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MASTER'S THESIS

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Abstract

In this thesis we present a method for calculating exact solutions to a system of equations for two phase flow. This solution is valid for a special case of initial condition called the Riemann problem. The system consists of three hyperbolic conservation laws including gas mass balance, liquid mass balance and total momentum balance. Also, there is a slip relation which relates the true velocities of each phase together. In solving the Riemann problem for the two-phase flow drift flux model, some assumptions have been taken like incompressible liquid. In this thesis the development of the final solution for the Riemann problem is presented. This development includes the star region parameter determination and the exact solution in the rarefaction waves. Using the Riemann exact method to solve the equation systems of hyperbolic conservation laws helps in making a fundamental understanding of the physics and characteristic behavior. Also determining the flow parameters in the interface by the Riemann solution builds a basis for numerical approaches in two phase flow drift flux models.

Acknowledgement

I want to thank Tore Flåtten (Professor at UiS) my supervisor, for the excellent supervision and guidance during my master thesis. He has been very enthusiastic and encouraged me to learn more, which has given me a lot of motivation. Then I am thankful to my family who have been supporting me even over such a long distance. Also, I want to thank my friends in UiS who have built an enjoyable stay for me in Stavanger. Finally, I'm thankful to Parham Barazesh who helped me in this thesis directly.

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1. INTRODUCTION

In well bores and transport pipelines, the possibility of two-phase flow is high. In this condition, usually drift flux models with associated numerical methods are used to find the important properties of the flow inside the well bore in the transient or steady state condition. Now in this thesis an exact solution for two phase flow will be used for determining the flow parameters. This solution is valid for a special case of initial condition called the Riemann problem. In the Riemann problem a single discontinuity is assumed at a location. Over this discontinuity the gas and oil properties change. Then by increasing time this discontinuity would change in shape.

1.1. Motivation

As said previously, multiphase flow may appear during drilling operations or the production phase. For example, during the drilling phase the interaction of cuttings and mud create a two-phase flow. Also, the underbalanced drilling is another example of two-phase flow. In underbalanced drilling the bottom hole pressure is less than the formation pressure, so the gas or oil would infuse into the well bore and create two phase flow. Also, during drilling, a gas kick or oil kick may be introduced into the well and the flow inside the well turns to two phase flow or multi-phase flow. In 2000, a study on two phase flow modeling was implemented for underbalanced drilling. In this paper , using the drift flux model, the transient condition in underbalanced drilling was investigated (Lage, Fjelde et al. 2000).

Understanding the dynamic behavior in multiphase flow lets us build more accurate mathematical models for future software development and simulators. In previous studies for two phase flow, mostly numerical methods with drift flux theory has been used for different drilling and production simulators software. In this study, an exact solution to the Riemann problem is going to be used for a two-phase flow mixture of oil and gas. This exact method can be a verification for numerical methods used previously for underbalanced drilling or other two-phase flow incidents. The Riemann problem, which is an initial value problem, has a variety of applications for fluid dynamics and gives good insights for understanding the characteristics and the whole concepts of flow dynamics. In the finite volume numerical method, the Riemann problem is used to determine flow parameters at the control volume interface.

1.2. Thesis structure

This thesis has been structured based on four chapters. In Chapter one which is the introduction, first the general definition of the Riemann problem is presented. Then the motivation and thesis structure are presented.

In chapter two which is the theory part, the related literature review about the exact solution for linear scalar equations, application of the Riemann problem in the Buckley-Leverett equation, the Riemann problem for Euler isothermal gas flow, the problem of shallow water flow, and the exact solution for two phase flow are presented. Then the necessary concepts, definitions and the development of equations and final solutions for the Riemann problem for linear scalar PDEs, the inviscid Burgers equation, the system of linear equations of conservation laws, the system of nonlinear equations of Euler for isothermal gas and the system of full Euler equations are presented .

In chapter three which is the methodology part, the steps toward the exact Riemann solution development for two phase flow of a gas-oil mixture are described. Then the flow chart and procedure for computer programming is presented in this chapter.

In chapter four which is the results and discussion part, the results of the programming for the exact Riemann solution for two-phase flow is presented. In this chapter several initial conditions will be tried to test all types and configurations of possible waves. These configurations are:

1) All shock waves
2) All rarefaction waves
3) Shock in first wave and rarefaction in second wave
4) Rarefaction in first wave and shock in second wave

2. THEORY

2.1. Literature review and background

A Riemann problem is an initial value problem made of a conservation equation or a system of conservation equations together with constant initial data which has a single discontinuity.

The Riemann problem is applicable for the understanding of equations like the Euler equations since all properties, like rarefaction and shock waves , appear as characteristics in the solution. It also gives an exact solution to some complex nonlinear equations.

The inviscid Burgers equation is a partial differential equation which is very practical in many engineering fields like fluid mechanics. It is a nonlinear scalar equation and understanding the Riemann solution for this equation is useful for developing the final exact solution for two phase flow. The Riemann problem has been described for the inviscid Burgers equation by (LeVeque 2002).

The Buckley-Leverett is a nonlinear equation and the Riemann problem for this equation is more complex than for linear equations. The concept of characteristics and entropy conditions for the Buckley‑Leverett Equation has been described by (LeVeque 2002).

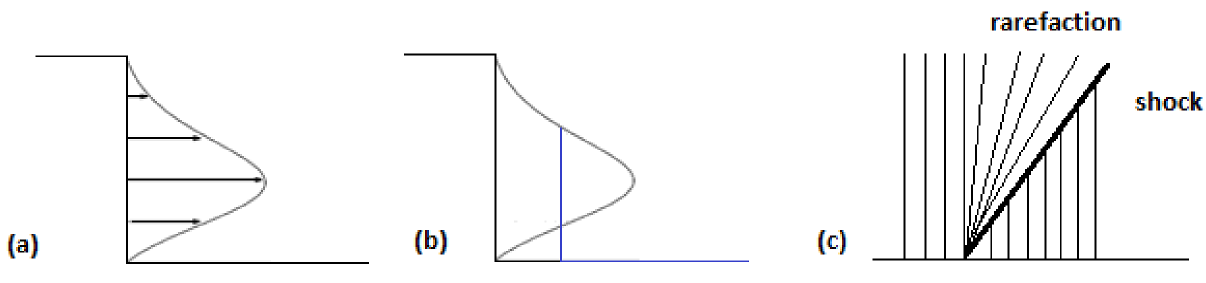

Figure 2-1: Buckley–Leverett Riemann solution. (a) triple-valued solution (b) area-preserving shock (c) Wave forms in x-t plane.

Another important application of the Riemann problem is the solution of the Euler equations for isothermal gas. The isothermal equations consist of two conservation laws of mass balance and momentum balance. For solving the Riemann problem for this kind of equations, the theory of Hugoniot and integral curves is applicable. In the book “Numerical Methods for Conservation Laws” the details of the Riemann problem for isothermal gas have been worked out (LeVeque 1992). The solution development procedure for isothermal gas is much like the shallow water problem. The solution for the shallow water problem has been described in the book “Finite volume methods for hyperbolic equations” (LeVeque 2002). In this method the theories of nonlinear scalar and system of linear equations are combined to produce the general theory of nonlinear systems.

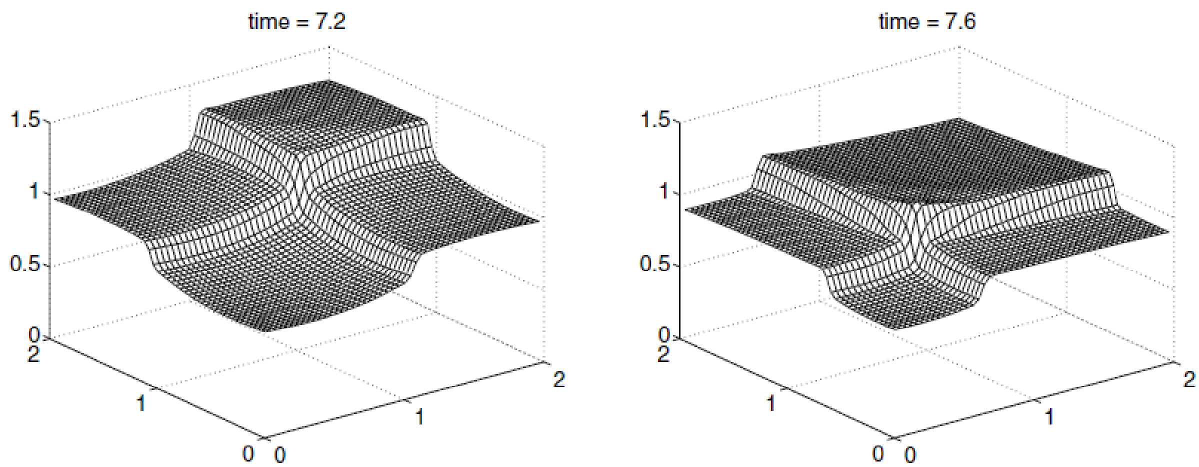

Figure 2-2:An example of water wave propagation in the problem of shallow water (LeVeque 2002)

For the context of two-phase flow, many analytical and numerical approaches have been used by several authors for example (Enwald, Peirano et al. 1996),(Stewart and Wendroff 1984),(Gonthier and Powers 2000),(Lahey Jr, Drew et al. 2001), (Saurel and Abgrall 1999), (Städtke 2006), (Zeidan, Slaouti et al. 2007),(Zeidan and Slaouti 2009),(Luke, Cinnella et al. 2007), (Zeidan 2011).

The Riemann problem for solving two phase flow is an initial value problem and has been developed for several flow models. The structure of equations for two phase flow is complex so proposed exact solutions were limited to some simplifications. For example, in the work of

(Andrianov and Warnecke 2004) a limited Riemann solution has been proposed for the Baer and Nunziato equations (Baer and Nunziato 1986). It is an indirect solution because a solution has been assumed and they look for corresponding initial data. The first direct Riemann solution for two phase flow models has been proposed by (Schwendeman, Wahle et al. 2006). A direct solution which is like the Baer and Nunziato in mathematical form has been proposed by (Castro and Toro 2006). Another Riemann solver for the two phase flow Baer and Nunziato equations has been proposed by (Deledicque and Papalexandris 2007) .

Also the Riemann solution structure for two phase flow was investigated by (Zeidan 2011). In his work, a hyperbolic conservative model without velocity equilibrium but with mechanical equilibrium has been used. In this paper it is assumed that the gas concentration is constant in the whole section which is a questionable assumption.

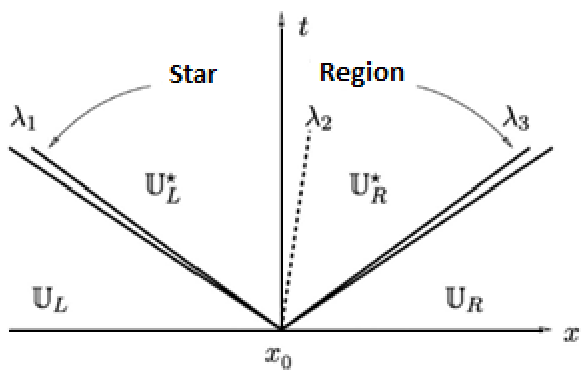

Figure 2-3:General Riemann solution for mixture of gas and liquid (Zeidan 2011)

As you see in the figure (2-3) the solution consists of 2 nonlinear waves which can be shock or rarefaction and one middle contact discontinuity which is linear.

2.2. The Riemann problem for linear hyperbolic equations

It is possible to construct the analytical solution of the general initial value problem for the linear advection equation. The linear advection equation is the simplest hyperbolic conservation law. The initial data which is shown by $u_0(x)$ is used to find the analytical solution for this equation. Now a special initial value problem is used here which is called the Riemann problem.

$$PDE: \qquad u_t + au_x = 0 \tag{2.1}$$

$$Initial\ Condition: \qquad u(x,0) = u_0(x) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x > 0 \end{cases}$$

In the equation (2.1), u_L is the left value and u_R is the right value which are constant. As you see there is a discontinuity at x = 0. If this initial value problem is solved by the characteristic method, then we have:

$$\frac{dx}{dt} = a \qquad x = at + x_0 \tag{2.2}$$

$$\frac{du(x(t),t)}{dt} = \frac{\partial u}{\partial x}a + \frac{\partial u}{\partial t} = 0 \tag{2.3}$$

$$du(x(t),t) = 0 \qquad u(x(t),t) = u(x(0),0) = u_0(x_0) \tag{2.4}$$

$$u(x,t) = u_0(x_0) \tag{2.5}$$

$$u(x,t) = u_0(x - at) \tag{2.6}$$

So, the Riemann problem with the initial condition (2.1) is a special case of initial value problem and satisfies (2.6). Now we expect points on the initial data to go forward with speed *a* and pass distance *d* in time *t*. It means that the initial discontinuity propagates with speed *a*. This special characteristic curve *x* = *at* separates the left and right characteristics. So, the left part of this characteristic takes the value of u_L and the right part of the characteristic takes the value of u_R.

$$u(x,t) = u_0(x - at) = \begin{cases} u_L & \text{if } x - at < 0 \\ u_R & \text{if } x - at > 0 \end{cases} \tag{2.7}$$

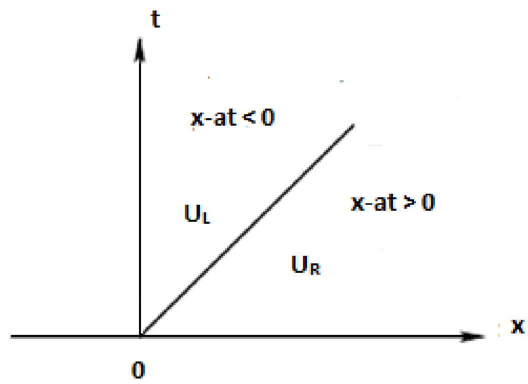

Figure 2-4:Illustration of the Riemann solution in the x–t plane

2.3. Non – linearities and wave formation

In this section some features of non–linear hyperbolic conservation laws, and the creation of waves are introduced. Here the focus is on the scalar nonlinear conservation law (Toro 2013).

$$u_t + f(u)_x = 0\ , u(x,0) = u_0(x) \tag{2.8}$$

$$u_t + \frac{\partial f}{\partial u} u_x = 0$$

$$u_t + \lambda(u) u_x = 0 \tag{2.9}$$

As you know the term of $\lambda(u)$ is defined by equation (2.10):

$$\lambda(u) = \frac{df}{du} = f'(u) \tag{2.10}$$

λ is the speed of the characteristic. For example, in the previous section $\lambda(u)$ for the linear hyperbolic law is constant and equal to *a*. The convexity of *f(u)* affects the structure of the solution *u(x,t).*

There are three possibilities for $\lambda(u)$ which include:

1. Monotone increasing function of *u*:

$$\frac{d\lambda(u)}{du} = \lambda'(u) = f''(u) > 0\ \ (convex\ flux) \tag{2.11}$$

2. Monotone decreasing function of *u*:

$$\frac{d\lambda(u)}{du} = \lambda'(u) = f''(u) < 0 \ \ (concave\ flux) \tag{2.12}$$

3. has extrema, for some *u*:

$$\frac{d\lambda(u)}{du} = \lambda'(u) = f''(u) = 0 \tag{2.13}$$

Now we consider the characteristic curves $x = x(t)$ which satisfy the initial value problem:

$$\frac{dx}{dt} = \lambda(u), \qquad x(0) = x_0 \tag{2.14}$$

If we assume *u* and *x* are both functions of *t*, the total derivative of *u* along the curve *x(t)* equals to:

$$\frac{du}{dt} = u_t + \lambda(u)u_x = 0 \tag{2.15}$$

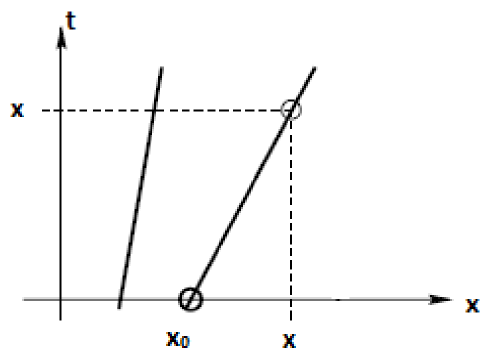

Figure 2-5:Typical characteristic curves for a non–linear hyperbolic conservation law

2.3.1. Shock wave

A shock wave is the result of collision of characteristics with different speed. So, the average speed between two characteristics would be obtained by a special method. In a control volume X_L to X_R, there is a line $s = s(t)$ which create a jump discontinuity for $U(x, t)$. The two fixed points X_L and X_R on the x–axis has been selected such that $X_L < L(t) < X_R$

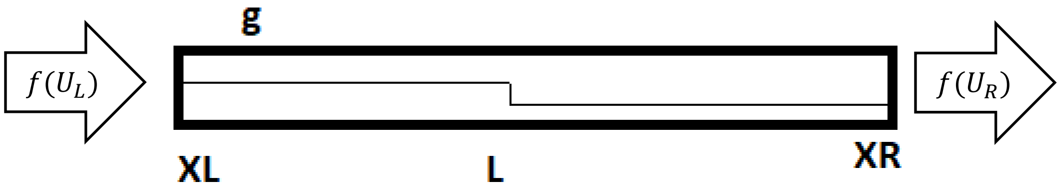

Figure 2-6:Schematic figure of conservation of U in the controlled volume

$$\alpha = \frac{g}{L} \tag{2.16}$$

At each time:

$$U = U_L L\,\alpha + U_R L(1-\alpha) \tag{2.17}$$

$$\frac{\partial U}{\partial t} = f(U_L) - f(U_R) \tag{2.18}$$

If we take the derivative of U and put α in the equation (2.18), we reach to a conclusion that the speed of the shock wave is:

$$s = \frac{\mathrm{f}(U_L) - \mathrm{f}(U_R)}{U_L - U_R} \tag{2.19}$$

The equation (2.19) is called the Rankine-Hugoniot condition.

$$s = \frac{\Delta f}{\Delta U} \tag{2.20a}$$

$$\lambda(U_L) > s > \lambda(U_R) \tag{2.20b}$$

For shock waves, the condition (2.20b) should be satisfied. This rule is called the Lax entropy condition (Lax 1957).

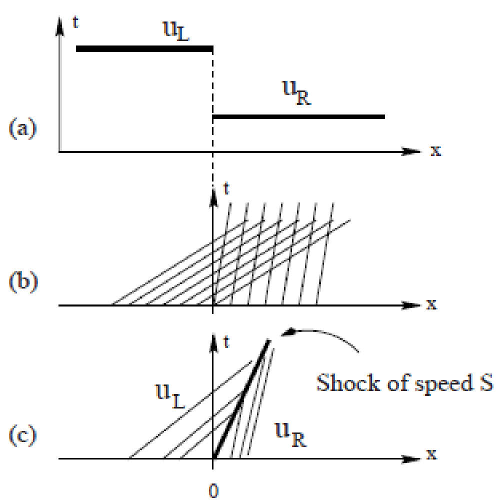

Figure 2-7: (a) initial data (b) picture of characteristics (c) solution on x–t plane (Toro 2013)

2.3.2.Rarefaction wave

We consider the initial value problem with convex flux function *f(u):*

$$u_t + f(u)_x = 0$$

$$u(x,0) = u_0(x) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x > 0 \end{cases} \qquad (2.21)$$

In the case of $u_L < u_R$, if we consider the wave as a shock one , the entropy condition would be violated. The entropy violating solution is like equation (2.7) but it is unstable. It means that small changes of initial data lead to large perturbations in the solution.

As you see in the figure (2-8), $u_L < u_R$ which makes the middle zone an expansive one. To find the solution in the middle zone we take another approach. As the first step, we assume the initial data has a linear transition between two points X_L and X_R (figure 2-9).

$$u_0(x) = \begin{cases} u_L & \text{if } x \le x_L \\ u_L + \frac{(u_R - u_L)}{(x_R - x_L)}(x - x_L) & \text{if } x_L < x < x_R \\ u_R & \text{if } x \ge x_R \end{cases} \qquad (2.22)$$

The solution for *u (x, t),* consists of two constant states, u_L and u_R, separated by a region of transition between the values of u_L and u_R. It is called rarefaction wave.

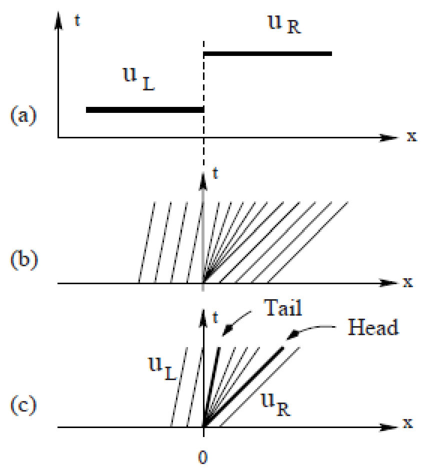

Figure 2-8:Centered rarefaction wave: (a) initial data (b) picture of characteristics (c) solution on x–t plane (Toro 2013)

The characteristics emanate at time = 0 as shown in the figure (2-9). The solution for *u(x, t)* is obtained by this characteristics which consists of constant states of u_L and u_R which are the tail and the head of the solution. There is a transition zone between u_L and u_R. So, this wave is called rarefaction.

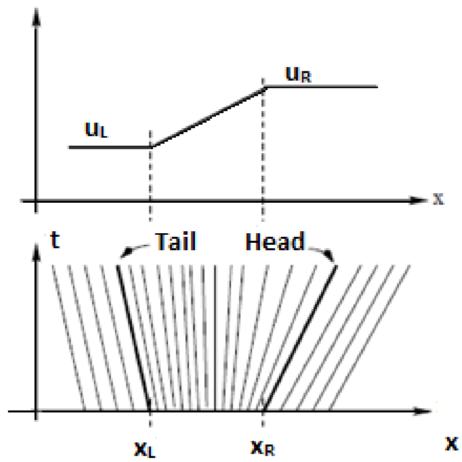

Figure 2-9:Non–centered rarefaction wave (Toro 2013)

In the figure of (2-9), X_R is called the head of the rarefaction wave and X_L is called the tail of the rarefaction wave.

From X_R we have:

$$x = x_R + \lambda(u_R)t \tag{2.23}$$

And from X_L we have:

$$x = x_L + \lambda(u_L)t \tag{2.24}$$

So, the entire solution becomes:

$$\begin{aligned} u(x,t) &= u_L && \text{if} \quad \frac{x-x_L}{t} \leq \lambda_L \\ \lambda(u) &= \frac{x-x_L}{t} && \text{if} \quad \lambda_L < \frac{x-x_L}{t} < \lambda_R \\ u(x,t) &= u_R && \text{if} \quad \frac{x-x_L}{t} \geq \lambda_R \end{aligned} \tag{2.25}$$

The size of ΔX of the interval over which the initial value has been spread over, is not effective on the final solution, in other words the solution just depends on term of *(x/t)*. The structure of the above solution is totally different from the entropy violating solution and is stable.

