SMOOTH INVARIANT MANIFOLDS AND NORMAL FORMS

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INTRODUCTION

This book is related to the qualitative theory of dynamical systems and is devoted to the study of flows and cascades in the vicinity of a smooth invariant manifold.

Much attention is given by specialists in differential equations to the investigation of invariant manifolds. There are several reasons for this. Firstly, the collection of all compact invariant manifolds (in particular, equilibria, periodic orbits, invariant tori etc.) constitutes, so to speak, the skeleton of the dynamical system. Therefore, one would like to know whether these manifolds persist under perturbations of the vector field, and what happens in their vicinity (for example, do the nearby solutions tend to the manifold, or stay nearby, or leave the neighbourhood?) Secondly, the existence, for example, of an exponentially stable invariant submanifold permits one to reduce the investigation of nearby motions to that of points in the manifold itself and thereby to lower the dimension of the phase space. Thirdly, the possibility of reducing a dynamical system to normal form is intimately related to the existence of invariant manifolds. The following simple observation serves as an illustration. Two differential equations $\dot{x} = f(x)$ and $\dot{y} = g(y)$ are conjugate via a smooth change of variables $y = h(x)$ if and only if the system $\dot{x} = f(x)$, $\dot{y} = g(y)$ admits a smooth invariant submanifold of the form $\{(x, y): \; y = h(x)\}$. Various interrelations between conjugacies of dynamical systems, on the one hand, and invariant sections of certain extensions, on the other hand, are repeatedly used in this work.

The main purpose of the book is to present, as fully as possible, the basic results concerning the existence of stable and unstable local manifolds and the recent achievements in the theory of finitely smooth normal forms of vector fields and diffeomorphisms in the vicinity of a rest point and a periodic trajectory. Besides, an attempt is made to summarize the not numerous results obtained so far in the investigation of dynamical systems near an arbitrary invariant submanifold. The choice of material is stipulated by the wish to reflect, in the first place, the *typical*, *generic* properties of dynamical systems. That is why we consider normal forms relative only to the *hyperbolic* variables (*i.e.*, in the direction *transverse* to the center manifold), whereas the subtle problem concerning further simplifications *along* the center manifold is beyond the scope of our considerations.

The first two chapters deal with dynamical systems near an equilibrium and a periodic orbit. Several important results are stated here without proof because they easily follow from more general theorems concerning arbitrary compact invariant manifolds which are presented (with full proofs) in the last four chapters. This way of presentation has allowed us to essentially shorten the text, but, as can be expected, it will not be approved by readers interested only in classical topics. Let us note in excuse that, suprisingly enough, the proofs for a rest point are not much easier as compared with the general case (cf., for example, the papers by Takens [1] and Robinson [1]).

There is a vast array of papers and books devoted to the questions touched upon in this book. When speaking about *invariant manifolds*, one should first of all mention the fundamental investigations of Lyapunov [1] and Poincaré [2] mainly devoted to the analytic case. Further progress was achieved by Hadamard [1], Bohl [1], and Perron [1-3]. Hadamard [1] proposed a highly useful method for proving the existence of invariant manifolds now called *the graph transform method*. Another approach close to the method of Green's functions was developed by Bohl [1] and Perron [1-3]. The *Hadamard-Bohl-Perron theory* was further elaborated and extended by Anosov [1-3], Smale [1], Kelley [1], Kupka [1], Neimark [1-3], Pliss [1-2], Reizins [1], Samoilenko [1,2], Takens [1] and many others. A great number of theorems about integral manifolds was established by applying *asymptotic methods* due to Bogolyubov and Mitropolskii (see the book by Mitropolskii and Lykova [1]). Grobman [1] and Hartman [1] have shown that a vector field near a hyperbolic singular point is topologically linearizable. This result was extented by Pugh and Shub [1] to the case of an arbitrary normally hyperbolic compact invariant submanifold. On the basis of previous results obtained by McCarthy [1], Kyner [1], Hale [1], Moser [1] and others, Sacker [1,2] proposed a rather general condition sufficient for a compact invariant manifold to persist under perturbations.

In the seventies, the Hadamard-Bohl-Perron theory was summed up and brought to its final form (see Hirsch, Pugh and Shub [1] and Fenichel [1-3]). Unfortunately, the style of presentation in these works can hardly be acknowledged as fully satisfactory because many proofs are only sketched and their accomplishment (left to the reader) needs in fact a deep insight into global analysis on manifolds.

