \frac{t^{\beta+1} - \tau^{\beta+1}}{\beta+1}, \quad i = 1, 2,$$

where $a_{i0} = \dfrac{a_i(t)}{t^\beta}, i = 1, 2$.

Finally, we have the estimate of $I_{1,1}$

$$|I_{1,1}| \le \frac{\sqrt{\beta+1}(C_6 t^{\frac{\beta-1}{2}} + 1)^3}{2\sqrt{\pi} H^3_{\min}(t)} \tilde{a}_{\max}(t) \int\limits_0^t \frac{f(0,\tau) - \mu_1'(\tau)}{\sqrt{t^{\beta+1} - \tau^{\beta+1}}} d\tau,$$

or if we use (13)

$$|I_{1,1}| \le \frac{(C_6 t^{\frac{\beta-1}{2}} + 1)^3}{2H^4_{\min}(t)} \frac{\mu_3(t)}{t^\beta} \tilde{a}_{\max}(t).$$

Other summands in the expression $|u_{2x}(0,t) - u_{1x}(0,t)|$ are estimated analogously to $I_{1,2}$. We continue to estimate (23) as follows:

$$\tilde{a}_{\max}(t) \le \frac{(C_6 t^{\frac{\beta-1}{2}} + 1)^3 H^4_{\max}(t)}{2H^4_{\min}(t)} \tilde{a}_{\max}(t) + S(t) \tilde{a}_{\max}(t), \tag{24}$$

where $S(t)$ is the sum of terms depending on t which vanishes for $t = 0$. The existence of the limit $\lim\limits_{t \to +0} H(t) > 0$ implies the existence of the limit $\lim\limits_{t \to +0} \dfrac{(C_6 t^{\frac{\beta-1}{2}} + 1)^3 H^4_{\max}(t)}{2H^4_{\min}(t)} = \dfrac{1}{2}$. Therefore, there exists $t_1 : 0 < t_1 \le T$ such that the following inequality holds:

$$\frac{(C_6 t^{\frac{\beta-1}{2}} + 1)^3 H^4_{\max}(t)}{2H^4_{\min}(t)} \le \frac{3}{4}, \quad t \in [0, t_1]. \tag{25}$$

Hence, we can rewrite (24) in the form:

$$\tilde{a}_{\max}(t)\left(\frac{1}{4}-S(t)\right)\leq 0,\quad t\in[0,t_1].$$

There exists $t_2 : 0 < t_2 \leq T$, such that $\frac{1}{4} - S(t) > 0$ for any $t \in [0, t_2]$. We come to the contradiction. This implies that $a_1(t) \equiv a_2(t)$ when $t \in [0, \tilde{t}]$ with $\tilde{t} = \min(t_1, t_2)$.

Now we establish the uniqueness of the solution problem (1)-(4) for any $t \in [0, T]$. The functions $(a(t), u(x, t))$ satisfy the conditions

$$u_t = a_1(t)u_{xx} + a(t)u_{2xx},\quad (x,t)\in Q_T, \tag{26}$$

$$u(x,0) = 0,\quad x\in[0,h], \tag{27}$$

$$u(0,t) = u(h,t) = 0,\quad t\in[0,T], \tag{28}$$

$$a_1(t)u_x(0,t) = -a(t)u_{2x}(0,t),\quad t\in[0,T]. \tag{29}$$

Denote by $G_1^{(i)}(x, t, \xi, \tau), i = 1, 2$, the Green functions of the equations $u_t = a_i(t)u_{xx}$ with the boundary condition (28). The solution of the (26)-(28) can be written with the aid of $G_1^{(1)}(x, t, \xi, \tau)$

$$u(x,t) = \int_0^t\int_0^h G_1^{(1)}(x,t,\xi,\tau)a(\tau)u_{2\xi\xi}(\xi,\tau)d\xi d\tau. \tag{30}$$

Substituting (30) into (29), we obtain the integral equation for $a(t)$

$$a(t)u_{2x}(0,t) = -a_1(t)\int_0^t\int_0^h G_{1x}^{(1)}(0,t,\xi,\tau)a(\tau)u_{2\xi\xi}(\xi,\tau)d\xi d\tau,\quad t\in[0,T]. \tag{31}$$