As you see in the figure (2-9), the initial data disintegrates because higher values move faster than lower values. In the case of a wave emanating from a single point, the wave is called a rarefaction wave and the solution becomes:

$$\begin{aligned} u(x,t) &= u_L && \text{if} \quad \frac{x}{t} \leq \lambda_L \\ \lambda(u) &= \frac{x-x_L}{t} && \text{if} \quad \lambda_L < \frac{x}{t} < \lambda_R \\ u(x,t) &= u_R && \text{if} \quad \frac{x}{t} \geq \lambda_R \end{aligned} \tag{2.26}$$

.

2.3.3.The Riemann Problem for the inviscid Burgers equation

One good example of the Riemann problem for the nonlinear hyperbolic equation is the inviscid Burgers equation which has been described by (Toro 2013).

$$\begin{aligned} &PDE: \; u_t + \left(\frac{u^2}{2}\right)_x = 0, \\ &Initial\ condition: \quad u(x,0) = \begin{cases} u_L, & \text{if} \quad x < 0 \\ u_R, & \text{if} \quad x > 0 \end{cases} \end{aligned} \tag{2.27}$$

According to what was discussed previously, the exact Riemann solution is a single wave. If we apply the entropy condition, the wave is shock or rarefaction. If $u_L > u_R$, it is a shock wave and if $u_L < u_R$, it is a rarefaction wave. So, the complete solution is:

$$\begin{aligned} &u(x,t) = \begin{cases} u_L \;\; if\ x - St < 0 \\ u_R \;\; if\ x - St > 0 \end{cases} \quad \text{if}\, u_L > u_R \;\; (shock\ wave) \\ &\qquad s = \frac{1}{2}(u_L + u_R) \;\; \text{according to equation (2 − 20a)} \\ &u(x,t) = \begin{cases} u_L & \text{if}\ \frac{x}{t} \leq u_L \\ \frac{x}{t} & \text{if} \quad u_L < \frac{x}{t} < u_R \\ u_R & \text{if}\ \frac{x}{t} \geq u_R \end{cases} \quad \text{if}\, u_L < u_R \;\; (Rarefaction\ wave) \end{aligned} \tag{2.28}$$

In the general solution of the inviscid Burgers equation (2.28), the entropy condition helps us to recognize the wave type. An implementation of the solution (2.28) for the inviscid Burgers equation has been done in Python and the results are below:

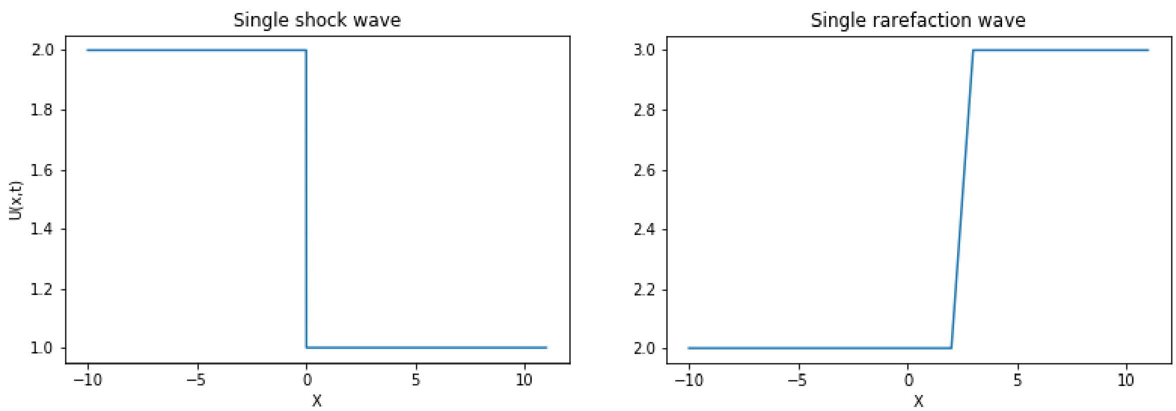

Figure 2-10:Simulation result of Inviscid Burgers Equation for different conditions of u_L and u_R

2.4. Non-linear systems of conservation laws

In nonlinear scalar equations, if the solution is dependent on speed change, waves do not propagate unchanged. In general, waves are developed in the shape of compression or expansion. The Riemann problem solution in the simplest case where the flux function is convex, is a shock wave or rarefaction wave.

In this section, the theory of linear systems of hyperbolic equations and nonlinear scalar equations are combined to develop the concept of nonlinear systems of equations. Like the linear system, a system of *m* equations requires splitting the jump into *m* separate waves. It means that for each wave there is a jump and the waves can be shock or rarefaction.

This general theory is developed by using the one-dimensional isothermal Euler gas equations as an example. The isothermal Euler system is a suitable example since it is a system of 2 equations and the nonlinear structure is simple.

2.4.1. The isothermal Euler equations

In the model of one-dimensional isothermal Euler, as an example we consider a tube where velocity and density of the gas are the two basic parameters and other parameters like pressure and momentum can be determined by them. We will denote the fluid density by the symbol of $\rho\,(x, t)$ and fluid velocity by $u(x,t)$. As the fluid inside the pipe is gas, the density is not constant because gas is compressible. The mass equation showing mass balance is below (Toro 2013).

$$\rho_t + (\rho u)_x = 0 \tag{2.29}$$

If the density is higher than nearby, we expect the gas to push into neighboring gas so the velocity would change because of the variation in density. If the velocity is constant, the equation (2.29) reduces to equation (2.30).

$$\rho_t + u\rho_x = 0 \tag{2.30}$$

The equation (2.29) is for density and we need a new equation for velocity. We consider the momentum as a conserved quantity because it is physically conserved. The momentum flux is expressed with equation (2.31).

$$momentum\ flux = \ \rho u^2 + p \tag{2.31}$$

Here p is pressure. By the assumption that ρ, p, u are smooth, the differential equation of conservation of momentum is obtained by equation (2.32).

$$(\rho u)_t + (\rho u^2 + p)_x \ = 0 \tag{2.32}$$

The combination of equations (2.29) and (2.32) would be a system of two conservation laws. As you see ρ and ρu both appear in the equations, so they are coupled equations and since the product of unknowns exist, they are nonlinear.

In the isothermal flow we can drop the conservation of energy equation and use a simple equation of state which relates pressure to density (2.33). In this equation, the constant a is the sound velocity in the ideal gas.

$$p = \ a^2\rho \tag{2.33}$$

So, the nonlinear system of conservation laws has the form of equation (2.34).

$$\begin{bmatrix} \rho \\ \rho u \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + a^2 \rho \end{bmatrix}_x = 0 \tag{2.34}$$

We can write the nonlinear system of equation (2.34) to another form.

$$q_t + f'(q) q_x = 0 \tag{2.35}$$

Where:

$$q(x,t) = \begin{bmatrix} \rho \\ \rho u \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$
$$f(q) = \begin{bmatrix} \rho u \\ \rho u^2 + a^2 \rho \end{bmatrix} \tag{2.36}$$

The jacobian matrix $f'(q)$ is:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ -\left(\frac{q_2}{q_1}\right)^2 + a^2 & \frac{2q_2}{q_1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -u^2 + a^2 & 2u \end{bmatrix} \tag{2.37}$$

The eigen values of $f'(q)$ are equal to:

$$\lambda^1 = u - a, \qquad \lambda^2 = u + a \tag{2.38}$$

And the related eigen vectors are equal to:

$$r^1 = \begin{bmatrix} 1 \\ u - a \end{bmatrix}, \quad r^2 = \begin{bmatrix} 1 \\ u + a \end{bmatrix} \tag{2.39}$$

2.4.1.1. Strategy for solving the Riemann problem

Solving the Riemann problem for the Euler isothermal model in a gas tube, one needs to find the solution for any arbitrary condition of q_L and q_R. For obtaining the exact solution we should consider the below steps:

1- Using an appropriate entropy condition, determine the type of each wave (shock or rarefaction)

2- The intermediate condition q_* should be identified.

3- In any rarefaction waves, the solution should be found.

2.4.1.2. Shock waves

In this section, a shock wave which separates two constant states is analyzed. In a shock wave there are some relations which connect the left and right state of the shock wave together. We assume that one side is constant, then we find the middle state. Still we use the isothermal gas tube as an example here. We assume that there are 2 shock waves in the gas tube. There is an intermediate condition q_* which is connected to q_R and q_L. Hence this q_* should satisfy the two sides.

According to section (2.3.1), the entropy condition is satisfied during a shock wave. So, the shock wave velocity equals to:

$$s(q_* - q) = f(q_* - q) \tag{2.40}$$

By using this formula for the gas tube, two equations would be obtained:

$$\begin{aligned} s(\rho_* - \rho) &= (\rho_* u_* - \rho u) \\ s(\rho_* u_* - \rho u) &= \rho_* u_*^2 - \rho u^2 + a^2(\rho_* - \rho) \end{aligned} \tag{2.41}$$

As said before ρ^* is a constant state. So (2.41) is a system of two equations with 3 unknowns. But for each wave there is a separate equation which satisfies the entropy condition. There are many ways to parametrize the system of (2.41). A simple method is to find *s* from the first equation and put it in the second equation.

$$s = \frac{(\rho_* u_* - \rho u)}{(\rho_* - \rho)} \tag{2.42}$$

By putting equation (2.42) to the second equation of (2.41) a formula would be obtained for *u* which is based on ρ.

$$u(\rho) = u_c \pm \sqrt{\left(a^2 \frac{\rho_c}{\rho} + a^2 \frac{\rho}{\rho_c} - 2a^2\right)} \tag{2.43}$$

The subscript c shows right or left value. For every $\rho \neq \rho_c$, 2 values of u exist which belong to a specific wave family. Based on the sign of the equation of (2.43) we find that (+) is for the wave with eigenvalue family of $\lambda^2 = u + a$ and (–) for the wave with eigenvalue family of $\lambda^1 = u - a$.

2.4.1.3. All shock solution

Now assume we have some arbitrary condition of q_L and q_R and we know that both waves are shock type. As said in the previous section there are two formulae which connect the middle state of q_* to the right and left values. So, the formula which connects the middle state to the right state is:

$$u(\rho) = u_R + \sqrt{\left(a^2 \frac{\rho_R}{\rho} + a^2 \frac{\rho}{\rho_R} - 2a^2\right)} \quad (2.44)$$

And the formula which connects the middle state to the left state is:

$$u(\rho) = u_L - \sqrt{\left(a^2 \frac{\rho_L}{\rho} + a^2 \frac{\rho}{\rho_L} - 2a^2\right)} \quad (2.45)$$

So, as you see there are two equations with two unknowns. This system of equations can be easily solved by substituting the middle state of *u* in one equation to another one. In this case there would be a nonlinear equation which can be solved by the Newton iterative method. The solution of two shock waves has been coded with python and the results are in figure (2-11).

2.4.1.4. Rarefaction wave

As said before the rarefaction waves are expansive, and they are not discontinuous like shock waves. In this condition there would be a function of $\tilde{q}(\frac{x}{t})$ which equals to *q (x, t)*. Keep in mind the $\tilde{q}(\frac{x}{t})$ should satisfy the relation of (2.46).

$$f'(\tilde{q}(x/t)) = (\frac{x}{t})\tilde{q}'(x/t) \quad (2.46)$$

We will consider how equation (2.46) is applied to find the q in the rarefaction wave and the Riemann problem is solved.

If we assume that $\tilde{q}(\varepsilon)$ is a smooth curve that is defined by the variable of ε then this curve is an integral curve of the wave vector r_p. Generally at each point in an integral curve of a vector field , the tangent vector is an eigenvector of $f'(\tilde{q})$(LeVeque 2002).In this condition at each point of $\tilde{q}$ the tangent is an eigenvector of $f'(\tilde{q}(\varepsilon))$.

So, we can use this theory on the example of isothermal Euler gas as below:

$$\tilde{q}'(\varepsilon) = r^1\big(\tilde{q}(\varepsilon)\big) = \begin{bmatrix} 1 \\ \frac{\tilde{q}_2}{\tilde{q}_1} - a \end{bmatrix} \quad (2.47)$$

$$\tilde{q}^{1\prime} = 1 \quad (2.48)$$

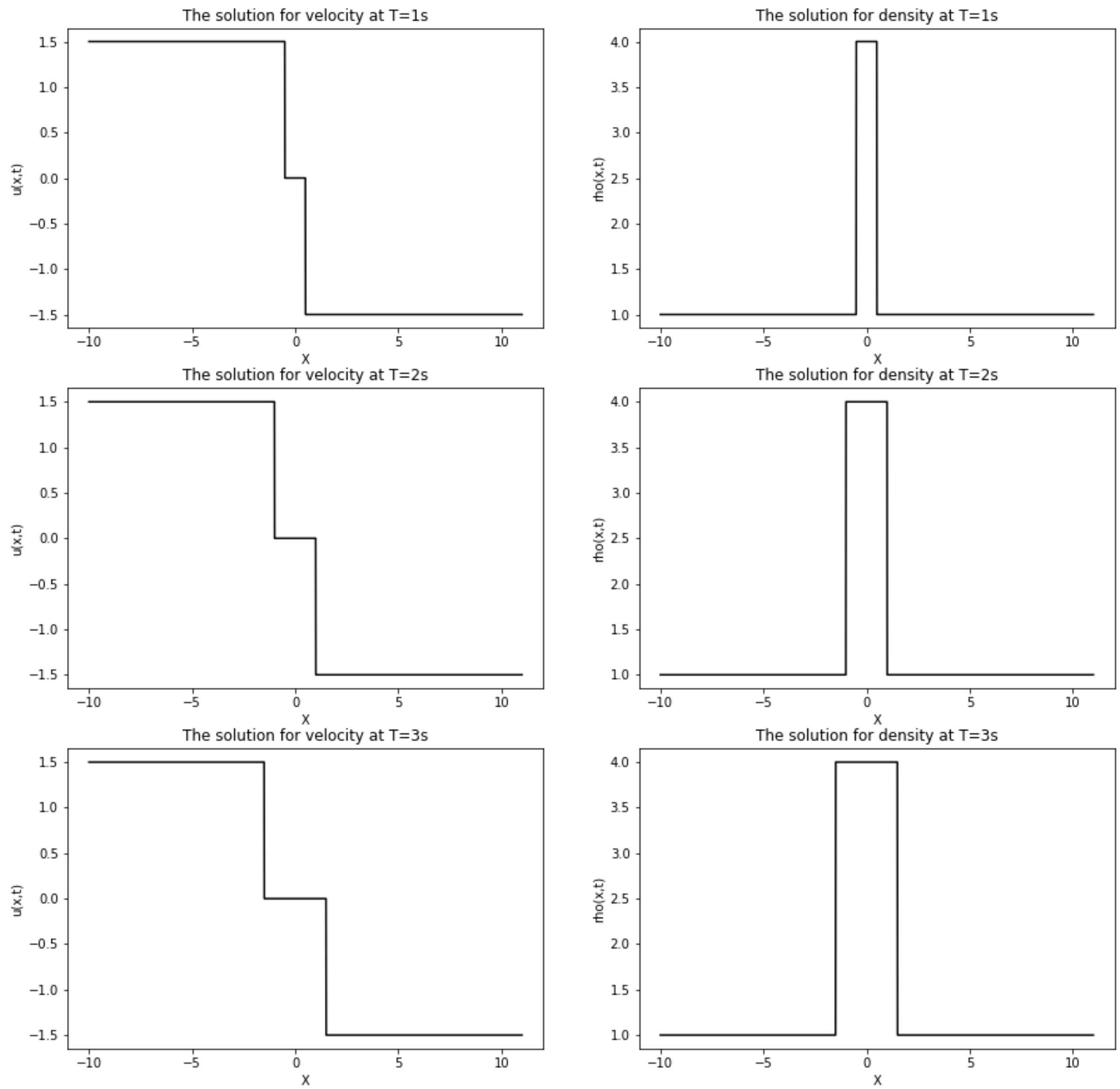

Figure 2-11: The solution for two shock wave at time = 1s,2s,3s with the inputs of $u_L = 1.5$, u_R=-1.5, a=1 , ρ_L=1 and ρ_R=1

And,

$$\tilde{q}^{2\prime} = \frac{\tilde{q}_2}{\tilde{q}_1} - a \tag{2.49}$$

$$\tilde{q}_1\,(\varepsilon) = \varepsilon \tag{2.50}$$

So $\rho = \varepsilon$ and according to the equation of (2.49),

$$\tilde{q}^{2\prime}(\varepsilon) = \frac{\tilde{q}_2}{\varepsilon} - a \tag{2.51}$$

Equation (2.51) is an ODE which can be solved analytically. After solving (2.51) and substituting the value of density and velocity, the final integral curve of (2.52) is obtained.

$$u(\rho) = -a\left[\ln\frac{\rho}{\rho_L} - \frac{u_L}{a}\right] \tag{2.52}$$

Also, we know that if $\lambda^1(q_L) < \lambda^1(q_*)$ the wave number 1 is a rarefaction wave and the smooth curve of q during a rarefaction wave is as below:

$$\rho(\varepsilon) = \rho_L . e^{\frac{u_L - a - \varepsilon}{a}} \tag{2.53}$$

If we follow the previous procedure for the second wave, we reach to the following equations. Also, like the first wave if $\lambda^2(q_r) < \lambda^2(q_*)$, the second wave is rarefaction.

$$u(\rho) = a\left[\ln\frac{\rho}{\rho_R} + \frac{u_R}{a}\right] \tag{2.54}$$

And,

$$\rho(\varepsilon) = \rho_R . e^{\frac{\varepsilon - u_R - a}{a}} \tag{2.55}$$

2.4.1.5. Two rarefactions

Now we consider two arbitrary values for q_L and q_R in a way that we know the two waves are rarefaction. So, the final form of the solution for one rarefaction wave is:

$$q(x,t) = \begin{cases} q_l & \text{if} \quad \frac{x}{t} < \varepsilon_1 \\ \tilde{q}\left(\frac{x}{t}\right) & \text{if}\ \varepsilon_1 < x/t < \varepsilon_2 \\ q_r & \text{if} \quad \frac{x}{t} > \varepsilon_2 \end{cases} \tag{2.56}$$

$$\varepsilon_1 = \lambda(q_l) \quad \varepsilon_2 = \lambda(q_r) \tag{2.57}$$

So, in a two-rarefaction wave we can find the middle state by solving the equations of (2.52) and (2.54). Like for two shock waves, this equation can be solved by the iterative Newton method. The solution of two rarefaction waves has been coded in Python and some results are below:

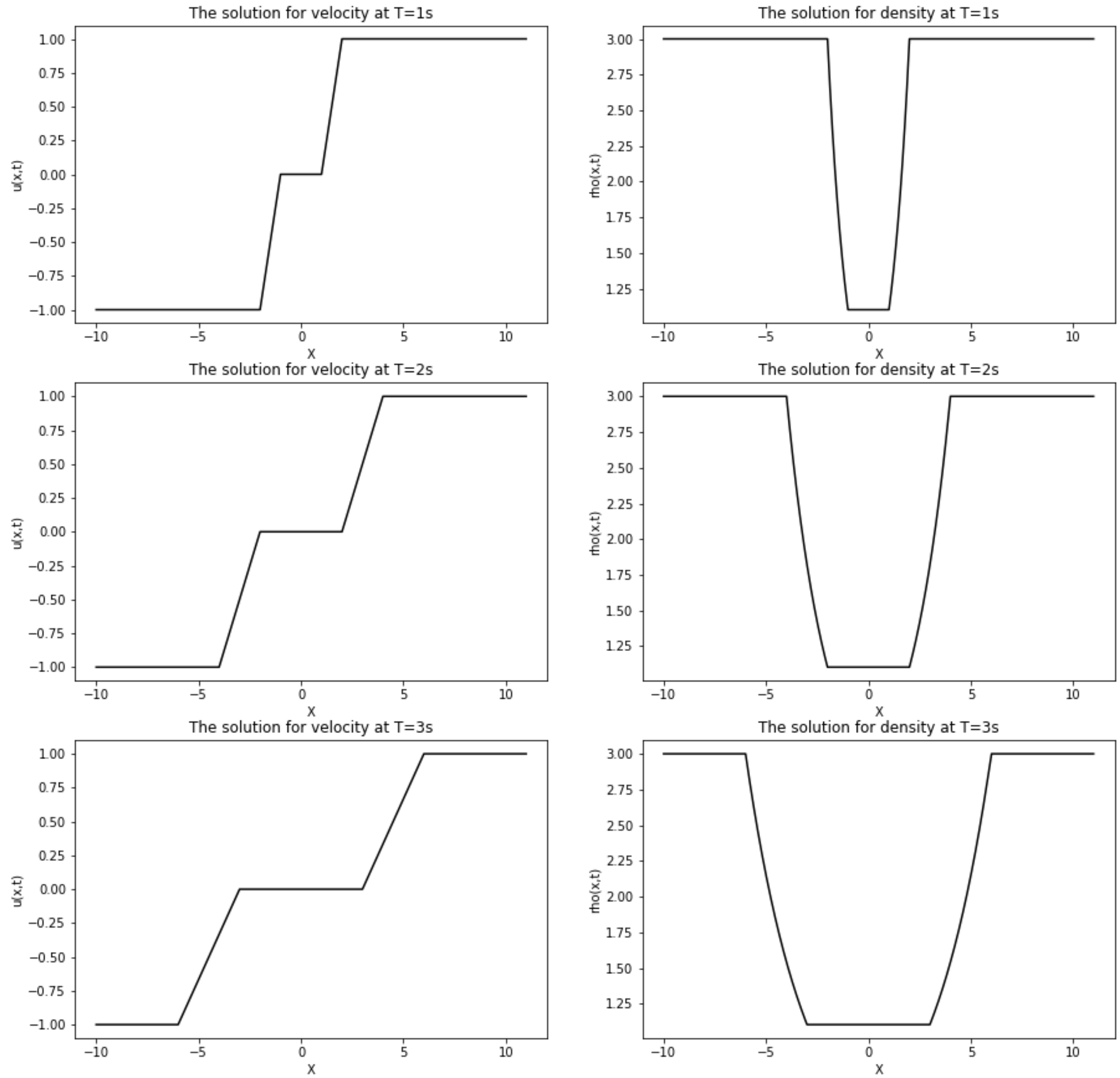

Figure 2-12: The solution for two rarefaction wave at time = 1s,2s,3s with the inputs of u_L = -1 , u_R=1, a=1 , ρ_L=3 and ρ_R=3