The method of *normal forms* founded by Poincaré [1] was further developed by Dulac, Siegel, Sternberg, Kolmogorov, Arnold, Moser, Bruno and others (see the books by Arnold

[4], Hartman [3] and Bruno [2]). These investigations are chiefly devoted to formal, analytic, and infinitely differentiable normal forms. The problem on finitely smooth normal forms was studied by Belitskii [1], Samovol [1-10] and Sell [1-3].

Let us briefly review the contents of the book. In the first chapter, we present the well-known facts on the structure of flows and cascades near an equilibrium and a closed orbit. In § 1, we recall the relationships between differential equations, vector fields, and phase flows. The second section, § 2, is devoted to the *Grobman-Hartman linearization theorem* in the vicinity of a hyperbolic singular point. § 3 is concerned with the *Floquet-Lyapunov normal form* of a vector field near a periodic orbit. The next section contains the main results on the existence of local smooth manifolds in the vicinity of an equilibrium and a periodic trajectory (the so-called *Hadamard-Bohl-Perron theory*). These results are used to derive some theorems on preliminary normal forms which serve as the starting point of the next chapter.

Chapter II, central to this book, deals with *normal forms* of vector fields and diffeomorphisms in the neighbourhood of a fixed point *with respect to the group of finitely smooth changes of variables*. In recent years, it was acknowledged that these normal forms are essential for the non-local *bifurcation theory* (see Arnold, Afrajmovich, Il'yashenko and Shil'nikov [1], Il'yashenko and Yakovenko [1]) because they are stable under perturbations, in contrast to the classical resonant normal forms.

The first section serves as an introduction. We pose here the problem on reducing vector fields near a hyperbolic equilibrium to normal form and sketch the research objects pursued in this chapter. In § 2, we present the classical results due to Poincaré and Dulac on normalization of jets of vector fields and diffeomorphisms at a rest point. The next section contains several important theorems on polynomial (weakly) resonant normal forms. We show, in particular, that if two vector fields have contact of a sufficiently high order at the equilibrium, then they are locally C^k conjugate with one another. In § 4, we discuss the possibility of further simplification of the resonant normal form and consider a number of examples which demonstrate that certain monomials entering the normal form can be killed by C^k changes of variables. In § 5, we propose a new, very general condition, $\mathfrak{A}(k)$, imposed on a monomial x^τ that enables one to delete x^τ out of the resonant normal form. This condition is used in § 6 to prove a deep theorem on C^k linearization. Because the condition $\mathfrak{A}(k)$ is rather involved, it is desirable to have some relatively simple conditions each implying $\mathfrak{A}(k)$. Several such conditions are established in § 7. The next section contains theorems on C^k normal forms expressed in terms of the condition $\mathfrak{A}(k)$. These results are supplemented in § 9 and § 10 by some theorems based, besides $\mathfrak{A}(k)$, on some other principles. The last section gives a survey of all the results obtained in Chapter II.

The third chapter is concerned with *linear extensions* of dynamical systems. Such objects occur, for example, when linearizing a dynamical system near an invariant submanifold. In §§ 1-3, we give a brief review of the main results obtained in this area (for a detailed exposition, the reader is referred to the book by Bronstein [4]). Although these results may appear somewhat far from our subject, they are basic to many constructions and proofs in the sequel. In order to describe two important classes of linear extensions (namely, linear extensions satisfying the *transversality condition* and those *with no non-trivial bounded motions*), we use in § 4 *quadratic Lyapunov functions* defined on the underlying vector bundle. Various kinds of *weak regularity* of linear extensions are investigated in § 5. Some relationships between weak regularity, transversality, hyperbolicity, and the existence of a *Green-Samoilenko function* are established. In particular, it is shown that a C^k Green-Samoilenko function exists if and only if the k-jet transversality condition is fulfilled.

