Efficient Rate-Constrained Nash Equilibrium in Collision Channels with State Information

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***Abstract*—We consider a wireless collision channel, shared by a finite number of users who transmit to a common base station. Users are self-optimizing, and each wishes to minimize its average transmission rate (or power investment), subject to minimum-throughput demand. The channel quality between each user and the base station is time-varying, and partially observed by the user in the form of channel state information (CSI) signals. We assume that each user can transmit at a fixed power level and that its transmission decision at each time slot is stationary in the sense that it can depend only on the current CSI. We are interested in properties of the Nash equilibrium of the resulting game between users.**

We define the feasible region of user's throughput demands, and show that when the demands are within this region, there exist exactly two Nash equilibrium points, with one strictly better than the other (in terms of invested power) for all users. We further provide some lower bounds on the channel capacity that can be obtained, both in the symmetric and non-symmetric case. Finally, we show that a simple greedy mechanism converges to the best equilibrium point without requiring any coordination between the users.

I. Introduction

A. Background and Motivation

The emerging use of wireless technologies (such as WIFI and WIMAX) for data communication has brought to focus novel system characteristics which are of less importance in wireline platforms. Power control and the effect of mobility on network performance are good examples of topics which are prominent in the wireless area. An additional distinctive feature of wireless communications is the possible time variation in the channel quality between the sender and the receiver, an effect known as channel fading [1].

As wireless networks grew larger, it became evident that centralized control would be impractical for coordinating all elements of the network, and in particular end-user transmissions. The celebrated Aloha protocol was designed at the early 70's as a distributed mechanism which can allow efficient media sharing. This protocol and its variants, such as CSMA-CD and tree-algorithms [2], are *cooperative* in the sense that each user is committed to perform his part of the protocol. Modern wireless network protocols are often based on Aloha-related concepts (for example, the 802.11 standards [3]). The design of such protocols raises novel challenges and difficulties, as the wireless arena becomes more involved.

An additional consideration is the possibly selfish behavior of users, who can bias their transmission decisions to accommodate their own best interest. Such behavior is to be expected in wireless networks, considering the dynamic and ad-hoc nature of such networks, and the scarce resources of mobile terminals. In many cases, an individual user can momentarily improve its Quality of Service (QoS) metrics, such as delay and throughput, by accessing the shared channel more frequently. Aggressiveness of even a single user may lead to a chain reaction, resulting in possible throughput collapse. Significant research has been recently dedicated to analyzing wireless random access networks shared by self-interested agents, applying non-cooperative game theoretic tools for the analysis [4]–[7].

Our work considers a shared uplink in the form of a collision channel, where a user's transmission can be successful only if no other user attempts transmission simultaneously. A basic assumption of our user model is that each user has some throughput requirement, which it wishes to sustain with a minimal power investment. The required throughput of each user may be dictated by its application (such as video or voice which may require fixed bandwidth), or mandated by the system. A distinctive feature of our model is that the channel quality between each user and the base station is stochastically varying. For example, the channel quality may evolve as a block fading process [1] with a general underlying state distribution (such as Rayleigh, Rice, and Nakagami-m, see [1]). A user may base its transmission decision upon available indications on the channel state, known as channel state information (CSI). This decision is selfishly made by the individual without any coordination with other users, giving rise to a non-cooperative game. Our focus in this paper is on *stationary* transmission strategies, in which the decision whether to transmit or not can depend (only) on the current CSI signal. Non-stationary strategies are naturally harder to analyze, and moreover, their advantage over stationary strategies is not clear in large, distributed and selfish environments[1].

The technological relevance for our work lies, for example, in WLAN systems, where underlying network users have

[1] Accordingly, our study centers on equilibrium points that are obtained in stationary strategies. We note however that the Nash equilibrium in stationary strategies remains an equilibrium point even within the larger class of general strategies.

diverse (application-dependent) throughput requirements. The leading standard, namely the 802.11x [3], employs a random access protocol, whose principles are based on the original Aloha. Interestingly, on-going IEEE standardization activity (the 802.11n standard) focuses on the incorporation of CSI for better network utilization. This last fact further motivates to study the use of CSI in distributed, self-optimizing user environments.

B. Related Work

Exploiting channel state information for increasing the network's capacity has been an on-going research topic within the information theory community (see [1] for a survey). Recent research (see [8], [9] and references therein) is dedicated to uplink decentralized approaches, in which each station's transmission decision can be based on private CSI only. Nodes are assumed to operate in a cooperative manner, thus willing to accept a unified throughput-maximizing transmission policy.

Game theoretic tools have been widely applied to analyze selfish behavior in communication networks (see [10] for a survey). Recently, some papers have considered Aloha-like random access networks from a non-cooperative perspective [4]–[7], [11]. Of specific relevance to our work is a paper by Jin and Kesidis [6], which considers a shared collision channel with users who have fixed throughput demands. Users dynamically adapt their transmission rates in order to obtain their required demands. Our work provides, as a special case, a comprehensive analysis of their model, and further extends it by incorporating channel state information as affecting the transmission policy.

Our user model that incorporates both channel-aware and self-interested mobiles is quite novel. A related model was considered in [11], [12], where users with long term power constraints determine the transmission power for given CSI to maximize their individual throughput. The reception rules which are considered are either single packet "capture" [11] or multi-reception [12]. These papers are mainly concerned with the existence of a Nash equilibrium point and some basic structural properties thereof.

C. Contribution and Paper Organization

This paper presents a comprehensive study of the non-cooperative game between the channel-aware, self-interested network users. The main contributions are summarized below.

• We provide a model for an uplink collision channel that incorporates stochastic channel variation and CSI, with which the interaction of selfish users may be studied.

• Our equilibrium analysis reveals that when the throughput demands are within the network capacity, there exist exactly two Nash equilibrium points in the resulting game.

• We show that one equilibrium is strictly better than the other in terms of power investment for all users. We further show that the performance gap (in terms of the total power investment) between the equilibrium points is potentially unbounded.

• A simple lower bound on the total channel throughput (or capacity) is provided. We relate this bound to the well-known result for the capacity of an Aloha network ($1/e$).