2.4.1.6. The general Riemann solution for Euler isothermal gas

As you see in the figures of (2.11) and (2.12) the density of the middle condition determines the type of waves. If the density of the middle condition is greater than the left density, the first wave is shock and if the density of the middle condition is less than the left density, the first wave is rarefaction. Also, if the density of middle condition is greater than the right density, the second wave is shock and if the density of middle condition is less than the right density, the second wave is rarefaction. So, we can define two functions which are dependent on the middle state density. According to the equations (2.58), if the middle state density is greater than the left density, the left velocity is greater than the middle velocity and according to the left eigenvalue the wave is shock. This statement is also correct for the right wave.

$$\varphi_l(\rho) = \begin{cases} u(\rho) = -a\left[\ln\frac{\rho}{\rho L} - \frac{uL}{a}\right] & \text{if } \rho < \rho_l \\ u(\rho) = u_L - \sqrt{\left(a^2\frac{\rho_L}{\rho} + a^2\frac{\rho}{\rho_L} - 2a^2\right)} & \text{if } \rho > \rho_l \end{cases} \tag{2.58}$$

$$\varphi_r(\rho) = \begin{cases} u(\rho) = a\left[\ln\frac{\rho}{\rho L} + \frac{uL}{a}\right] & \text{if } \rho < \rho_r \\ u(\rho) = u_R + \sqrt{\left(a^2\frac{\rho_R}{\rho} + a^2\frac{\rho}{\rho_R} - 2a^2\right)} & \text{if } \rho > \rho_r \end{cases} \tag{2.59}$$

In the case of $\varphi_l(\rho) = \varphi_r(\rho)$, we can find the middle state of density. Then by the appropriate formula the middle state of the velocity is determined. The whole procedure of isothermal Euler gas tube has been coded in python and the results are below:

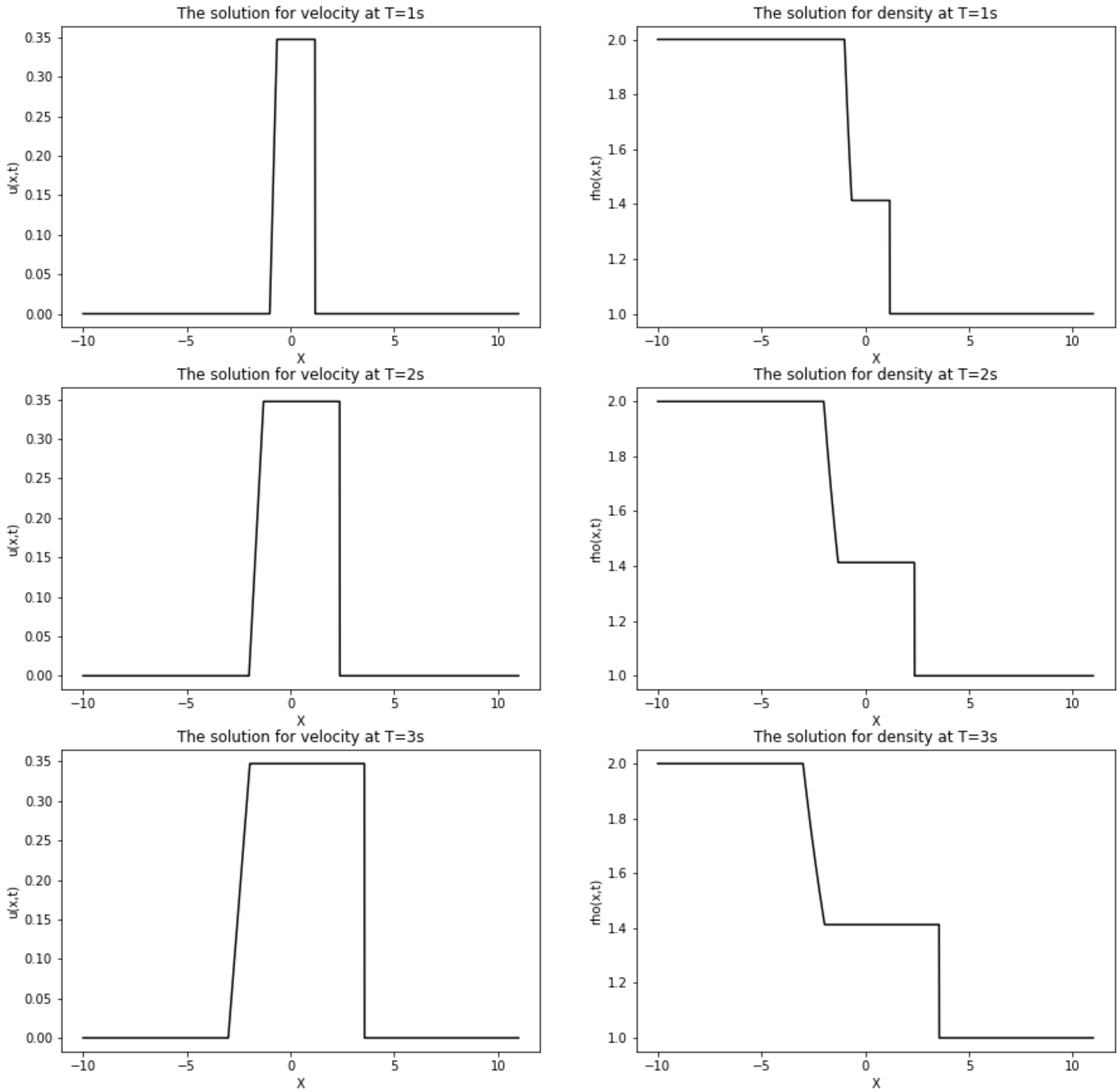

Figure 2-13:The solution for 1-rarefaction 2-shock wave at time = 1s,2s,3s with the inputs of u_L = 0, u_R=0, a=1 , ρ_L=2 and ρ_R=1

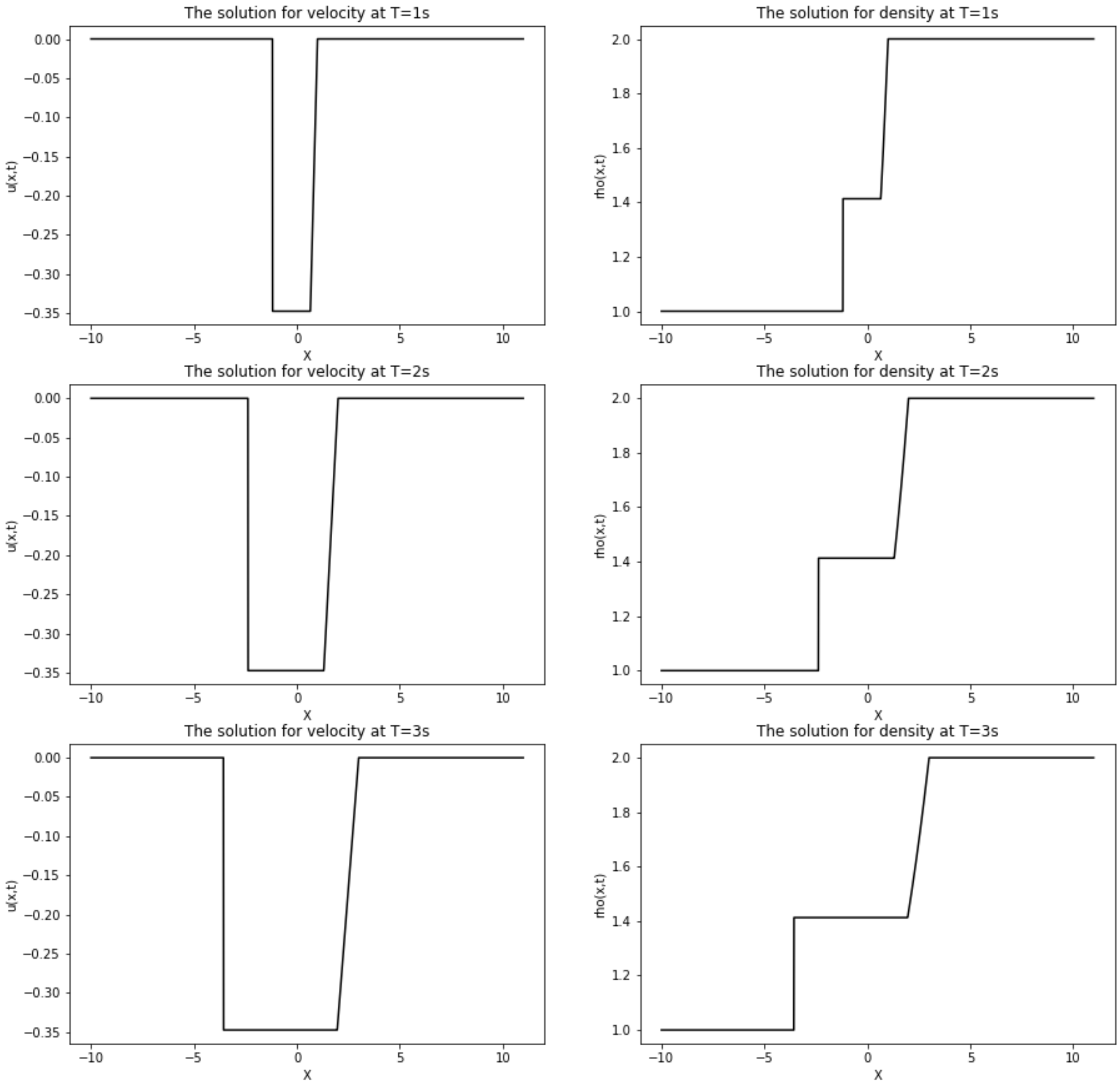

Figure 2-14: The solution for 1-shock 2-rarefaction wave at time = 1s,2s,3s with the inputs of $u_L = 0$, $u_R=0$, $a=1$, $\rho_L=1$ and $\rho_R=2$

2.4.1.7. Numerical verification for Euler isothermal gas

Numerical modeling for Euler isothermal gas flow has been done by Parham Barazesh in his master thesis(Barazesh 2019). He has used a method called X force scheme which consists of predictor and corrector steps. The initial condition which has been used for the Riemann exact solution is the same as figure (2-14).

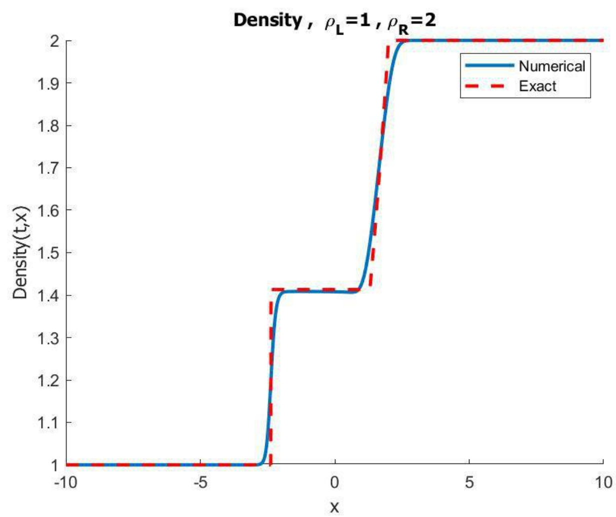

Figure 2-15:Numerical validation for density with the initial condition of figure (2-14)

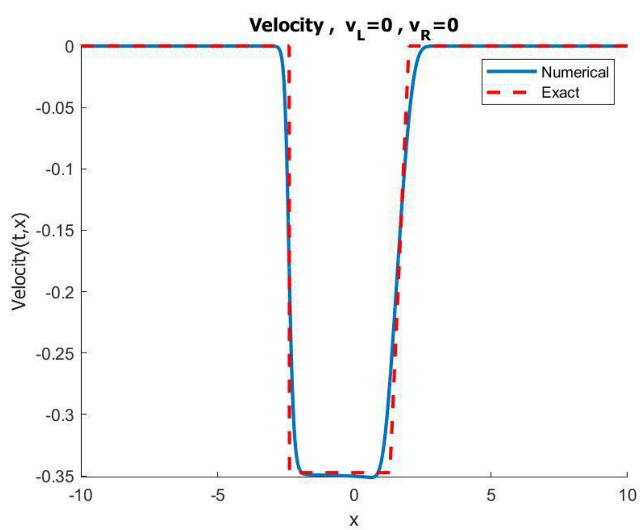

Figure 2-16:Numerical validation for velocity with the initial condition of figure (2-14)

2.5. Riemann Problem for linear systems

In this topic, the characteristic structure of a linear system will be investigated. Further the Riemann problem of linear hyperbolic systems is studied. The understanding of this problem develops a basis for more complex nonlinear system of equations.

The linear hyperbolic system is in the form of equation (2.60) which is like equation (2.21) with this difference that in the Riemann problem for equation (2.21) there is only one jump but for a system of equations there are several jumps simultaneously.

$$q_t + Aq_x = 0 \tag{2.60}$$

The equation of (2.60) is hyperbolic as long as $A \in R^{m\times m}$ can be diagonalized with real eigenvalues, so:

$$A = R\,\Lambda\,R^{-1} \tag{2.61}$$

In the equation of (2.61), R is the matrix of eigenvectors. so, we can introduce a new variable which is defined as:

$$\omega = R^{-1}\,q \tag{2.62}$$

So, equation (2.62) lets us to reduce equation (2.60) to equation (2.63)

$$\omega_t + \Lambda\,\omega_x = 0 \tag{2.63}$$

Now we consider the Euler equation with constant gas speed as a linear system. There are two conservation equations for isothermal gas: mass balance and momentum balance. The procedure for developing the nonlinear Euler isothermal gas equations has been presented in section (2.4.1). Now by considering the constant gas speed, the equation system (2.64) becomes linear.

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_t + \begin{bmatrix} 0 & 1 \\ c^2 - v^2 & 2v \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_x = 0 \tag{2.64}$$

And the initial value for Riemann problem is:

$$q^0 = \begin{cases} q_L \text{ if } x < 0 \\ q_R \text{ if } x > 0 \end{cases} \tag{2.65}$$

In the equation of (2.64), c and v constants are symbols for sound velocity and gas velocity respectively.

So, $A = \begin{bmatrix} 0 & 1 \\ c^2 - v^2 & 2v \end{bmatrix}$ and $\lambda_{1=} v - c$ and $\lambda_{2=} v - c$.

$$R = \begin{bmatrix} 1 & 1 \\ v-c & v+c \end{bmatrix} \tag{2.66}$$

The inverse of matrix R is:

$$R^{-1} = \begin{bmatrix} v+c & -1 \\ c-v & 1 \end{bmatrix} \tag{2.67}$$

According to the equation of (2.62):

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \frac{1}{2c}\begin{bmatrix} (v+c)q_1 - q_2 \\ (c-v)q_1 + q_2 \end{bmatrix} \tag{2.68}$$

According to the characteristic solution of linear hyperbolic scalar:

$$\omega = \omega^0 \, (x - \lambda t) \tag{2.69}$$

So according to the equation of (2.68) the solution for ω is:

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \dfrac{v+c}{2c} \, q_1^0(x-\lambda_1 t) - \dfrac{1}{2c} q_2^0(x-\lambda_1 t) \\ \dfrac{c-v}{2c} q_1^0(x-\lambda_2 t) + \dfrac{1}{2c} q_2^0(x-\lambda_2 t) \end{bmatrix} \tag{2.70}$$

According to (2.62):

$$q(x,t) = R\omega(x,t) \tag{2.71}$$

So, the final solution for q_1 and q_2 is as below:

$$q(x,t) = \begin{bmatrix} \frac{v+c}{2c} \, q_1^0(x-\lambda_1 t) - \frac{1}{2c} q_2^0(x-\lambda_1 t) + \frac{c-v}{2c} q_1^0(x-\lambda_2 t) + \frac{1}{2c} q_2^0(x-\lambda_2 t) \\ \frac{v^2-c^2}{2c} \, q_1^0(x-\lambda_1 t) - \frac{v-c}{2c} q_2^0(x-\lambda_1 t) + \frac{c^2-v^2}{2c} q_1^0(x-\lambda_2 t) + \frac{v+c}{2c} q_2^0(x-\lambda_2 t) \end{bmatrix} \tag{2.72}$$

The solution of equation (2.72) is coded in python and the results for a Riemann initial condition is as below:

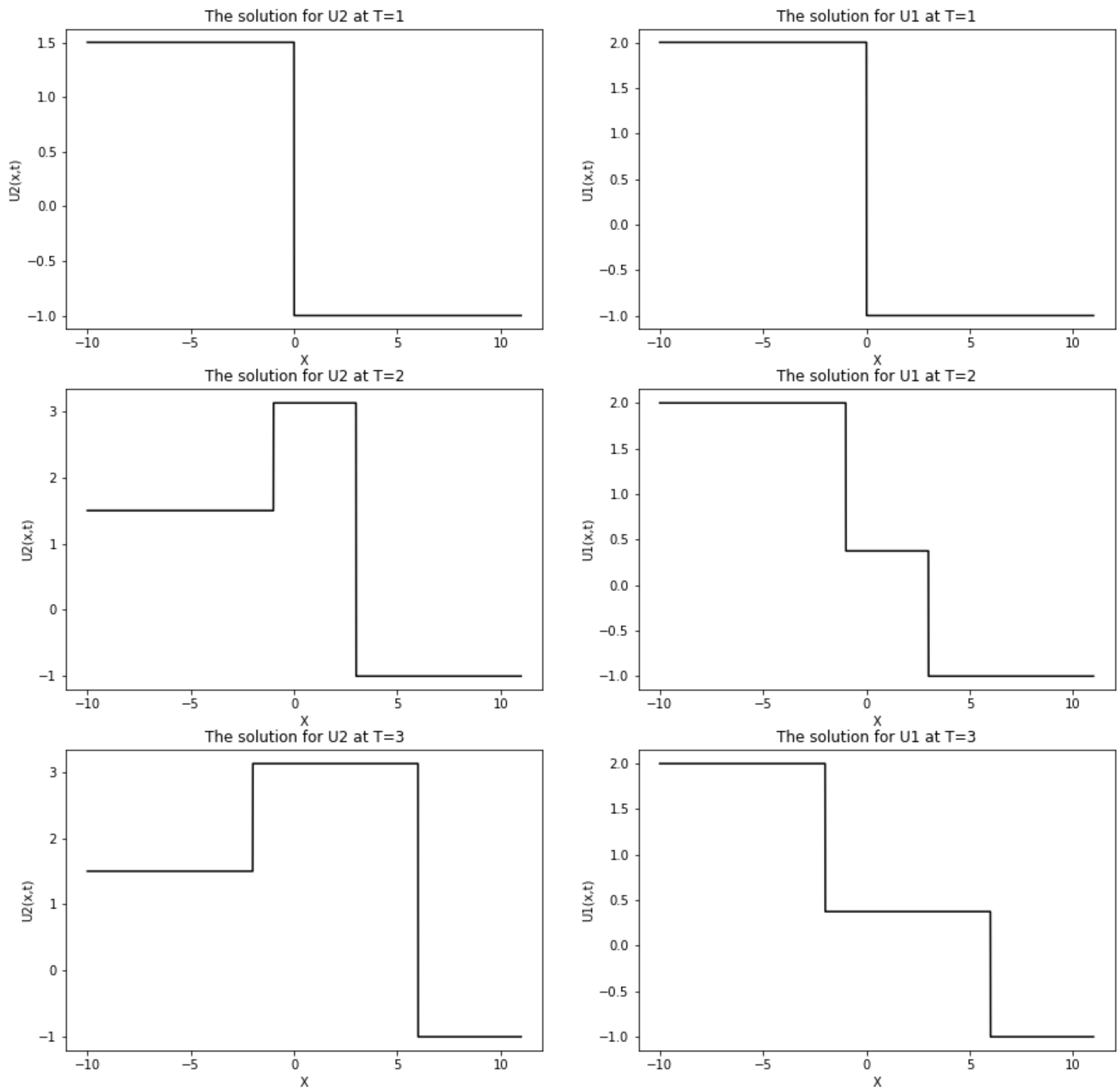

Figure 2-17:The corresponding results for initial condition of U_{1L}=2, U_{1R}= -1, U_{2L} = 1.5, U_{2R}= -1, and constants of c=2, v=1. The graphs are plotted at T_1 = 1, T_2=2, T_3=3

2.5.1.The Riemann problem for full Euler equation

In the previous section we assumed that the gas tube is immersed in a water bath and the temperature is constant. So, the model name was isothermal Euler equation. In this section the term of energy will be introduced, and the full Euler equation will be solved by Riemann problem. The procedure and theory behind is from the book of Riemann solvers and numerical methods for fluid dynamics (Toro 2013).

The initial value problem of full Euler equation in a way that there would be a discontinuity in *x=0*, and left and right values for gas properties, is Riemann problem. The conservation laws and initial conditions are as below:

$$U_t + F(U)_x = 0 \tag{2.73}$$

$$U = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} \quad , F = \begin{bmatrix} \rho u \\ \rho u^2 + P \\ u(E+P) \end{bmatrix} \tag{2.74}$$

Where E is the energy of gas,

The initial condition of Riemann problem is as below:

$$U(x,0) = \begin{cases} U_L & \text{if } x < 0 \\ U_R & \text{if } x > 0 \end{cases} \tag{2.75}$$

In solving this nonlinear system, it is better to use the primitive variables *(ρ, u, p)* where u is the particle velocity, p is the pressure and ρ is the density. As you see the initial condition (2.75) consists of two constants of the primitive variables for two sides (U_L and U_R).Hence for shock waves the Rankine-Hugoniot condition should be determined on the conservation equations but the final formula is again based on the primitive variables. Like the previous section the physical interpretation of Riemann problem is the gas inside a tube separated by a membrane. When the membrane is torn, this discontinuity would deform, and each wave propagates with own speed and shape.

2.5.1.1. Pressure and velocity equations

In this section, the related equations for determining the middle value of pressure and velocity are presented.

The root of the following equation gives the middle value pressure.

$$f(p, U_L, U_R) = f_L(p, U_L) + f_R(p, U_R) + \Delta u = 0 , \qquad \Delta u = u_R - u_L \tag{2.76}$$

The function for f_L is given by (2.77):

$$f_L(p, UL) = \begin{cases} (p - p_L)\left[\frac{A_L}{p + B_L}\right]^{\frac{1}{2}} & \text{if } p > p_L \ (shock) \\ \frac{2a_L}{(\gamma - 1)}\left[\left(\frac{p}{p_L}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] & \text{if } p < p_L \ (Rarefaction) \end{cases} \tag{2.77}$$

And the function for f_R is given by (2.78):

$$f_R(p, UR) = \begin{cases} (p - p_R)\left[\frac{A_R}{p + B_R}\right]^{\frac{1}{2}} & \text{if } p > p_R \ (shock) \\ \frac{2a_R}{(\gamma - 1)}\left[\left(\frac{p}{p_R}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] & \text{if } p < p_R \ (Rarefaction) \end{cases} \tag{2.78}$$

γ , a_L and a_R are gas heat capacity and sound velocity constants in the left and right sides respectively.

