LMU-TPW-00-19
hep-th/0007231
July, 2000

arXiv:hep-th/0007231v1 28 Jul 2000

Nonlinear Self–Duality and Supersymmetry*

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Abstract

We review self-duality of nonlinear electrodynamics and its extension to several Abelian gauge fields coupled to scalars. We then describe self-duality in supersymmetric models, both $\mathcal{N}=1$ and $\mathcal{N}=2$. The self-duality equations, which have to be satisfied by the action of any self-dual system, are found and solutions are discussed. One important example is the Born-Infeld action. We explain why the $\mathcal{N}=2$ supersymmetric actions proposed so far are not the correct world-volume actions for D3 branes in $d=6$.

*Based on talks given at the XII Workshop 'Beyond the Standard Model' (February 2000, Bad Honnef, Germany) and at the Erwin Schrödinger International Institute for Mathematical Physics (March 2000, Vienna, Austria).

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1 Introduction

The simplest and best known example of a self-dual system is electrodynamics in vacuum. The set of Maxwell's equations is invariant under the simultaneous replacements $\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$. While being a symmetry of the Hamiltonian $H = \vec{E}^2 + \vec{B}^2$, the Lagrangian does transform: $L = \vec{E}^2 - \vec{B}^2 \to -L$. The generalization to a $(p-1)$-form potential C in $d = 2p$ dimensions with action $S = \int \mathrm{d}C \wedge *\mathrm{d}C$ is immediate.

These theories are free systems with linear equations of motion. The interesting question is whether one can construct interacting self-dual systems. The main goal of these notes is to discuss the conditions (*self-duality equations*) which have to be satisfied by the action of a dynamical system in order to be self-dual, in the sense to be specified below. Apparently Schrödinger was the first to discuss nonlinear self-duality. In [1] he reformulated the Born-Infeld (BI) theory [2] in such a way that it was manifestly invariant under U(1) duality rotations. We will mainly be interested in four-dimensional nonlinear systems of gauge fields coupled to matter. For non-supersymmetric systems the results have been obtained, as a generalization of patterns of duality in extended supergravity [3, 4] (see also [5]), in [6, 7, 8, 9, 10, 11] and reviewed and extended in [12, 13]. Our special emphasis is on manifestly $\mathcal{N} = 1, 2$ supersymmetric generalizations.

As will be discussed below, self-dual theories possess quite remarkable properties. Our main concern, however, in pursuing the study of such systems lies in the fact that self-duality turns out to be intimately connected with spontaneous breaking of supersymmetry (for still not completely understood reasons). Recently several models for partial breaking of $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ in four dimensions [14, 15, 16, 17] have been constructed. Two most prominent models – described by the Goldstone-Maxwell multiplet [14, 16] and by the tensor Goldstone multiplet [15, 16] – are self-dual $\mathcal{N} = 1$ supersymmetric theories; the other Goldstone multiplets are dual superfield version of the tensor one (as we will describe, self-duality may be consistent with the existence of dual formulations). In our opinion, this cannot be accidental.

It may look curious but the fact that the nonlinear superfield constraint, which underlies the Goldstone-Maxwell construction of [14, 16], has turned out to be fruitful for nontrivial generalizations. This constraint was used in [18, 19] to derive nonlinear U(n) duality invariant models, both in non-supersymmetric and supersymmetric cases. In the present paper, we apply the nonlinear constraint, which is at the heart of the tensor Goldstone construction of [15, 16], to derive new self-dual systems.

These notes are organized as follows. In sect. 2 we review nonlinear electrodynamics:

we define the notion of self-duality and state the self-duality equation which has to be satisfied by the action. The derivation can be found in Appendix A. We also discuss various properties of self-dual nonlinear electrodynamics, e.g. when coupled to a complex scalar field. We then proceed with a description of the general structure of self-dual Lagrangians, of which the Born-Infeld action is but a particular example, with very special properties, though. In sect. 3 we present, following Refs. [6, 7, 12, 13], the generalization to a collection of U(1) vector-fields, coupled to an arbitrary number of scalar fields. Sect. 4, which is based on Ref. [20], is the $\mathcal{N}=1$ supersymmetric version of sect. 2. In sect. 5 we discuss properties of the supersymmetric Born-Infeld action and make contact with the work of Bagger and Galperin [14], where this action was obtained as a model of partial $\mathcal{N}=2 \to \mathcal{N}=1$ supersymmetry breaking. In the next section we supersymmetrize the analysis of sect. 4. In sect. 7 we discuss self-dual models with tensor multiplets. In sect. 8, we temporarily leave supersymmetry and derive the self-duality equations and determine the maximal duality group of a d-dimensional system with n Abelian $(p-1)$-form potentials and m Abelian $(d-p-1)$-form potentials, with and without coupling to scalar fields. In sect. 9 we turn to $\mathcal{N}=2$ supersymmetric models. We find the duality equation and demonstrate that the $\mathcal{N}=2$ Born-Infeld action proposed in Ref. [21] is indeed self-dual. This action correctly reduces to the $\mathcal{N}=1$ Born-Infeld action when the $(0,1/2)$ part of the $\mathcal{N}=2$ vector multiplet is switched off. However, there are in fact infinitely many manifestly $\mathcal{N}=2$ generalization of the $\mathcal{N}=1$ Born-Infeld action with this property [20]. Within the context of the D3-brane world-volume action, one has to impose additional properties (beyond self-duality), in particular the action should be invariant under translations in the transverse directions in the embedding space, or, in other words, it should contain only derivatives of the scalar fields. We show that even when allowing for nonlinear field redefinitions, the action of Ref. [21, 20] does not satisfy this property. It is therefore not the correct model for partial $\mathcal{N}=4 \to \mathcal{N}=2$ supersymmetry breaking, based on the $\mathcal{N}=2$ Goldstone-Maxwell multiplet. We should mention that we know of no à priori reason why such a theory should be automatically self-dual. However this is the case for partial breaking of $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$. In any case, the manifestly $\mathcal{N}=2$ supersymmetric world-volume action of a D3 brane in $d=6$ is still unknown (as well as the manifestly $(1,0)$ supersymmetric BI action in $d=6$, from which it might be derived via dimensional reduction).

As already mentioned, Appendix A contains the derivation of the self-duality equation in the simplest context, namely of pure nonlinear electrodynamics.

At the end of the introduction we want to mention that all our considerations are classical. The systems we study should be considered as effective theories. That they are relevant is demonstrated by the appearance of the Born-Infeld action as the world-volume action of D-branes [22, 23]. However the study of nonlinear self-dual systems might also be interesting in its own right.

Any nonlinear theory must possess a dimensionful parameter. Within the context of (open) string theory this is the string scale α'. We will always set this parameter to unity.

2 Self-duality in nonlinear electrodynamics

We begin with a review [6, 7, 8, 9, 10, 11] of self-dual models of a single U(1) gauge field with field strength $F_{ab} = \partial_a A_b - \partial_b A_a$. The dynamics of such a model is determined by a nonlinear Lagrangian $L(F_{ab}) = -\frac{1}{4}F^{ab}F_{ab} + \mathcal{O}(F^4)$. With the definition[1]

$$\tilde{G}_{ab}(F) \;\equiv\; \frac{1}{2}\,\varepsilon_{abcd}\,G^{cd}(F) \;=\; 2\,\frac{\partial L(F)}{\partial F^{ab}}\;, \qquad G(F) = \tilde{F} + \mathcal{O}(F^3)\;, \tag{2.1}$$

the Bianchi identity and the equation of motion read

$$\partial^b \tilde{F}_{ab} = 0\;, \qquad \partial^b \tilde{G}_{ab} = 0\;. \tag{2.2}$$

Since these differential equations, satisfied by F, have the same form, one may consider *duality transformations*[2]

$$\begin{pmatrix} G'(F') \\ F' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} G(F) \\ F \end{pmatrix}\;, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})\;, \tag{2.3}$$

such that the transformed quantities F' and G' also satisfy the equations (2.2). For G' one should require

$$\tilde{G'}_{ab}(F') \;=\; 2\,\frac{\partial L'(F')}{\partial F'^{ab}}\;, \tag{2.4}$$

and the transformed Lagrangian, $L'(F)$, exists (in general, $L'(F) \neq -\frac{1}{4}F\cdot F + \mathcal{O}(F^4)$) and can be determined for any $\mathrm{GL}(2,\mathbb{R})$-matrix entering the transformation (2.3). In particular, for an infinitesimal duality transformation[3]

$$\delta \begin{pmatrix} G \\ F \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix}\;, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{gl}(2,\mathbb{R}) \tag{2.5}$$

[1]We are working in $d=4$ Minkowski space, where $\tilde{\tilde{F}} = -F$, and often use the notation $F\cdot G = F^{ab}G_{ab}$ implying $F\cdot\tilde{G} = \tilde{F}\cdot G$ and $\tilde{F}\cdot\tilde{G} = -F\cdot G$.

[2]In the case of Maxwell's electrodynamics, the field strength transforms into its Hodge dual $\tilde{F}$, hence the name 'duality transformations'.

[3]Throughout this paper, small Latin letters from the beginning of the alphabet denote finite duality transformation parameters, capital letters are used for infinitesimal transformations.

one finds

$$\Delta L = L'(F) - L(F) = (A + D)\, L(F) - \frac{1}{2} D\, \tilde{G} \cdot F + \frac{1}{4} B\, F \cdot \tilde{F} - \frac{1}{4} C\, G \cdot \tilde{G}\ ; \tag{2.6}$$

c.f. also sect. 3, eq. (3.27).

The above considerations become nontrivial if one requires the model to be *self-dual*, *i.e.*

$$L'(F) \ = \ L(F)\ . \tag{2.7}$$

The requirement of self-duality implies:
(i) only U(1) duality rotations can be consistently defined in the nonlinear case, although Maxwell's case is somewhat special (see sect. 3 for details)

$$\begin{pmatrix} G'(F') \\ F' \end{pmatrix} \ = \ \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} G(F) \\ F \end{pmatrix}\ ; \tag{2.8}$$

(ii) the Lagrangian solves the *self-duality equation* [8, 10, 11]

$$G^{ab}\, \tilde{G}_{ab} \ + \ F^{ab}\, \tilde{F}_{ab} \ = \ 0. \tag{2.9}$$

A derivation of the self-duality equation is presented in Appendix A.

Due to the definition of $G(F)$, the self-duality equation severely constrains the possible functional form of $L(F)$. Any solution of the self-duality equation defines a self-dual model.

Self-dual theories possess several remarkable properties:

I. *Duality-invariance of the energy-momentum tensor*
Given an invariant parameter g in the self-dual theory, the observable $\partial L(F,g)/\partial g$ is duality invariant [6]. Indeed, using eq. (A.6) and the duality invariance of g, one gets

$$\delta\, \frac{\partial}{\partial g}\, L = \frac{\partial}{\partial g}\, \delta\, L = \frac{1}{2}\lambda\, \frac{\partial}{\partial g} \left(\tilde{G} \cdot G \right) = \frac{1}{2}\lambda\, \frac{\partial}{\partial g} \left(\tilde{G} \cdot G + \tilde{F} \cdot F \right) = 0\ , \tag{2.10}$$

since F is g-independent. Any self-dual theory can be minimally coupled to the gravitational field g_{mn} such that the duality invariance remains intact, and g_{mn} does not change under the curved-space duality transformations. Therefore, the energy-momentum tensor is duality invariant.

II. SL$(2,\mathbb{R})$ *duality invariance in the presence of dilaton and axion*
Given a self-dual model $L(F)$, its compact U(1) duality group can be enlarged [9, 10, 11]

to the non-compact $\mathrm{SL}(2,\mathbb{R})$, by suitably coupling the electromagnetic field to the dilaton φ and axion a,

$$\mathcal{S} = \mathcal{S}_1 + \mathrm{i}\,\mathcal{S}_2 = a + \mathrm{i}\,\mathrm{e}^{-\varphi} \ . \tag{2.11}$$

Non-compact duality transformations read

$$\begin{pmatrix} G' \\ F' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix} \ , \qquad \mathcal{S}' = \frac{a\mathcal{S}+b}{c\mathcal{S}+d} \ , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \ , \tag{2.12}$$

and the duality invariant Lagrangian is

$$L(F,\mathcal{S},\partial\mathcal{S}) \ = \ L(\mathcal{S},\partial\mathcal{S}) \ + \ L(\sqrt{\mathcal{S}_2}\,F) \ + \ \frac{1}{4}\,\mathcal{S}_1 F\cdot\tilde{F} \ . \tag{2.13}$$

with $L(\mathcal{S},\partial\mathcal{S})$ the $\mathrm{SL}(2,\mathbb{R})$ invariant Lagrangian for the scalar fields,

$$L(\mathcal{S},\partial\mathcal{S}) \ = \ \frac{\partial\bar{\mathcal{S}}\,\partial\mathcal{S}}{(\mathcal{S}-\bar{\mathcal{S}})^2} \ . \tag{2.14}$$

A derivation of the self-dual model (2.13) will be described in sect. 3.