• We describe a fully distributed mechanism which converges to the better equilibrium point. The suggested mechanism is natural in the sense that it relies on the user's best response to given network conditions.

We emphasize that all our results are valid under general assumptions on the channel state distribution and CSI signals. We also note that our game model is related (but not identical) to S-modular games [13], [14]. However, the results we obtain here are stronger than these implied by the general theory.

The structure of the paper is as follows. We first present the general model (Section II), and identify basic properties related to stationary transmission strategies. A detailed equilibrium analysis is provided in Section III. Section IV focuses on the achievable network capacity. In Section V we present a mechanism which converges to the better equilibrium. We discuss several aspects of our results in Section VI. Conclusion and further research direction are drawn in Section VII. Due to lack of space, several proofs are omitted from the main text, and can be found in an accompanying technical report [15][2].

II. The Model and Preliminaries

We consider a wireless network, shared by a finite set of mobile users $\mathcal{I} = \{1, \ldots, n\}$ who transmit at a fixed power level to a common base station over a shared collision channel. Time is slotted, so that each transmission attempt takes place within slot boundaries that are common to all. A transmission can be successful only if no other user attempts transmission simultaneously. Thus, at each time slot, at most one user can successfully transmit to the base station. To further specify our model, we start with a description of the channel between each user and the base station (Section II-A), ignoring the possibility of collisions. In Section II-B, we formalize the user objective and formulate the non-cooperative game which arises in a multi-user shared network.

A. The Single-User Channel

Our model for the channel between each user and the base station is characterized by two basic quantities.

a. Channel state information. At the beginning of each time slot k, every user i obtains a channel state information (CSI) signal $\zeta_{i,k} \in \mathcal{Z}_i \subset \mathbb{R}^+$, which provides an indication (possibly partial) of the quality of the current channel between the user and the base station (a larger number corresponds to a better channel quality). We assume that each set $\mathcal{Z}_i$ of possible CSI signals for user i is finite[3] and denote its elements by $\{z_{i1}, z_{i2}, \ldots, z_{ix_i}\}$, with $z_{i1} < z_{i2} < \cdots < z_{ix_i}$.

b. Expected data rate. We denote by $R_i(z_i) > 0$ the expected data rate (say, in bits per second) that user i can sustain at any

[2] A preliminary version of our work, which focuses on a simplified model with no CSI, was presented at the Net-Coop'07 workshop, Avignon [16].

[3] This is assumed for convenience only. Note that the channel quality may still take continuous value, which the user reasonably classifies into a finite number of information states.

given slot as a function of the current CSI signal $z_i \in \mathcal{Z}_i$. We assume that the function $R_i(z_i)$ strictly increases in z_i.

Throughout this paper we make the following assumption:

Assumption 1: (i) $Z_i = \{\zeta_{i,k}\}_{k=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables; the probability of observing a particular CSI signal $z_i \in \mathcal{Z}_i$ in any given slot is denoted by $P_i(z_i) > 0$ (signals with zero probability are excluded from the set $\mathcal{Z}_i$). (ii) The sequences Z_i and Z_j are independent for $i \neq j$.

The above model applies to the following network scenario. The quality (or state) of the channel between user i and the base station may vary over time. Instead of the exact channel state, user i observes a CSI signal z_i. After observing the CSI at the beginning of a slot, user i may respond by adjusting its coding scheme in order to maximize its data throughput on that slot. The expected data rate $R_i(z_i)$ thus takes into account the actual channel state distribution (conditioned on z_i), as well as the coding scheme employed by the user. Our modeling assumptions accommodate, in particular, the so-called block-fading channel, which is broadly studied in the literature (see [1], [8] and references therein). More details on the interpretation of our model are provided in [15].

B. User Objective and Game Formulation

In this subsection we describe the user objective and the non-cooperative game which arises as a consequence of the user interaction over the collision channel. In Section II-B1 we define the Nash equilibrium of the game, and also characterize stationary transmission strategies, which are central in this paper. Some basic properties of these strategies are highlighted in Section II-B2.

1) Basic Definitions: We associate with each user i a throughput demand ρ_i (in bits per slot) which it wishes to deliver over the network. The objective of each user is to minimize its average transmission power (which is equivalent in our model to the average rate of transmission attempts, as users transmit at a fixed power level), while maintaining the effective data rate at (or above) this user's throughput demand. We further assume that users always have packets to send, yet they may delay transmission to a later slot to accommodate their required throughput with minimal power investment.

Our focus in this paper is on *stationary* transmission strategies, in which the decision whether to transmit or not can depend (only) on the current CSI signal. A formal definition is provided below.

Definition 2.1 (stationary strategies): A stationary strategy for user i is a mapping from $\mathcal{Z}_i$ to $[0,1]$. Equivalently, a stationary strategy will be represented by an x_i-dimensional vector $\mathbf{s}_i = (s_{i1}, \ldots, s_{ix_i}) \in [0,1]^{x_i}$, where the m-th entry corresponds to user i's transmission probability when the observed CSI signal is z_{im}.

For example, the vector $(0, \ldots, 0, 1)$ represents the strategy of transmitting (w.p. 1) only when the CSI signal is the highest possible. Note that the transmission probability in a slot (or the transmission rate), which is a function of $\mathbf{s}_i$ only, is given by

$$p_i(\mathbf{s}_i) = \sum_{m=1}^{x_i} s_{im} P_i(z_{im}). \tag{1}$$

We use the term multi-strategy when referring to a collection of user strategies, and denote by $\mathbf{s} \triangleq (\mathbf{s}_1, \ldots, \mathbf{s}_n)$ the multi-strategy comprised of all users' strategies. The notation $\mathbf{s}_{-i}$ would be used for the transmission strategies of all users but for the i-th one. Evidently, the probability that no user from the set $\mathcal{I}\backslash i$ transmits in a given slot is given by $\prod_{j\neq i}(1-p_j(\mathbf{s}_j))$. Since the transmission decision of each user is independent of the decisions of other users, the expected data rate of user i, denoted $r_i(\mathbf{s}_i, \mathbf{s}_{-i})$, is given by

$$r_i(\mathbf{s}_i, \mathbf{s}_{-i}) = \left[\sum_{m=1}^{x_i} s_{im} P_i(z_{im}) R_i(z_{im})\right] \prod_{j\neq i} \left(1 - p_j(\mathbf{s}_j)\right),$$

where the expression $\sum_{m=1}^{x_i} s_{im} P_i(z_{im}) R_i(z_{im})$ stands for the average rate which is obtained in a collision-free environment under the same strategy $\mathbf{s}_i$.