And the constants of A_R , B_R , A_L and B_L are determined as below:

$$A_L = \frac{2}{(\gamma + 1)\rho_L} , B_L = \frac{(\gamma - 1)}{(\gamma + 1)} p_L \tag{2.79}$$

$$A_R = \frac{2}{(\gamma + 1)\rho_R} , B_R = \frac{(\gamma - 1)}{(\gamma + 1)} p_R \tag{2.80}$$

And the velocity for middle state is determined by (2.81):

$$u = \frac{1}{2}(u_L + u_R) + \frac{1}{2}[f_R(p_m) - f_L(p_m)] \tag{2.81}$$

As depicted in the figure (2-18), there are two possibilities for the right and left waves. There are shock or rarefaction. Also, there would be a contact discontinuity between these two waves showing the eigen value of u. Hence during the middle discontinuity just density changes.

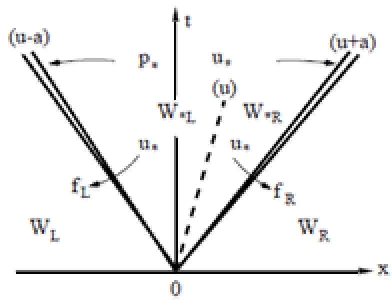

Figure 2-18:Structure of waves in Riemann solution for full Euler equations(Toro 2013)

For obtaining the formula for the function of *f* based on the wave type, shock or rarefaction the below conditions should be considered:

If function *f* is for shock wave, the Rankine-Hugoniot condition should be satisfied on the conservation laws then the necessary relations for function *f* can obtained based on the primitive variables.

If function *f* is for rarefaction wave, the Riemann invariants should be implemented on the primitive variables. So, by the integral curve the necessary formula for function *f* are obtained.

2.5.1.2. function of f_L for left shock

For the proof of left shock wave equation of f_L (2.77) you can read (Toro 2013). The middle left density is obtained by the equation of (2.82).

$$\rho_{mL} = \rho_L \left[\frac{\frac{(\gamma-1)}{(\gamma+1)} + \frac{p_m}{p_L}}{\frac{(\gamma-1)}{(\gamma+1)}\left(\frac{p_m}{p_L}\right)+1} \right] \tag{2.82}$$

And the middle value of velocity is obtained by (2.83):

$$u_m = u_L - f_L(p_m, UL) \tag{2.83}$$

2.5.1.3. function of f_L for left rarefaction

For the proof of left rarefaction wave equation of f_L (2.74) you can read (Toro 2013). The middle left density is obtained by the equation of (2.81).

$$\rho_{mL} = \rho_L \left(\frac{p_m}{p_L}\right)^{\frac{1}{\gamma}} \quad (2.84)$$

$$a_{mL} = a_L \left(\frac{p_m}{p_L}\right)^{\frac{\gamma-1}{2\gamma}} \quad (2.85)$$

a_{mL} is the sound velocity constant in middle left region and the middle value of velocity is obtained by (2.80).

2.5.1.4. function of f_R for right shock

For the proof of left shock wave equation of f_R (2.78) you can read(Toro 2013). The middle right density is obtained by the equation of (2.86).

$$\rho_{mR} = \rho_R \left[\frac{\frac{(\gamma-1)}{(\gamma+1)} + \frac{p_m}{p_R}}{\frac{(\gamma-1)}{(\gamma+1)}\left(\frac{p_m}{p_R}\right) + 1}\right] \quad (2.86)$$

And the middle value of velocity is obtained by (2.87):

$$u_m = u_R + f_R(p_m, UR) \quad (2.87)$$

2.5.1.5.function of f_R for right rarefaction

For the proof of right rarefaction wave equation of f_R (2.78) you can read (Toro 2013). The middle right density is obtained by the equation of (2.88):

$$\rho_{mR} = \rho_R \left(\frac{p_m}{p_R}\right)^{\frac{1}{\gamma}} \quad (2.88)$$

And the middle value of velocity is obtained by (2.87).

2.5.1.6. Pressure numerical solution

There are several methods for solving the equation of (2.76). A reliable and fast solution is obtained by newton iterative method. In this method the behavior of the function is important because in some case the solution may not be convergent. Also, the initial guess should be selected wisely since the function may diverge by some initial guesses. The related formula for newton iterative method is as below:

$$p_1 = p_0 - \frac{f(p_0)}{\acute{f}(p_0)} \tag{2.89}$$

The derivative of equation (2.77) and (2.78) is as below:

$$f'_K = \begin{cases} \left(\frac{A_K}{B_K + p}\right)^{1/2}\left[1 - \frac{p - p_K}{2(p_K + p)}\right] & \text{if } p > p_K \quad \mathit{Shock\ wave} \\ \frac{1}{\rho_K a_K}\left(\frac{p}{p_K}\right)^{-(\gamma+1)/2\gamma} & \text{if } p < p_K \quad \mathit{Rarefaction\ wave} \end{cases} \tag{2.90}$$

As you see in the equation of (2.90) you can find the f'_L *and* f'_R by substitution of K by R or L.

2.5.1.7. Complete solution for left wave

The left wave is either a shock wave or a rarefaction wave. In the case of shock wave, the speed of shock is obtained by (2.91):

$$S_L = u_L - a_L\left[\frac{\gamma+1}{2\gamma}\frac{p_m}{p_L} + \frac{\gamma-1}{2\gamma}\right]^{\frac{1}{2}} \tag{2.91}$$

If the left wave is rarefaction, there would be 2 points which move with different speeds. One point is the wave head and the other one is the wave tail. These two points are obtained as below:

$$S_H = u_L - a_L \text{ and } S_T = u_m - a_{mL} \tag{2.92}$$

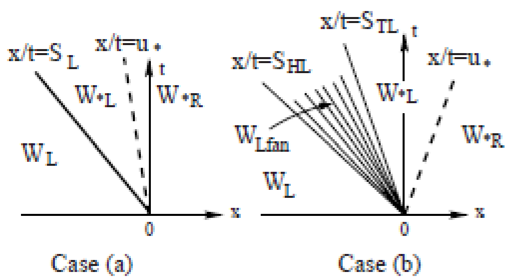

Figure 2-19:Sampling in two possibilities for left wave. (a) is shock condition and (b) is rarefaction condition(Toro 2013)

In the case of rarefaction, the solution inside the wave should be determined by the integral curve.

$$U_{Lfan} = \begin{cases} \rho = \rho_L \left[\frac{2}{(\gamma+1)} + \frac{(\gamma-1)}{(\gamma+1)a_L}(u_L - \frac{x}{t})\right]^{\frac{2}{\gamma-1}} \\ u = \frac{2}{\gamma+1}\left[a_L + \frac{(\gamma-1)}{2}u_L + \frac{x}{t}\right] \\ p = p_L \left[\frac{2}{(\gamma+1)} + \frac{(\gamma-1)}{(\gamma+1)a_L}(u_L - \frac{x}{t})\right]^{\frac{2\gamma}{\gamma-1}} \end{cases} \quad (2.93)$$

2.5.1.8. Complete solution for right wave

Like left wave, right wave is shock wave or rarefaction wave. For the shock wave the shock speed is obtained by (2.94):

$$S_R = u_R + a_R \left[\frac{\gamma+1}{2\gamma}\frac{p_m}{p_R} + \frac{\gamma-1}{2\gamma}\right]^{\frac{1}{2}} \quad (2.94)$$

If the right wave is a rarefaction wave, there would be two point which move with different speeds. One point would be the head and the other one is the tail of the rarefaction wave.

$$S_H = u_R + a_R \ and \ S_T = u_m + a_{mR} \quad (2.95)$$

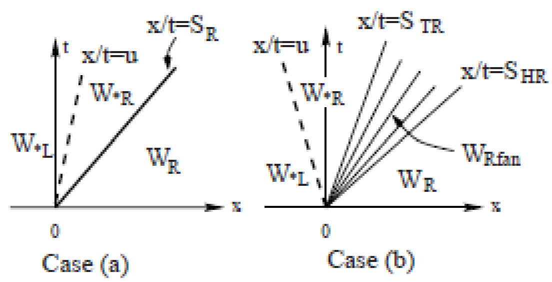

Figure 2-20:Sampling in two possibilities for right wave. (a) is shock condition and (b) is rarefaction condition(Toro 2013)

In the case of right rarefaction wave, the solution inside the wave is obtained by (2.96):

$$U_{Rfan} = \begin{cases} \rho = \rho_R \left[\frac{2}{(\gamma+1)} - \frac{(\gamma-1)}{(\gamma+1)a_L}\left(u_L - \frac{x}{t}\right)\right]^{\frac{2}{\gamma-1}} \\ u = \frac{2}{\gamma+1}\left[-a_R + \frac{(\gamma-1)}{2}u_R + \frac{x}{t}\right] \\ p = p_R \left[\frac{2}{(\gamma+1)} - \frac{(\gamma-1)}{(\gamma+1)a_L}\left(u_R - \frac{x}{t}\right)\right]^{\frac{2\gamma}{\gamma-1}} \end{cases} \quad (2.96)$$

The procedure for solving full Euler equation has been coded in python and the results for all wave type are as below:

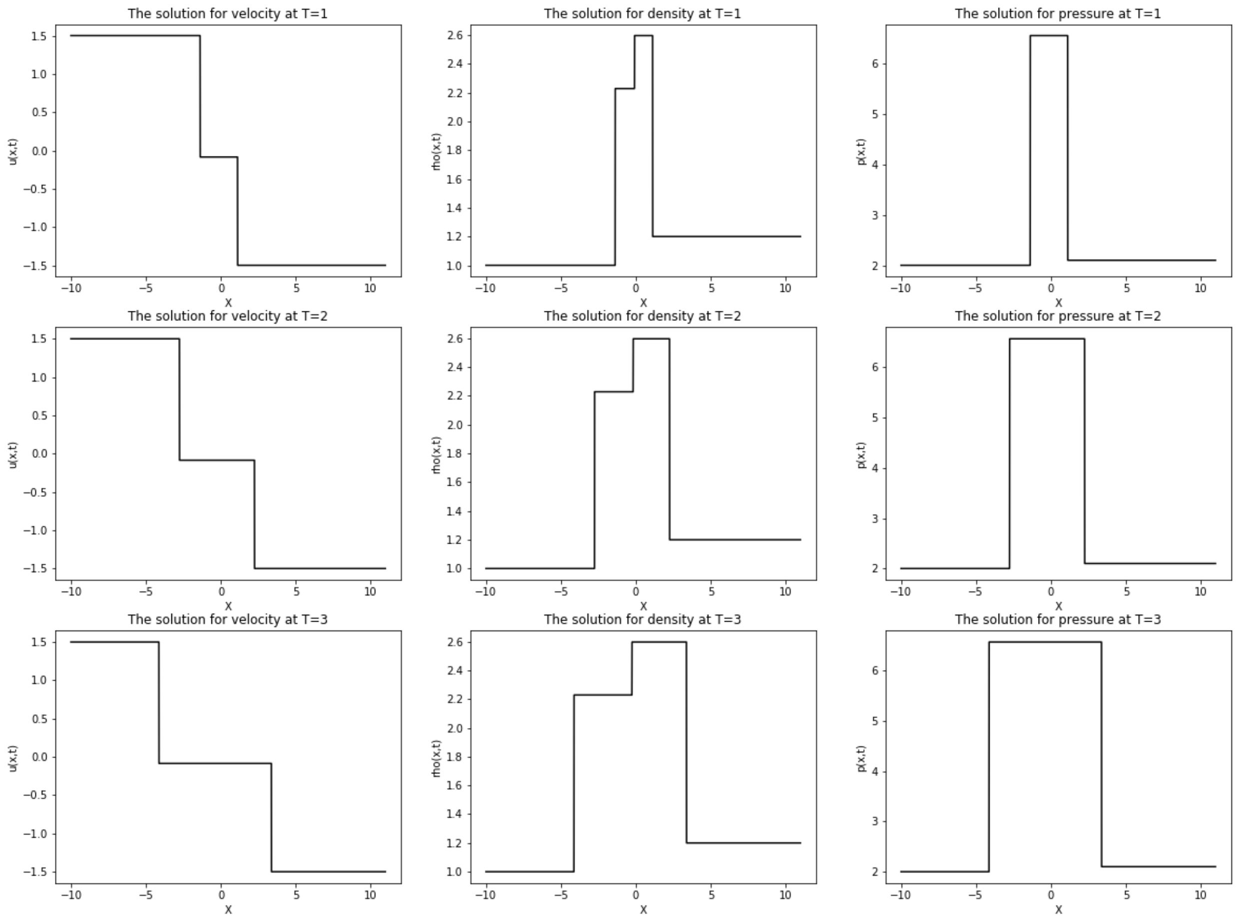

Figure 2-21:All shock Riemann solution for full Euler model. The Initial condition is : u_L=1.5 , u_R= -1.5,p_L=2, p_R=2.1, rho_$_L$= 1 , rho_$_R$= 1.2, $\gamma = 1.4$

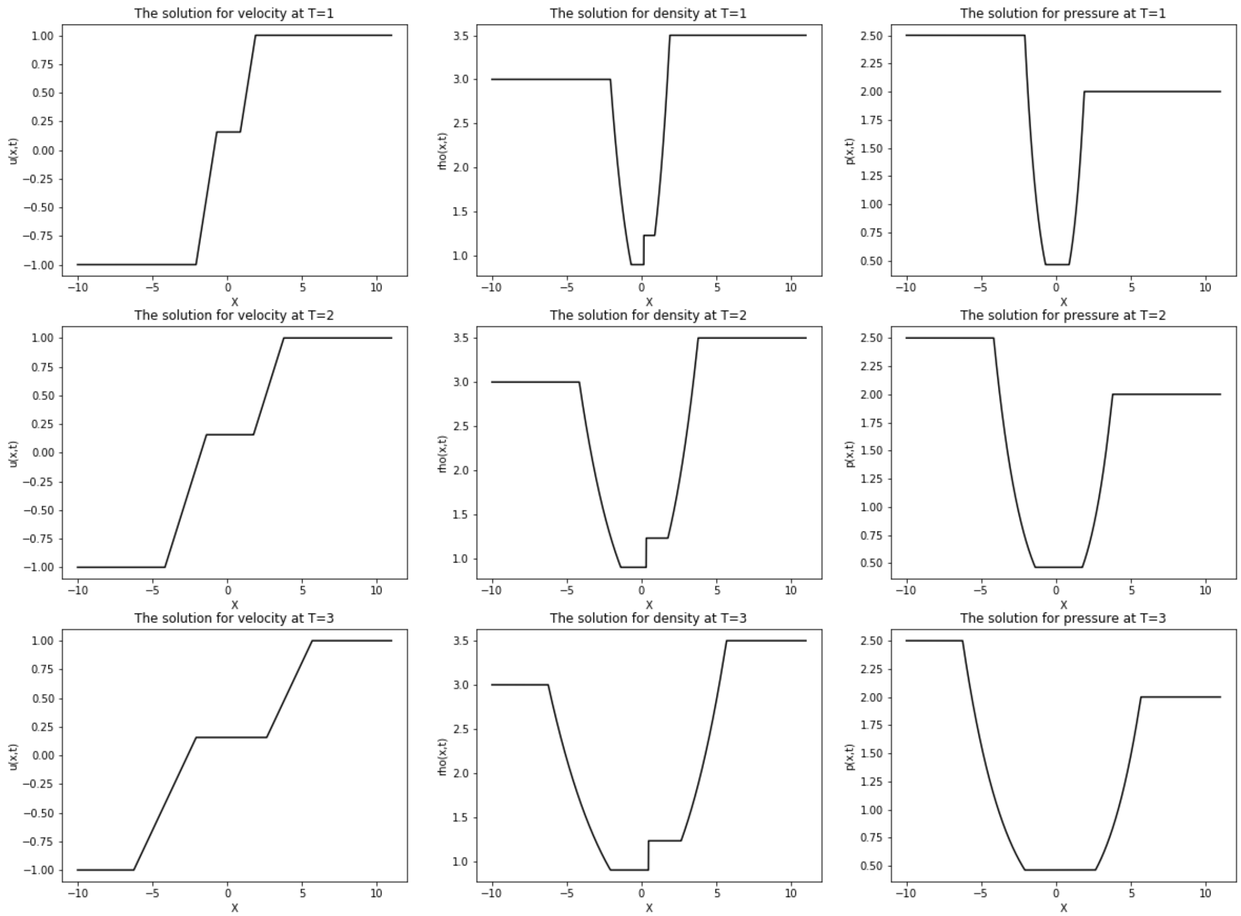

Figure 2-22:All Rarefaction Riemann solution for full Euler model. The Initial condition is: u_L=-1, u_R= 1, p_L=2.5, p_R=2, ρ_L= 3 , ρ_R= 3.5, $\gamma = 1.4$

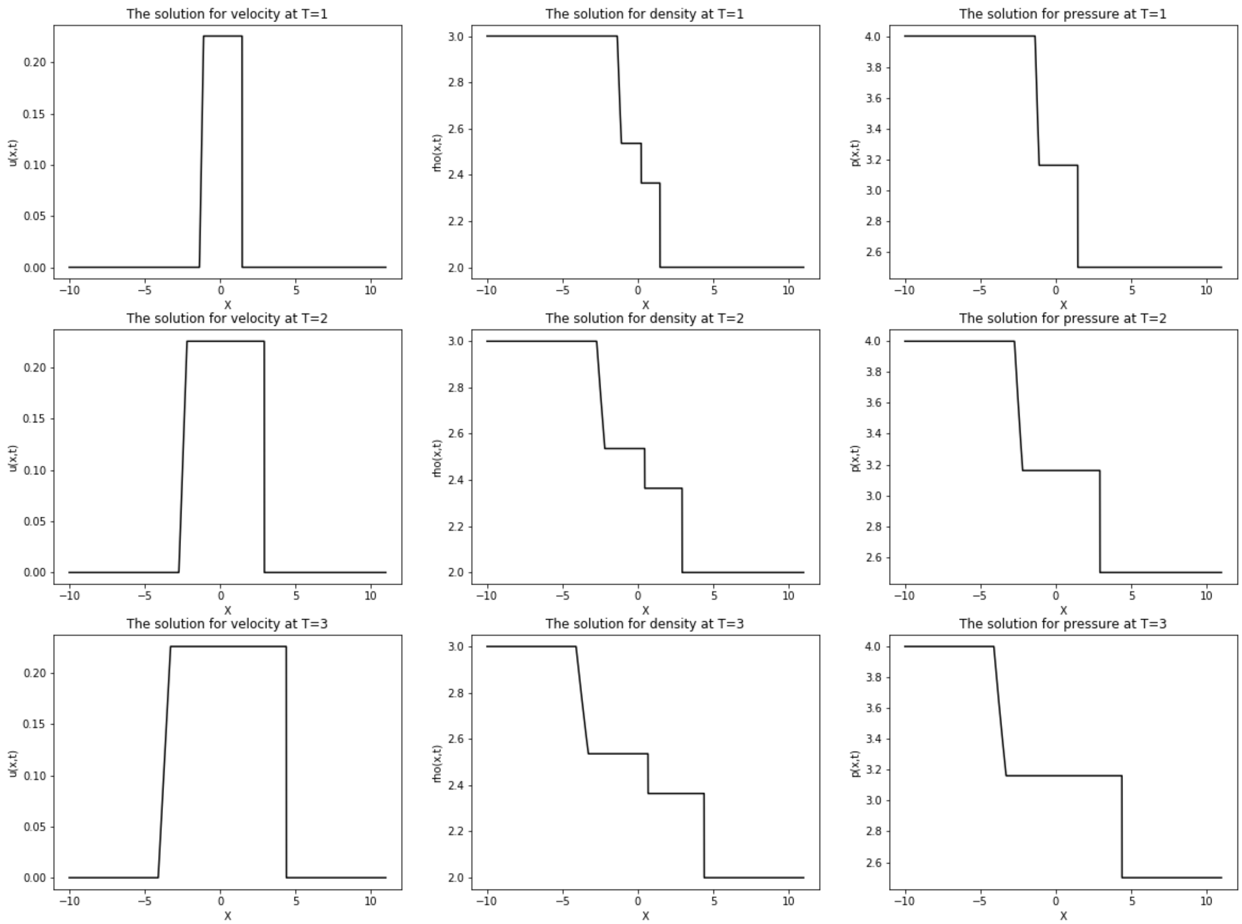

Figure 2-23:1-Rarefaction 2- shock Riemann solution for full Euler model. The Initial condition is: $u_L=0$, $u_R= 0$, $p_L=4$, $p_R=2.5$, $\rho_L= 3$, $\rho_R= 2$, $\gamma = 1.4$

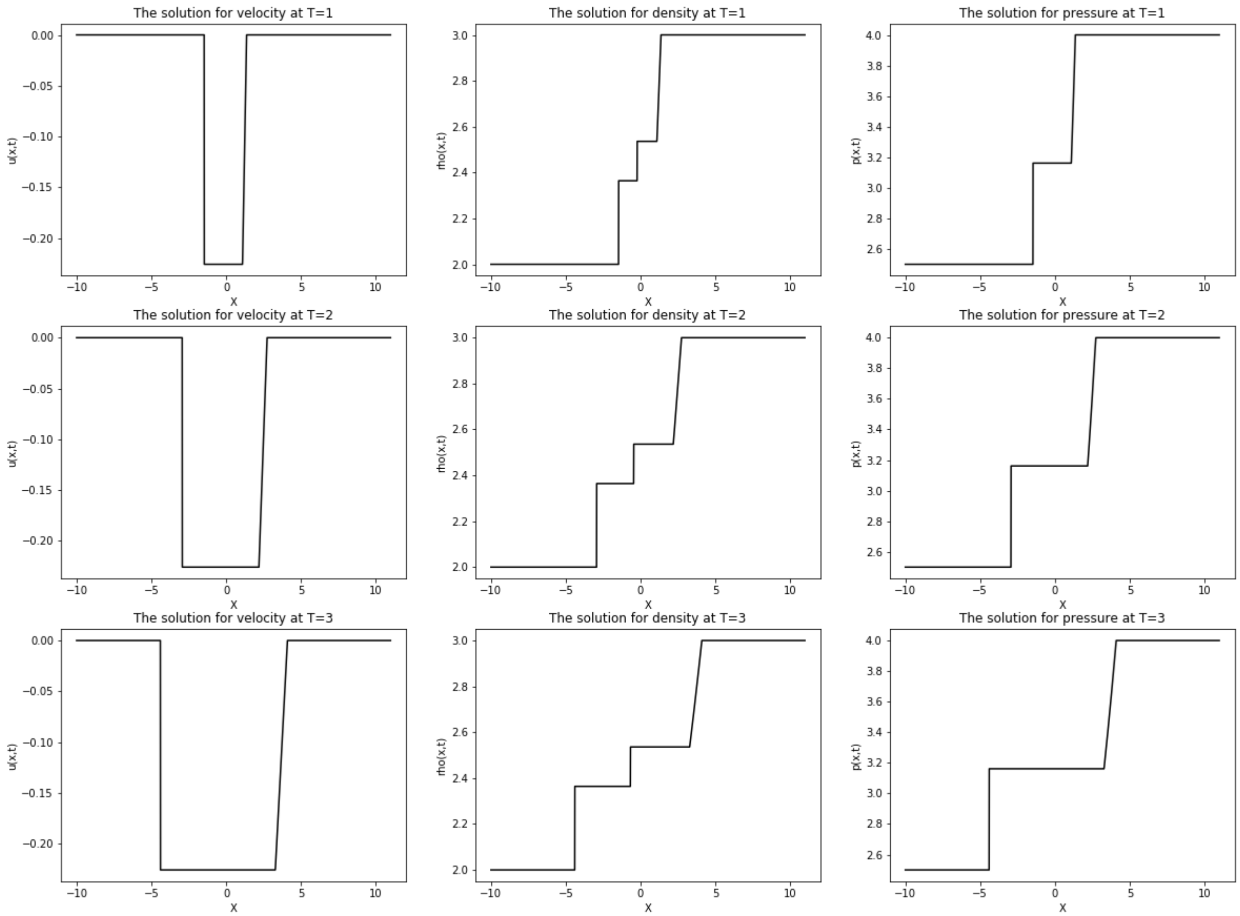

Figure 2-24:1-shock 2- rarefaction Riemann solution for full Euler model. The Initial condition is: u_L=0, u_R= 0, p_L=2.5, p_R=4, ρ_L= 2 , ρ_R= 3, $\gamma = 1.4$

2.5.1.9. Numerical validation for full Euler model

The numerical solution for full Euler model has been done by Parham Barazesh in his master thesis. He has used a method called X force scheme consists of predictor and corrector steps (Barazesh 2019). The initial condition which has been used for the Riemann exact solution is the same as figure (2-24).