III. *Self-duality under Legendre transformation*

What is usually meant by '*duality transformations*' in field theory, more precisely for models of gauge differential forms of which electrodynamics is one example, are Legendre transformations. We now show that any system which solves the self-duality equation is automatically invariant under Legendre transformation.

Let us recall the definition of Legendre transformation in the case of a generic model of nonlinear electrodynamics specified by $L(F)$. One associates with $L(F)$ an auxiliary model $L(F,F_{\mathrm{D}})$ defined by

$$L(F,F_{\mathrm{D}}) = L(F) - \frac{1}{2}\,F\cdot\tilde{F}_{\mathrm{D}} \ , \qquad {F_{\mathrm{D}}}^{ab} = \partial^a {A_{\mathrm{D}}}^b - \partial^b {A_{\mathrm{D}}}^a \ . \tag{2.15}$$

F is now an unconstrained antisymmetric tensor field, A_{D} a Lagrange multiplier field and F_{D} the dual electromagnetic field. This model is equivalent to the original one. Indeed, the equation of motion for A_{D} implies $\partial_b\tilde{F}^{ab} = 0$ and therefore the second term in $L(F,F_{\mathrm{D}})$ is a total derivative, that is $L(F,F_{\mathrm{D}})$ reduces to $L(F)$. On the other hand, one can first consider the equation of motion for F:

$$G(F) \ = \ F_{\mathrm{D}} \ . \tag{2.16}$$

It is solved by expressing F as a function of the dual field strength, $F = F(F_{\mathrm{D}})$. Inserting this solution into $L(F,F_{\mathrm{D}})$, one gets the dual model

$$L_{\mathrm{D}}(F_{\mathrm{D}}) \equiv \Big(L(F) - \frac{1}{2}F\cdot\tilde{F}_{\mathrm{D}}\Big)\Big|_{F=F(F_{\mathrm{D}})} \ . \tag{2.17}$$

It remains to show that for any solution L of the self-duality equation, its Legendre transform $L_{\rm D}$ satisfies:

$$L_{\rm D}(F) \;=\; L(F)\;. \tag{2.18}$$

It follows from the results of Appendix A that the combination $L - \frac{1}{4}\,F\cdot\tilde{G}$ is invariant under arbitrary duality rotations, *i.e.*

$$L(F) - \frac{1}{4}\,F\cdot\tilde{G}(F) = L(F') - \frac{1}{4}\,F'\cdot\tilde{G}'(F')\;. \tag{2.19}$$

For a finite U(1) duality rotation (2.8) by $\lambda = \pi/2$ this relation reads

$$L(F) - \frac{1}{2}\,F\cdot\tilde{F}_{\rm D} \;=\; L(F_{\rm D})\;, \qquad\qquad F_{\rm D} \;\equiv\; G(F)\;. \tag{2.20}$$

Comparing with (2.17) this proves (2.18).

Let us turn to a more detailed discussion of the self-duality equation (2.9). Since in four dimensions the electromagnetic field has only two independent invariants

$$\alpha = \frac{1}{4}\,F^{ab}F_{ab}\;, \qquad\qquad \beta = \frac{1}{4}\,F^{ab}\tilde{F}_{ab}\;, \tag{2.21}$$

its Lagrangian $L(F_{ab})$ can be considered as a real function of one complex variable

$$L(F_{ab}) = L(\omega,\bar{\omega})\;, \qquad\qquad \omega = \alpha + {\rm i}\,\beta\;. \tag{2.22}$$

The theory is parity invariant iff $L(\omega,\bar{\omega}) = L(\bar{\omega},\omega)$.

One calculates $\tilde{G}$ (2.1) to be

$$\tilde{G}_{ab} = \Big(F_{ab} + {\rm i}\,\tilde{F}_{ab}\Big)\,\frac{\partial L}{\partial\omega} \;+\; \Big(F_{ab} - {\rm i}\,\tilde{F}_{ab}\Big)\,\frac{\partial L}{\partial\bar{\omega}}\;, \tag{2.23}$$

and the self-duality equation (2.9) takes the form

$${\rm Im}\,\left\{\omega - 4\,\omega\,\left(\frac{\partial L}{\partial\omega}\right)^2\right\} = 0\;. \tag{2.24}$$

In the literature one finds alternative forms of the self-duality equation [8, 11] but it is eq. (2.24) which turns out to be most convenient for supersymmetric generalizations. If one splits L into the sum of Maxwell's part and an interaction,

$$L = -\frac{1}{2}\,\Big(\omega + \bar{\omega}\Big) \;+\; L_{\rm int}\;, \qquad\qquad L_{\rm int} = \mathcal{O}(\omega^2)\;, \tag{2.25}$$

(2.24) becomes a condition on $L_{\rm int}$:

$${\rm Im}\,\left\{\omega\,\frac{\partial L_{\rm int}}{\partial\omega} - \omega\,\left(\frac{\partial L_{\rm int}}{\partial\omega}\right)^2\right\} = 0\;. \tag{2.26}$$

We restrict $L_{\rm int}$ to a real analytic function of ω and $\bar\omega$. Then, every solution of eq. (2.26) is of the form[4]

$$L_{\rm int}(\omega,\bar\omega) \ = \ \omega\,\bar\omega\;\Lambda(\omega,\bar\omega)\ , \qquad\qquad \Lambda = {\rm const}\ +\ \mathcal{O}(\omega)\ , \tag{2.27}$$

where Λ satisfies

$$\mathrm{Im}\,\left\{\frac{\partial(\omega\,\Lambda)}{\partial\omega} - \bar\omega\,\left(\frac{\partial(\omega\,\Lambda)}{\partial\omega}\right)^2\right\} = 0\ . \tag{2.28}$$

Note that for any solution $L_{\rm int}(\omega,\bar\omega)$ of (2.26), or any solution $\Lambda(\omega,\bar\omega)$ of (2.28), the functions

$$\hat L_{\rm int}(\omega,\bar\omega) = \frac{1}{g^2}\,L_{\rm int}(g^2\,\omega,g^2\,\bar\omega)\ , \qquad \hat\Lambda(\omega,\bar\omega) = g^2\,\Lambda(g^2\,\omega,g^2\,\bar\omega) \tag{2.29}$$

are also solutions for arbitrary real parameter g^2.

In perturbation theory one looks for a parity invariant solution of the self-duality equation by considering the Ansatz

$$\Lambda(\omega,\bar\omega) \ = \ \sum_{n=0}^{\infty}\ \sum_{p+q=n} C_{p,q}\,\omega^p\bar\omega^q\ , \qquad\qquad C_{p,q} = C_{q,p}\ \in\ \mathbb{R}\ , \tag{2.30}$$

where $n=p+q$ is the level of the coefficient $C_{p,q}$. It turns out that for odd level the self-duality equation uniquely expresses all coefficients recursively. If, however, the level is even, the self-duality equation uniquely fixes the level-n coefficients $C_{p,q}$ with $p\neq q$ through those at lower levels, while $C_{r,r}$ remain undetermined. This means that a general solution of the self-duality equation involves an arbitrary real analytic function of one real argument, $f(\omega\bar\omega)$.

There are a few exact solutions of the self-duality equation known, the most prominent one being the BI Lagrangian [2]

$$\begin{aligned} L_{\rm BI} &= \frac{1}{g^2}\Big\{\ 1 - \sqrt{-\det(\eta_{ab}+gF_{ab})}\ \Big\} \\ &= \frac{1}{g^2}\Big\{\ 1 - \sqrt{1+g^2(\omega+\bar\omega)+\frac{1}{4}g^4(\omega-\bar\omega)^2}\ \Big\}\ , \\ \Lambda_{\rm BI} &= \frac{g^2}{1+\frac{1}{2}g^2(\omega+\bar\omega)+\sqrt{1+g^2(\omega+\bar\omega)+\frac{1}{4}g^4(\omega-\bar\omega)^2}}\ , \end{aligned} \tag{2.31}$$

[4] In the Euclidean formulation of self-dual theories, it is the form (2.27) of $L_{\rm int}$ which allows for (anti)self-dual solutions $\tilde F = \pm F$ [24].

with g the coupling constant. In the limit $g \to 0$, $L_{\rm BI}$ reduces to the Maxwell Lagrangian. Some other exact solutions of the self-duality equation were constructed in Ref. [25].

It is worth noting that the BI Lagrangian can be given in the form [14, 16]

$$L_{\rm BI} = -\frac{1}{2}\,(\chi + \bar{\chi})\ , \tag{2.32}$$

where the complex field χ is a functions of ω and $\bar{\omega}$ which satisfies the nonlinear constraint

$$\chi + \frac{1}{2}\,g^2 \chi\bar{\chi} - \omega \ = \ 0\ . \tag{2.33}$$

As will be discussed below, this form of the BI Lagrangian admits nontrivial generalizations [18, 19].

We close this section with a comment. While we have limited our discussion to Lagrangians which depend on F but not on its derivatives, the latter case can also be treated easily if one considers the action rather than the Lagrangian and if one defines

$$\tilde{G}[F] \ = \ 2\,\frac{\delta S[F]}{\delta F}\ ,$$

etc.. This procedure is mandatory when we treat supersymmetric models.

3 Theory of duality invariance I: non-supersymmetric models

This section has mainly review character. We discuss the theory of duality invariance of non-supersymmetric models with Abelian gauge fields [6, 7, 12, 13], coupled to scalar and antisymmetric tensor fields. Supersymmetric models will be treated in sects. 4-6.

3.1 Fundamentals

We consider a theory of n Abelian gauge fields coupled to matter fields ϕ^μ. The gauge fields enter the Lagrangian only via their field strengths F^i_{ab}, where $i = 1, 2, \ldots, n$,

$$L \ = \ L(F^i_{ab},\, \phi^\mu,\, \partial_a \phi^\mu) \ \equiv \ L(\varphi)\ . \tag{3.1}$$

As in sect. 2, we introduce the dual fields

$$\tilde{G}^i_{ab}(\varphi) \ \equiv \ 2\,\frac{\partial L(\varphi)}{\partial F^{i\,ab}} \tag{3.2}$$

which arise in the equations of motion $\partial^b \tilde{G}^i_{ab} = 0$ for the gauge fields.

Our aim is to analyze the general conditions for the equations of motion (including the Bianchi identities) of the theory to be invariant under infinitesimal duality transformations

$$\delta \begin{pmatrix} G \\ F \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix} , \qquad \delta\phi^\mu = \xi^\mu(\phi) . \tag{3.3}$$

Here A, B, C and D are real constant $n \times n$ matrices, and ξ^μ are some unspecified functions of the matter fields. The variation δG is understood as follows

$$\delta G = G'(\varphi') - G(\varphi) , \qquad \tilde{G}'(\varphi') = 2\,\frac{\partial L(\varphi')}{\partial F'} = 2\,\frac{\partial L(\varphi)}{\partial F'} + 2\,\frac{\partial}{\partial F}\,\delta L , \tag{3.4}$$

where

$$\delta L \;=\; L(\varphi') - L(\varphi) . \tag{3.5}$$

Using the definitions $F' = F + C\,G + D\,F$ and $\phi' = \phi + \xi(\phi)$ of the transformed fields, one can express the derivative $\partial/\partial F'$ in (3.4) in terms of those w.r.t. the original fields. This gives

$$\delta\tilde{G}^i_{ab} = 2\,\frac{\partial}{\partial F^{i\,ab}}\,\delta L - C^{jk}\,\tilde{G}^j \cdot \frac{\partial G^k}{\partial F^{i\,ab}} - D^{ji}\,\tilde{G}^j_{ab} , \tag{3.6}$$

where we have used the definition (3.2). The latter variation should coincide with $\delta\tilde{G}$ that follows from (3.3) and their consistency is equivalent to the relation

$$\begin{aligned} &\frac{\partial}{\partial F^{i\,ab}}\left[2\delta L - \frac{1}{2}B^{jk}\,F^j \cdot \tilde{F}^k - \frac{1}{2}C^{jk}\,G^j \cdot \tilde{G}^k\right] \\ &= 2\Big(A^{ij} + D^{ji}\Big)\frac{\partial L}{\partial F^{j\,ab}} + \frac{1}{2}\Big(B^{ij} - B^{ji}\Big)\tilde{F}^j_{ab} + \frac{1}{2}\Big(C^{kj} - C^{kj}\Big)\tilde{G}^j \cdot \frac{\partial G^k}{\partial F^{i\,ab}} . \end{aligned} \tag{3.7}$$

Here the left-hand side is a partial derivative of some function with respect to F. The right-hand side satisfies the same property iff

$$D + A^{\mathrm{T}} = \kappa\,\mathbf{1} , \qquad B^{\mathrm{T}} = B , \qquad C^{\mathrm{T}} = C , \tag{3.8}$$

for some real κ. As a result, we find

$$\frac{\partial}{\partial F^i}\left[\delta L - \frac{1}{4}B^{jk}\,F^j \cdot \tilde{F}^k - \frac{1}{4}C^{jk}\,G^j \cdot \tilde{G}^k - \kappa\,L\right] = 0 . \tag{3.9}$$

This relation expresses the fact that the Bianchi identities and equations of motion of the gauge fields are invariant under the duality transformation (3.3), (3.8).