The basic assumption of our model is that users are self-optimizing and are free to determine their own transmission schedule in order to fulfill their objectives. Furthermore, users are unable to coordinate their transmission decisions. This situation is modeled and analyzed in our paper as a non-cooperative game [17] between the n users. In particular, we are interested in the Nash equilibrium point of the game. A Nash equilibrium point (NEP) for our model is a multi-strategy $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_n)$, which is self-sustaining in the sense that all throughput constraints are met, and neither user can lower its transmission rate by unilaterally modifying its transmission strategy. Formally,

Definition 2.2 (Nash equilibrium point): A multi-strategy $\mathbf{s} \triangleq (\mathbf{s}_1, \ldots, \mathbf{s}_n)$ is a Nash equilibrium point if

$$\mathbf{s}_i \in \operatorname*{argmin}_{\tilde{\mathbf{s}}_i} \left\{ p_i(\tilde{\mathbf{s}}_i) : r_i(\tilde{\mathbf{s}}_i, \mathbf{s}_{-i}) \geq \rho_i \right\}. \tag{2}$$

The transmission rate p_i can be regarded as the cost which the user wishes to minimize. Using game-theoretic terminology, a Nash equilibrium is a multi-strategy $\mathbf{s} \triangleq (\mathbf{s}_1, \ldots, \mathbf{s}_n)$ so that each $\mathbf{s}_i$ is a *best response* of user i to $\mathbf{s}_{-i}$, in the sense that the user's cost is minimized.

2) Threshold Strategies: A subclass of stationary strategies which is central in our analysis is defined below.

Definition 2.3 (threshold strategies): A threshold strategy is a stationary strategy of the form $\mathbf{s}_i = (0, 0, \ldots, 0, s_{im_i}, 1, 1 \ldots, 1)$, $s_{im_i} \in (0, 1]$, where z_{im_i} is a threshold CSI level above which user i always transmits, and below which it never transmits.

We next state some properties related to threshold strategies. Their simple proof is given in [15]. The first property suggests that users should always prefer threshold strategies.

Lemma 1: Assume that all users access the channel using a stationary strategy. Then a best response strategy of any user i is always a threshold strategy.

As a result of the above lemma, we may analyze the non-cooperative game by restricting the strategies of each user i

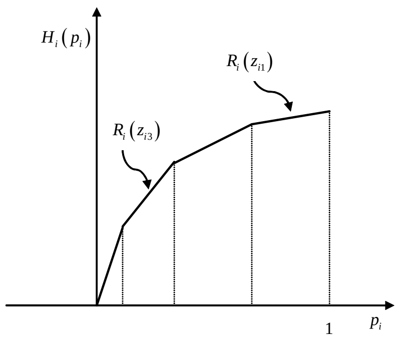

Fig. 1. An example of the collision-free rate function $H_i(p_i)$. In this example there are four CSI signals. Note that the slope of $H_i(p_i)$ is exactly the rate of the threshold CSI which corresponds to p_i.

to the set of threshold strategies, denoted by T_i. We proceed by noting that every threshold strategy can be identified with a unique *scalar* value $p_i \in [0,1]$, which is the transmission probability in every slot, i.e., $p_i \equiv p_i(\mathbf{s}_i)$. More precisely:

Lemma 2: The mapping

$$\mathbf{s}_i = (0,\ldots,0,s_{im_i},1,\ldots,1) \in T_i \mapsto p_i \equiv p_i(\mathbf{s}_i) \in [0,1],$$

is a surjective (one-to-one and onto) mapping from the set of threshold strategies T_i to the interval $[0,1]$.

Given this mapping, the stationary policy of each user will be henceforth represented by a scalar $p_i \in [0,1]$, which uniquely determines the CSI threshold and its associated transmission probability, denoted by $z_{im_i}(p_i)$ and $s_{im_i}(p_i)$ respectively. Consequently, the user's expected throughput per slot in a collision free environment, denoted by H_i, can be represented as a function of p_i only, namely

$$H_i(p_i) \stackrel{\triangle}{=} s_{im_i}(p_i)P_i(z_{im_i}(p_i))R_i(z_{im_i}(p_i)) + \sum_{m=m_i(p_i)+1}^{x_i} P_i(z_{im})R_i(z_{im}). \quad (3)$$

This function will be referred to as the *collision-free rate function*. Using this function, we may obtain an explicit expression for the user's average throughput, as a function of $\mathbf{p} = (p_1,\ldots,p_n)$, namely

$$r_i(p_i,\mathbf{p}_{-i}) = H_i(p_i)\prod_{j\neq i}(1-p_j). \quad (4)$$

Example: No CSI. A special important case is when no CSI is available. This corresponds to $x_i = 1$ in our model. In this case the collision-free rate function is simply $H_i(p_i) = \bar{R}_i p_i$, where $\bar{R}_i = R_i(z_{i1})$ is the expected data rate that can be obtained in any given slot.

III. Equilibrium Analysis

In this section we analyze the Nash equilibrium point (2) of the network under stationary transmission strategies. For the analysis, we require the following properties of the rate function (3):

Lemma 3: The collision-free rate function H_i satisfies the following properties.

(i) $H_i(0) = 0$.
(ii) $H_i(p_i)$ is a continuous and strictly increasing function over $p_i \in [0,1]$.
(iii) $H_i(p_i)$ is concave.

Proof: Noting (1), $p_i = 0$ means no transmission at all, thus an average rate of zero. It can be easily seen that $H_i(p_i)$ in (3) is a piecewise-linear (thus continuous), strictly increasing function. As to concavity, note that the slope of H_i is determined by $R_i(z_{im_i})$ which decreases with p_i (see Figure 1), as m_i decreases in p_i (from Eq. (3)) and z_{im} is increasing in m (by definition). □

A key observation which is useful for the analysis is that every Nash equilibrium point can be represented via a set of n equations in the n variables $\mathbf{p} = (p_1,\ldots,p_n)$. This is summarized in the next proposition.