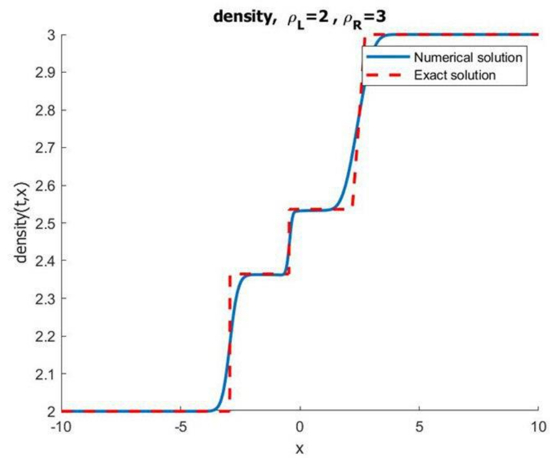

Figure 2-25:Numerical validation for density with initial condition of figure (2-24)

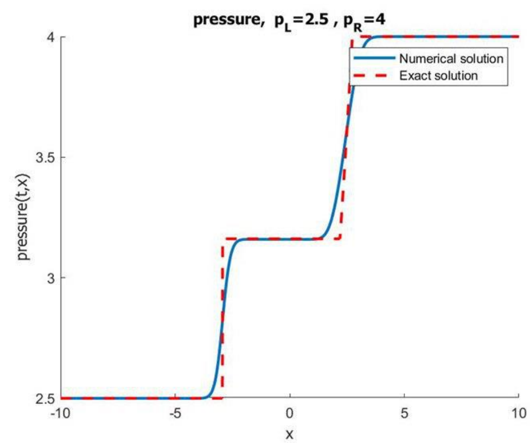

Figure 2-26:Numerical validation for pressure with initial condition of figure (2-24)

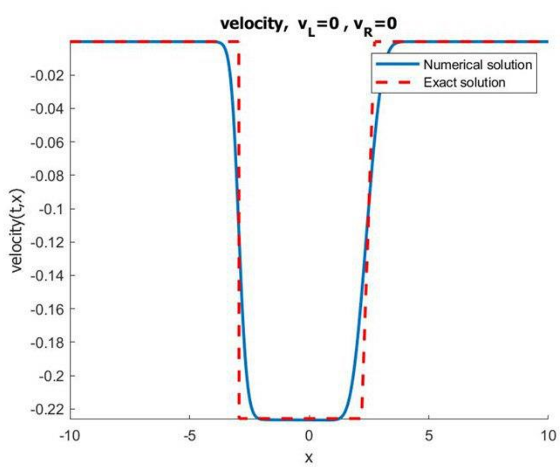

Figure 2-27:Numerical validation for pressure with initial condition of figure (2-24)

In the previous chapter, the Riemann problem was presented for scalar linear equation, system of linear equations, system of nonlinear equations including isothermal gas and full Euler equations. In this chapter the Riemann problem for two phase flow drift flux model is going to be developed and solved. For two phase flow, three conservation equations including conservation of gas mass, liquid mass and total momentum exist.

$$\frac{\partial(\rho_g\alpha_g)}{\partial t}+\frac{\partial(\rho_g\alpha_g u)}{\partial x}=0 \tag{3.1}$$

$$\frac{\partial(\rho_l\alpha_l)}{\partial t}+\frac{\partial(\rho_l\alpha_l u)}{\partial x}=0 \tag{3.2}$$

$$\frac{\partial\left[(\rho_g\alpha_g+\rho_l\alpha_l)u\right]}{\partial t}+\frac{\partial\left[(\rho_g\alpha_g+\rho_l\alpha_l)u^2+p\right]}{\partial x}=0 \tag{3.3}$$

The volumetric concentration of gas is obtained by equation (3.4):

$$y=\frac{\rho_g\alpha_g}{\rho} \tag{3.4}$$

Here α_g is gas volume percentage and y is the gas mass concentration.

By substituting equation (3.4) in equation (3.1) a new equation would be obtained:

$$\frac{\partial(\rho y)}{\partial t}+\frac{\partial(\rho y u)}{\partial x}=0 \tag{3.5}$$

The mixture density is obtained by equation (3.6):

$$\rho=\ \rho_g\alpha_g+\rho_l\alpha_l \tag{3.6}$$

Here α_l is liquid volume percentage.

By substituting the equation of (3.6) in equation (3.2) a new equation would be obtained:

$$\frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{3.7}$$

Also, by substituting the equation of (3.6) in equation of (3.3) a new equation would be obtained:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} = 0 \tag{3.8}$$

Now the equations of (3.5), (3.7) and (3.8) are another form for representing two phase flow conservation equations. Now we can define the function of *f* and *q* like in section (2-4-1).

$$q = \begin{bmatrix} \rho y \\ \rho \\ \rho u \end{bmatrix} \tag{3.9}$$

$$f = \begin{bmatrix} \rho y u \\ \rho u \\ \rho u^2 + p \end{bmatrix} \tag{3.10}$$

For Riemann problem, solving for the vector of primitive variables make the problem easier. The primitive variables are pressure, velocity and density.

$$w = [\rho, u, p] \tag{3.11}$$

It is possible to reform the two-phase flow conservations laws to rewrite the equations in terms of primitive variables.

If we expand (3.5), a new equation would be obtained for y:

$$y\left(\frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho u)}{\partial x}\right) + \rho\left(\frac{\partial y}{\partial t} + u\frac{\partial y}{\partial x}\right) = 0 \tag{3.12}$$

According to the equation of (3.7), the first term of equation (3.12) is equal to zero. So, the equation of (3.12) turns to:

$$\frac{\partial y}{\partial t} + u\frac{\partial y}{\partial x} = 0 \tag{3.13}$$

By expanding the equation of (3.8), a new equation will be obtained for velocity:

$$u\left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x}\right] + \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right] + \frac{\partial p}{\partial x} = 0 \tag{3.14}$$

Based on equation (3.7), Again the first term of equation (3.14) is equal to zero. So, a new equation for the primitive variable of *u* is obtained:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{3.15}$$

For obtaining an equation for pressure we should introduce a new function (3.16).

$$D_t = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \tag{3.16}$$

$$D_t y = 0 \tag{3.17}$$

$$D_t p = \left(\frac{\partial p}{\partial \rho}\right)_y D_t \rho + \left(\frac{\partial p}{\partial y}\right)_\rho D_t y \tag{3.18}$$

$$D_t p = c^2 D_t \rho \tag{3.19}$$

$$D_t \rho + \rho\frac{\partial u}{\partial x} = 0 \tag{3.20}$$

By mixing the equations of (3.19) and (3.20), the equation of (3.21) would be obtained:

$$D_t p + \rho c^2\frac{\partial u}{\partial x} = 0 \tag{3.21}$$

If we expand the equation of (3.21), another equation will be obtained for the primitive variable of p.

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho c^2\frac{\partial u}{\partial x} = 0 \tag{3.22}$$

So, three equations of two-phase flow can be written as equation (3.33):

$$\begin{bmatrix} u \\ p \\ y \end{bmatrix}_t + \begin{bmatrix} u & \rho & 0 \\ c\rho^2 & u & 0 \\ 0 & 0 & u \end{bmatrix}\begin{bmatrix} u \\ p \\ y \end{bmatrix}_x = 0 \tag{3.23}$$

The jacobian matrix for equation (3.23) is the second matrix which is multiplied by q_x and the eigen values for this matrix is as below:

$$\lambda_1 = u - c \quad \lambda_2 = u \quad \lambda_3 = u + c \tag{3.24}$$

3.1. The Riemann problem

The Riemann problem for two phase flow equations is the initial value problem for conservation laws of (3.10).

$$U_t + F(U)_x = 0 \tag{3.25}$$

$$U(x,0) = U_0(x) = \begin{cases} U_L & \text{if } x < 0 \\ U_R & \text{if } x > 0 \end{cases} \tag{3.26}$$

As mentioned previously, in Riemann problem it is better to use primitive variables instead of conserved variables.

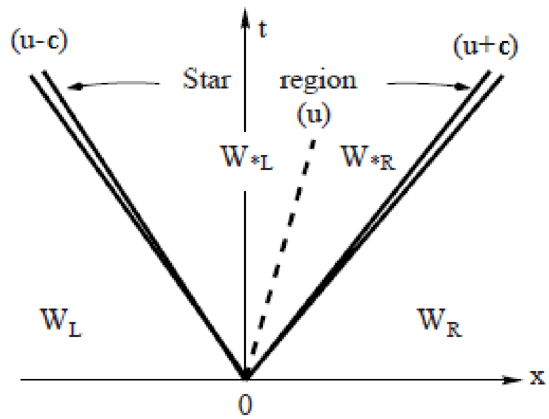

Figure 3-1:Riemann solution structure for two phase flow in x-t plan

The exact Riemann solution for two phase flow has three waves and the eigen values of these wave were determined as equations (3.24). According to figure (3-1), theses waves separate the space into four zones. W_R is data for right hand side, W^*_R is the data for second region from right, W^*_L is the data for third region and W_L is the data for left hand side. The middle wave creates two sub regions by a contact discontinuity however the first and third waves are either shock or rarefaction wave. By further analyzing the eigen-structure it can be shown that the pressure and velocity doesn't change across the middle contact discontinuity. So just the gas mass concentration and the mixture density change across the middle wave. So, the unknowns are p*, u*, ρ^*_R and ρ^*_L.

3.1.1.Equations for pressure and velocity

To find the pressure and velocity in the star zone we should find all equations which relate the right state and left state to the middle state. For finding these equations, we should consider all conditions which may happen including all shock wave, all rarefaction wave, 1-shock 2-rarefaction and 1-rarefaction 2- shock.

For this problem we have assumed that the liquid is incompressible and the velocities of two phase are the same, so the slip velocity is zero.

$$p = a^2 \rho_g \tag{3.27}$$

So, by substituting the equation of (3.27) in term of ρ_g in the equation of (3.4) a new equation for pressure would be obtained:

$$p = \frac{\rho \rho_l a^2 y}{\rho_l - \rho(1-y)} \tag{3.28}$$

Here ρ_l is the density of incompressible fluid.

Also, by taking derivation of equation (3.28) by ρ , an equation for c is obtained:

$$c = \frac{\partial p}{\partial \rho}$$
$$c = \frac{\rho_l a \sqrt{y}}{\rho_l - \rho(1-y)} \tag{3.29}$$

3.1.1.1. The velocity equation across left shock wave

For left shock wave with the speed of S_L we assume that the left shock is an immobile boundary which the parameters in each side can be defined as figure (3.2).

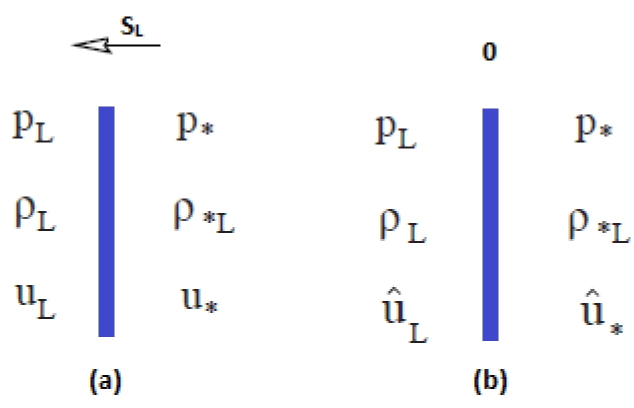

Figure 3-2 Left Shock wave (a) the speed of wave is S_L (b) the new frame in which shock speed is zero and relative velocities are defined

$$\hat{u}_L = u_L - S_L \quad \text{and} \quad \hat{u}_* = u_* - S_L \tag{3.30}$$

By using Rankine-Hugoniot condition as described in section (2.3.1) for the conservation equations of (3.5) ,(3.7) and (3.8) the following equations are obtained:

$$\rho_L \hat{u}_L = \rho_* \hat{u}_* \tag{3.31}$$

$$\rho_L \hat{u}_L^2 + p_L = \rho_* \hat{u}_*^2 + p_* \tag{3.32}$$

$$y_L = y_* \tag{3.33}$$

Now mass flux rate across left shock can be introduced as:

$$Q_L = \rho_L \hat{u}_L = \rho_* \hat{u}_* \tag{3.34}$$

Now we can use equation (3.32) to solve mass flux based on pressure:

$$Q_L = -\frac{p_* - p_L}{\hat{u}_* - \hat{u}_L} \tag{3.35}$$

From equation (3.30):

$$Q_L = -\frac{p_* - p_L}{u_* - u_L} \tag{3.36}$$

From equation (3.36) we obtain:

$$u_* = u_L - \frac{p_* - p_L}{Q_L} \tag{3.37}$$

Now we are close to obtain the middle velocity just based on middle pressure and left-hand side values. We can use the following relations:

$$\hat{u}_L = \frac{Q_L}{\rho_L} \;, \quad \hat{u}_* = \frac{Q_L}{\rho_*}$$

The equation of (3.35) would turn to:

$$Q_L^2 = -\frac{p_* - p_L}{\dfrac{1}{\rho_{*L}} - \dfrac{1}{\rho_L}} \tag{3.38}$$

Using the equation of (3.39) the density in the star zone can be related to the density behind the shock wave.

$$\rho_{*L} = \frac{p_* \rho_L \rho_l}{p_L \rho_l - p_L \rho_L (1 - y_L) + p_* \rho_L (1 - y_L)} \tag{3.39}$$

Here ρ_l is the incompressible fluid density. By substituting (3.39) in (3.38), then (3.37), a new equation for velocity in star zone would be accomplished .

$$u_* = u_L - \sqrt{\left[\left(\frac{p_L}{p_*\rho_L} - \frac{p_L(1-y_L)}{p_*\rho_l} + \frac{1-y_L}{\rho_l} - \frac{1}{\rho_L}\right)(p_L - p_*)\right]} \tag{3.40}$$

Now we define a new function which describe the star region velocity:

$$u_* = u_L + f_L(p_*, W_L) \tag{3.41}$$

So f_L for left shock is determined by equation (3.40) and (3.41):

$$f_L(p_*, W_L) = -\sqrt{\left[\left(\frac{p_L}{p_*\rho_L} - \frac{p_L(1-y_L)}{p_*\rho_l} + \frac{1-y_L}{\rho_l} - \frac{1}{\rho_L}\right)(p_L - p_*)\right]} \tag{3.42}$$

3.1.1.2. The velocity equation for left rarefaction wave

For the case of left rarefaction wave, the generalized Riemann invariant is used to make connection between unknown star region and the left-hand side value.

Any curves which connect the left-hand side to the star region should satisfy the following equation:

$$\frac{\partial u}{\partial \rho} = \pm \frac{c(\rho)}{\rho} \tag{3.43}$$

The plus or minus depends on which wave family is considered. By substituting (3.29) in (3.43), for left rarefaction wave the Riemann invariant would be:

$$u_L - u_* = -\int_{\rho_*}^{\rho_L} \frac{\rho_l a\sqrt{y_L}}{[\rho_l - \rho(1-y_L)]\rho} \partial\rho \tag{3.44}$$

The integral of (3.44) can be solved analytically:

$$u_* = u_L + a\sqrt{y}\ln\left[\left(\frac{1}{\rho_*} - \frac{1-y_L}{\rho_l}\right)\left(\frac{\rho_L}{1-\frac{1-y_L}{\rho_l}\rho_L}\right)\right] \tag{3.45}$$

By substituting (3.39) in (3.45), the final equation for velocity across left rarefaction wave is obtained.

$$u_* = u_L + a\sqrt{y}\ln\left[\left(\frac{p_L}{p_*\rho_L} - \frac{p_L}{p_*\rho_l}(1-y_L)\right)\left(\frac{\rho_L}{1-\frac{1-y_L}{\rho_l}\rho_L}\right)\right] \tag{3.46}$$

So f_L for left rarefaction is determined by equation (3.46) and (3.41):

$$f_L(p_*, W_L) = a\sqrt{y_L}\ln\left[\left(\frac{p_L}{p_*\rho_L} - \frac{p_L}{p_*\rho_l}(1-y_L)\right)\left(\frac{\rho_L}{1-\frac{1-y_L}{\rho_l}\rho_L}\right)\right] \tag{3.47}$$

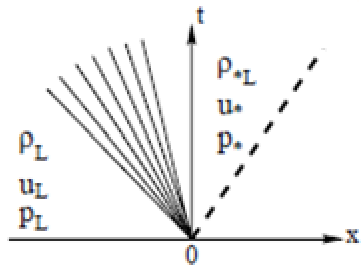

Figure 3-3:Left rarefaction wave which connect left hand side condition to unknown left star zone

3.1.1.3. The velocity equation for right shock wave

Like left shock wave, for right rarefaction wave we consider a frame in which the right shock speed is zero and relative velocities are defined as below:

$$\hat{u}_R = u_R - S_R \quad \text{and} \quad \hat{u}_* = u_* - S_R \tag{3.48}$$

Also, by using Rankine-Hugoniot condition as described in section of (2.3.1) for the conservation equations of (3.5) ,(3.7) and (3.8) the following equations are obtained:

$$\rho_R\hat{u}_R = \rho_*\hat{u}_* \tag{3.49}$$

$$\rho_R\hat{u}_R^2 + p_R = \rho_*\hat{u}_*^2 + p_* \tag{3.50}$$

$$y_R = y_* \tag{3.51}$$

Now mass flux rate across left shock can be introduced as:

$$Q_R = -\rho_R\hat{u}_R = -\rho_*\hat{u}_* \tag{3.52}$$

Now by using equation (3.50), an expression for mass flux rate is obtained:

$$Q_R = \frac{p_* - p_R}{u_* - u_R} \tag{3.53}$$

So,

$$u_* = u_R + \frac{p_* - p_R}{Q_R} \tag{3.54}$$

The equation of (3.53) can be written like equation (3.55):

$$Q_R^2 = \frac{p_* - p_R}{-\frac{1}{\rho_{*L}} + \frac{1}{\rho_L}} \tag{3.55}$$

The density of right star zone can be related to the right value of density by following equation:

$$\rho_{*R} = \frac{p_* \rho_R \rho_l}{p_R \rho_l - p_R \rho_R (1 - y_R) + p_* \rho_R (1 - y_R)} \tag{3.56}$$

By substituting equation of (3.56) in (3.55) then the result in (3.54) , the final equation for star region velocity is obtained.

$$u_* = u_R + \sqrt{\left[\left(\frac{p_R}{p_* \rho_R} - \frac{p_R(1 - y_R)}{p_* \rho_l} + \frac{1 - y_R}{\rho_l} - \frac{1}{\rho_R}\right)(p_R - p_*)\right]} \tag{3.57}$$

$$u_* = u_R + f_R(p_*, W_R) \tag{3.58}$$

So, the f_R for right shock wave is as below:

$$f_R(p_*, W_R) = \sqrt{\left[\left(\frac{p_R}{p_* \rho_R} - \frac{p_R(1 - y_R)}{p_* \rho_l} + \frac{1 - y_R}{\rho_l} - \frac{1}{\rho_R}\right)(p_R - p_*)\right]} \tag{3.59}$$

3.1.1.4. The velocity equation for right rarefaction wave

For the case of right rarefaction wave, the generalized Riemann invariant is used to make connection between unknown star region and the left-hand side value like left rarefaction wave. Any curves which connect the right-hand side to the star region should satisfy the following equation:

$$\frac{\partial u}{\partial \rho} = \pm \frac{c(\rho)}{\rho} \tag{3.60}$$

The plus or mine depends on which wave family is considered. By substituting (3.29) in (3.60), for right rarefaction wave the Riemann invariant would be:

$$u_R - u_* = \int_{\rho_*}^{\rho_R} \frac{\rho_l a \sqrt{y_R}}{[\rho_l - \rho(1 - y_R)]\rho} \partial \rho \tag{3.61}$$

The integral of (3.61) can be solved analytically:

$$u_* = u_R - a\sqrt{y}\ln\left[\left(\frac{1}{\rho_*} - \frac{1-y_R}{\rho_l}\right)\left(\frac{\rho_R}{1-\frac{1-y_R}{\rho_l}\rho_R}\right)\right] \quad (3.62)$$

By substituting (3.39) in (3.62), the final equation for velocity across right rarefaction wave is obtained.

$$u_* = u_R - a\sqrt{y_R}\ln\left[\left(\frac{p_R}{p_*\rho_R} - \frac{p_R}{p_*\rho_l}(1-y_R)\right)\left(\frac{\rho_R}{1-\frac{1-y_R}{\rho_l}\rho_R}\right)\right] \quad (3.63)$$

So f_R for right rarefaction is determined by equation (3.62) and (3.63).

$$f_R(p_*, W_R) = -a\sqrt{y_R}\ln\left[\left(\frac{p_R}{p_*\rho_R} - \frac{p_R}{p_*\rho_l}(1-y_R)\right)\left(\frac{\rho_R}{1-\frac{1-y_R}{\rho_l}\rho_R}\right)\right] \quad (3.64)$$

So, the final equation by which the star region pressure can be determined is:

$$f(p_*, W_L, W_R) = f_L(p_*, W_L) - f_R(p_*, W_R) + \Delta u \quad (3.65)$$

Where,

$$\Delta u = u_L - u_R \quad (3.66)$$

The solution for $f(p_*, W_L, W_R) = 0$, gives the star region pressure. after the star region pressure is determined, star region velocity is determined by the following equation:

$$u_* = \frac{1}{2}(u_L + u_R) + \frac{1}{2}[f_L(p_*, W_L) + f_R(p_*, W_R)] \quad (3.67)$$

3.1.2.Pressure numerical solution

Like the Riemann problem for isothermal gas and full Euler equation, for determining the middle state pressure there are several methods. The newton iterative method is practical and efficient which has had accurate results for previous Riemann problems. So, for two phase flow Riemann problem this method is used.