Now let us turn to the transformation of the matter equation of motion:

$$E_\mu \;=\; \frac{\delta}{\delta\phi^\mu}\, S[F,\phi] \;=\; \Big(\frac{\partial}{\partial\phi^\mu} - \partial_a\, \frac{\partial}{\partial(\partial_a\phi^\mu)}\Big)\, L \;. \tag{3.10}$$

By definition, its variation reads (it is simpler to work with the action)

$$\begin{aligned} \delta E &= \frac{\delta}{\delta\phi'}\, S[F',\phi'] - \frac{\delta}{\delta\phi}\, S[F,\phi] \\ &= \frac{\delta}{\delta\phi'}\, S[F,\phi] + \frac{\delta}{\delta\phi}\, \delta S \;. \end{aligned} \tag{3.11}$$

Using $F' = F + C\,G + D\,F$ and $\phi' = \phi + \xi(\phi)$ one can express the derivative $\delta/\delta\phi'$ in the second line in terms of those w.r.t. the original fields. This leads to

$$\delta E_\mu \;=\; \frac{\delta}{\delta\phi^\mu}\left[\delta S - \frac{1}{4}\int \mathrm{d}^4x\, C^{ij}\tilde{G}^i\cdot G^j\right] \;-\; \frac{\partial\xi^\nu}{\partial\phi^\mu}\, E_\nu \;. \tag{3.12}$$

From here it is clear that E_μ will transform covariantly under duality transformations,

$$\delta E_\mu \;=\; -\,\frac{\partial\xi^\nu}{\partial\phi^\mu}\, E_\nu \;, \tag{3.13}$$

if we require

$$\frac{\delta}{\delta\phi^\mu}\left[\delta S - \frac{1}{4}\int \mathrm{d}^4x\, C^{jk} G^j\cdot\tilde{G}^k\right] \;=\; 0 \;. \tag{3.14}$$

The relations (3.9) and (3.14) are compatible with each other provided $\kappa = 0$ and hence

$$\delta L \;=\; \frac{1}{4}B^{ij}\, F^i\cdot\tilde{F}^j + \frac{1}{4}C^{ij}\, G^i\cdot\tilde{G}^j \;. \tag{3.15}$$

It is easy to check that the combination (the 'interaction Hamiltonian') $L - \frac{1}{4}F^i\cdot\tilde{G}^i$ is duality invariant,

$$\delta\Big(L - \frac{1}{4}\, F^i\cdot\tilde{G}^i\Big) \;=\; 0 \;. \tag{3.16}$$

Eq. (3.15) can be rewritten in an equivalent, but more useful, form if one directly varies L as a function of its arguments. This leads to the self-duality equation

$$\delta_\phi L \;=\; \frac{1}{4}B^{ij}\, F^i\cdot\tilde{F}^j - \frac{1}{4}C^{ij}\, G^i\cdot\tilde{G}^j + \frac{1}{2}A^{ij}\, F^i\cdot\tilde{G}^j \;, \tag{3.17}$$

where

$$\delta_\phi L = \left(\xi^\mu\, \frac{\partial}{\partial\phi^\mu} + (\partial_a\phi^\nu)\, \frac{\partial\xi^\mu}{\partial\phi^\nu}\, \frac{\partial}{\partial(\partial_a\phi^\mu)}\right) L \;. \tag{3.18}$$

Since $\kappa = 0$, the condition (3.8) on the matrix parameters in (3.3) can be rewritten in matrix notation as

$$X^{\mathrm{T}}\Omega \;+\; \Omega\, X \;=\; 0 \;, \tag{3.19}$$

where

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \qquad \Omega = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} . \tag{3.20}$$

We conclude that $\mathrm{Sp}(2n, \mathbb{R})$ is the maximal group of duality transformations, although in specific models the duality group G may actually be smaller. It should be pointed out that $\mathrm{Sp}(2n, \mathbb{R})$ or its non-compact subgroup G may appear as the group of duality symmetries if the set of matter fields ϕ^μ include scalar fields parameterizing the coset space G/H, with H the maximal compact subgroup of G (see [6, 13] for a more detailed discussion). Any self-dual theory without matter, $L(F)$, can be understood as a self-dual model with matter, $L(F, \phi, \partial\phi)$, with the matter fields frozen, $\phi(x) = \phi_0 \in G/H$. The duality transformations preserving this background must thus be a subgroup of $\mathrm{U}(n)$, the maximal compact subgroup of $\mathrm{Sp}(2n, \mathbb{R})$. If one treats the matter fields ϕ^μ as coupling constants, then non-compact duality transformations relate models with different coupling constants. It is worth recalling that for the maximal compact subgroup of $\mathrm{Sp}(2n, \mathbb{R})$ the relations (3.19) and (3.20) should be supplemented by $X^{\mathrm{T}} = -X$ and hence

$$D = A \ , \quad C = -B \ , \quad A^{\mathrm{T}} = -A \ , \quad B^{\mathrm{T}} = B \quad \Longrightarrow \quad (B + \mathrm{i}A)^\dagger = (B + \mathrm{i}A) \ . \tag{3.21}$$

3.2 U(n) duality invariant models

Let us analyze the conditions of self-duality for pure gauge theories with maximal duality group $\mathrm{U}(n)$. Because of (3.21) and since $\delta_\phi L = 0$ in the absence of matter, the self-duality equation (3.17) reduces to [13, 19]

$$B^{ij}\,(F^i \cdot \tilde{F}^j + G^i \cdot \tilde{G}^j) \ + \ 2A^{ij}\,F^i \cdot \tilde{G}^j \ = \ 0.$$

Since the matrices A and B satisfy eq. (3.21) and otherwise arbitrary, the latter relation leads to the self-duality equations

$$G^i \cdot \tilde{G}^j + F^i \cdot \tilde{F}^j = 0 \ , \tag{3.22}$$

$$(F^i \cdot \frac{\partial}{\partial F^j} - F^j \cdot \frac{\partial}{\partial F^i})\, L = 0 \ . \tag{3.23}$$

The first equation is a natural generalization of the self-duality equation (2.9). The second equation requires manifest $\mathrm{SO}(n)$ invariance of the Lagrangian when F^i transforms in the fundamental representation of $\mathrm{SO}(n)$.

The $\mathrm{U}(n)$ duality invariant models possess quite remarkable properties. In particular, they are self-dual under a Legendre transformation which acts on a single Abelian gauge field while keeping the other $n-1$ fields invariant. The proof is similar to that given in sect. 2. Another property is that any $\mathrm{U}(n)$ duality invariant model can be lifted to a model with the maximal non-compact duality symmetry $\mathrm{Sp}(2n,\mathbb{R})$ by coupling the gauge fields to scalar fields ϕ^μ parameterizing the quotient space $\mathrm{Sp}(2n,\mathbb{R})\,/\,\mathrm{U}(n)$ [10, 11, 13]. The case $n=1$ will be discussed in the next subsection.

Nonlinear $\mathrm{U}(n)$ duality invariant models with $n>1$ were first constructed in [18, 19] as a generalization of the special algebraic representation for the BI action reviewed in sect. 2. The Lagrangian reads

$$L = -\frac{1}{2}\,\mathrm{tr}\,(\chi+\bar{\chi})\ , \tag{3.24}$$

where the complex $n\times n$ matrix χ is a function of F^i which satisfies the nonlinear constraint

$$\chi^{ij} + \frac{1}{2}\,\chi^{ik}\bar{\chi}^{jk} \;=\; \omega^{ij}\ , \qquad \omega^{ij} \;=\; \frac{1}{4}(F^i\cdot F^j + \mathrm{i}\,F^i\cdot\tilde{F}^j)\ . \tag{3.25}$$

We refer the reader to [18, 19] for the proof of self-duality. The explicit solution of above constraint on χ was provided in Ref. [26].

One might feel uneasy with above derivation of the self-duality equations (3.22) and (3.23) in pure gauge theory $L(F)$ as it was essentially based on the relation (3.15) which is valid in the presence of matter. Without using the matter consistency condition (3.14) we could not have set $\kappa=0$ and, therefore, the variation of L should be

$$\delta L = \frac{1}{4}B^{ij}\,F^i\cdot\tilde{F}^j + \frac{1}{4}C^{ij}\,G^i\cdot\tilde{G}^j + \kappa\,L\ . \tag{3.26}$$

However, practically all conclusions turn out to remain unchanged if we make use of additional physical requirements (the use of matter fields in the previous consideration simply allows to streamline the derivation). Let us consider for simplicility the case of a single gauge field, $n=1$. Then eq. (3.26) implies ($\kappa=A+D$) (*c.f.* eq. (2.6) with $\Delta L=0$)

$$\frac{1}{4}B\,F\cdot\tilde{F} - \frac{1}{4}C\,G\cdot\tilde{G} = D\frac{\partial L}{\partial F}\cdot F - (A+D)\,L\ . \tag{3.27}$$

Assuming that L is parity even, the expressions on both sides have different parities and should vanish separately

$$B\,F\cdot\tilde{F} - C\,G\cdot\tilde{G} = 0\ , \tag{3.28}$$

$$D\frac{\partial L}{\partial F}\cdot F = (A+D)\,L\ . \tag{3.29}$$

Let us also assume that L reduces to Maxwell's Lagrangian in the weak field limit, $L=-\frac{1}{4}F\cdot F+\mathcal{O}(F^4)$, hence $G=\tilde{F}+\mathcal{O}(F^3)$, $\tilde{G}=-F+\mathcal{O}(F^3)$, and therefore eq. (3.28) means

$$(B+C)\,F\cdot\tilde{F} + \mathcal{O}(F^4) \;=\; 0\ . \tag{3.30}$$

To the lowest order, this is satisfied iff $B = -C$. Eq. (3.29) means that $L(F)$ is a homogeneous function provided $D \neq 0$. This equation requires $D = A$ if $L = -\frac{1}{4} F \cdot F$ and $D = A = 0$ otherwise. We see that only U(1) duality rotations are possible in nonlinear electrodynamics, while in Maxwell's theory one can also allow scale transformations. The latter are however forbidden if one requires invariance of the energy-momentum tensor under duality transformations.

3.3 Coupling to dilaton and axion

We are going to prove that any U(1) duality invariant model $L(F)$ can be uniquely coupled to the dilaton and axion such that the resulting model $L(F, \mathcal{S})$ is invariant under $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{Sp}(2,\mathbb{R})$ duality transformations [9, 10, 11]. This property was stated in sect. 2.

Following the notation of subsect. 3.1, the case under consideration corresponds to $n = 1$ and $\phi^\mu = (\mathcal{S},\ \bar{\mathcal{S}})$. In accordance with eq. (2.12), the infinitesimal transformation of $\mathcal{S}$ reads

$$\delta \mathcal{S} \ = \ B + 2A\,\mathcal{S} - C\,\mathcal{S}^2 \ . \tag{3.31}$$

To describe the interaction of the dilaton and axion with the gauge field, we assume that the total Lagrangian is of the form $L(\mathcal{S}, \partial \mathcal{S}) + L(F, \mathcal{S})$ where the duality invariant kinetic term was given in (2.14). The self-duality equation (3.17) is now equivalent to the following three equations on $L(F, \mathcal{S})$:

$$\begin{aligned} 2\mathcal{S}\,\frac{\partial L}{\partial \mathcal{S}} + 2\bar{\mathcal{S}}\,\frac{\partial L}{\partial \bar{\mathcal{S}}} &= F \cdot \frac{\partial L}{\partial F} \ , \\ \frac{\partial L}{\partial \mathcal{S}} + \frac{\partial L}{\partial \bar{\mathcal{S}}} &= \frac{1}{4} F \cdot \tilde{F} \ , \\ \mathcal{S}^2\,\frac{\partial L}{\partial \mathcal{S}} + \bar{\mathcal{S}}^2\,\frac{\partial L}{\partial \bar{\mathcal{S}}} &= \frac{1}{4} G \cdot \tilde{G} \ . \end{aligned} \tag{3.32}$$

Inspection of these equations shows that $L(F, \mathcal{S})$ is

$$L(F, \mathcal{S}) \ = \ L_{(}\sqrt{\mathcal{S}_2}\,F) \ + \ \frac{1}{4}\,\mathcal{S}_1 F \cdot \tilde{F} \ , \tag{3.33}$$

where $L(F)$ solves the self-duality equation (2.9). Since $L(F, \mathcal{S})$ is self-dual, the combination $L - \frac{1}{4}\,F \cdot \tilde{G}$ is duality invariant. Its invariance under a finite duality rotation by $\pi/2$ is equivalent to the fact that the Legendre transform of the Lagrangian is

$$L(F, \mathcal{S}) - \frac{1}{2}\,F \cdot \tilde{F}_{\rm D} \ = \ L(F_{\rm D}, -\frac{1}{\mathcal{S}}) \ , \qquad F_{\rm D} \ \equiv \ G(F) \ , \tag{3.34}$$

c.f. eq. (2.20).