Using the notation (12), it is easy to see that

$$a_0(t) = -\frac{a_1(t)}{t^\beta u_{2x}(0,t)}\int_0^t\int_0^h G_{1x}^{(1)}(0,t,\xi,\tau)a_0(\tau)\tau^\beta u_{2\xi\xi}(\xi,\tau)d\xi d\tau, \tag{32}$$

or

$$a_0(t) = \int_0^t K(t,\tau)a_0(\tau)d\tau,\quad t\in[0,T], \tag{33}$$

where

$$K(t,\tau) \equiv -\frac{a_1(t)\tau^\beta}{t^\beta u_{2x}(0,t)}\int_0^h G_{1x}^{(1)}(0,t,\xi,\tau)u_{2\xi\xi}(\xi,\tau)d\xi. \tag{34}$$

Establish that the kernel $K(t, \tau)$ is integrable, using the fact that $u_2(x, t)$ is the solution of the problem (1)-(4). From (6) we find the derivative

$$u_{2xx}(x,t) = \int_0^h G_1^{(2)}(x,t,\xi,0)\varphi''(\xi)d\xi + \int_0^t G_{1\xi}^{(2)}(x,t,0,\tau)(\mu_1'(\tau)-f(0,\tau))d\tau$$

$$+\int_0^t G_{1\xi}^{(2)}(x,t,h,\tau)(f(h,\tau)-\mu_2'(\tau))d\tau + \int_0^t\int_0^h G_1^{(2)}(x,t,\xi,\tau)f_{\xi\xi}(\xi,\tau)d\xi d\tau. \tag{35}$$

Substituting (35) into (34), we obtain:

$$K(t,\tau)=-\frac{a_1(t)\tau^\beta}{t^\beta u_{2x}(0,t)}\int\limits_0^h G_{1x}^{(1)}(0,t,\xi,\tau)\left(\int\limits_0^h G_1^{(2)}(\xi,\tau,\eta,0)\varphi''(\eta)d\eta\right.$$

$$+\int\limits_0^\tau G_{1\eta}^{(2)}(\xi,\tau,0,\sigma)(\mu_1'(\sigma)-f(0,\sigma))d\sigma+\int\limits_0^\tau G_{1\eta}^{(2)}(\xi,\tau,h,\sigma)(f(h,\sigma)-\mu_2'(\sigma))d\sigma$$

$$\left.+\int\limits_0^\tau\int\limits_0^h G_1^{(2)}(\xi,\tau,\eta,\sigma)f_{\eta\eta}(\eta,\sigma)d\eta d\sigma\right)d\xi\equiv-\frac{a_1(t)\tau^\beta}{t^\beta u_{2x}(0,t)}\sum_{i=1}^4 K_i(t,\tau).$$

Consider the summand $K_2(t,\tau)$, using the explicit representation of the Green functions

$$K_2(t,\tau)=\frac{1}{4\pi}\int\limits_0^h\int\limits_0^\tau\frac{\mu_1'(\sigma)-f(0,\sigma)}{((\theta_1(t)-\theta_1(\tau))(\theta_2(\tau)-\theta_2(\sigma)))^{3/2}}$$

$$\times\sum_{n,m=-\infty}^{\infty}(\xi+2nh)(\xi+2mh)\exp\left(-\frac{(\xi+2nh)^2}{4(\theta_1(t)-\theta_1(\tau))}-\frac{(\xi+2mh)^2}{4(\theta_2(\tau)-\theta_2(\sigma))}\right)d\sigma d\xi.$$

Separating out of the series the term which corresponds to $n=0$ and $m=0$, we shall estimate the expression

$$K_{2,0}=\frac{1}{4\pi((\theta_1(t)-\theta_1(\tau))(\theta_2(\tau)-\theta_2(\sigma)))^{3/2}}\times$$

$$\times\int\limits_0^h\xi^2\exp\left(-\frac{\xi^2}{4(\theta_1(t)-\theta_1(\tau))}-\frac{\xi^2}{4(\theta_2(\tau)-\theta_2(\sigma))}\right)d\xi.$$

Changing the variable in the latter integral $z=\frac{\xi}{2}\sqrt{\frac{\theta_1(t)-\theta_1(\tau)+\theta_2(\tau)-\theta_2(\sigma)}{(\theta_1(t)-\theta_1(\tau))(\theta_2(\tau)-\theta_2(\sigma))}}$, we have

$$K_{2,0}=\frac{2}{\pi(\theta_1(t)-\theta_1(\tau)+\theta_2(\tau)-\theta_2(\sigma))^{3/2}}\int\limits_0^{T(t,\tau,\sigma)}z^2\exp(-z^2)dz,$$

where $T(t,\tau,\sigma)=\frac{h}{2}\sqrt{\frac{\theta_1(t)-\theta_1(\tau)+\theta_2(\tau)-\theta_2(\sigma)}{(\theta_1(t)-\theta_1(\tau))(\theta_2(\tau)-\theta_2(\sigma))}}$.