Proposition 1 (The equilibrium equations): A multi strategy $\mathbf{p} = (p_1,\ldots,p_n)$ is a Nash equilibrium point if and only if it solves the following set of equations

$$r_i(p_i,\mathbf{p}_{-i}) = H_i(p_i)\prod_{j\neq i}(1-p_j) = \rho_i, \quad i \in \mathcal{I}. \quad (5)$$

Proof: Adapting the Nash equilibrium definition (2) to stationary threshold strategies, a NEP is a multi-strategy $\mathbf{p} = (p_1,\ldots,p_n)$ such that

$$p_i = \min\left\{\tilde{p}_i \in [0,1]:\ r_i(\tilde{p}_i,\mathbf{p}_{-i}) \geq \rho_i\right\}, \quad i \in \mathcal{I}, \quad (6)$$

where r_i is defined in (4). Since $r_i(\tilde{p}_i,\mathbf{p}_{-i})$ is strictly increasing in $\tilde{p}_i$ (by Lemma 3), (6) is equivalent to $r_i(p_i,\mathbf{p}_{-i}) = \rho_i$, $i \in \mathcal{I}$, which is just (5). □

Due to the above result, we shall refer to the set of equations (5) as the *equilibrium equations*.

A. Two Equilibria or None

We next address the *number* of equilibrium points in our system. Obviously, if the overall throughput demands of the users are too high there cannot be an equilibrium point, since the network naturally has limited traffic capacity (the capacity of the network will be considered in Section IV). When throughput demands are within the feasible region, we establish that there are exactly *two* Nash equilibria.

Denote by $\rho = (\rho_1,\ldots,\rho_n)$ the vector of throughput demands, and let Ω be the set of feasible vectors ρ, for which there exists at least one Nash equilibrium point (equivalently, for which there exists a solution to (5) with $p_i \in [0,1]$). Figure 2 illustrates the set of feasible throughput demands for a simple two-user case, with $H_i(p_i) = p_i$. It may be verified that Ω is a closed set with nonempty interior. More details on the structure of Ω are given in [15]. We specify below the number of equilibrium points for any throughput demand vector $\rho = (\rho_1,\ldots,\rho_n)$ at the interior of Ω.

Theorem 2: Consider the non-cooperative game model under stationary transmission strategies. Let Ω be the set of feasible throughput demand vectors $\rho = (\rho_1,\ldots,\rho_n)$, and let

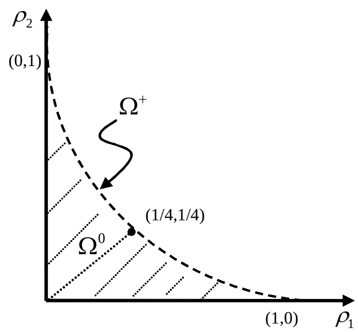

Fig. 2. The set of feasible throughput demands for a two user network with $H_i(p_i) = p_i$, $i = 1, 2$.

Ω^0 be its interior. Then for each $\rho \in \Omega^0$ there exist exactly two Nash equilibria.

Proof: See appendix.

In [15] we show that there exists a unique equilibrium point for strictly-positive throughput vectors ρ which are on the boundary of Ω (the set Ω^+ in Figure 2). However, since the boundary has measure zero, we can exclude the single equilibrium case from our discussion.

B. The Energy Efficient Equilibrium

Going beyond the basic questions of existence and number of equilibrium points, we wish to further characterize the properties of the equilibrium points. In particular, we are interested here in the following question: How do the two equilibrium points compare: is one "better" than the other? The next theorem shows that indeed one equilibrium point is power-superior for all users.

Theorem 3: Assume that the throughput demand vector ρ is within the feasible region Ω^0, so that there exist two equilibria in stationary strategies. Let $\mathbf{p}$ and $\tilde{\mathbf{p}}$ be these two equilibrium points. If $p_i < \tilde{p}_i$ for some user i, then $p_j < \tilde{p}_j$ for every $j \in \mathcal{I}$.

Proof: Define $a_{ik} \triangleq \frac{\rho_i}{\rho_k}$. For every user $k \neq i$ divide the ith equation in the set (5) by the kth one. We obtain

$$a_{ik} = \frac{H_i(p_i)(1-p_k)}{H_k(p_k)(1-p_i)} < \frac{H_i(\tilde{p}_i)(1-p_k)}{H_k(p_k)(1-\tilde{p}_i)}, \tag{7}$$

since H_i is increasing. Now since $\frac{H_i(\tilde{p}_i)(1-\tilde{p}_k)}{H_k(\tilde{p}_k)(1-\tilde{p}_i)} = a_{ik}$, it follows that $\frac{(1-\tilde{p}_k)}{H_k(\tilde{p}_k)} < \frac{(1-p_k)}{H_k(p_k)}$. Since H_k is increasing in p_k, we conclude from the last inequality that $p_k < \tilde{p}_k$. □

The last result is significant from the network point of view. It motivates the design of a network mechanism that will avoid the inferior equilibrium point, which is wasteful for *all* users. This will be our main concern in Section V. Henceforth, we identify the better equilibrium point as the *Energy Efficient Equilibrium (EEE)*.

We now turn to examine the quality of the EEE relative to an appropriate social cost. Recall that each user's objective is to minimize its average transmission rate subject to a throughput demand. Thus, a natural performance criterion for evaluating any multi-strategy $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_n)$ (in particular, an equilibrium multi-strategy) is given by the sum of the user's average transmission rates induced by $\mathbf{s}$, namely

$$Q(\mathbf{s}) = \sum_i p_i(\mathbf{s}_i). \tag{8}$$

The next theorem addresses the quality of the EEE with respect to that criterion.

Theorem 4: Let $\mathbf{p}$ be an EEE. Then $\sum_i p_i \leq 1$.

Proof: See appendix.

An immediate conclusion from the above theorem is that the overall power investment at the EEE is bounded, as the sum of transmission probabilities is bounded. This means, in particular, that the average transmission power of all users is bounded by the maximal transmission power of a single station.