3.4 Coupling to NS B-field and RR fields

Within the context of type IIB string theory, one is interested in duality-invariant couplings of the model (3.33) to the NS and RR two-forms, B_{ab} and C_{ab}, and the RR four-form, C_{abcd} (which are possible bosonic background fields). *E.g.* the self-duality of the world-volume theory of a D3-brane is inherited from the $\mathrm{SL}(2,\mathbb{R})$ symmetry of type IIB supergravity [27] (see also [28]). These fields transform under $\mathrm{SL}(2,\mathbb{R})$ as

$$\begin{pmatrix} C' \\ B' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C \\ B \end{pmatrix} ,$$
$$\tilde{C}_4{}' = \tilde{C}_4 + \frac{1}{4} bd\, B \cdot \tilde{B} + \frac{1}{2} bc\, B \cdot \tilde{C} + \frac{1}{4} ac\, C \cdot \tilde{C} \ . \tag{3.35}$$

The transformation of $\tilde{C}_4$ provides a nonlinear representation of $\mathrm{SL}(2,\mathbb{R})$.[5] In the presence of B_2, C_2 and C_4, the Lagrangian (3.33) is extended to

$$L(F, \mathcal{S}, B, C, \tilde{C}_4) \ = \ L(\sqrt{\mathcal{S}_2}\, \mathcal{F}) \ + \ \frac{1}{4} \mathcal{S}_1 \mathcal{F} \cdot \tilde{\mathcal{F}} + \tilde{C}_4 - \frac{1}{2} C \cdot \tilde{\mathcal{F}} \ , \tag{3.36}$$

where

$$\mathcal{F}_{ab} = F_{ab} + B_{ab} \ . \tag{3.37}$$

The theory is invariant under standard gauge transformations of the gauge forms B_2, C_2 and C_4. Moreover, the theory is indeed $\mathrm{SL}(2,\mathbb{R})$ duality invariant. Given the set of matters fields $\phi^\mu = (\mathcal{S}, \bar{\mathcal{S}}, B_{ab}, C_{ab}, \tilde{C}_4)$ it is an instructive exercise to check that the self-duality equation (3.17) is satisfied.

4 Self-duality in $\mathcal{N} = 1$ supersymmetric nonlinear electrodynamics

Gaillard and Zumino conclude their paper [11] by posing the following problem: "When the Lagrangian is self-dual, it is natural to ask whether its supersymmetric extension possesses a self-duality property that can be formulated in a supersymmetric way." The problem was solved in [29] for the case when the Lagrangian is quadratic in the U(1) field strengths coupled to supersymmetric matter. The solution in the nonlinear case was obtained in [20] for a single vector multiplet and will be extended in the sect. 6 to any

[5] Note that the combination $\tilde{C}_4 - \frac{1}{4} C \cdot \tilde{B}$ is $\mathrm{SL}(2,\mathbb{R})$ invariant.

number of vector multiplets coupled to scalar multiplets. In the present section we are going to review the $\mathcal{N}=1$ supersymmetric results of [20].

Let $S[W,\bar{W}]$ be the action generating the dynamics of a single $\mathcal{N}=1$ vector multiplet. The (anti) chiral superfield strengths $\bar{W}_{\dot\alpha}$ and W_α,[6]

$$W_\alpha = -\frac{1}{4}\,\bar{D}^2 D_\alpha V \ , \qquad \bar{W}_{\dot\alpha} = -\frac{1}{4}\,D^2 \bar{D}_{\dot\alpha} V \ , \tag{4.1}$$

are defined in terms of a real unconstrained prepotential V. As a consequence, the strengths are constrained superfields, that is they satisfy the Bianchi identity

$$D^\alpha W_\alpha \ = \ \bar{D}_{\dot\alpha} \bar{W}^{\dot\alpha} \ . \tag{4.2}$$

Suppose that $S[W,\bar{W}] \equiv S[v]$ can be unambiguously defined[7] as a functional of *unconstrained* (anti) chiral superfields $\bar{W}_{\dot\alpha}$ and W_α. Then, one can define *(anti) chiral* superfields $\bar{M}_{\dot\alpha}$ and M_α as

$$\mathrm{i}\, M_\alpha\,[v] \equiv 2\,\frac{\delta}{\delta W^\alpha}\,S[v] \ , \qquad -\mathrm{i}\,\bar{M}^{\dot\alpha}\,[v] \equiv 2\,\frac{\delta}{\delta \bar{W}_{\dot\alpha}}\,S[v] \ , \tag{4.3}$$

with the functional derivatives defined in the standard way

$$\begin{aligned} \delta S &= \int \mathrm{d}^6 z\, \delta W^\alpha\, \frac{\delta S}{\delta W^\alpha} + \int \mathrm{d}^6 \bar{z}\, \delta \bar{W}_{\dot\alpha}\, \frac{\delta S}{\delta \bar{W}_{\dot\alpha}} \ , \\ \frac{\delta}{\delta W^\alpha(z)}\, W^\beta(z') &= \delta_\alpha{}^\beta \Big(-\frac{1}{4}\bar{D}^2\Big)\, \delta^4(x-x')\,\delta^2(\theta-\theta')\,\delta^2(\bar\theta-\bar\theta') \ . \end{aligned} \tag{4.4}$$

The vector multiplet equation of motion following from the action $S[W,\bar{W}]$ reads

$$D^\alpha M_\alpha \ = \ \bar{D}_{\dot\alpha} \bar{M}^{\dot\alpha} \ . \tag{4.5}$$

Since the Bianchi identity (4.2) and the equation of motion (4.5) have the same functional form, one may consider, similar to the non-supersymmetric case, U(1) duality rotations

$$\begin{pmatrix} M'_\alpha\,[v'] \\ W'_\alpha \end{pmatrix} \ = \ \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} M_\alpha\,[v] \\ W_\alpha \end{pmatrix} \ , \tag{4.6}$$

[6]Our $\mathcal{N}=1$ conventions are those of [30, 31]. In particular, $z=(x^a,\theta^\alpha,\bar\theta_{\dot\alpha})$ are the coordinates of $\mathcal{N}=1$ superspace, $\mathrm{d}^8 z = \mathrm{d}^4 x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta$ is the full superspace measure, and $\mathrm{d}^6 z = \mathrm{d}^4 x\,\mathrm{d}^2\theta$ is the measure in the chiral subspace.

[7]This is always possible if $S[W,\bar{W}]$ does not involve the combination $D^\alpha W_\alpha$ as an independent variable.

where M' should be

$$\mathrm{i}\, M'_\alpha\,[v'] = 2\,\frac{\delta}{\delta W'^\alpha}\,S[v']~. \tag{4.7}$$

In order for such duality transformations to be consistently defined, the action $S[W,\bar W]$ must satisfy a generalization of the self-duality equation (2.9). Its derivation follows essentially the same steps as described in Appendix A, but with a proper replacement of partial derivatives by functional derivatives. To preserve the definition (4.3) of M_α and its conjugate, the action should transform under an infinitesimal duality rotation as

$$\delta S = S[v'] - S[v] = \frac{\mathrm{i}}{4}\,\lambda \int \mathrm{d}^6 z\,\{M^\alpha M_\alpha - W^\alpha W_\alpha\} ~+~ \text{c.c.} \tag{4.8}$$

On the other hand, S is a functional of W_α and $\bar W_{\dot\alpha}$ only, and therefore its variation is

$$\delta S = \frac{\mathrm{i}}{2}\,\lambda \int \mathrm{d}^6 z\, M^\alpha M_\alpha ~+~ \text{c.c.} \tag{4.9}$$

Since these two variations must coincide, we arrive at the following reality condition

$$\mathrm{Im} \int \mathrm{d}^6 z \,\Big(W^\alpha W_\alpha ~+~ M^\alpha M_\alpha\Big) ~=~ 0~. \tag{4.10}$$

In eq. (4.10), the superfield M_α was defined in (4.3), and W_α should be considered as an *unconstrained* chiral superfields. Eq. (4.10) is the condition for the $\mathcal{N}=1$ supersymmetric theory to be self-dual. We call it the $\mathcal{N}=1$ self-duality equation.

With proper modifications, the properties of self-dual theories, which we described in sect. 2, also hold for $\mathcal{N}=1$ self-dual models. In particular, the derivative of the self-dual action with respect to an invariant parameter is always duality invariant. This implies duality invariance of the $\mathcal{N}=1$ supercurrent, i.e. the multiplet of the energy-momentum tensor (see [31] for a review). Duality invariant couplings to the dilaton-axion multiplet will be discussed in sect. 6. Here we would like to concentrate on self-duality under $\mathcal{N}=1$ Legendre transformation, defined as follows. Given a vector multiplet model $S[W,\bar W]$, we introduce the auxiliary action

$$S[W,\bar W, W_{\rm D}, \bar W_{\rm D}] = S[W,\bar W] - \frac{\mathrm{i}}{2}\int \mathrm{d}^6 z\, W^\alpha W_{{\rm D}\,\alpha} + \frac{\mathrm{i}}{2}\int \mathrm{d}^6 \bar z\, \bar W_{\dot\alpha} \bar W_{\rm D}{}^{\dot\alpha}~, \tag{4.11}$$

where W_α is now an unconstrained chiral spinor superfield, and $W_{{\rm D}\,\alpha}$ the dual field strength

$$W_{{\rm D}\,\alpha} = -\frac{1}{4}\,\bar D^2 D_\alpha\, V_{\rm D}~, \qquad \bar W_{{\rm D}\,\dot\alpha} = -\frac{1}{4}\,D^2 \bar D_{\dot\alpha}\, V_{\rm D}~. \tag{4.12}$$

This model is equivalent to the original model, since the equation of motion for $W_{\rm D}$ implies that W satisfies the Bianchi identity (4.2), and the action (4.11) reduces to $S[W,\bar W]$. On the other hand, the equation of motion for W is $M[W,\bar{W}] \;=\; W_{\rm D}$, with M defined in (4.3). Solving this equation, $W = W[W_{\rm D},\bar{W}_{\rm D}]$, and inserting the solution back into the action (4.11), one gets the dual model $S_{\rm D}[W_{\rm D},\bar{W}_{\rm D}]$ or, what is the same, the Legendre transform of $S[W,\bar{W}]$. For all $\mathcal{N}=1$ self-dual theories, $S_{\rm D}=S$. This follows from the fact that the combination

$$S - \frac{{\rm i}}{4}\int {\rm d}^6 z\, W^\alpha M_\alpha + \frac{{\rm i}}{4}\int {\rm d}^6 \bar{z}\, \bar{W}_{\dot\alpha}\bar{M}^{\dot\alpha} \tag{4.13}$$

is invariant under arbitrary U(1) duality rotations.

We now present a family of $\mathcal{N}=1$ supersymmetric self-dual models with actions of the general form

$$S \;=\; \frac{1}{4}\int {\rm d}^6 z\, W^2 + \frac{1}{4}\int {\rm d}^6 \bar{z}\, \bar{W}^2 + \frac{1}{4}\int {\rm d}^8 z\, W^2\,\bar{W}^2\,\Lambda\Big(\frac{1}{8}D^2\,W^2\,,\,\frac{1}{8}\bar{D}^2\,\bar{W}^2\Big)\;, \tag{4.14}$$

where $\Lambda(u,\bar{u})$ is a real analytic function of the complex variable

$$u \;\equiv\; \frac{1}{8}D^2\,W^2\;. \tag{4.15}$$

Functionals of this type naturally appear as low-energy effective actions in quantum supersymmetric gauge theories; by 'low-energy action' we mean here the part of the full effective action independent of the derivatives of the U(1) field strength F. In fact, the low-energy effective actions usually have the more general form (see, for instance, [32, 33, 34]):

$$S_{\rm eff} \;=\; \frac{1}{4}\int {\rm d}^6 z\, W^2 + \frac{1}{4}\int {\rm d}^6 \bar{z}\, \bar{W}^2 + \int {\rm d}^8 z\, W^2\,\bar{W}^2\,\Omega\Big(D^2\,W^2, \bar{D}^2\,\bar{W}^2, D^\alpha\,W_\alpha\Big)\;. \tag{4.16}$$

However, the combination $D^\alpha\,W_\alpha$ is nothing but the free equation of motion of the $\mathcal{N}=1$ vector multiplet. Contributions to effective action, which contain factors of the classical equations of motion, are ambiguous. They are often ignored. It is worth pointing out that there is no unique way to define the action (4.16) as a functional of unconstrained chiral superfield W_α and its conjugate (what is required in the framework of our approach to supersymmetric self-dual theories) when Ω depends on $D^\alpha\,W_\alpha = \bar{D}_{\dot\alpha}\,\bar{W}^{\dot\alpha}$.