So, the related equations for right and left waves are as below:

$$f_L(p_*,W_L)=\begin{cases}-\sqrt{\left[\left(\frac{p_L}{p_*\rho_L}-\frac{p_L(1-y_L)}{p_*\rho_l}+\frac{1-y_L}{\rho_l}-\frac{1}{\rho_L}\right)(p_L-p_*)\right]} & if\ p_*>p_L\\ a\sqrt{y_L}\ln\left[\left(\frac{p_L}{p_*\rho_L}-\frac{p_L}{p_*\rho_l}(1-y_L)\right)\left(\frac{\rho_L}{1-\frac{1-y_L}{\rho_l}\rho_L}\right)\right] & if\ p_*<p_L\end{cases} \tag{3.68}$$

$$f_R(p_*,W_R)=\begin{cases}\sqrt{\left[\left(\frac{p_R}{p_*\rho_R}-\frac{p_R(1-y_R)}{p_*\rho_l}+\frac{1-y_R}{\rho_l}-\frac{1}{\rho_R}\right)(p_R-p_*)\right]} & if\ p_*>p_R\\ -a\sqrt{y_R}\ln\left[\left(\frac{p_R}{p_*\rho_R}-\frac{p_R}{p_*\rho_l}(1-y_R)\right)\left(\frac{\rho_R}{1-\frac{1-y_R}{\rho_l}\rho_R}\right)\right] & if\ p_*<p_R\end{cases} \tag{3.69}$$

As said previously the equation of (3.65) can be solved by the iterative method. For this approach the derivative of (3.65) should be obtained.

$$p_1 = p_0-\frac{f(p_0)}{f'(p_0)} \qquad Newton\ iterative\ method$$

$$f'(p_*,W_L,W_R)=f_L{}'(p_*,W_L)-f_R{}'(p_*,W_R) \tag{3.70}$$

$$f_L{}'(p_*,W_L)=\begin{cases}-\frac{1}{2}\left[\left(\frac{p_L}{p_*\rho_L}-\frac{p_L(1-y_L)}{p_*\rho_l}+\frac{1-y_L}{\rho_l}-\frac{1}{\rho_L}\right)(p_L-p_*)\right]^{-\frac{1}{2}} * \\ \left[\left[-\frac{1}{p_*^2}\left(\frac{p_L}{\rho_L}-\frac{p_L(1-y_L)}{\rho_l}\right)(p_L-p_*)\right]-\left[\left(\frac{p_L}{p_*\rho_L}-\frac{p_L(1-y_L)}{p_*\rho_l}+\frac{1-y_L}{\rho_l}-\frac{1}{\rho_L}\right)\right]\right] & if\ p_*>p_L\\ a\sqrt{y_L}\left[\frac{-\frac{1}{p_*^2}\left(\frac{p_L}{\rho_L}-\frac{p_L}{\rho_l}(1-y_L)\right)}{\left(\frac{p_L}{p_*\rho_L}-\frac{p_L}{p_*\rho_l}(1-y_L)\right)}\right] & if\ p_*<p_L\end{cases} \tag{3.71}$$

$$f_R'(p_*, W_R) = \begin{cases} \frac{1}{2}\left[\left(\frac{p_R}{p_*\rho_R} - \frac{p_R(1-y_R)}{p_*\rho_l} + \frac{1-y_R}{\rho_l} - \frac{1}{\rho_R}\right)(p_R - p_*)\right]^{-\frac{1}{2}} * \\ \left[\left[-\frac{1}{p_*^2}\left(\frac{p_R}{\rho_R} - \frac{p_R(1-y_R)}{\rho_l}\right)(p_R - p_*)\right] - \left[\left(\frac{p_R}{p_*\rho_R} - \frac{p_R(1-y_R)}{p_*\rho_l} + \frac{1-y_R}{\rho_l} - \frac{1}{\rho_R}\right)\right]\right] if\ p_* > p_R \\ -a\sqrt{y_R}\left[\frac{-\frac{1}{p_*^2}\left(\frac{p_R}{\rho_R} - \frac{p_R}{\rho_l}(1-y_R)\right)}{\left(\frac{p_R}{p_*\rho_R} - \frac{p_R}{p_*\rho_l}(1-y_R)\right)}\right] \quad if\ p_* < p_R \end{cases} \tag{3.72}$$

3.1.3.The complete solution

In the previous section we found the algorithm to find the star region pressure and velocity. By equation of (3.39) and (3.65) the value of density in right star region and left star region can be determined. The next task is finding the left and right waves completely. For shock wave, the shock velocity and density in the star region should be determined. For the case of rarefaction wave, it is more complex. First the head and tail of rarefaction wave should be determined then the primitive variables value should be determined inside the rarefaction fan.

3.1.3.1. Left shock wave

If star region pressure is more than left pressure, the wave is shock. Previously we determined the velocity and pressure in the star region. For determining density in the left star region, equation (3.39) is used. The shock speed can be determined by applying Rankine-Hugoniot condition for the conservation equation of (3.7).

$$S_L = \frac{\rho_L u_L - \rho_{*L} u_*}{\rho_L - \rho_{*L}} \tag{3.73}$$

So, we have completely determined the solution for the entire region by the contact discontinuity in the case of left shock wave.

3.1.3.2. Left rarefaction wave

If the star region pressure is less than the left pressure, the wave is rarefaction. As said previously, first the tail and head of a rarefaction wave should be determined as below:

$$S_{HL} = u_L - c_L \tag{3.74}$$

$$S_{TL} = u_* - c_{*L} \tag{3.75}$$

Where c can be determined through equation of (3.29).

The next step would be determining pressure, velocity and density in the rarefaction fan. We consider a characteristic ray from the origin and an arbitrary point of (x, t) in the fan.

$$\frac{dx}{dt} = \frac{x}{t} = u - c \tag{3.76}$$

Also, there is a relation between velocity and density by using Riemann invariant from section (3.1.1.2). By substituting equation (3.29) in (3.76), an expression for density is obtained:

$$\rho = \left(\rho_l - \frac{\rho_l a\sqrt{y_L}}{u - \frac{x}{t}}\right)\left(\frac{1}{1 - y_L}\right) \tag{3.77}$$

By substituting (3.77) in (3.45) an expression for velocity in the rarefaction fan is obtained.

$$u = u_L + a\sqrt{y_L}\ln\left[\left(\frac{1}{\left(\rho_l - \frac{\rho_l a\sqrt{y_L}}{u - \frac{x}{t}}\right)\left(\frac{1}{1 - y_L}\right)} - \frac{1 - y_L}{\rho_l}\right)\left(\frac{\rho_L}{1 - \frac{1 - y_L}{\rho_l}\rho_L}\right)\right] \tag{3.78}$$

The equation of (3.78) can be solved iteratively for velocity. In this method an initial guess for velocity is considered and u is solved iteratively until the difference between guessed and the solved u is less than desirable amount.

After finding velocity at point (x, t), density can be determined through equation of (3.77). Then pressure can be determined by equation (3.28).

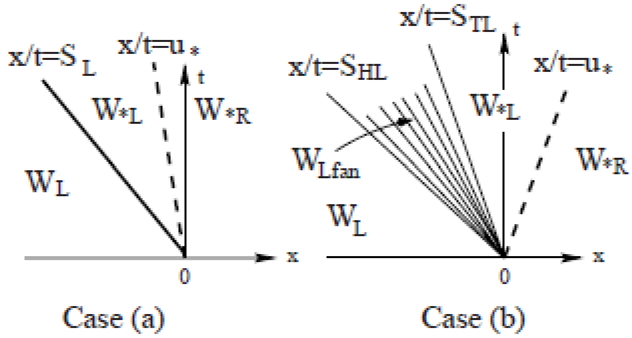

Figure 3-4:Solution sampling at the left hand side of contact discontinuity: (a) : left shock wave (b): left rarefaction wave(Toro 2013)

3.1.3.3. Right shock wave

If star region pressure is more than right pressure, the wave is shock. Previously we determined the velocity and pressure in the star region. For determining density in the right star region, equation (3.56) is used. The shock speed can be determined by applying Rankine-Hugoniot condition in the conservation equation of (3.7).

$$S_R = \frac{\rho_R u_R - \rho_{*R} u_*}{\rho_R - \rho_{*R}} \tag{3.79}$$

So, we have completely determined the solution for the entire region by the contact discontinuity in the case of right shock wave.

3.1.3.4. Right rarefaction wave

If the star region pressure is less than the right pressure, the wave is rarefaction. As said previously, first the tail and head of a rarefaction wave should be determined as below:

$$S_{HR} = u_R - c_R \tag{3.80}$$

$$S_{TR} = u_* - c_{*R} \tag{3.81}$$

Where *c* can be determined through equation of (3.29).

The next step would be determining pressure, velocity and density in the rarefaction fan. We consider a characteristic ray from the origin and an arbitrary point of (x, t) in the fan.

$$\frac{dx}{dt} = \frac{x}{t} = u + c \tag{3.82}$$

Also, there is a relation between velocity and density by using Riemann invariant from section of (3.1.1.4). By substituting equation (3.29) in (3.82), an expression for density is obtained:

$$\rho = \left(\rho_l + \frac{\rho_l a\sqrt{y_R}}{u - \frac{x}{t}}\right)\left(\frac{1}{1 - y_R}\right) \tag{3.83}$$

By substituting (3.83) in (3.62) an expression for velocity in the rarefaction fan is obtained.

$$u = u_R - a\sqrt{y_R}\ln\left[\left(\frac{1}{\left(\rho_l + \frac{\rho_l a\sqrt{y_R}}{u - \frac{x}{t}}\right)\left(\frac{1}{1 - y_R}\right)} - \frac{1 - y_R}{\rho_l}\right)\left(\frac{\rho_R}{1 - \frac{1 - y_R}{\rho_l}\rho_R}\right)\right] \tag{3.84}$$

The equation of (3.84) can be solved iteratively for velocity. In this method an initial guess for velocity is considered and *u* is solved iteratively until the difference between guessed and the solved *u* is less than desirable amount.

After finding velocity at point (x, t), density can be determined through equation of (3.83). Then pressure can be determined by equation (3.28).

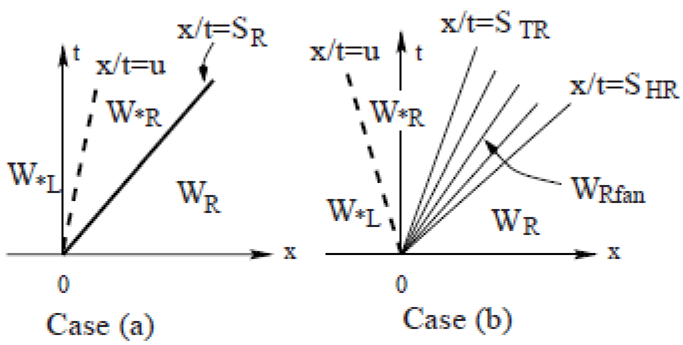

Figure 3-5:Solution sampling at the right hand side of contact discontinuity: (a) : right shock wave (b): right rarefaction wave(Toro 2013)

3.1.4. Solution sampling

In previous section the complete solver and wave structure for two phase flow Riemann problem at any point (x, t) was presented. In this section an approach for solution sampling is presented which is useful for computer programming. Assume we want to find the solution for vector of primitive variable $W(\rho, u, p)$ in an arbitrary point of (x, t). The sampling is performed in term of S = x/t. In this situation two condition may happen. Left side of contact and right side of contact.

3.1.4.1. If $S \leq {}^{x}/_{t}$ which means the left side of the contact discontinuity

As mentioned previously, there are two possibilities for left wave: shock and rarefaction. In the case of shock wave, the complete solution for left side of contact is as below:

$$W(x,t) = \begin{cases} W_{*L} & if \quad S_L \leq \frac{x}{t} \leq u_* \\ W_L & if \quad \frac{x}{t} \leq S_L \end{cases} \tag{3.85}$$

Where W_{*L} equals to the primitive variables value in the left star region and S_L is the left shock speed. If the left wave is rarefaction, the complete solution for left hand side of contact is as below:

$$W(x,t) = \begin{cases} W_L & if \quad \frac{x}{t} \leq S_{HL} \\ W_{Lfan} & if \quad S_{HL} \leq \frac{x}{t} \leq S_{TL} \\ W_{*L} & if \quad S_{TL} \leq \frac{x}{t} \leq u_* \end{cases} \tag{3.86}$$

Where W_{Lfan} is the solution in left rarefaction fan which is obtained by the procedure explained in section (3.1.3.2).

3.1.4.2. If $S \geq {}^{x}/_{t}$ which means the right side of the contact discontinuity

Like left side of contact, there are two possibilities for the right wave: shock and rarefaction. In the case of shock wave, the complete solution in the right side of contact is as below:

$$W(x,t) = \begin{cases} W_{*R} & if \quad u_* \leq \frac{x}{t} \leq S_R \\ W_R & if \quad \frac{x}{t} \geq S_R \end{cases} \tag{3.87}$$

Where W_{*R} equals to the primitive variables value in the right star region and S_R is the right shock speed. If the right wave is rarefaction, the complete solution for right hand side of contact is as below:

$$W(x,t) = \begin{cases} W_{*R} & if \quad u_* \leq \frac{x}{t} \leq S_{TR} \\ W_{Rfan} & if \quad S_{TR} \leq \frac{x}{t} \leq S_{HR} \\ W_R & if \quad \frac{x}{t} \geq S_{HR} \end{cases} \tag{3.88}$$

Where W_{Rfan} is the solution in right rarefaction fan which is obtained by the procedure explained in section (3.1.3.4).

3.2. Computer programming for drift flux Riemann problem

In this thesis the computer coding has been done in Python and the codes can be fined in the GitHub link provided. The section (3.1.4) has been provided for determining the main algorithm for computer coding. For left of contact discontinuity below algorithm has been used:

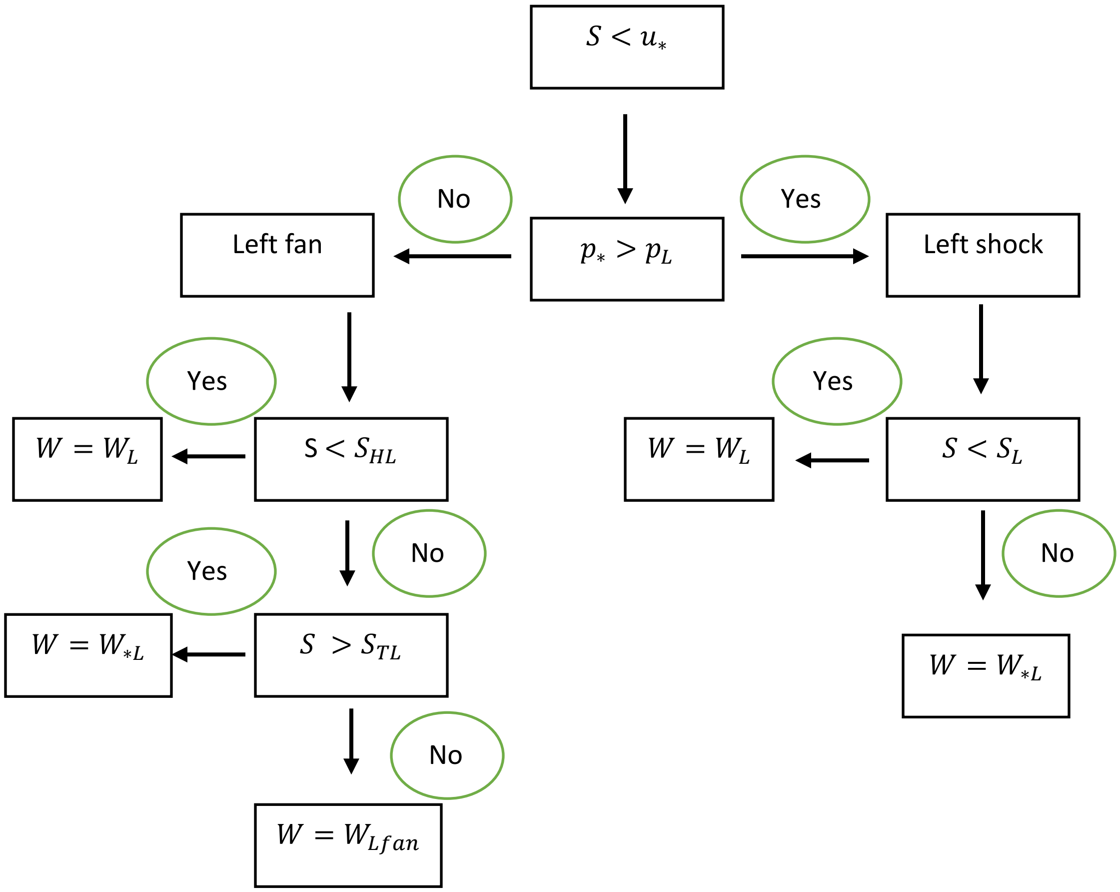

Figure 3-6:Flow chart for solution sampling at an arbitrary point of (x, t) in left of contact discontinuity

For the right side of the contact, below flow chart has been used:

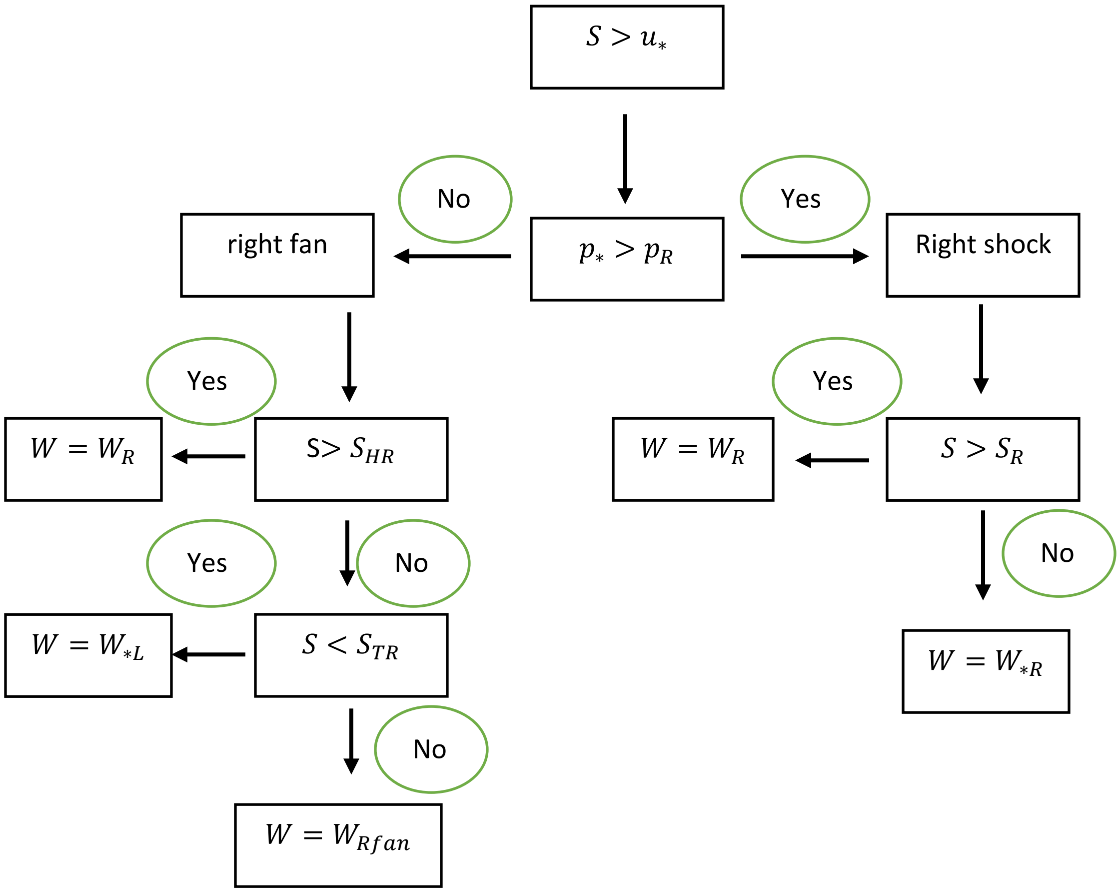

Figure 3-7: Flow chart for solution sampling at an arbitrary point of (x, t) in right of contact discontinuity

4. RESULTS AND DISCUSSION

4.1. Numerical test

In this section some Riemann problems with initial condition are selected to investigate the Riemann solver presented in section of (3.1.2). The data of four tests in terms of primitive variables is shown in table (4-1). In all cases the density of incompressible liquid is 1 kg/liter and the sound velocity in gas is 340 m/s.

Table 4-1:data for initial condition for the Riemann problem

Test	$\boldsymbol{\rho_L}(^{\boldsymbol{kg}}/_{\boldsymbol{m^3}})$	$\boldsymbol{u_L}(^{\boldsymbol{m}}/_{\boldsymbol{s}})$	$\boldsymbol{p_L(pa)}$	$\boldsymbol{\rho_R}(^{\boldsymbol{kg}}/_{\boldsymbol{m^3}})$	$\boldsymbol{u_R}(^{\boldsymbol{m}}/_{\boldsymbol{s}})$	$\boldsymbol{p_R(pa)}$
1	**850**	**1**	**110000**	**950**	**-1**	**120000**
2	**900**	**-1**	**170000**	**800**	**1**	**160000**
3	**950**	**0**	**160000**	**850**	**0**	**120000**
4	**850**	**0**	**120000**	**950**	**0**	**160000**

The computed values for pressure, velocity and density in star region for mentioned 4 tests have been shown in table (4-2). The pressure in the star zone has been determined by solving $f(p) = 0$ by iterative newton Rapson method and velocity and densities have been determined by the related formula presented in chapter 3.