C. Social Optimality and Efficiency Loss

We proceed to examine the extent to which selfish behavior affects system performance. That it, we are interested to compare the quality of the obtained equilibrium points to the centralized, system-optimal solution (still restricted to stationary strategies). Recently, there has been much work in quantifying the "efficiency loss" incurred by the selfish behavior of users in networked systems (see [18] for a comprehensive review). The two concepts which are most commonly used in this context are the *price of anarchy* (PoA), which is (an upper bound on) the performance ratio (in terms of a relevant social performance measure) between the global optimum and the *worst* Nash equilibrium, and *price of stability* (PoS), which is (an upper bound on) the performance ratio between the global optimum and the *best* Nash equilibrium.

Returning to our specific network scenario, consider the case where a central authority, which is equipped with user characteristics $\mathbf{H} = (H_1, \ldots, H_n)$ and $\rho = (\rho_1, \ldots, \rho_n)$ can enforce a stationary transmission strategy for every user $i \in \mathcal{I}$. We consider (8) as the system-wide performance criterion, and compare the performance of this optimal solution to the performance at the Nash equilibria. A socially optimal multi-strategy denoted $\mathbf{s}^*(\mathbf{H}, \rho)$, is a strategy that minimizes (8), while obeying all user throughput demands ρ_i. Similarly, denote by $\mathbf{s}^{\mathbf{b}}(\mathbf{H}, \rho)$ and $\mathbf{s}^{\mathbf{w}}(\mathbf{H}, \rho)$ the multi-strategies at the better NEP and at the worse NEP, respectively. Then the PoA and PoS are given by

$$PoA = \sup_{\mathbf{H},\rho} \frac{Q\big(\mathbf{s}^{\mathbf{w}}(\mathbf{H},\rho)\big)}{Q\big(\mathbf{s}^*(\mathbf{H},\rho)\big)}, \quad PoS = \sup_{\mathbf{H},\rho} \frac{Q\big(\mathbf{s}^{\mathbf{b}}(\mathbf{H},\rho)\big)}{Q\big(\mathbf{s}^*(\mathbf{H},\rho)\big)}. \tag{9}$$

We next show that the PoA is generally unbounded, while the PoS is always one.

Theorem 5: Consider the non-cooperative game, the NEP of which is defined in (2). Then (i) The PoS is always one, and (ii) The PoA is generally unbounded.

Proof: (i) This claim follows immediately, noting that 1) the socially optimal stationary strategy is a threshold strategy (by applying a similar argument to the one used in Lemma 1), and 2) the socially optimal stationary strategy obeys the equilibrium equations (5) (following a similar argument to the one used in Proposition 1). Hence, by Proposition 1 the optimal solution is also an equilibrium point. Equivalently, this means that $PoS = 1$.

(ii) We establish that the price of anarchy is unbounded by means of an example. Consider a network with n identical users with $H_i(p_i) = \bar{R}p_i$ (this collision-free rate function corresponds to users who cannot obtain any CSI). Each user's throughput demand is $\rho_i = \epsilon \to 0$. Recall that the throughput demands are met with equality at every equilibrium point (Proposition 1). Then, by symmetry, we obtain a single equilibrium equation, namely $\bar{R}p(1-p)^{n-1} = \epsilon$. As ϵ goes to zero, the two equilibria are $p_a \to 1$ and $p_b \to 0$. Obviously, the latter point is also a social optimum; it is readily seen that the price of anarchy here equals in the limit to $\frac{p_a}{p_b} \to \infty$. □

The above theorem clearly motivates the need for a mechanism that will induce the EEE, as this equilibrium point coincides with the socially-optimal solution, while the gap between the two equilibria could be arbitrarily large.

We conclude this section with a comment on the computational properties of an equilibrium point, which may be directly observed from the proof of Theorem 2. It can be seen that verifying the existence of an equilibrium point (for a given throughput demand vector ρ) is computationally equivalent to finding an extremum point for a scalar unimodal function. Similarly, the equilibrium points themselves are computed by finding the zeros of that function. This suggests that the computation of the Nash equilibrium can be efficiently accomplished by standard search techniques, such as the bisection method or the golden section search method (see, e.g., [19]). More details are given in [15]. An alternative way for verifying the existence of the equilibrium point, as well as calculating the EEE is obtained by simulating the best-response dynamics, which is considered in Section V. Indeed, we show in that section that this dynamics either converges to the EEE, or obtains probabilities larger than 1 in case that no equilibrium exists.

IV. Achievable Channel Capacity

The aim of this section is to provide explicit lower bounds for the achievable channel capacity. The term "capacity" is used here for the total throughput (normalized to successful transmission per slot) which can be obtained in the network. We focus here on the case where users have no CSI, and then relate our result to general CSI.

Consider the null-CSI model, where no user can observe any CSI (see the example at the end of Section II). Recall that the collision-free rate in this case is given by $H_i(p_i) = \bar{R}_i p_i$, where $\bar{R}_i$ is the expected data rate in case of a successful transmission. Define $y_i \triangleq \frac{\rho_i}{\bar{R}_i}$, which we identify henceforth as the *normalized throughput* demand for user i: indeed, y_i stands for the required rate of successful transmissions. Then the equilibrium equations (5) become

$$p_i \prod_{j \neq i}(1 - p_j) = y_i, \quad 1 \leq i \leq n. \tag{10}$$

We shall first consider the symmetric case, i.e., $y_i = y$ for every user i, and then relate the results to the general non-symmetric case. The theorem below establishes the conditions for the existence of an equilibrium point in the symmetric null-CSI case.

Theorem 6 (Symmetric users): Let $y_i = y$ for every $1 \leq i \leq n$. Then (i) A Nash equilibrium exists if and only if

$$ny \leq (1 - \frac{1}{n})^{n-1}. \tag{11}$$

(ii) In particular, a Nash equilibrium exists if $ny \leq e^{-1}$.