Let us analyze the conditions for the model (4.14) to be self-dual. One finds

$${\rm i}\,M_\alpha \;=\; W_\alpha\,\Big\{\,1 - \frac{1}{4}\,\bar{D}^2\,\Big[\bar{W}^2\,\Big(\Lambda + \frac{1}{8}\,D^2\Big(W^2\,\frac{\partial\Lambda}{\partial u}\Big)\Big)\Big]\,\Big\}\;. \tag{4.17}$$

Then, eq. (4.10) leads to

$${\rm Im}\int {\rm d}^8 z\, W^2\,\bar{W}^2\,\Big(\Gamma \;-\; \bar{u}\,\Gamma^2\Big) \;=\; 0\;, \tag{4.18}$$

where

$$\Gamma \equiv \Lambda + \frac{1}{8}(D^2 W^2) \frac{\partial \Lambda}{\partial u} = \frac{\partial (u\,\Lambda)}{\partial u} . \tag{4.19}$$

In deriving eq. (4.18) we have used the following property of the $\mathcal{N}=1$ vector multiplet:

$$W_\alpha W_\beta W_\gamma = 0 . \tag{4.20}$$

Since the functional relation (4.18) must be satisfied for arbitrary (anti) chiral superfields $\bar{W}_{\dot\alpha}$ and W_α, we arrive at the following differential equation for $\Lambda(u, \bar{u})$:

$$\mathrm{Im} \left\{ \frac{\partial (u\,\Lambda)}{\partial u} - \bar{u} \left(\frac{\partial (u\,\Lambda)}{\partial u} \right)^2 \right\} = 0 . \tag{4.21}$$

This equation is identical to the self-duality equation (2.28).

To obtain the component form of (4.14), one applies the reduction rules

$$\int \mathrm{d}^8 z\, U = \frac{1}{16} \int \mathrm{d}^4 x\, D^2 \bar{D}^2\, U\,|_{\theta=0} , \qquad \int \mathrm{d}^6 z\, U_{\rm c} = -\frac{1}{4} \int \mathrm{d}^4 x\, D^2\, U_{\rm c}\,|_{\theta=0} . \tag{4.22}$$

We also introduce the component fields of the $\mathcal{N}=1$ vector multiplet, $\{\lambda_\alpha,\, \bar{\lambda}_{\dot\alpha},\, F_{ab},\, D\}$, in the standard way [30, 31]:

$$\begin{aligned} \lambda_\alpha(x) &= W_\alpha|_{\theta=0} , \\ F_{\alpha\beta}(x) &= -\frac{\mathrm{i}}{4}(D_\alpha W_\beta + D_\beta W_\alpha)|_{\theta=0} , \\ D(x) &= -\frac{1}{2} D^\alpha W_\alpha|_{\theta=0} , \end{aligned} \tag{4.23}$$

with

$$F_{\alpha\dot\alpha\,\beta\dot\beta} \equiv (\sigma^a)_{\alpha\dot\alpha} (\sigma^a)_{\beta\dot\beta} F_{ab} = 2\varepsilon_{\alpha\beta}\, \bar{F}_{\dot\alpha\dot\beta} + 2\varepsilon_{\dot\alpha\dot\beta}\, F_{\alpha\beta} . \tag{4.24}$$

Here we are interested only in the bosonic sector of the model and therefore set $\lambda_\alpha = 0$ in what follows. Under this assumption one can readily compute the component Lagrangian

$$L(F_{ab},\, D) = -\frac{1}{2}(\mathbf{u} + \bar{\mathbf{u}}) + \mathbf{u}\bar{\mathbf{u}}\, \Lambda(\mathbf{u}, \bar{\mathbf{u}}) , \qquad \mathbf{u} \equiv \frac{1}{8} D^2 W^2|_{\theta=0} = \omega - \frac{1}{2} D^2 , \tag{4.25}$$

with ω defined in eq. (2.22). Since only even powers of the auxiliary field D appear in L, its equation of motion has the solution $D = 0$. If we take this solution, the duality equation (4.21) implies that the non-supersymmetric model $L(F) = L(F,\, D = 0)$ is self-dual.

We arrive at the conclusion: every non-supersymmetric self-dual model of the type considered in sect. 2 admits an $\mathcal{N}=1$ supersymmetric extension which is self-dual under manifestly supersymmetric duality rotations. The procedure of constructing such a supersymmetric extension is constructive: given a self-dual Lagrangian $L(F)$, one should first derive $\Lambda(\omega, \bar\omega)$ defined by eqs. (2.25) and (2.27), and then use this function to generate the action (4.14).

5 Properties of the $\mathcal{N}=1$ supersymmetric BI action

We use the results of sect. 3 to obtain the unique $\mathcal{N}=1$ supersymmetric self-dual extension of the BI theory (2.31). With the use of $\Lambda_{\rm BI}$ one immediately gets

$$S_{\rm BI} = \frac{1}{4}\int {\rm d}^6 z\, W^2 + \frac{1}{4}\int {\rm d}^6 \bar{z}\, \bar{W}^2 + \frac{g^2}{4}\int {\rm d}^8 z\, \frac{W^2\,\bar{W}^2}{1+\frac{1}{2}\,A + \sqrt{1+A+\frac{1}{4}\,B^2}}~,$$
$$A = \frac{g^2}{8}\Big(D^2\,W^2 + \bar{D}^2\,\bar{W}^2\Big)~, \qquad B = \frac{g^2}{8}\Big(D^2\,W^2 - \bar{D}^2\,\bar{W}^2\Big)~. \tag{5.1}$$

In what follows, for convenience we fix the coupling constant to $g^2 = 4$.

The above action was first introduced in [35, 36] as a super extension of the BI theory. However, only much later it was realized that the theory encodes a remarkably reach structure. Bagger and Galperin [14], and later Roček and Tseytlin [16] discovered that (5.1) is the action for a Goldstone multiplet associated with $\mathcal{N}=2 \to \mathcal{N}=1$ partial supersymmetry breaking. Using a reformulation of (5.1) with auxiliary superfields, Brace, Morariu and Zumino [18] demonstrated that the theory is invariant under U(1) duality rotations. The latter property has turned out to be a simple consequence of the approach developed in [20] and reviewed in the previous section. Below we give a concise review of the results of [14] on partial $\mathcal{N}=2 \to \mathcal{N}=1$ supersymmetry breaking.

Bagger and Galperin noticed that the Cecotti-Ferrara action (5.1) can be represented in the form

$$S ~=~ \frac{1}{4}\int {\rm d}^6 z\, X + \frac{1}{4}\int {\rm d}^6 \bar{z}\, \bar{X}~, \tag{5.2}$$

where the *chiral* superfield X is a functional of W and $\bar{W}$ such that it satisfies the nonlinear constraint

$$X ~+~ \frac{1}{4}\, X\, \bar{D}^2\, \bar{X} ~=~ W^2~. \tag{5.3}$$

Indeed, using the action rule

$$\int {\rm d}^8 z\, U = -\frac{1}{4}\int {\rm d}^6 z\, \bar{D}^2\, U \tag{5.4}$$

and the constraint (5.3), one can rewrite (5.2) in the form

$$S = \frac{1}{4}\int {\rm d}^6 z\, W^2 + \frac{1}{4}\int {\rm d}^6 \bar{z}\, \bar{W}^2 + \frac{1}{2}\int {\rm d}^8 z\, X\,\bar{X}~. \tag{5.5}$$

Using the constraint (5.3) once more, we can represent $X\,\bar{X}$ as

$$X\,\bar{X} = \frac{W^2\,\bar{W}^2}{(1+\frac{1}{4}\,\bar{D}^2\bar{X})(1+\frac{1}{4}\,D^2 X)}~. \tag{5.6}$$

Since $W^3 = 0$, on the right-hand side we can safely take D^2X in an effective form $D^2X_{\rm eff}$ determined by the equation

$$D^2X_{\rm eff} = \frac{D^2W^2}{1+\frac{1}{4}\,\bar{D}^2\bar{X}_{\rm eff}} \ . \tag{5.7}$$

Using this in (5.5) one reproduces (5.1).

The dynamical system defined by eqs. (5.2) and (5.3) is manifestly $\mathcal{N}=1$ supersymmetric. Remarkably, it turns out to be invariant under a second, nonlinearly realized, supersymmetry transformation

$$\delta X = 2\epsilon^\alpha\, W_\alpha \ , \tag{5.8}$$

$$\delta W_\alpha = \epsilon_\alpha + \frac{1}{4}\,\bar{D}^2\bar{X}\,\epsilon_\alpha + {\rm i}\,\partial_{\alpha\dot\alpha}X\,\bar\epsilon^{\dot\alpha} \ , \tag{5.9}$$

with ϵ_α a constant parameter. Such transformations commute with the first, linearly realized, supersymmetry, and altogether they generate the $\mathcal{N}=2$ algebra without central charge. There is a simple way to derive the supersymmetry transformations (5.8) and (5.9). One first observes that the variation (5.8) leaves the action (5.2) invariant, as a consequence of the explicit form of the field strength W_α, see eq. (4.1). Due to (5.3), the variation δX must be induced by a variation of W_α of the form

$$\delta W_\alpha = \epsilon_\alpha + \frac{1}{4}\,\bar{D}^2\bar{X}\,\epsilon_\alpha + \hat\delta W_\alpha \ , \tag{5.10}$$

where $\hat\delta W$ should satisfy

$$W^\alpha\,\hat\delta W_\alpha = \frac{1}{4}\,X\,\bar{D}^2\bar{W}_{\dot\alpha}\,\bar\epsilon^{\dot\alpha} = -{\rm i}\,X\,\partial_{\alpha\dot\alpha}W^\alpha\,\bar\epsilon^{\dot\alpha} \ . \tag{5.11}$$

Since

$$W^\alpha\,X \ = \ 0 \ , \tag{5.12}$$

the latter relation can be rewritten as follows

$$W^\alpha\,\hat\delta W_\alpha = {\rm i}\,W^\alpha\,\partial_{\alpha\dot\alpha}X\,\bar\epsilon^{\dot\alpha} \ , \tag{5.13}$$

and we thus arrive at the variation (5.9). But this is not yet the end of the story, since one still has to check that the variation (5.9) is consistent with the Bianchi identity (4.2). Indeed it is. However, in sect. 9 we will see that the above procedure cannot be directly generalized to the case of $\mathcal{N}=2$ supersymmetry.

In [14] Bagger and Galperin proved that the action (5.1) is self-dual under the $\mathcal{N}=1$ Legendre transformation. Their proof is ingenious but rather involved. The results of sect. 4 make this property obvious. The $\mathcal{N}=1$ super BI theory (5.1) is invariant under U(1) duality rotations, and therefore it is automatically self-dual under the $\mathcal{N}=1$ Legendre transformation.

6 Theory of self-duality II: $\mathcal{N}=1$ supersymmetric models

In this section we develop a general formalism of duality invariance for $\mathcal{N}=1$ supersymmetric theories of n Abelian vector multiplets, described by chiral spinor strengths W^i_α and their conjugates $\bar{W}^i_{\dot\alpha}$, in the presence of supersymmetric matter – chiral superfields Φ^μ and their conjugates $\bar\Phi^\mu$. We will use the condensed notation $S[v] = S[W^{\alpha\, i}, \bar{W}^i_{\dot\alpha}, \Phi^\mu, \bar\Phi^\mu]$ for the action functional and, as in sect. 4, introduce *(anti) chiral* superfields $\bar{M}^{\dot\alpha\, i}$ and M^i_α dual to $\bar{W}^i_{\dot\alpha}$ and $W^{\alpha\, i}$:

$$\mathrm{i}\, M^i_\alpha\,[v] \equiv 2\,\frac{\delta}{\delta W^{\alpha\, i}}\, S[v]\ , \qquad -\mathrm{i}\, \bar{M}^{\dot\alpha\, i}\,[v] \equiv 2\,\frac{\delta}{\delta \bar{W}^i_{\dot\alpha}}\, S[v]\ . \tag{6.1}$$

To simplify notation, we introduce

$$M^i \cdot M^j = \int \mathrm{d}^6 z\, M^{\alpha\, i}\, M^j_\alpha\ , \qquad \bar{M}^i \cdot \bar{M}^j = \int \mathrm{d}^6 \bar{z}\, \bar{M}^i_{\dot\alpha}\, \bar{M}^{\dot\alpha\, j} \tag{6.2}$$

and similarly for superspace contractions of (anti) chiral scalar superfields.