Table 4-2:Exact solution for pressure, velocity and densities in the star region

Test	$\boldsymbol{p_*(pa)}$	$\boldsymbol{u_*}(^{\boldsymbol{m}}/_{\boldsymbol{s}})$	$\boldsymbol{\rho_{*L}}(^{\boldsymbol{kg}}/_{\boldsymbol{m^3}})$	$\boldsymbol{\rho_{*R}}(^{\boldsymbol{kg}}/_{\boldsymbol{m^3}})$
1	**151403.14**	**-0.41**	**886.39**	**959.96**
2	**135988.51**	**-0.029**	**878**	**772.67**
3	**134118.49**	**0.51**	**940.90**	**863.65**
4	**134118.49**	**-0.51**	**863.65**	**940.90**

4.2. All shock waves, test No 1

According to the table 4.2 and 4.1, the star zone pressure in test 1 is larger than the left and right-sides pressures so all shock wave happens in this situation. Figure (4-1) shows the graphs of pressure and velocity against location in different times. Although the solution depends only on (x/t), it is plotted for 3 different times to illustrate how the discontinuity deforms over the time

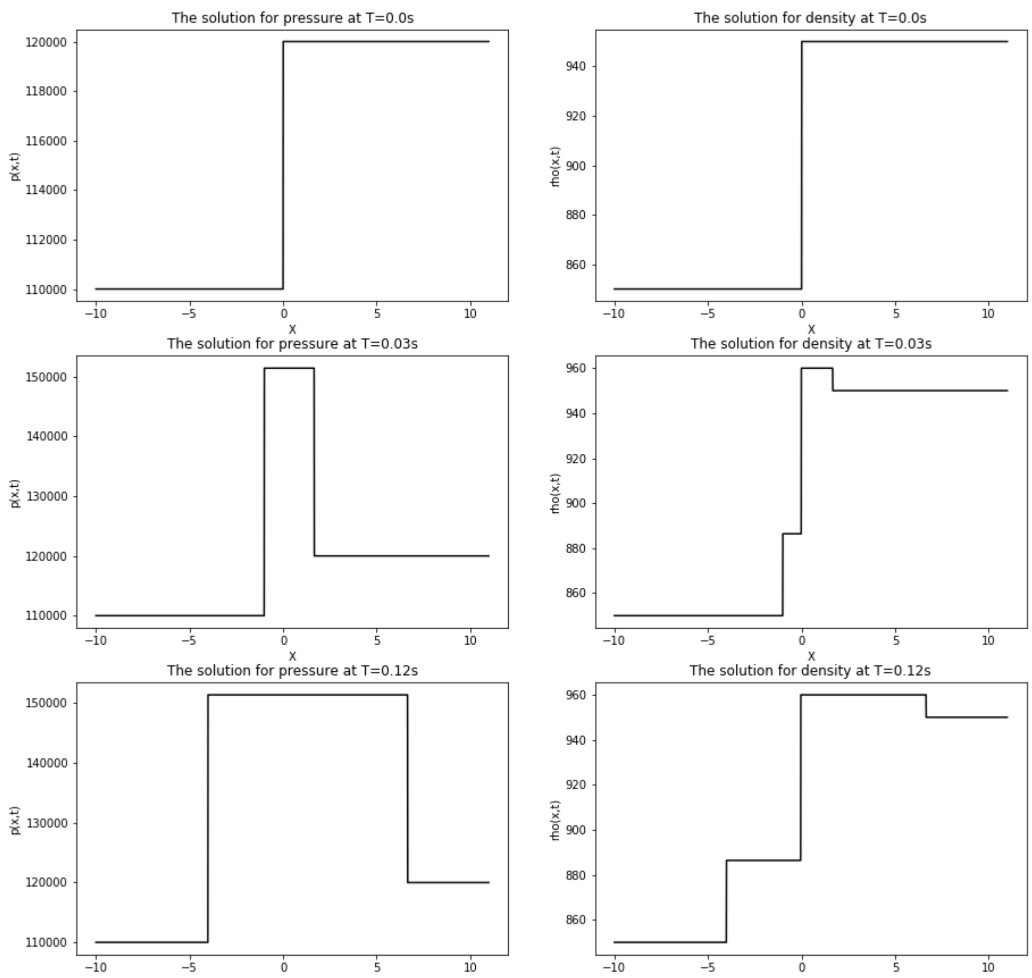

Figure 4-1:Illustration of pressure and density against location in different times of t1=0 , t2=0.03 and t3=0.12 for test number 1.

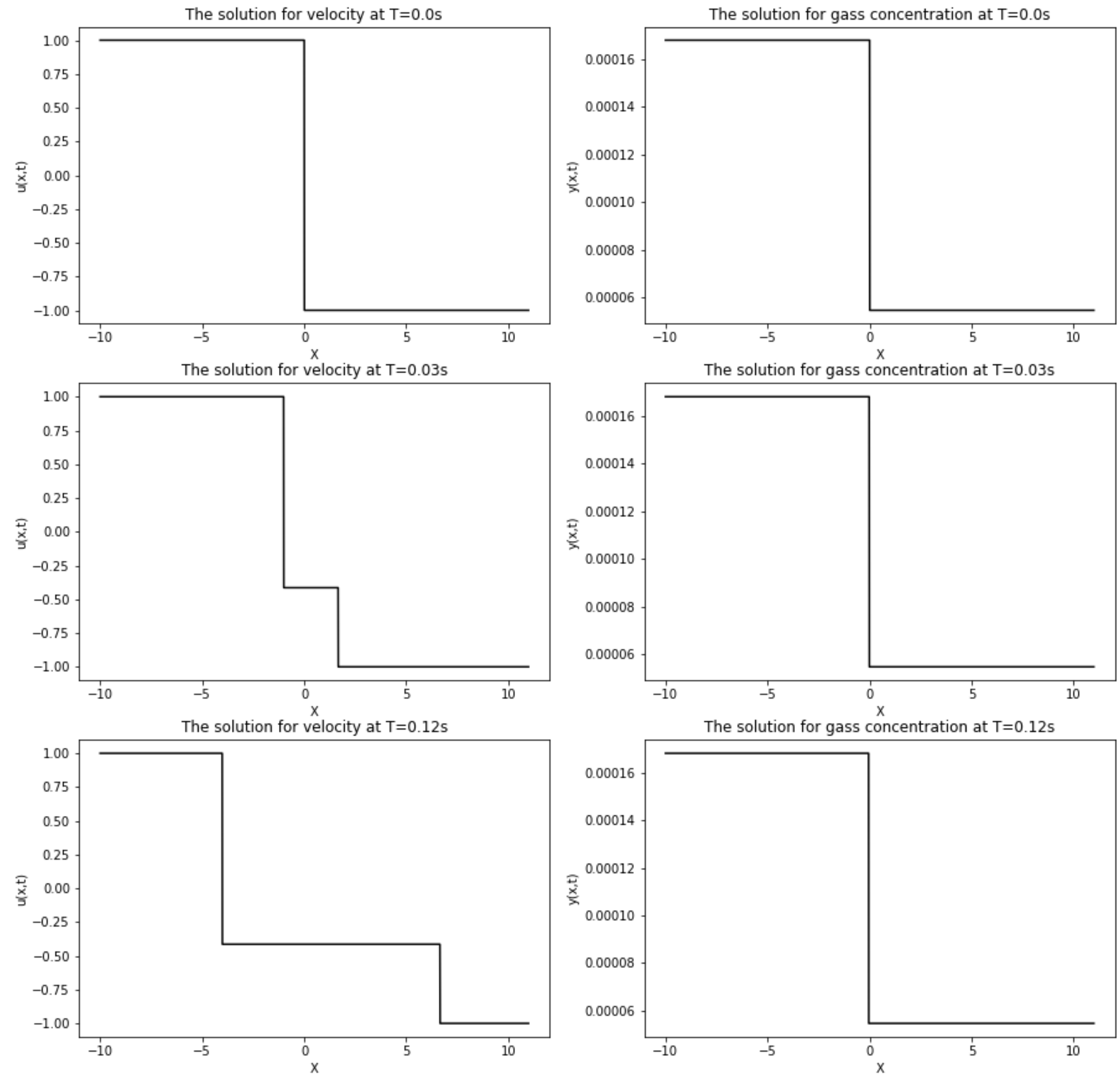

Figure 4-2::Illustration of velocity and gas concentration against location in different times of t1=0 , t2=0.03 and t3=0.12 for test number 1

4.3. All rarefaction waves, test No 2

According to the table 4.2 and 4.1, the star zone pressure in test 2 is smaller than the left and right-side pressures so all rarefaction wave happens in this situation. Figure (4-3) shows the graphs of pressure and velocity against location in different times.

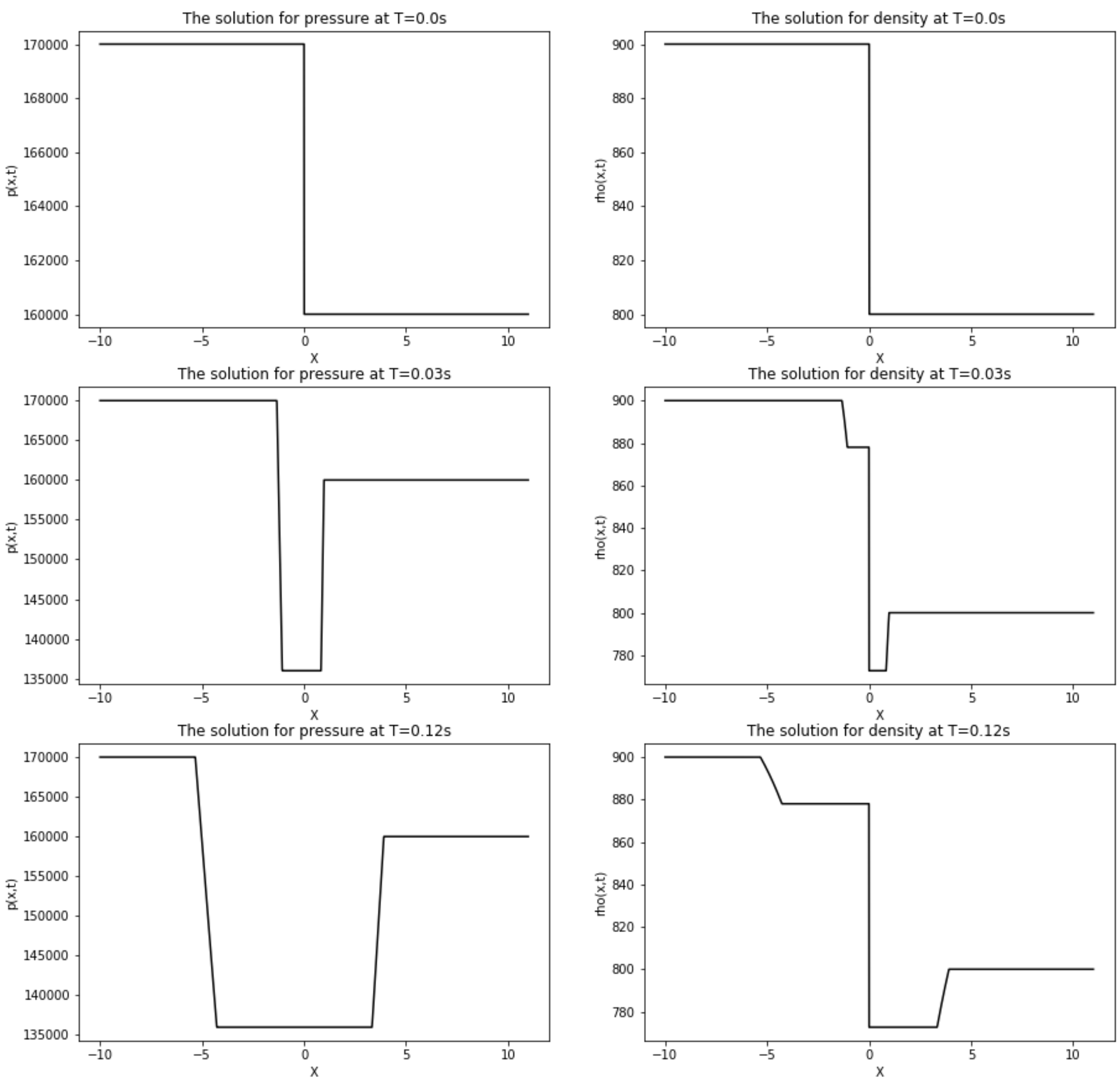

Figure 4-3: Illustration of pressure and density against location in different times of t1=0 , t2=0.03 and t3=0.12 for test number 2.

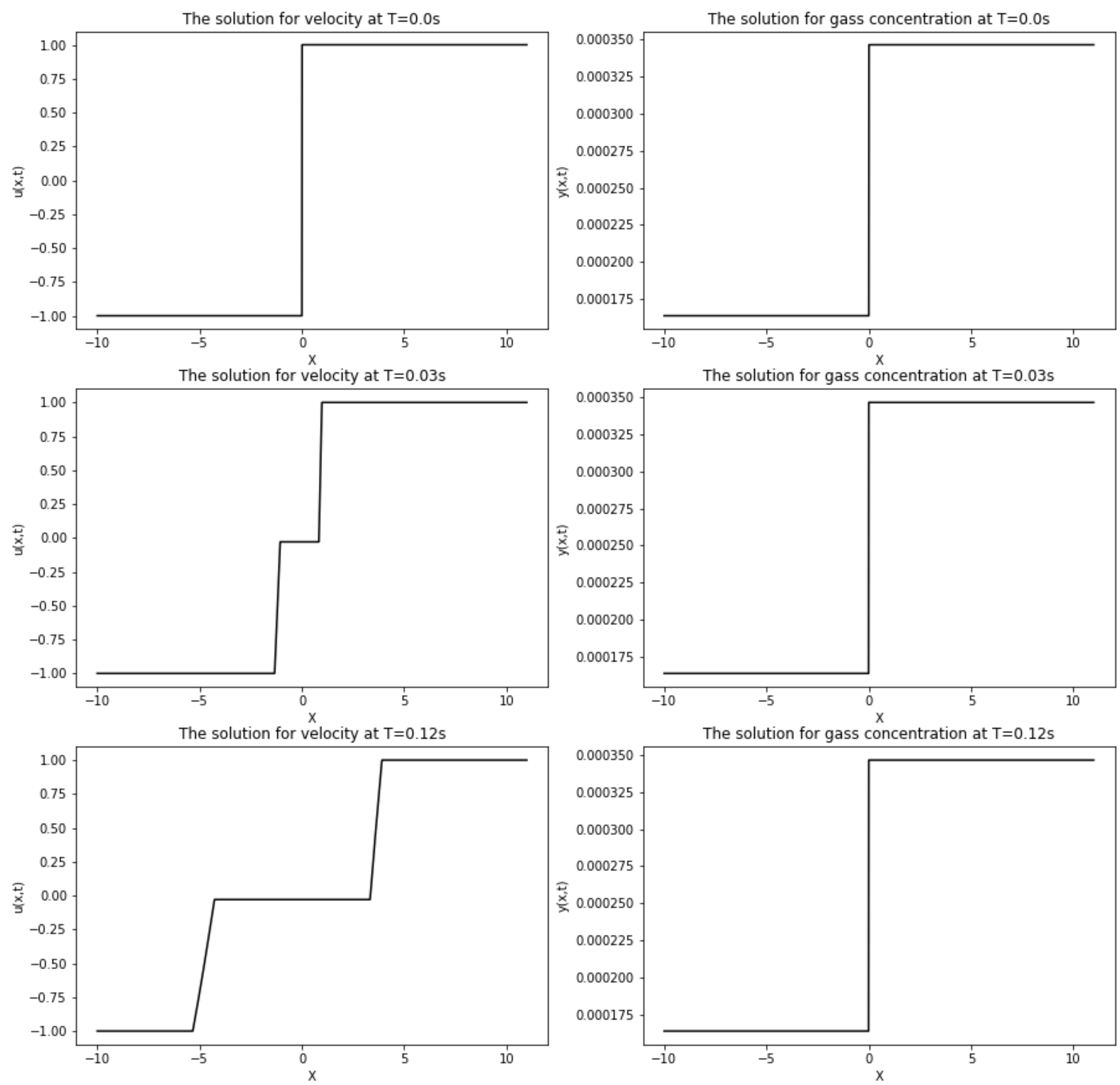

Figure 4-4: Illustration of velocity and gas concentration against location in different times of t1=0 , t2=0.03 and t3=0.12 for test number 2.

4.4. 1_{th}-Rarefaction 2_{th}- shock waves, test No 3

According to the table 4.2 and 4.1, the star zone pressure in test no 3 is smaller than the left side pressure and larger than the right-side pressure so the first wave is rarefaction and the second wave is shock wave. Figure (4-5) shows the graphs of pressure and velocity against location in different times.

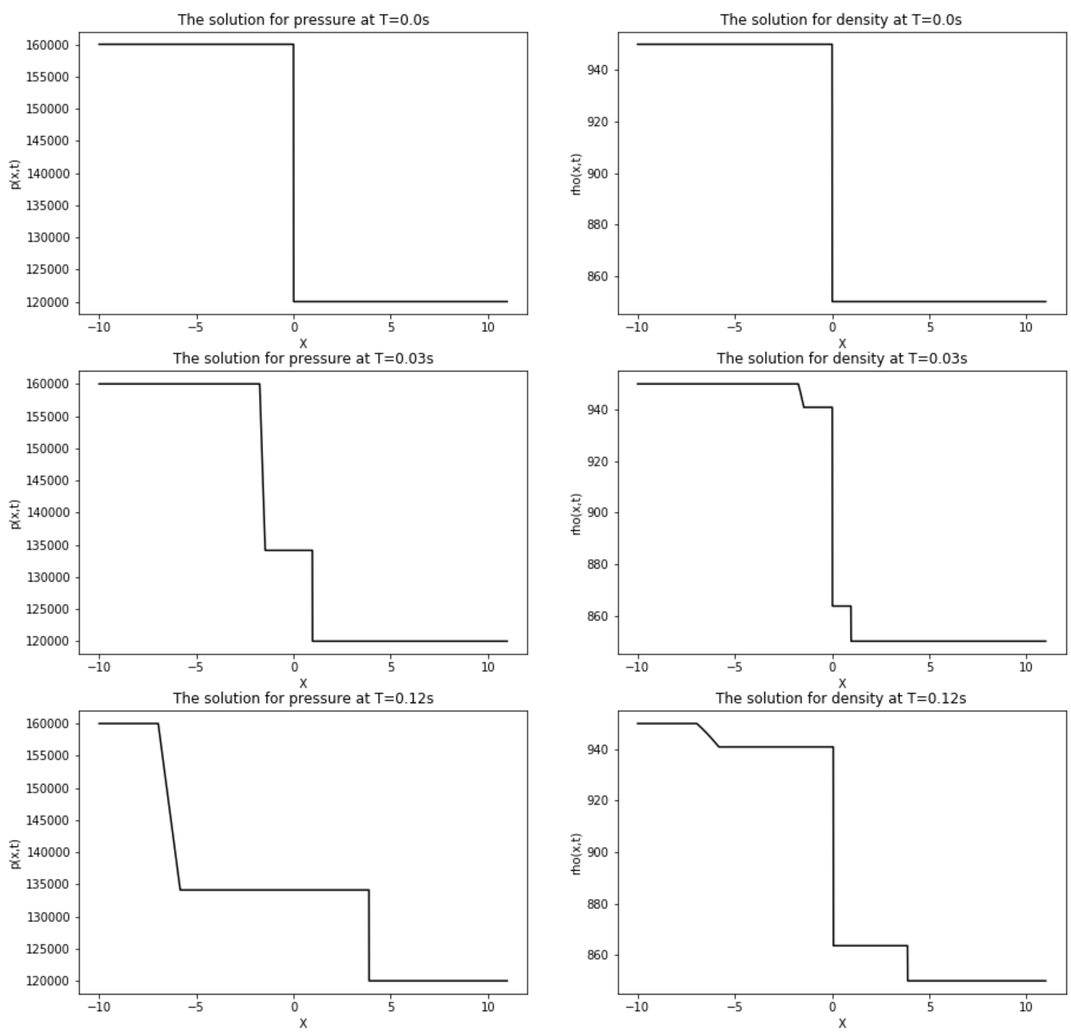

Figure 4-5:Illustration of pressure and density against location in different times of t1=0 , t2=0.03 and t3=0.12 for test number 3.

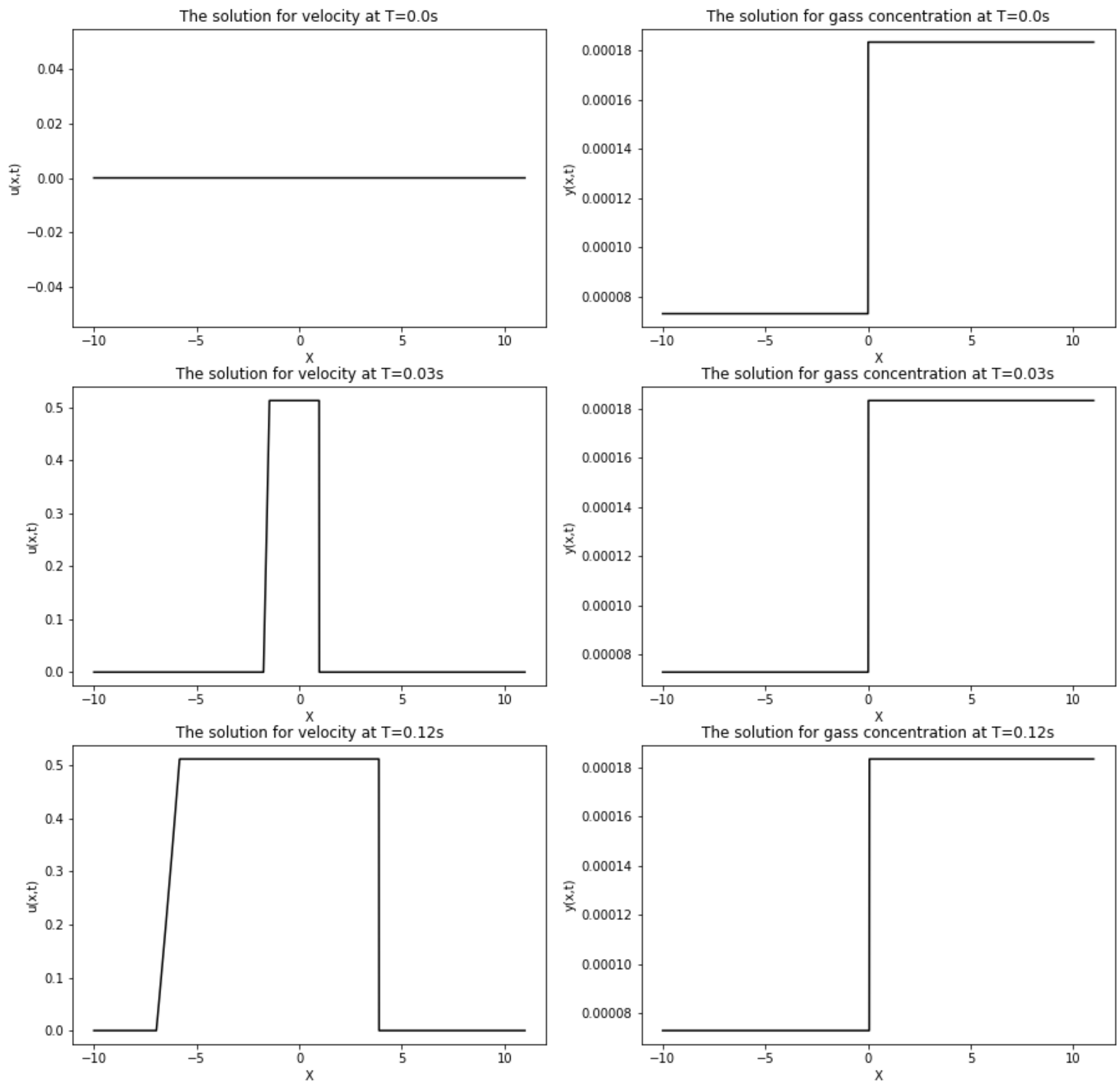

Figure 4-6:Illustration of velocity and gas concentration against location in different times of t1=0 , t2=0.03 and t3=0.12 for test number 3.