Integrating by parts we reduce the previous expression to the form

$$K_{2,0}=-\frac{h}{2\pi(\theta_1(t)-\theta_1(\tau)+\theta_2(\tau)-\theta_2(\sigma))\sqrt{(\theta_1(t)-\theta_1(\tau))(\theta_2(\tau)-\theta_2(\sigma))}}$$

$$\times\exp\left(-\frac{h^2}{4}\left(\frac{1}{\theta_1(t)-\theta_1(\tau)}+\frac{1}{\theta_2(\tau)-\theta_2(\sigma)}\right)\right)+$$

$$+\frac{1}{\pi(\theta_1(t)-\theta_1(\tau)+\theta_2(\tau)-\theta_2(\sigma))^{3/2}}\int\limits_0^{T(t,\tau,\sigma)}\exp(-z^2)dz.$$

Estimating $K_{2,0}$, we obtain the inequality

$$|K_{2,0}|\le\frac{C_{16}}{(\theta_1(t)-\theta_1(\tau)+\theta_2(\tau)-\theta_2(\sigma))^{3/2}}.$$

Applying (19) we have

$$|K_{2,0}|\le\frac{C_{17}}{(t^{\beta+1}-\sigma^{\beta+1})^{3/2}}.$$

Obviously, the estimates of other summands in $K_2(t,\tau)$ are similar. Returning to the estimate of $K_2(t,\tau)$ and using the notation (14), we obtain

$$\left|\frac{a_1(t)\tau^\beta}{t^\beta u_{2x}(0,t)}K_2(t,\tau)\right| \le \frac{C_{18}A_1\tau^\beta}{\int\limits_0^t \frac{d\tau}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}}} \int\limits_0^\tau \frac{d\sigma}{(t^{\beta+1}-\sigma^{\beta+1})^{3/2}}$$
$$\le C_{19}\tau^\beta \frac{t^{(\beta-1)/2}}{I_1} \int\limits_0^\tau \frac{d\sigma}{(t^{\beta+1}-\sigma^{\beta+1})^{3/2}}.$$

Consider the integral in the latter inequality:

$$\int\limits_0^\tau \frac{d\sigma}{(t^{\beta+1}-\sigma^{\beta+1})^{3/2}} = \frac{1}{t^{3(\beta+1)/2}} \int\limits_0^\tau \frac{d\sigma}{(1-(\sigma/t)^{\beta+1})^{3/2}} \le \frac{1}{t^{3\beta/2}} \int\limits_0^\tau \frac{d\sigma}{(t-\sigma)^{3/2}} \le$$
$$\le \frac{2}{t^{3\beta/2}\sqrt{t-\tau}}.$$

Finally we have

$$\left|\frac{a_1(t)\tau^\beta}{t^\beta u_{2x}(0,t)}K_2(t,\tau)\right| \le \frac{C_{20}\tau^\beta t^{(\beta-1)/2}}{t^{3\beta/2}\sqrt{t-\tau}} \le \frac{C_{21}}{\sqrt{t(t-\tau)}}.$$

Other summands $K_i(t,\tau)$ are estimated analogously. It follows that we get such inequality for $K(t,\tau)$:

$$|K(t,\tau)| \le \frac{C_{22}}{\sqrt{t(t-\tau)}}.$$

Taking into account that $a_0(t)\equiv 0$ for $t\in[0,\tilde t]$, we finally have

$$|K(t,\tau)| \le \frac{C_{22}}{\sqrt{\tilde t(t-\tau)}} \le \frac{C_{23}}{\sqrt{t-\tau}}, \quad t\in[0,T].$$

This means that the equation (33) as a homogeneous Volterra integral equation of the second kind has only trivial solution $a_0(t)\equiv 0$. Then $a(t)\equiv 0, t\in[0,T]$ and $u(x,t)\equiv 0, (x,t)\in\overline{Q}_T$ as a consequence of the uniqueness of the solution of direct problem (26)-(28).