Proof: (i) By dividing the equilibrium equations (10) of any two users, it can be seen that every symmetric-users equilibrium satisfies $p_i = p_j = p$ ($\forall$ i, j). Thus, the equilibrium equations (10) reduce to a single (scalar) equation:

$$h(p) \triangleq p(1 - p)^{n-1} = y. \tag{12}$$

We next investigate the function $h(p)$. Its derivative is given as $h'(p) = (1 - p)^{n-2}(1 - np)$. It can be seen that the maximum value of the function $h(p)$ is obtained at $p = 1/n$. An equilibrium exists if and only if the maximal value of $h(p)$ is greater than y. Substituting the maximizer $p = 1/n$ in (12) implies the required result.

(ii) It may be easily verified that the right hand side of (11) decreases with n. Since $\lim_{n\to\infty}(1 - \frac{1}{n})^{n-1} = e^{-1}$, the claim follows from (i). □

We now show that the simple bound obtained above holds for non-symmetric users as well, implying that the symmetric case is worst in terms feasible channel utilization.

Theorem 7 (Asymmetric users): For any set of n null-CSI users with normalized throughput demands $\{y_i\}$, an equilibrium point exists if

$$\sum_{i=1}^{n} y_i \leq (1 - \frac{1}{n})^{n-1}. \tag{13}$$

Proof: (outline) A key property, which is unique to the null-CSI case is that the following relation holds for every $i, j \in \mathcal{I}$:

$$p_j = \frac{a_{ji}p_i}{1 - p_i + a_{ji}p_i}, \tag{14}$$

where $a_{ji} \triangleq y_j/y_i$. This relation is obtained by dividing the equilibrium equation of the ith user by the equation of the jth one. The idea of the proof of the theorem is to fix p_i and to obtain an expression for the total normalized throughput $\sum_{i=1}^{n} y_i$ by using (14). It can be shown that the minimal value of $\sum_{i=1}^{n} y_i$ is obtained when $a_{ji} = 1$ for every $j \in \mathcal{I}$, which is essentially the symmetric case. Since the above holds for every p_i, the maximal total normalized throughput is the lowest at the symmetric case. A detailed proof is provided in [15]. □

Note that the quantity e^{-1} is also the well-known maximal throughput of a slotted Aloha system with Poisson arrivals and an infinite set of nodes [2]. In our context, if the normalized throughput demands do not exceed e^{-1}, an equilibrium point is guaranteed to exist. Thus, in a sense, we may conclude that noncooperation of users, as well restricting users to stationary strategies, do not reduce the capacity of the collision channel. In [15] we show that the network capacity can only increase when users obtain channel state information. Hence, equation (13) with e^{-1} as its right hand side limit, serves as a global sufficient condition for the existence of an equilibrium point, which holds with general CSI.

V. Best-Response Dynamics

A Nash equilibrium point for our system represents a strategically stable working point, from which no user has incentive to deviate unilaterally. Still, the question of if and how the system arrives at an equilibrium remains open. Furthermore, since our system has two Nash equilibria with one (the EEE) strictly better than the other, it is of major importance (from the system viewpoint, as well as for each individual user) to employ mechanisms that converge to the better equilibrium rather than the worse.

The distributed mechanism we consider here relies on a user's best-response, which is generally the optimal user reaction to a given network condition (see [17]). Specifically, the best response of a given user is a transmission probability which brings the obtained throughput of that user (given other user strategies) to its throughput demand ρ_i. Accordingly, observing (4), the best response of user i for any multi-strategy $\mathbf{p} = (p_1, \ldots, p_n)$ is given by

$$p_i := H_i^{-1}\left(\frac{\rho_i}{\prod_{j\neq i}(1-p_j)}\right), \tag{15}$$

where H_i^{-1} is the inverse function of the collision-free rate function H_i (if the argument of H_i^{-1} is larger than maximal value of H_i, p_i can be chosen at random). Note that H_i^{-1} is well defined, since H_i is continuous and monotone (Lemma 3). It is important to notice that each user is not required to be aware of the transmission probability of every other user. Indeed, only the overall idle probability of other users $\prod_{j\neq i}(1-p_j)$ is required in (15). Our mechanism can be described as follows – *Each user updates its transmission probability from time to time through its best response (15). The update times of each user need not be coordinated with other users*.

This mechanism reflects what greedy, self-interested users would naturally do: Repeatedly observe the current network situation and react to bring their costs to a minimum. For the analysis of best-response dynamics we assume the following.

Assumption 2:
(i) The user population is fixed.
(ii) Users repeatedly update their transmission probabilities (i.e., an infinite number of updates for each user) using Eq. (15) .
(iii) The effective elements $\prod_{j\neq i}(1-p_j)$ and H_i^{-1} are perfectly estimated by the user before each update.
(iv) The transmission probabilities of each user are initialized to zero ("slow start").

Our convergence result is summarized below.

Theorem 8 (Convergence to the EEE): Under Assumption 2, best response dynamics asymptotically converges to the EEE, in case that a Nash equilibrium point exists.

The proof of the above result relies on showing that the vector of user probabilities $\mathbf{p}$ monotonously increases until convergence. A detailed proof is given in [15]. We note that the initialization of the transmission probabilities to zero (or any other value smaller than the EEE) is essential for this result.

In [15], we also show that best-response dynamics reconverge to the EEE in case that users join or leave the network. Resilience to changes in the user population is meaningful especially in wireless mobile systems such as adhoc networks, where stations connect and disconnect quite frequently.

VI. Discussion

We briefly discuss here some consequences of our results, emphasizing network management aspects. Our equilibrium analysis has revealed that within the feasible region the system has two Nash equilibrium points with one strictly better than the other. The better equilibrium (the EEE) is socially optimal, hence the network should ensure that users indeed operate at that equilibrium. An important step in this direction is the above suggested distributed mechanism which converges to the EEE. It should be mentioned however that fluctuations in the actual system might clearly bring the network to an undesired equilibrium. Hence, centralized management (based on user feedbacks) may still be required to identify the possible occurrence of the worse equilibria, and then direct the network to the EEE. Possible mechanisms for this purpose remain a research direction for the future.