In Chapter IV, we investigate invariant subbundles of *weakly non-linear extensions* of dynamical systems. Some results on the existence of invariant subbundles of extensions close to exponentially splitted linear extensions are presented in § 1. In particular, a theorem which generalizes the classical result of Hadamard [1] is proved. In § 2, we show that any non-linear extension sufficiently close to an exponentially separated linear extension can be decomposed into a Whitney sum of two extensions. The Grobman-Hartman linearization theorem is generalized in § 3 to weakly non-linear hyperbolic extensions. In § 4, we examine the question on *smoothness* of invariant subbundles. The proof of the main theorem is based on the now traditional graph transform method. It also makes use of the smooth invariant section theorem which is presented (with a detailed proof) in the Appendix. The application of global analysis methods and results enables us to avoid the use of local coordinates and to control all stages of the proof.

Chapter V deals with smooth invariant submanifolds satisfying the so-called *normal k-hyperbolicity* condition introduced in § 1. We prove in § 2 that such a submanifold is C^k *persistent* under perturbations. We also establish the existence of its *stable and unstable local manifolds*. These results constitute the kernel of the general Hadamard-Bohl-Perron theory. Besides that, we present a theorem on topological linearization near the given submanifold which is a direct generalization of the Grobman-Hartman theorem. As it is shown in § 3, normal k-hyperbolicity is not only sufficient, but also necessary for a submanifold to be C^k persistent. The notion of *asymptotic phase* for an exponentially stable invariant submanifold is studied in § 4, and some theorems on smoothness of the asymptotic phase are proved. Besides, it is shown that the stable manifold W^s of a normally k-hyperbolic compact invariant submanifold Λ is invariantly fibered by C^k submanifolds W^s_x, $x \in \Lambda$ (of course, a similar result is valid for the unstable manifold W^u). These statements may be considered as a supplement to the Hadamard-Bohl-Perron theory. In the final part of this section, we present proofs of several theorems stated (but not proved) in Chapter I.

Chapter VI is concerned with the question of whether two dynamical systems are *smoothly conjugate* to one another in the vicinity of their common smooth invariant submanifold. It is assumed that these systems have contact of high order at all points of the submanifold. In § 1, we consider the case when this submanifold is exponentially stable and prove a generalization of Sternberg's [1] theorem on linearization of contractions. The general case is handled in § 2, and a theorem due to Robinson [1] is presented which extends some results previously obtained by Sternberg [1,2], Chen [1] and Takens [1]. We deduce from these theorems some results (stated without proof in Chapter II) concerning polynomial resonant normal forms of dynamical systems near an equilibrium and a periodic orbit with respect to finitely smooth changes of coordinates.

The book is adressed to specialists in the qualitative theory of differential equations and, especially, in bifurcation theory. Although written for mathematicians, it may prove to be helpful to all those who use normal forms when investigating concrete differential equations.

While research workers will find in the book an up-to-date account of recent developments in the theory of finitely smooth normal forms, the authors have tried to make the first part (Chapters I and II) accessible to non-specialists in this field. Such readers should use Chapter I as a summary (or, rather, a glossary) and take the classical results presented in this chapter on faith. The background material needed to understand Chapter II is differential calculus of several variables and ordinary differential equations. To be more precise, the main tools used here are the Taylor expansion formula and Banach's fixed point theorem for contractions (applied to operators in some special functional spaces). As to the second part, the reader is assumed to be familiar with the fundamentals of *global analysis* on manifolds (Bourbaki

[1], Leng [1], Hirsch [1]) and *fiber bundle theory* (Husemoller [1]). For the reader's convenience, at the end of the book an Appendix is given which contains some definitions and facts from differential calculus and the theory of smooth manifolds, as well as some more special results repeatedly used in the course of the book.

We adopt standart notation: the group of real numbers is denoted by $\mathbb{R}$, the group of integers is denoted by $\mathbb{Z}$, $\mathbb{Z}_+$ is the set of non-negative integers. Given a mapping $f\colon X \to Y$, graph(f) denotes the set $\{(x, f(x)) \colon x \in X\}$. The symbol ▶ marks the beginning and ■ marks the end of a proof.

The sections are divided into subsections, each numbered (within a chapter) by a pair of numbers, where the first one refers to the number of section and the second one refers to the number of the subsection in this section. If necessary, we add, in front of these two numbers, the number of the chapter. So, the triple III.2.4, for example, denotes subsection 4 of section 2 of chapter III. The Appendix is also divided in subsections numbered consecutively and marked with a capital A.

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