6.1 General analysis

We are interested in determining the conditions for the theory to be self-dual under *chiral* superfield duality transformations

$$\delta \begin{pmatrix} M \\ W \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} M \\ W \end{pmatrix}\ , \qquad \delta\Phi^\mu = \xi^\mu(\Phi^\nu)\ , \tag{6.3}$$

with ξ^μ a *holomorphic* functions of the chiral matter fields. Here A, B, C and D are constant real $n \times n$ matrices; these matrices have to be real, since the Bianchi identities $D^\alpha W^i_\alpha = \bar{D}_{\dot\alpha} \bar{W}^{\dot\alpha\, i}$ and the equations of motion $D^\alpha M^i_\alpha = \bar{D}_{\dot\alpha} \bar{M}^{\dot\alpha\, i}$ are special reality conditions.

By self-duality we understand the following:

I. We require

$$\mathrm{i}\, M'[v'] = 2\frac{\delta}{\delta W'}\, S[v'] = 2\frac{\delta}{\delta W'}\, S[v] + 2\frac{\delta}{\delta W}\, \delta S\ , \tag{6.4}$$

where $\delta S = S[v'] - S[v]$.

II. The Φ-equation of motion

$$E_\mu[v] = \frac{\delta}{\delta \Phi^\mu} S[v] \tag{6.5}$$

transforms covariantly under duality transformations

$$\delta E_\mu = -\frac{\partial \xi^\nu(\Phi)}{\partial \Phi^\mu}\, E_\nu \;, \tag{6.6}$$

where

$$\delta E = E'[v'] - E[v] \;, \qquad E'[v'] = \frac{\delta}{\delta \Phi'}\, S[v'] = \frac{\delta}{\delta \Phi'}\, S[v] + \frac{\delta}{\delta \Phi}\, \delta S \;. \tag{6.7}$$

Analysis of the self-duality conditions is similar to the non-supersymmetric case described in sect. 3. The transformation law (6.3) and condition I are consistent provided

$$\begin{aligned} &\frac{\delta}{\delta W^{\alpha\, i}} \Big[\delta S - \frac{\mathrm{i}}{4}\, B^{jk} \Big(W^j \cdot W^k - \bar{W}^j \cdot \bar{W}^k \Big) - \frac{\mathrm{i}}{4}\, C^{jk} \Big(M^j \cdot M^k - \bar{M}^j \cdot \bar{M}^k \Big) \Big] \\ = \;\; &+ \frac{\mathrm{i}}{4}\, (C^{jk} - C^{kj}) \Big((\frac{\delta}{\delta W^{\alpha\, i}} M^k) \cdot M^j - (\frac{\delta}{\delta W^{\alpha\, i}} \bar{M}^k) \cdot \bar{M}^j \Big) \\ &+ \; \frac{\mathrm{i}}{4}\, (B^{ij} - B^{ji}) W^j_\alpha \; + \; (D^{ji} + A^{ij})\, \frac{\delta}{\delta W^{\alpha\, j}} S[v] \;. \end{aligned} \tag{6.8}$$

Since the left-hand side is a total variational derivative, the matrices A, B, C and D should be constrained as in eq. (3.8). Then, the above relation turns into

$$\begin{aligned} \frac{\delta}{\delta W^{\alpha\, i}} \Big[\delta S &- \frac{\mathrm{i}}{4}\, B^{jk} \Big(W^j \cdot W^k - \bar{W}^j \cdot \bar{W}^k \Big) \\ &- \frac{\mathrm{i}}{4}\, C^{jk} \Big(M^j \cdot M^k - \bar{M}^j \cdot \bar{M}^k \Big) - \kappa S[v] \Big] = 0 \;. \end{aligned} \tag{6.9}$$

Furthermore, the Φ-equation of motion can be shown to change under duality transformations as

$$\begin{aligned} \delta E_\mu = -\frac{\partial \xi^\nu}{\partial \Phi^\mu}\, E_\nu \\ + \frac{\delta}{\delta \Phi^\mu} \Big[\delta S - \frac{\mathrm{i}}{4}\, B^{jk} \Big(W^j \cdot W^k - \bar{W}^j \cdot \bar{W}^k \Big) - \frac{\mathrm{i}}{4}\, C^{jk} \Big(M^j \cdot M^k - \bar{M}^j \cdot \bar{M}^k \Big) \Big] \;. \end{aligned} \tag{6.10}$$

Consequently, condition II is satisfied if we impose the condition

$$\frac{\delta}{\delta \Phi^\mu} \Big[\delta S - \frac{\mathrm{i}}{4}\, B^{jk} \Big(W^j \cdot W^k - \bar{W}^j \cdot \bar{W}^k \Big) - \frac{\mathrm{i}}{4}\, C^{jk} \Big(M^j \cdot M^k - \bar{M}^j \cdot \bar{M}^k \Big) \Big] = 0 \;. \tag{6.11}$$

The latter is consistent with (6.9) provided $\kappa = 0$. Therefore, $\mathrm{Sp}(2n, \mathbb{R})$ is the maximal duality group (see sect. 3), and the action transforms as

$$\begin{aligned} \delta S &= \frac{\mathrm{i}}{4}\, \delta \Big(W^i \cdot M^i - \bar{M}^i \cdot \bar{W}^i \Big) \\ &= \frac{\mathrm{i}}{4}\, B^{ij} \Big(W^i \cdot W^j - \bar{W}^i \cdot \bar{W}^j \Big) + \frac{\mathrm{i}}{4}\, C^{ij} \Big(M^i \cdot M^j - \bar{M}^i \cdot \bar{M}^j \Big) \;. \end{aligned} \tag{6.12}$$

Equation (6.12) contains nontrivial information. The point is that the action can be varied directly,

$$\begin{aligned}\delta S &= S[v'] - S[v] \\ &= \frac{\mathrm{i}}{2}\Big(\delta W^i \cdot M^i - \delta \bar{W}^i \cdot \bar{W}^i\Big) + \delta\Phi^\mu \cdot \frac{\delta S}{\delta \Phi^\mu} + \delta\bar{\Phi}^\mu \cdot \frac{\delta S}{\delta\bar{\Phi}^\mu} \ ,\end{aligned} \tag{6.13}$$

and the two results should coincide. This gives

$$\begin{aligned}\delta\Phi^\mu \cdot \frac{\delta S}{\delta \Phi^\mu} &+ \delta\bar{\Phi}^\mu \cdot \frac{\delta S}{\delta\bar{\Phi}^\mu} \\ &= \frac{\mathrm{i}}{4}\, B^{ij}\Big(W^i \cdot W^j - \bar{W}^i \cdot \bar{W}^j\Big) - \frac{\mathrm{i}}{4}\, C^{ij}\Big(M^i \cdot M^j - \bar{M}^i \cdot \bar{M}^j\Big) \\ &+ \frac{\mathrm{i}}{2}\, A^{ij}\Big(W^i \cdot M^j - \bar{W}^i \cdot \bar{M}^j\Big) \ .\end{aligned} \tag{6.14}$$

This is the self-duality equation in the presence of matter.

In the absence of matter, the maximal duality group is $\mathrm{U}(n)$ and the transformation parameters in (6.14) are constrained by $B = -C = B^{\mathrm{T}}$, $A^{\mathrm{T}} = -A$. If the duality group is $\mathrm{U}(n)$, then eq. (6.14) leads to the following self-duality equations

$$\mathrm{Im}\,\Big(W^i \cdot W^j + M^i \cdot M^j\Big) = 0 \ , \tag{6.15}$$

$$\mathrm{Im}\,\Big(W^i \cdot M^j - W^j \cdot M^i\Big) = 0 \ . \tag{6.16}$$

Eq. (6.16) requires the theory to be invariant under $\mathrm{SO}(n)$ which acts linearly on W^i. For $n = 1$, eq. (6.15) reduces to (4.10).

Similar to the non-supersymmetric case [9, 10, 11, 13], a $\mathrm{U}(n)$ duality invariant theory of n Abelian vector multiplets can be lifted to an $\mathrm{Sp}(2n, \mathbb{R})$ duality invariant model by coupling the vector multiplets to scalar multiplets Φ^μ parameterizing the quotient space $\mathrm{Sp}(2n, \mathbb{R}) \,/\, \mathrm{U}(n)$. Below we give a proof for $n = 1$.

6.2 Coupling to the dilaton-axion multiplet

Our aim here is to couple the system (4.14), (4.21) to the dilaton-axion multiplet Φ such that the resulting model be $\mathrm{SL}(2, \mathbb{R})$ duality invariant. The $\mathrm{SL}(2, \mathbb{R})$-transformation of Φ coincides with the $\mathcal{S}$-transformation (2.12). Its infinitesimal form is

$$\delta\Phi = B + 2A\,\Phi - C\,\Phi^2 \ . \tag{6.17}$$

The self-duality equation (6.14) is now equivalent to the following requirements on the action functional $S = S[W, \Phi]$:

$$2\Phi \cdot \frac{\delta S}{\delta \Phi} + 2\bar{\Phi} \cdot \frac{\delta S}{\delta \bar{\Phi}} = W \cdot \frac{\delta S}{\delta W} + \bar{W} \cdot \frac{\delta S}{\delta \bar{W}} \ , \tag{6.18}$$

$$\frac{\delta S}{\delta \Phi} \cdot 1 + \frac{\delta S}{\delta \bar{\Phi}} \cdot 1 = \frac{\mathrm{i}}{4} \left(W \cdot W - \bar{W} \cdot \bar{W} \right) , \tag{6.19}$$

$$\Phi^2 \cdot \frac{\delta S}{\delta \Phi} + \bar{\Phi}^2 \cdot \frac{\delta S}{\delta \bar{\Phi}} = \frac{\mathrm{i}}{4} \left(M \cdot M - \bar{M} \cdot \bar{M} \right) \ . \tag{6.20}$$

We are interested in a solution of these equations which for $\Phi = -\mathrm{i}$ reduces to the self-dual system given by eqs. (4.14) and (4.21). A direct analysis of the self-duality equations gives the solution

$$\begin{aligned} S[W, \Phi] = {} & \frac{\mathrm{i}}{4} \int \mathrm{d}^6 z \, \Phi \, W^2 - \frac{\mathrm{i}}{4} \int \mathrm{d}^6 \bar{z} \, \bar{\Phi} \, \bar{W}^2 \\ & - \frac{1}{16} \int \mathrm{d}^8 z \, (\Phi - \bar{\Phi})^2 \, W^2 \, \bar{W}^2 \, \Lambda\Big(\frac{\mathrm{i}}{16} (\Phi - \bar{\Phi}) \, D^2 \, W^2 \, , \, \frac{\mathrm{i}}{16} (\Phi - \bar{\Phi}) \, \bar{D}^2 \, \bar{W}^2 \Big) \ . \end{aligned} \tag{6.21}$$

To this action one can add the dilaton-axion kinetic term $\int \mathrm{d}^8 z \, K(\Phi, \bar{\Phi})$, with $K(\Phi, \bar{\Phi})$ the Kähler potential of the Kähler manifold $\mathrm{SL}(2, \mathbb{R}) \, / \, \mathrm{U}(1)$. It is worth pointing out that the dilaton and axion (2.11) are related to Φ by the rule $\bar{\mathcal{S}} = \Phi|_{\theta=0}$. For the $\mathcal{N} = 1$ super BI action (5.1), the coupling to the dilaton-axion multiplet was described in [18, 19].

6.3 Coupling to NS and RR supermultiplets

The model (6.21) can be generalized by coupling it to supermultiplets containing the NS and RR two-forms, B_2 and C_2, and the RR four-form, C_4. The extended action is

$$S[W, \Phi, \beta, \gamma, \Omega] \ = \ S[\mathcal{W}, \Phi] \ + \ \left\{ \int \mathrm{d}^6 z \left(\Omega + \frac{1}{2} \gamma^\alpha \mathcal{W}_\alpha \right) \ + \ \text{c.c.} \right\} \ , \tag{6.22}$$

where

$$\mathcal{W}_\alpha \ = \ W_\alpha + \mathrm{i} \, \beta_\alpha \ . \tag{6.23}$$

is the supersymmetrization of $F + B$. Here β_α, γ_α and Ω are unconstrained chiral superfields which include, among their components, the fields B_2, C_2 and C_4, respectively. The action is invariant under the following gauge transformations

$$\delta \beta_\alpha = \mathrm{i} \, \delta W_\alpha = \mathrm{i} \, \bar{D}^2 D_\alpha K_1 \ , \tag{6.24}$$

$$\delta \gamma_\alpha = \mathrm{i} \, \bar{D}^2 D_\alpha K_2, \qquad \delta \Omega = -\tfrac{\mathrm{i}}{2} \, \mathcal{W}^\alpha \bar{D}^2 D_\alpha K_2 \ , \tag{6.25}$$

$$\delta \Omega = \mathrm{i} \, \bar{D}^2 K_3 \ , \tag{6.26}$$

with K_i real unconstrained superfields. Note that $\mathcal{W}_\alpha$ is invariant under (6.24). The transformations of β and γ imply that these superfields describe two tensor multiplets; *c.f.* also sect. 7. Eq. (6.26) implies that all components of Ω but $\mathrm{Re}\, D^2\Omega|_{\theta=0}$ can be algebraically gauged away; the remaining component transforms as a four-form and is identified with $\tilde{C}_4$.