Remark 1. The theorem of existence and uniqueness of solution for the problem (1)-(4) may be expanded to the problem with another boundary and overdetermination conditions. Really, consider the analogous problem for equation (1) with initial condition (2), boundary conditions $u_x(0,t)=\mu_1(t)$, $u_x(h,t)=\mu_2(t)$ and overdetermination condition $u(0,t)=\mu_3(t)$. Then by change of unknown function $u_x\equiv v$ the given problem is reduced to the following one:

$$v_t = a(t)v_{xx} + f_x(x,t), \quad (x,t)\in Q_T,$$
$$v(x,0) = \varphi'(x), \quad x\in[0,h],$$
$$v(0,t)=\mu_1(t), \quad v(h,t)=\mu_2(t), \quad t\in[0,T],$$
$$a(t)v_x(0,t) = \mu_3'(t) - f(0,t), \quad t\in[0,T].$$

Remark 2. The assumptions of the theorem on functions φ and f may be relaxed and reduced to the following conditions: $\varphi\in C^1[0,h], f\in C^{1,0}(\overline{Q}_T)$. To check this statement, it is sufficient to study the behavior of the corresponding summands in (35).

References

1. Waid M C 1974 The first initial-boundary value problem for some nonlinear time degenerate parabolic equation *Proc. Amer. Math. Soc.* **42** 61-84

2. Caffarelli L A, Friedman A 1979 Continuity of the density of a gas flow in a porous medium *Trans. Amer. Math. Soc.* **252** 99-113

3. Podgayev A G 1987 On initial-boundary value problems for some quasilinear parabolic equations with nonclassical degeneration *Sibirsk. mat. journ.* **28** 129-139

4. Grebenev V.N. 1994 On a system of degenerate parabolic equations that arises in fluid dynamics *Sibirsk. mat. journ.* **35** 753-767

5. Berestycki H, Busca J, Florent I 2000 An inverse parabolic problem arising in finance *C. R. Acad. Sci. Paris* **331** 965-969

6. Glushko V P 1968 Degenerate linear differential equations *Diff. Uravneniya* **4** 1957-1966

7. Kalashnikov A S 1968 On growing solutions of second order linear equations with non-negative characteristic form *Mat. Zametki* **3** 171-178.

8. Glushak A V, Shmulevich S D 1986 On some well-posed problems for higher order parabolic equations which degenerates on time variable *Diff. Uravneniya* **22** 1065-1068

9. Matsuzawa T. 1971 Sur une classe d'equations paraboliques degenerees *Ann. sci. Ecole norm. super.* **104** 1-19

10. Malyshev I 1991 On the parabolic potentials in degenerate-type heat equation *J. Appl. Math. Stochastic Anal* **4** 147-160

11. DiBenedetto E 1993 *Degenerate Parabolic Equations* (New York: Springer)

12. Beland Y 2001 Time-vanishing properties of solutions of some degenerate parabolic equations with strong absorption *Adv. Nonlinear Stud.* **1** 117-152

13. Isakov V 1997 *Inverse Problems for Partial Differential Equations* (New York: Springer)

14. Prilepko A I, Solovyev V V 1987 On solvability of inverse problems for determination of lower order coefficient in a parabolic equation *Diff. Uravneniya* **23** 136-143

15. Pilant M, Rundell W 1988 Fixed point method for a nonlinear parabolic inverse coefficient problem *Commun. Part. Dif. Equat.* **13** 469-493

16. Kirsch A 1996 *An Introduction to the Mathematical Theory of Inverse Problems* (New York: Springer)

17. Ivanchov M 2003 *Inverse Problems for Equations of Parabolic Type* (Lviv: VNTL Publishers)

18. Eldesbaev T 1987 An inverse problem for a second order degenerate hyperbolic equation *Izv. Acad. Nauk Resp. Kazakhstan. Ser. Fiz. Mat.* **3** 27-29

19. Gadjiyev M M 1987 An inverse problem for degenerate elliptic equation *Primeneniya Metodov Funk. Anal. v Uravneniyah Mat. Fiz.* (Novosibirsk) 66-71

20. Eldesbaev T 1995 On an inverse problem for a parabolic equation whose order degenerates. I *Izv. Nats. Acad. Nauk Resp. Kazakhstan. Ser. Fiz. Mat.* **3** 37-41

21. Ivanchov M, Saldina N An inverse problem for the degenerate heat equation (to appear)

22. Jones B F 1962 The determination of a coefficient in a parabolic differential equation. Part I. Existence and uniqueness *J. Math. Mech.* **11** 907-918