In this paper we mainly considered the throughput demands ρ_i as determined by the user itself. Alternatively, ρ_i may be interpreted as a bound on the allowed throughput which is imposed by the network (as part of a resource allocation procedure). The advantage of operating in this "allocated-rate" mode is twofold. First, the network can ensure that user demands do not exceed the network capacity (e.g., by restricting the allocated rate, or through call admission control). Second, users can autonomously reach an efficient working point without network involvement, as management overhead is reduced to setting the user rates only. The rate allocation phase (e.g., through service level agreements) is beyond the scope of the present model.

A final comment relates to elastic users that may lower their throughput demand based on a throughput–power tradeoff. An obvious effect of demand elasticity would be to lower the throughput at inefficient equilibria. It remains to be verified whether other properties established here remain valid in this case.

VII. Conclusion

We have investigated in this paper the interaction between self-interested wireless users, each wishing to sustain a given throughput requirement, while making use of available CSI. We have characterized the set of feasible throughput requirements for which a Nash equilibrium exists, and shown that within the feasible region there exist two distinct NEPs, with one being power-superior for *all* users. We further demonstrated that the performance gap between these two equilibria (in terms of power investment) could be arbitrarily large. Consequently, network users should be willing to accept a mechanism which ensures convergence to the better equilibrium. We have suggested a simple and natural mechanism based on each user's best response. This mechanism is shown to converge to the better equilibrium point (within a simplified, dynamic model) without requiring any coordination between the users.

In an extended version of this paper [15], we consider in greater detail the utility in obtaining higher-quality channel state information (i.e., CSI at better resolution). In particular, we show that if even a single user obtains better CSI, the power investment of *all* users is reduced. This result indicates that wireless platforms can benefit from technological enhancements which would lead to higher quality CSI, even when available to some users and not others, and under fully distributed and self-interested user environments.

The framework and results of this paper may be extended in several ways. One direction is to extend the reception model beyond the collision model studied in this paper. In particular, capture models (which sometimes better represent WLAN systems) and multi-packet reception models [20] (as in CDMA systems) are of obvious interest. Another extension of interest to the channel model is CSI signals that are *correlated* in time (i.e., subsequent CSI signals are statistically dependent) and/or in space (i.e., CSI signals of neighboring users are statistically dependent). Last, we intend to consider non-stationary user strategies. A central question is whether the system benefits from the use of more complex policies by selfish individuals. The incorporation of non-stationary strategies and correlated CSI seems to add considerable difficulty to the analysis, and may require more elaborate game theoretic tools than the ones used here.

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Appendix

Proof of Theorem 2: The idea of the proof is to reduce the equation set (5) to a single scalar equation in a single variable p_i, for some arbitrarily chosen user i. The right hand side of the equation remains ρ_i; unimodality of the left hand side will establish the required result.

Construction of the scalar function. Consider an equilibrium point with throughput demands $\rho = (\rho_1, \dots, \rho_n)$. Then by dividing the ith equilibrium equation (5) by the jth one, we obtain

$$\frac{H_i(p_i)}{1-p_i} = a_{ij}\frac{H_j(p_j)}{1-p_j}, \tag{16}$$

where $a_{ij} \triangleq \frac{\rho_i}{\rho_j}$. Note that for each $p_i \in [0,1]$ there exists a unique $p_j \in [0,1]$ such that (16) holds. This follows since the function $h_j(p_j) \triangleq \frac{H_j(p_j)}{1-p_j}$ is continuous, strictly increasing and ranges from 0 to ∞ over $p_j \in [0,1]$. Consequently, its inverse h_j^{-1} is a well defined, continuous increasing function, and

$$p_j(p_i, a_{ij}) = h_j^{-1}\left(\frac{H_i(p_i)}{1-p_i}a_{ij}\right). \tag{17}$$

By substitution, the ith equilibrium equation can be regarded as function of p_i only, namely $H_i(p_i)\prod_{j\neq i}(1-p_j(p_i,a_{ij})) = \rho_i$, where $p_j(p_i, a_{ij})$ is defined in (17). Let $\mathbf{a} = (a_{i1}, \dots, a_{in})$, where $a_{ii} \equiv 1$. It follows that an equilibrium exists for a given throughput vector ρ if and only if there exists some $p_i \in [0,1]$ such that

$$f_i(p_i, \mathbf{a}) \triangleq H_i(p_i) \prod_{j \neq i} (1 - p_j(p_i, a_{ij})) = \rho_i. \quad (18)$$

A key property which is required for the proof, is the continuity of the function $f_i(p_i, \mathbf{a})$, which is shown below.

Lemma 4: The function $f_i(p_i, \mathbf{a})$ defined in (18) is continuous in p_i.

Proof: Continuity in p_i follows straightforwardly by the continuity of the functions H_i and $p_j(p_i, a_{ij})$. □

Let $\rho_i^{\max}(\mathbf{a}) = \max_{p_i \in [0,1]} f_i(p_i, \mathbf{a})$ (where the maximum is attained by continuity of f_i). We next show that $f_i(p_i, \mathbf{a})$ is unimodal in p_i. Consequently, fixing $\mathbf{a}$, it follows by the continuity of f_i that every value of in the range $[0, \rho_i^{\max}(\mathbf{a}))$ corresponds to two equilibrium points, where the value $\rho_i^{\max}(\mathbf{a})$ corresponds to a single equilibrium. Details are provided below.

We fix $\mathbf{a}$, and let $g_i(p_i) \triangleq \log\big(f_i(p_i, \mathbf{a})\big)$. Unimodality of $g_i(p_i)$ would clearly imply unimodality of $f_i(p_i, \mathbf{a})$. The function $g_i(p_i)$ is given by

$$g_i(p_i) = \log H_i(p_i) + \sum_{j \neq i} \log(1 - p_j(p_i, a_{ij})). \quad (19)$$

For simplicity of notations we shall henceforth write p_j instead of $p_j(p_i, a_{ij})$, yet recall that p_j is a function of p_i.

Step 1: The function $g_i(p_i)$ is continuous in p_i by the continuity of $f_i(p_i, \mathbf{a})$. Furthermore, $g_i(0) = g_i(1) = -\infty$, and $g_i(p_i) > -\infty$ for $p_i \in (0,1)$ by the corresponding values of $f_i(p_i, \mathbf{a})$. In the sequel we claim that there exists a unique extremum point for the function $g_i(p_i)$. Furthermore this extremum lies in $(0,1)$, hence it is the maximizer of $g_i(p_i)$. We proceed to compute the derivative $g'(p_i)$ (Step 2), and then show that $g'(p_i)$ changes sign exactly once (Step 3).