The theory (6.22) is SL$(2,\mathbb{R})$ duality invariant provided the superfields β_α, γ_α and Ω transform as

$$\begin{pmatrix} \gamma' \\ \beta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \gamma \\ \beta \end{pmatrix} , \qquad \Omega' = \Omega - \frac{\mathrm{i}}{4} bd\, \beta^2 - \frac{\mathrm{i}}{2} bc\, \beta\gamma - \frac{\mathrm{i}}{4} ac\, \gamma^2 \ . \tag{6.27}$$

One can check that the self-duality equation (6.14) is satisfied, with $\Phi^\mu = (\Phi, \beta_\alpha, \gamma_\alpha, \Omega)$ the set of matter chiral superfields.

6.4 Example of U(n) self-dual supersymmetric theory

To conclude, we give an example of U(n) duality invariant model [18, 19] describing the dynamics of n interacting Abelian vector multiplets W^i_α. The action is

$$S \;=\; \frac{1}{4} \int \mathrm{d}^6 z \,\mathrm{tr}\, X + \frac{1}{4} \int \mathrm{d}^6 \bar{z} \,\mathrm{tr}\, \bar{X} \ , \tag{6.28}$$

where the *chiral* matrix superfield X is a functional of W^i_α and $\bar{W}^i_{\dot\alpha}$ such that it satisfies the nonlinear constraint

$$X^{ij} \;+\; \frac{1}{4} X^{ik}\, \bar{D}^2\, \bar{X}^{jk} \;=\; W^i\, W^j \ . \tag{6.29}$$

The proof of self-duality of this theory can be found in [18, 19]. Obviously, this system is a natural generalization of the Bagger–Galperin construction for the $\mathcal{N}=1$ super BI action, which we discussed in sect. 4.

Since for several vector multiplets $W^3 \neq 0$, after solving constraint (6.29) the action will have a more complicated form than (5.1).

7 Self-dual models with $\mathcal{N}=1$ tensor multiplet

In [15, 16] it was shown that partial breaking of $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$ can be described with the $\mathcal{N}=1$ tensor multiplet as the Goldstone multiplet. The construction of Bagger and Galperin [15] was based on an analogy between the $\mathcal{N}=1$ vector and tensor multiplets. Here we will pursue the same analogy to generalize the formalism of sect. 4 to construct nonlinear self-dual models of the $\mathcal{N}=1$ tensor multiplet.

We start with a brief description of the $\mathcal{N}=1$ tensor multiplet [37] (see [31] for more details). The multiplet is described by a real linear superfield L

$$D^2 L = \bar{D}^2 L = 0 \ , \qquad\qquad L = \bar{L} \ . \tag{7.1}$$

The general solution of this constraint is

$$L = \frac{1}{2}\,(D^\alpha \eta_\alpha + \bar{D}_{\dot\alpha}\bar{\eta}^{\dot\alpha}) \ , \qquad\qquad \bar{D}_{\dot\alpha}\eta_\alpha = 0 \ . \tag{7.2}$$

The chiral spinor superfield η_α is a gauge field defined modulo transformations

$$\delta\eta_\alpha = \mathrm{i}\,\bar{D}^2 D_\alpha K \ , \tag{7.3}$$

with K a real unconstrained superfield, and L is the gauge invariant field strength. The independent components of L are a scalar $\varphi = L|_{\theta=0}$, a Weyl spinor $\psi_\alpha = D_\alpha L|_{\theta=0}$ and its conjugate, and a vector $\tilde{V}_{\alpha\dot\alpha} = \frac{1}{2}\,[D_\alpha, \bar{D}_{\dot\alpha}]L|_{\theta=0}$ constrained by $\partial_a \tilde{V}^a = 0$. The constraint means that $\tilde{V}$ is the dual field strength of an antisymmetric tensor field, $\tilde{V}^a = \frac{1}{2}\,\varepsilon^{abcd}\,\partial_b B_{cd}$.

For generic models of the tensor multiplet, the gauge invariant action is a functional of L, $S[L]$. Here our consideration will be restricted to those models with actions of the form $S[\Psi, \bar{\Psi}]$, where

$$\Psi_\alpha = D_\alpha L \ , \qquad\qquad D_\beta \Psi_\alpha = 0 \ . \tag{7.4}$$

For example, for the free tensor multiplet we have

$$S_{\rm free} = -\int \mathrm{d}^8 z\, L^2 = \frac{1}{4}\int \mathrm{d}^6 \bar{z}\, \Psi^2 + \frac{1}{4}\int \mathrm{d}^6 z\, \bar{\Psi}^2 \ . \tag{7.5}$$

The antichiral spinor Ψ_α is a constrained superfield

$$-\frac{1}{4}\,\bar{D}^2 \Psi_\alpha + \mathrm{i}\,\partial_{\alpha\dot\alpha}\bar{\Psi}^{\dot\alpha} = 0 \ . \tag{7.6}$$

This constraint can be treated as the Bianchi identity. Its general solution is (7.4). The bosonic components of Ψ_α are field strengths of the zero-form and two-form, $U_a = \partial_a \varphi$ and $\tilde{V}_a$, respectively.

For the theory with action $S[\Psi, \bar{\Psi}]$, we introduce *antichiral* Υ_α and *chiral* $\bar{\Upsilon}^{\dot\alpha}$ superfields as follows

$$\mathrm{i}\,\Upsilon_\alpha \equiv 2\,\frac{\delta}{\delta\Psi^\alpha}\,S \ , \qquad\qquad -\mathrm{i}\,\bar{\Upsilon}^{\dot\alpha} \equiv 2\,\frac{\delta}{\delta\bar{\Upsilon}_{\dot\alpha}}\,S \ . \tag{7.7}$$

Then one can check that the equation of motion reads

$$-\frac{1}{4}\,\bar{D}^2\Upsilon_\alpha + \mathrm{i}\,\partial_{\alpha\dot\alpha}\bar{\Upsilon}^{\dot\alpha} = 0 \tag{7.8}$$

which has the same form as the Bianchi identity (7.6). Therefore, in analogy with sect. 4, one may consider U(1) duality rotations

$$\begin{pmatrix}\Upsilon' \\ \Psi'\end{pmatrix} = \begin{pmatrix}\cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda\end{pmatrix}\begin{pmatrix}\Upsilon \\ \Psi\end{pmatrix} . \tag{7.9}$$

The theory proves to be duality invariant iff the self-duality equation

$$\mathrm{Im}\int \mathrm{d}^6\bar{z}\,\Big(\Psi^\alpha\Psi_\alpha \ + \ \Upsilon^\alpha\Upsilon_\alpha\Big) \ = \ 0 \tag{7.10}$$

is satisfied.

Under duality rotations, the following functional

$$S - \frac{\mathrm{i}}{4}\int \mathrm{d}^6\bar{z}\,\Psi^\alpha\Upsilon_\alpha + \frac{\mathrm{i}}{4}\int \mathrm{d}^6 z\,\bar{\Psi}_{\dot\alpha}\bar{\Upsilon}^{\dot\alpha} \tag{7.11}$$

remains invariant. As in sect. 4, this property implies self-duality under a superfield Legendre transformation which is defined by replacing the action $S[\Psi,\bar{\Psi}]$ with

$$S[\Psi,\bar{\Psi},\Psi_{\mathrm{D}},\bar{\Psi}_{\mathrm{D}}] = S[\Psi,\bar{\Psi}] - \frac{\mathrm{i}}{2}\int \mathrm{d}^6\bar{z}\,\Psi^\alpha\Psi_{\mathrm{D}\,\alpha} + \frac{\mathrm{i}}{2}\int \mathrm{d}^6 z\,\bar{\Psi}_{\dot\alpha}\bar{\Psi}_{\mathrm{D}}{}^{\dot\alpha} , \tag{7.12}$$

where Ψ_α is now an unconstrained antichiral spinor superfield, and $\Psi_{\mathrm{D}\,\alpha}$ the dual field strength

$$\Psi_{\mathrm{D}\,\alpha} = D_\alpha L_{\mathrm{D}} , \qquad D^2 L_{\mathrm{D}} = 0 , \qquad \bar{L}_{\mathrm{D}} = L_{\mathrm{D}} . \tag{7.13}$$

Since above considerations are very similar to those in sect. 4, one can make use of the previous results to derive nonlinear self-dual models of the tensor multiplet. This is achieved by substituting $W^2 \to \bar{\Psi}^2$ in the action (4.14). The results of sec. 6 can also be generalized to the case of self-dual systems with several tensor multiplets.

8 Self-duality and gauge field democracy

The general theory of duality invariance in four space-time dimensions, which was reviewed in sect. 3, admits a natural higher-dimensional generalization [12, 13, 19]. In even dimensions $d = 2p$, one considers theories of n gauge $(p-1)$-forms $B^i_{a_1 \ldots a_{p-1}}$ coupled to matter fields ϕ^μ such that the Lagrangian is a function of the field strengths $F^i_{a_1 \ldots a_p} = p\,\partial_{[a_1} B^i_{a_2 \ldots a_p]}$, [8] matter fields and their derivatives, $L = L(F,\, \phi,\, \partial\phi)$. The action is invariant under the Abelian gauge symmetries $B^i \to B^i + d\Lambda^i$ where Λ^i is any $(p-2)$-form. In complete analogy with the four-dimensional case, one can introduce duality transformations and analyze the conditions for self-duality. The maximal duality group turns out to depend on the dimension of the space-time. For $d = 4k$ the maximal duality group is $\mathrm{Sp}(2n, \mathbb{R})$, while for $d = 4k+2$ it is $\mathrm{O}(n,n)$. In the absence of matter, the maximal duality group is compact: $\mathrm{U}(n)$ in $d = 4k$ dimensions, and $\mathrm{O}(n) \times \mathrm{O}(n)$ for $d = 4k+2$. The fact that the maximal duality group depends on the dimension of space-time was also discussed in [38, 39, 5, 40].

A natural question is what happens to a self-dual system upon dimensional reduction? The answer is that one finds a self-dual system with $(p-1)$-forms and $(d-p-1)$-forms in d space-time dimensions, where d is not necessarily even. We now discuss the general properties of such models. In $d = 4$ such models also appear as the bosonic sector of the self-dual systems of the $\mathcal{N} = 1$ tensor multiplet we discussed in sect. 7. In fact, the analysis of this section was inspired by self-duality of the tensor Goldstone multiplet [15].

In d space-time dimensions, we consider a theory of n gauge $(p-1)$-forms $B^i_{a_1 \ldots a_{p-1}}$ and m gauge $(d-p-1)$-forms $C^I_{a_1 \ldots a_{d-p-1}}$ coupled to matter fields ϕ^μ. We introduce the gauge invariant field strengths

$$U^i_{a_1 \ldots a_p} = p\,\partial_{[a_1} B^i_{a_2 \ldots a_p]}~, \qquad V^I_{a_1 \ldots a_{d-p}} = (d-p)\,\partial_{[a_1} C^I_{a_2 \ldots a_{d-p}]}~. \tag{8.1}$$

Without loss of generality, we assume $p < [d/2]$ [9] and then introduce the Hodge-dual of V,

$$\tilde{V}^I_{a_1 \ldots a_p} = \frac{1}{(d-p)!}\,\varepsilon_{a_1 \ldots a_p b_1 \ldots b_{d-p}}\,V^{I\,b_1 \ldots b_{d-p}}~, \tag{8.2}$$

which is of lower rank than V. The Bianchi identities read

$$\partial_{[b}\,U^i_{a_1 \ldots a_p]} = 0~, \qquad \partial^b\,\tilde{V}^I_{b a_1 \ldots a_{p-1}} = 0~. \tag{8.3}$$

The Lagrangian is required to be a function of the field strengths, matter fields and their derivatives

$$L = L(U,\, \tilde{V},\, \phi,\, \partial\phi) \equiv L(\varphi)~. \tag{8.4}$$

[8] Our normalization is $\partial_{[a_1} B_{a_2 \ldots a_p]} = \frac{1}{p!}(\partial_{a_1} B_{a_2 \ldots a_p} \pm \ldots)$

[9] $[.]$ denotes the integer part. The case $p = [d/2]$ for even d is special and was mentioned at the beginning of this section.