4.5. 1th-shock 2th- rarefaction waves, test No 4

According to the table 4.2 and 4.1, the star zone pressure in test no 4 is larger than the left side pressure and smaller than the right-side pressure so the first wave is shock and the second wave is rarefaction wave. Figure (4-7) shows the graphs of pressure and velocity against location in different times.

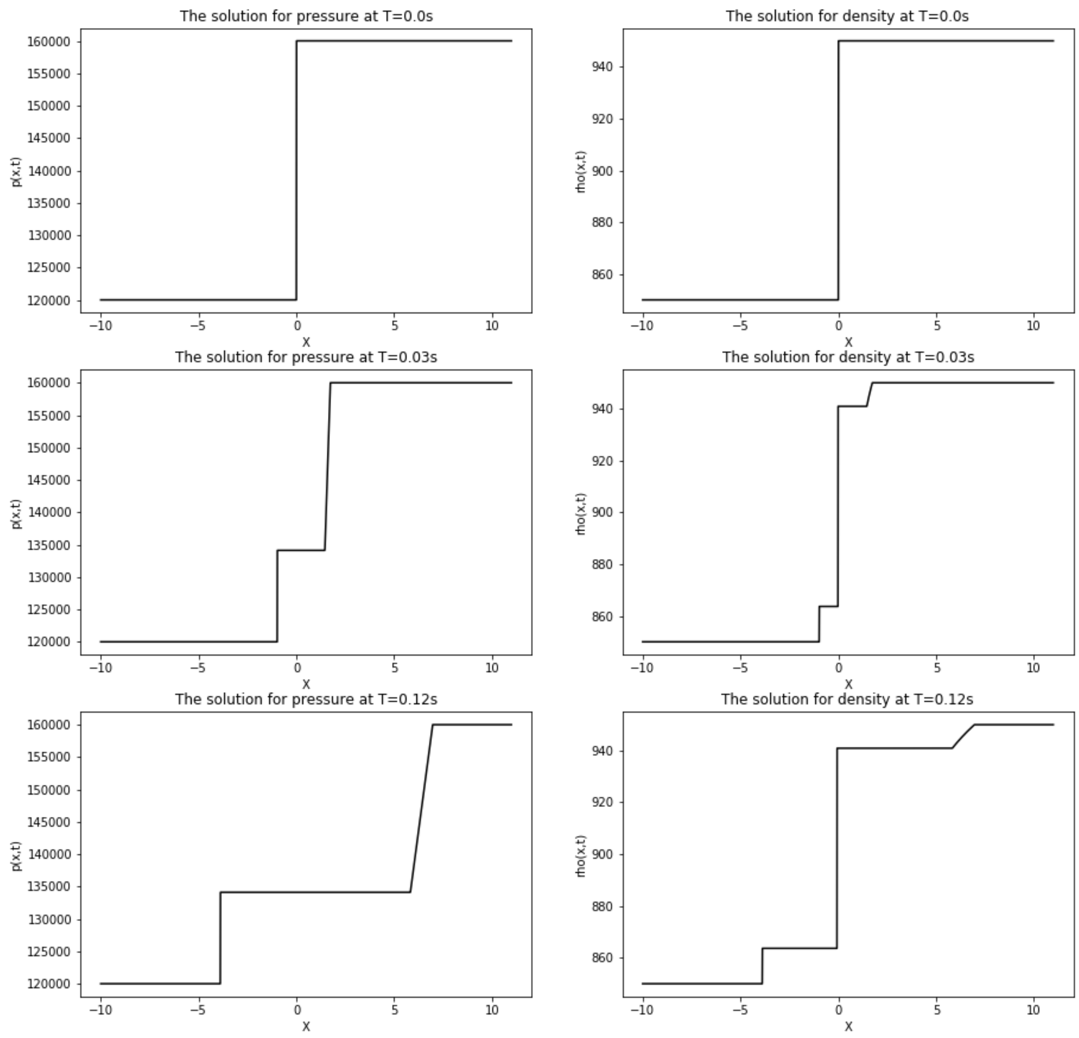

Figure 4-7: Illustration of pressure and density against location in different times of t1=0 , t2=0.03 and t3=0.12 for test number 4.

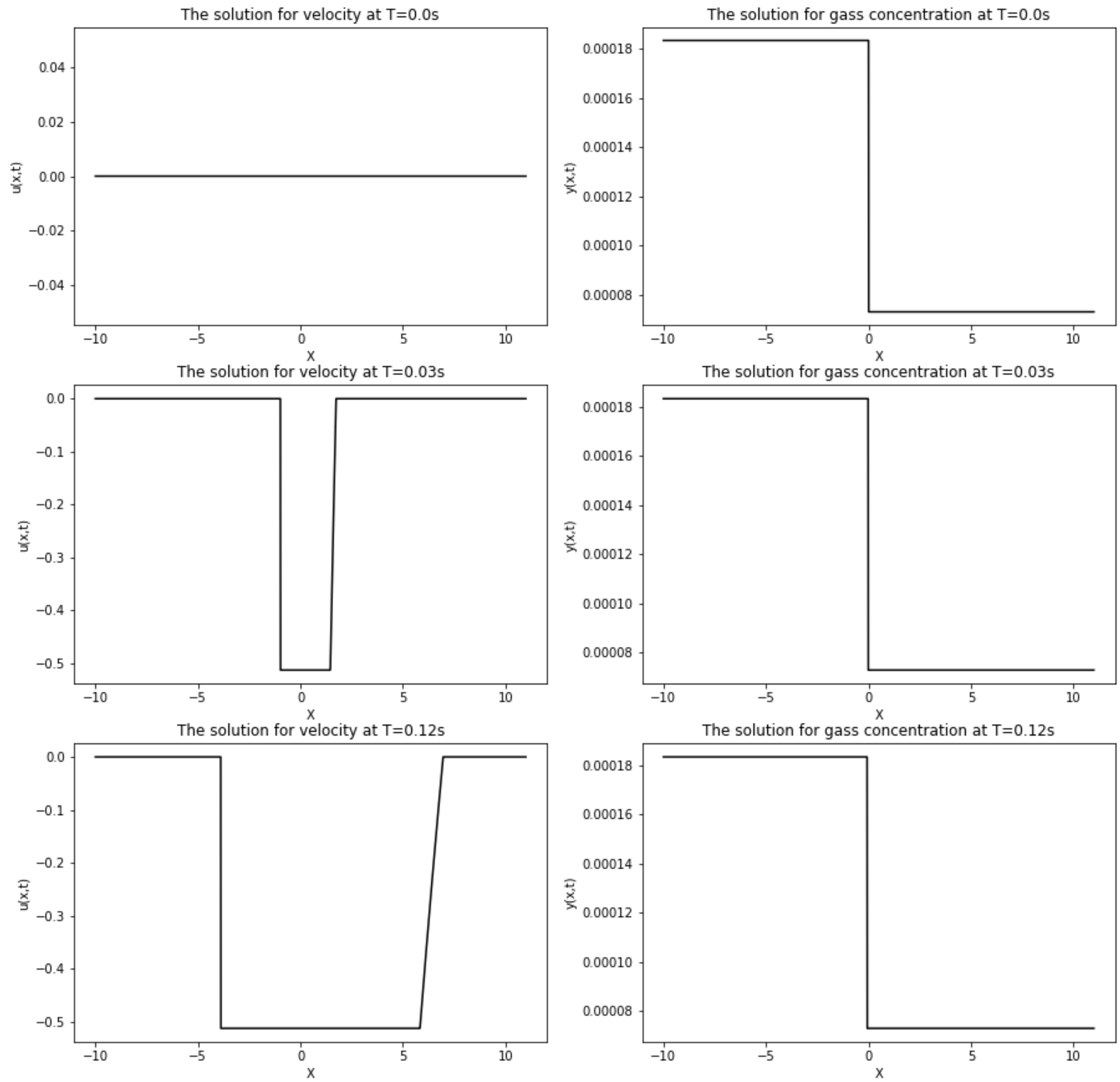

Figure 4-8: Illustration of velocity and gas concentration against location in different times of t1=0 , t2=0.03 and t3=0.12 for test number 4.

As the contact discontinuity moves with much lower speed than shock or rarefaction waves, the plot of gas mass concentration seems to be immobile.

4.6. The Riemann solution and computer coding verification

The Riemann solution and computer coding verification can be implemented by various methods. Using of mirroring technique and equivalent numerical solution are famous among these methods. In the mirroring technique the Riemann initial condition becomes exactly the opposite and the results should be the same however different in sign. For example, the test number 3 and number 4 are exactly mirror of each other. According to figures (4-5), (4-6) , (4-7) and (4-8) and table (4-2), the middle value of middle pressure and the absolute value of middle velocity are the same. Also, the middle values of densities are the same with opposite locations.

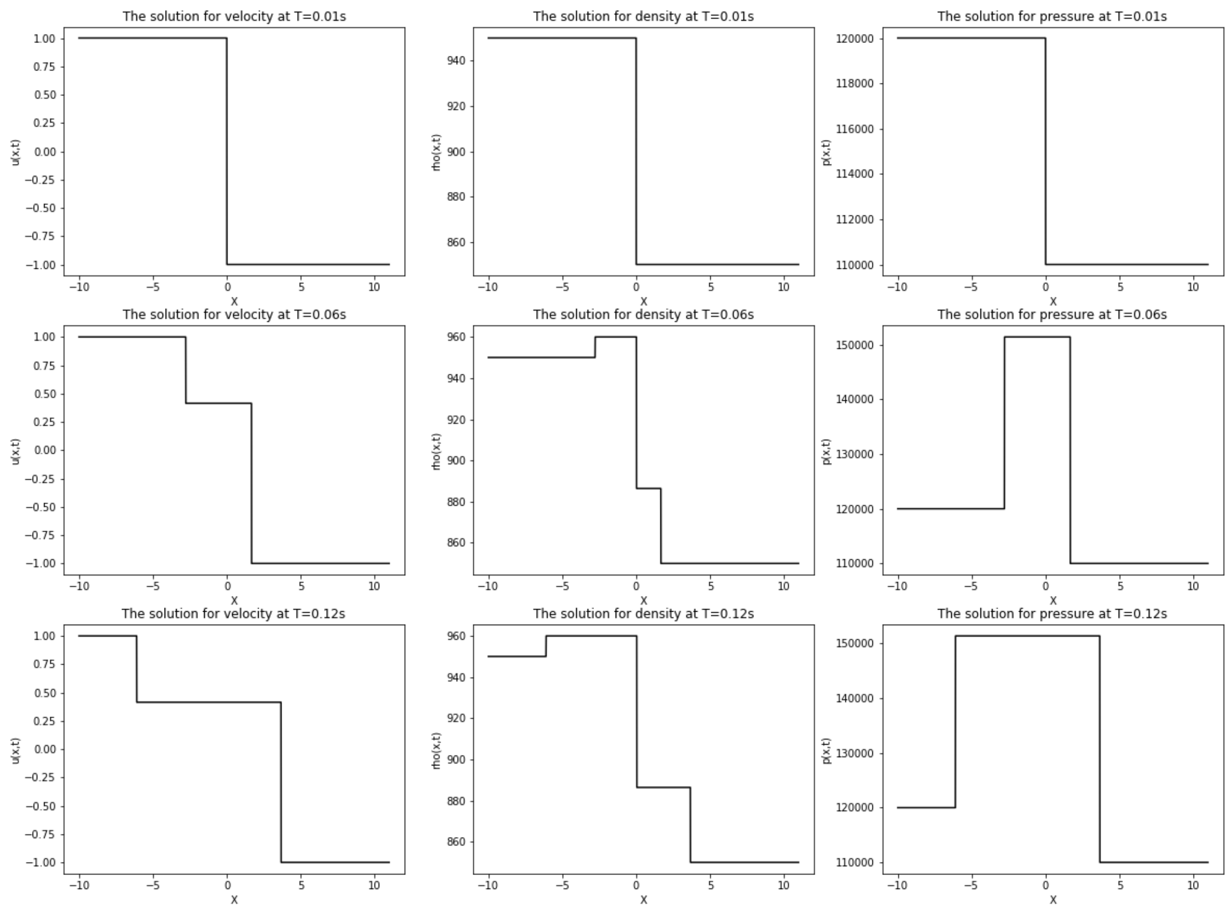

Figure 4-9:The Riemann solution for the mirroring initial condition of test number 1

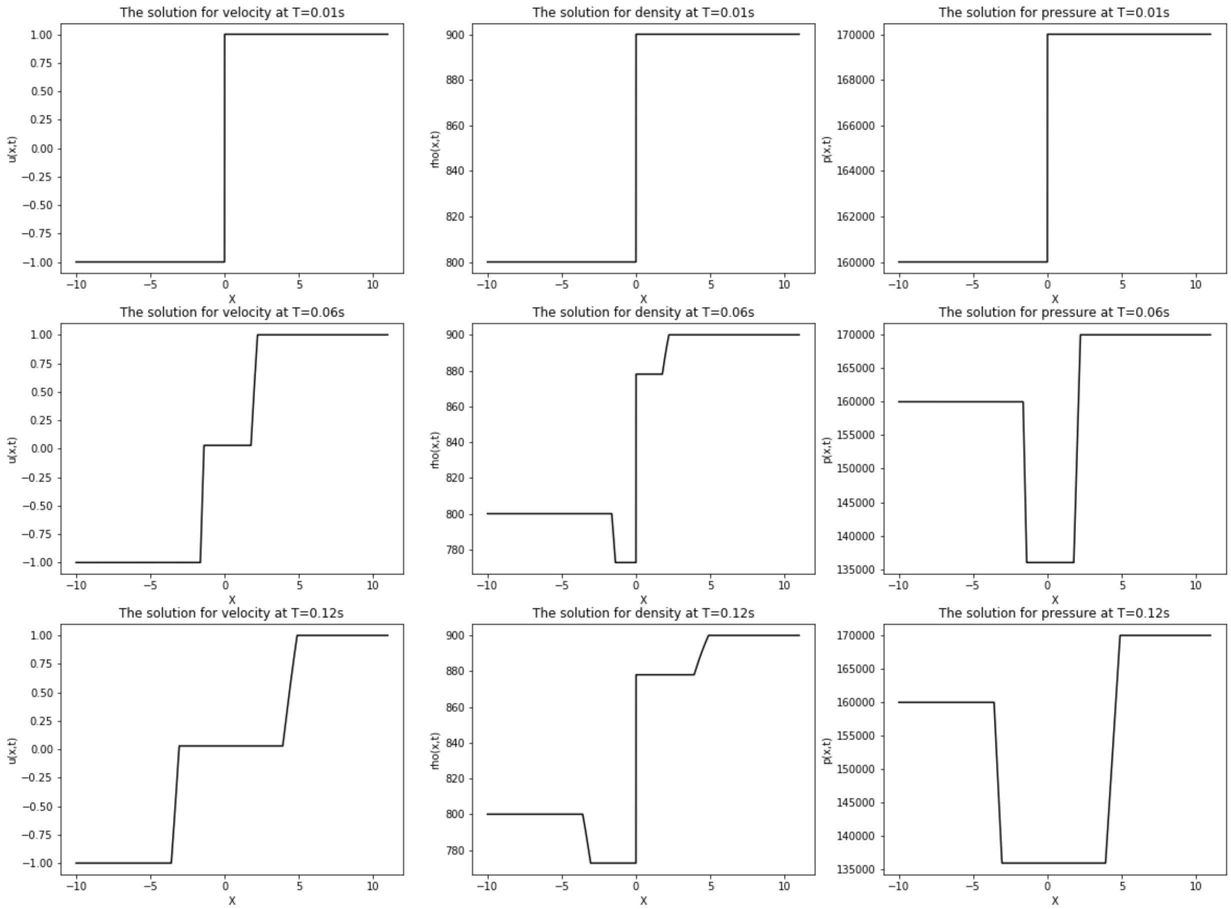

Figure 4-10:The Riemann solution for the mirroring initial condition of test number 2.

4.6.1.Numerical validation

In 2005 a study was implemented on numerical solution of Riemann problem for two phase flow drift flux model(Baudin, Berthon et al. 2005). In one of the experiments the liquid is assumed to be incompressible and the relative velocity between each phase is zero. These assumptions are like the assumption have been used for the Riemann exact solution here. So, the exact solution results can be verified by the numerical results for the same initial condition.

Table 4-3: Initial condition for the Riemann problem (comparison of numerical and exact solution)

Test	$\rho_L(^{kg}/_{m^3})$	$u_L(^{m}/_{s})$	y_L	$\rho_R(^{kg}/_{m^3})$	$u_R(^{m}/_{s})$	y_R
	500	**10**	**0.2**	**400**	**-10.4261**	**0.4**

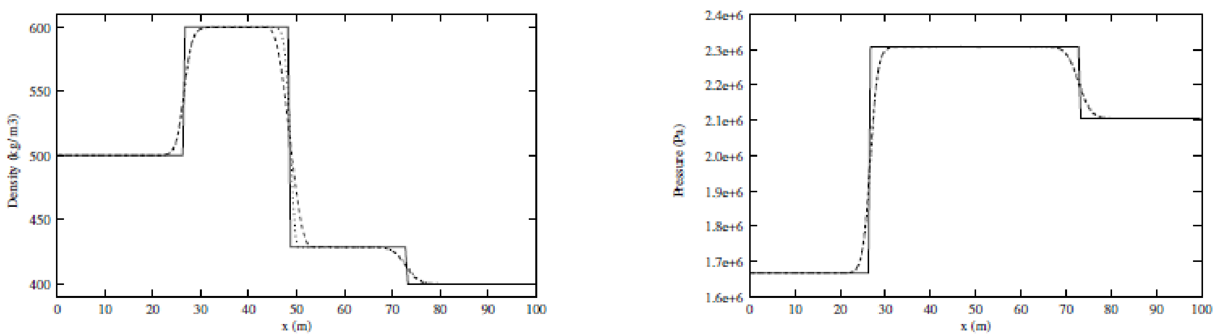

Figure 4-11:Numerical result for the Riemann problem for two phase flow with the initial condition in table(4-3) by (Baudin, Berthon et al. 2005)

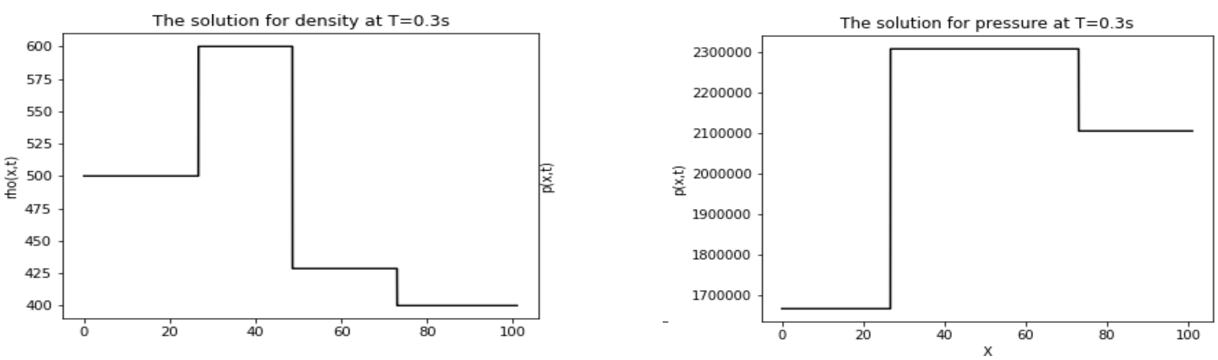

Figure 4-12:Exact solution for the Riemann problem for two phase flow with the initial condition in table(4-3)

4.7. Conclusion

The most important achievement in this thesis has been learning and understanding the characteristic behavior of two-phase flow system of equation, the Riemann problem and the wave structure during which the primitive variable behaves accordingly.

The development of Riemann exact solution for two phase flow has been a step by step process to let the readers understand it easily. First the Riemann problem for linear hyperbolic equation then the Riemann problem for nonlinear hyperbolic equation were investigated. After that the

Riemann problem for nonlinear and linear system of equation was implemented. Finally, the exact Riemann solution for two phase flow drift flux model was developed.

For two phase flow model the fluid was assumed to be incompressible and the drift flux is zero which produce unchanged gas mass coefficient during any rarefaction or shock waves. But this coefficient changes only across the contact discontinuity.

For verification of the results obtained from the exact solution for the Euler isothermal gas flow and the full Euler gas flow , the mentioned numerical solutions obtained by (Barazesh 2019) were used. Also for verification of the Riemann exact solution for the two phase flow drift flux model , numerical results obtained by (Baudin, Berthon et al. 2005) were used.

4.8. Future study

In this thesis, for the Riemann exact solution it has been assumed that the fluid is incompressible, and the drift flux is zero. although it is more complex, Developing the Riemann solution for non-zero drift flux and compressible fluid, produces more realistic results.

Using finite volume method and Riemann solution builds a practical method for determining two phase flow parameters during transient or steady state conditions in drilling well bore. Drilling incidents like gas kick and under balanced operation can be modeled efficiently by the finite volume method.

One simple example of finite volume method application is determining two phase flow parameters in underbalanced drilling. One method to estimate the bottom hole pressure in steady state condition is the shooting method. Also, the bottom hole pressure can be estimated by finite volume method and the result can be compared with the shooting method results. So, the efficiency of the finite volume method and the Riemann solution for two phase flow can be investigated.

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