Step 2: Taking the logarithm from both sides of (16) we obtain

$$\log H_i(p_i) + \log(1-p_j) = \tilde{a}_{ij} + \log H_j(p_j) + \log(1-p_i) \quad (20)$$

(where $\tilde{a}_{ij} = \log a_{ij}$). Denote by

$$H_i'(p_i) = \frac{dH_i(p_i)}{dp_i} \quad (21)$$

the derivative of H_i w.r.t. p_i. Recalling that H_i is piecewise-linear (see Theorem 3), at a finite set of points in which the standard derivative is undefined, we take the left derivative instead. Differentiating both sides of (20) yields the following equation

$$\frac{H_i'(p_i)}{H_i(p_i)} + \frac{1}{1-p_i} = \left(\frac{H_j'(p_j)}{H_j(p_j)} + \frac{1}{1-p_j}\right) \frac{dp_j}{dp_i}. \quad (22)$$

Thus,

$$\frac{dp_j}{dp_i} = \frac{\frac{H_i'(p_i)}{H_i(p_i)} + \frac{1}{1-p_i}}{\frac{H_j'(p_j)}{H_j(p_j)} + \frac{1}{1-p_j}}. \quad (23)$$

The derivative of $g_i(p_i)$ w.r.t. p_i is given by

$$g_i'(p_i) = \frac{H_i'(p_i)}{H_i(p_i)} - \sum_{j \neq i} \frac{1}{1-p_j} \frac{dp_j}{dp_i}. \quad (24)$$

Using (23), it follows that $\frac{1}{1-p_j}\frac{dp_j}{dp_i} =$

$$= \left(\frac{H_i'(p_i)}{H_i(p_i)} + \frac{1}{1-p_i}\right) \left[(1-p_j)\left(\frac{H_j'(p_j)}{H_j(p_j)} + \frac{1}{1-p_j}\right)\right]^{-1}.$$

Thus, $g_i'(p_i) = \frac{H_i'(p_i)}{H_i(p_i)} - \left(\frac{H_i'(p_i)}{H_i(p_i)} + \frac{1}{1-p_i}\right) \sum_{j \neq i} v_j(p_j)$, where $v_j(p_j) \triangleq \frac{H_j(p_j)}{(1-p_j)H_j'(p_j) + H_j(p_j)}$.

Step 3: The function $g_i(p_i)$ increases if and only if $g_i'(p_i) > 0$. Equivalently,

$$\sum_{j \neq i} v_j(p_j) < \frac{\frac{H_i'(p_i)}{H_i(p_i)}}{\frac{H_i'(p_i)}{H_i(p_i)} + \frac{1}{1-p_i}} = 1 - \frac{\frac{1}{1-p_i}}{\frac{H_i'(p_i)}{H_i(p_i)} + \frac{1}{1-p_i}} \quad (25)$$

$$= 1 - \frac{H_i(p_i)}{(1-p_i)H_i'(p_i) + H_i(p_i)} = 1 - v_i(p_i). \quad (26)$$

To summarize, $g_i(p_i)$ increases at p_i if and only if

$$\sum_{j \in \mathcal{I}} v_j(p_j) < 1. \quad (27)$$

Similarly g_i decreases if and only if $\sum_{j \in \mathcal{I}} v_j(p_j) > 1$.

Since H_i is concave increasing (Lemma 3), it may be verified that v_i is strictly increasing in p_i. This can be seen by noting that $\frac{1}{v_i(p_i)} = 1 + \frac{H_i'(p_i)(1-p_i)}{H_i(p_i)}$ strictly decreases with p_i (by Lemma 3). Consequently, since p_j strictly increases with p_i due to (17), then $\sum_{j \in \mathcal{I}} v_j(p_j)$ strictly increases with p_i. Note further that if $p_i = 0$ then $\sum_{j \in \mathcal{I}} v_j(p_j) = 0$; additionally, for $p_i = 1$, $p_j = 1$ for every j, hence $\sum_{j \in \mathcal{I}} v_j(p_j) = n$. Hence, the function $g_i(p_i)$ strictly increases up to some value $p_i^* \in (0,1)$ and then decreases.

Step 4: Due to the above and by the continuity of $f_i(p_i, \mathbf{a})$, the function $g_i(p_i)$ is a unimodal function. Hence $f_i(p_i, \mathbf{a})$ is unimodal. Fixing $\mathbf{a}$, this means that if the graph of f_i cross a value the vertical line of ρ_i , it does so exactly twice (or just once at the non-generic case where $\rho_i = \rho_i^{\max}(\mathbf{a})$). □

Proof of Theorem 4: We adopt in this proof the notations used for the proof of Theorem 2. Consider a reference user i. Note that an equilibrium point is obtained as a solution to the equation $g_i(p_i) = \log \rho_i$. By the unimodality of $g_i(p_i)$ (see proof of Theorem 2), it follows that the better equilibrium point is obtained at some $p_i \in [0,1]$ for which $g_i(p_i)$ is increasing. Hence,

$$\sum_{j \in \mathcal{I}} v_j(p_j) \leq 1, \quad (28)$$

by (27) (equality holds when the two equilibria coincide). The proof will be completed by showing that $v_j(p_j) \geq p_j$ for all $p_j \in [0,1]$ and each j. For convenience, we omit the user index j in the sequel. To see the latter, note that by the concavity of H (Lemma 3) it follows that $H(0) \leq H(p) + H'(p)(0-p)$, using the gradient inequality ([21], p. 69), where H' is defined in (21). Hence $H(p) \geq pH'(p)$, or equivalently $H(p)(1-p) \geq pH'(p)(1-p)$. Thus, $H(p) \geq pH'(p)(1-p) + pH(p) = p\big[H(p) + H'(p)(1-p)\big]$, or $v(p) \equiv \frac{H(p)}{(1-p)H'(p)+H(p)} \geq p$. The result of the theorem is now established by summing the last inequality on all users, combined with (28). □