In terms of the dual variables

$$\tilde{G}^i_{a_1\dots a_p}(\varphi) = p!\,\frac{\partial L(\varphi)}{\partial U^{i\,a_1\dots a_p}}\ , \qquad H^I_{a_1\dots a_p}(\varphi) = p!\,\frac{\partial L(\varphi)}{\partial \tilde{V}^{I\,a_1\dots a_p}}\ , \tag{8.5}$$

the equations of motion for the gauge fields read

$$\partial^b\,\tilde{G}^i_{ba_1\dots a_{p-1}} = 0\ , \qquad \partial_{[b}\,H^I_{a_1\dots a_p]} = 0\ . \tag{8.6}$$

The explicit structure of the Bianchi identities and equations of motion implies that one may consider duality transformations of the form

$$\begin{aligned}\delta\begin{pmatrix}H\\U\end{pmatrix} &= \begin{pmatrix}A & B\\C & D\end{pmatrix}\begin{pmatrix}H\\U\end{pmatrix}\ , \qquad \delta\begin{pmatrix}\tilde{V}\\\tilde{G}\end{pmatrix} = \begin{pmatrix}M & N\\R & S\end{pmatrix}\begin{pmatrix}\tilde{V}\\\tilde{G}\end{pmatrix}\ ,\\ \delta\phi^\mu &= \xi^\mu(\phi)\ .\end{aligned} \tag{8.7}$$

Here $A,\ B,\ C,\ D$ and $M,\ N,\ R,\ S$ are real constant matrices, and ξ^μ are some unspecified functions of the matter fields. We have suppressed the indices i, I. Compatibility of the duality transformations with self-duality now imposes the conditions

$$N = C^{\mathrm{T}}\ , \quad R = B^{\mathrm{T}}\ , \quad M + A^{\mathrm{T}} = \kappa\mathbf{1}\ , \quad S + D^{\mathrm{T}} = \kappa\mathbf{1}\ , \tag{8.8}$$

with κ some real constant, as well as the following functional relations

$$\begin{aligned}\frac{\partial}{\partial \tilde{V}^{I\underline{a}}}\left[\delta L - \frac{1}{p!}B^{Jj}\,\tilde{V}^J\cdot U^j - \frac{1}{p!}C^{jJ}\,\tilde{G}^j\cdot H^J - \kappa\,L\right] &= 0\ ,\\ \frac{\partial}{\partial U^{i\underline{a}}}\left[\delta L - \frac{1}{p!}B^{Jj}\,\tilde{V}^J\cdot U^j - \frac{1}{p!}C^{jJ}\,\tilde{G}^j\cdot H^J - \kappa\,L\right] &= 0\ ,\end{aligned} \tag{8.9}$$

where we have introduced the notation

$$\tilde{G}^i\cdot H^J = \tilde{G}^{i\,a_1\dots a_p}H^J_{a_1\dots a_p} \equiv \tilde{G}^{i\,\underline{a}}H^J_{\underline{a}}\ . \tag{8.10}$$

Furthermore, the matter equation of motion transforms covariantly if one requires

$$\frac{\delta}{\delta\phi^\mu}\left[\delta S - \frac{1}{p!}\int \mathrm{d}^4x\,C^{iI}\tilde{G}^i\cdot H^I\right] \;=\; 0\ . \tag{8.11}$$

Eqs. (8.9) and (8.11) are then seen to be compatible if $\kappa = 0$ and if the Lagrangian transforms as

$$\begin{aligned}\delta L &= \frac{1}{p!}B^{Ii}\,\tilde{V}^I\cdot U^i + \frac{1}{p!}C^{iI}\,\tilde{G}^i\cdot H^I\\ &= \delta\left(\frac{1}{p!}\,U^i\cdot\tilde{G}^i\right) = \delta\left(\frac{1}{p!}\,\tilde{V}^I\cdot H^I\right)\ .\end{aligned} \tag{8.12}$$

Since $\kappa = 0$, eq. (8.8) means that (8.7) becomes

$$\delta \begin{pmatrix} H \\ U \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} H \\ U \end{pmatrix} , \qquad \delta \begin{pmatrix} \tilde{V} \\ \tilde{G} \end{pmatrix} = \begin{pmatrix} -A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & -D^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \tilde{V} \\ \tilde{G} \end{pmatrix} . \tag{8.13}$$

One easily shows that both variations satisfy the same algebra, namely $\mathrm{gl}(n+m, \mathbb{R})$. The maximal connected duality group is therefore $\mathrm{GL}_0(n+m, \mathbb{R})$. The finite form for duality transformations is

$$\begin{pmatrix} H' \\ U' \end{pmatrix} = g \begin{pmatrix} H \\ U \end{pmatrix} , \qquad \begin{pmatrix} \tilde{V}' \\ \tilde{G}' \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} (g^{\mathrm{T}})^{-1} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \tilde{V} \\ \tilde{G} \end{pmatrix} , \tag{8.14}$$

with $g \in \mathrm{GL}_0(n+m, \mathbb{R})$.

Equation (8.12) can be rewritten in a different, more useful, form if one directly computes δL. This gives the self-duality equation

$$\begin{aligned} p!\, \delta_\phi L = {} & B^{Ii}\, \tilde{V}^I \cdot U^i - C^{iI}\, \tilde{G}^i \cdot H^I \\ & + A^{IJ}\, \tilde{V}^I \cdot H^J - D^{ij}\, \tilde{G}^i \cdot U^j \ , \end{aligned} \tag{8.15}$$

with $\delta_\phi L$ as in eq. (3.18).

In the absence of matter, the maximal connected duality group becomes $\mathrm{SO}(n+m)$, the maximal compact subgroup of $\mathrm{GL}_0(n+m, \mathbb{R})$; *i.e.* $A^{\mathrm{T}} = -A$, $D^{\mathrm{T}} = -D$, $B^{\mathrm{T}} = -C$. Then, the self-duality equation (8.15) is equivalent to

$$U^i \cdot \tilde{V}^I + \tilde{G}^i \cdot H^I = 0 \ , \tag{8.16}$$

$$\left(U^i \cdot \frac{\partial}{\partial U^j} - U^j \cdot \frac{\partial}{\partial U^i} \right) L = 0 \, , \qquad \left(\tilde{V}^I \cdot \frac{\partial}{\partial \tilde{V}^J} - \tilde{V}^J \cdot \frac{\partial}{\partial \tilde{V}^I} \right) L = 0 \, . \tag{8.17}$$

Eq. (8.17) says that the theory is manifestly $\mathrm{SO}(n) \times \mathrm{SO}(m)$ invariant.

By analogy with the results of [10, 11, 13], any $\mathrm{SO}(n+m)$ duality invariant model $L(U^i, \tilde{V}^I)$ can be lifted to a model with the non-compact duality symmetry $\mathrm{GL}_0(n+m, \mathbb{R})$ by coupling the gauge fields to scalar fields ϕ^μ parameterizing the quotient space $\mathrm{GL}_0(n+m, \mathbb{R}) \,/\, \mathrm{SO}(n+m)$.

Any $\mathrm{SO}(n+m)$ duality invariant model $L(U^i, \tilde{V}^I)$, where $U^i_p = \mathrm{d}B^i_{p-1}$ and $V^I_{d-p} = \mathrm{d}C^I_{d-p-1}$, with $n \neq 0$ and $m \neq 0$, enjoys self-duality under Legendre transformation which dualizes two given forms B^i_{p-1} and C^I_{d-p-1} into a $(d-p-1)$-form and a $(p-1)$-form, respectively. This is a simple consequence of the duality invariance, see sect. 2 for more details. On the other hand, one can apply a Legendre transformation which, say, leaves the gauge $(p-1)$-forms invariant but dualizes all gauge $(d-p-1)$-forms into $(p-1)$-forms. One then obtains a model of $(n+m)$ gauge $(p-1)$-forms. Remarkably, the $\mathrm{SO}(n+m)$ duality symmetry of the original model turns into a manifest (linear) $\mathrm{SO}(n+m)$ symmetry of the dualized model. This is a consequence of the self-duality equations (8.16) and (8.17) and the standard properties of Legendre transformation. Therefore, in the models that we have considered here, all fields are on the same footing, hence the title of this subsection. The $\mathrm{SO}(n+m)$ duality symmetry is linearly realized if all form are of the same degree.

The self-duality equations (3.22) and (3.23) are difficult to solve. However, for (8.16) and (8.17), there exists a simple scheme to derive their general solution. One starts with an $\mathrm{SO}(n+m)$ invariant model of $(n+m)$ gauge $(p-1)$-forms in d dimensions, and then simply dualize m of the fields into gauge $(d-p-1)$-forms by applying the proper Legendre transformation. The dualized model is invariant under the duality transformations.

If $n=m$, there are systems (we will give examples below) which are invariant under $\mathrm{Sp}(2n,\mathbb{R})$ rather than the maximal duality group $\mathrm{GL}(2n,\mathbb{R})$. This is the case if the matrix parameterizing the infinitesimal transformation of $\tilde{V}$ and $\tilde{G}$, written in the form

$$\delta\begin{pmatrix}\tilde{G}\\ \tilde{V}\end{pmatrix}=\begin{pmatrix}-D^{\mathrm{T}} & B^{\mathrm{T}}\\ C^{\mathrm{T}} & -A^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}\tilde{G}\\ \tilde{V}\end{pmatrix}\ , \tag{8.18}$$

is required to coincide with the transformation of H and U (*c.f.* (8.13)). In the absence of matter, the duality group of these systems reduces to $\mathrm{U}(n)$ (see sect. 3) and the self-duality equations take the form (from now on, we do not distinguish between indices i and I)

$$U^{(i}\cdot\tilde{V}^{j)}+\tilde{G}^{(i}\cdot H^{j)}=0\ , \tag{8.19}$$

$$\left(U^i\cdot\frac{\partial}{\partial U^j}+\tilde{V}^i\cdot\frac{\partial}{\partial\tilde{V}^j}\right)L\ -\ (i\leftrightarrow j)=0\ . \tag{8.20}$$

Eq. (8.20) means that the Lagrangian is manifestly $\mathrm{SO}(n)$ invariant. Any $\mathrm{U}(n)$ duality invariant model can be made $\mathrm{Sp}(2n,\mathbb{R})$ duality invariant by coupling the gauge fields to scalars valued in $\mathrm{Sp}(2n,\mathbb{R})\,/\,\mathrm{U}(n)$. For $n=1$ the result reads

$$L(U,\tilde{V},\mathcal{S})\ =\ \frac{1}{p!}\,\mathcal{S}_1\,U\cdot\tilde{V}\ +\ L(\sqrt{\mathcal{S}_2}\,U,\sqrt{\mathcal{S}_2}\,\tilde{V})\ , \tag{8.21}$$

with $\mathcal{S}$ the dilaton-axion field (2.11) transforming by the rule (2.12) under the duality group $\mathrm{SL}(2,\mathbb{R})$.

In contrast to $\mathrm{U}(1)$ duality invariant models of a single gauge $(2p-1)$-form in even dimensions $d=4p$, $\mathrm{U}(1)$ duality invariant models of a gauge $(p-1)$-form and a gauge

$(d-p-1)$-form in arbitrary dimensions d can be considered as reducible, since they involve two independent fields. However, the latter models possess 'self-dual' solutions

$$U_{a_1 \dots a_p} = \gamma \, H_{a_1 \dots a_p}(U, \tilde{V}) \ , \qquad \tilde{V}_{a_1 \dots a_p} = -\frac{1}{\gamma} \, \tilde{G}_{a_1 \dots a_p}(U, \tilde{V}) \ , \tag{8.22}$$

with γ a constant parameter. The explicit dependence of γ is dictated by the self-duality equation (8.19). Such solutions of the equations of motion describe the dynamics of a single field.

To conclude, we give an example of a U(1) duality invariant model. The Lagrangian reads

$$L \ = \ \frac{1}{p!} \ - \ \frac{1}{p!} \sqrt{1 + U \cdot U - \tilde{V} \cdot \tilde{V} - (U \cdot \tilde{V})^2} \ . \tag{8.23}$$

It is easy to check that L solves the self-duality equation (8.19), and therefore the theory is U(1) duality invariant. The theory can be equivalently represented in the form

$$L = -\frac{1}{2\,p!} \, (\chi + \bar{\chi}) \ , \tag{8.24}$$

where the complex field χ is a functions of U and $\tilde{V}$ which satisfies the nonlinear constraint

$$\chi + \frac{1}{2} \, \chi \bar{\chi} - \psi \ = \ 0 \ , \qquad \psi \ = \ \frac{1}{2} \, (U + \mathrm{i} \, \tilde{V})^2. \tag{8.25}$$

This representation is analogous to that for the BI theory described in sect. 2.

The above duality invariant system has a supersymmetric origin. Let us choose $d = 4$ and then $p = 1$ is the only interesting choice. The dynamical fields are a scalar φ and an antisymmetric gauge field B_{ab} which should enter the Lagrangian only via their field strengths $U_a = \partial_a \, \varphi$ and $\tilde{V}^a = \frac{1}{2} \, \varepsilon^{abcd} \, \partial_b \, B_{cd}$. Then, the Lagrangian (8.23) describes the bosonic sector of a model for partial $\mathcal{N} = 2 \to \mathcal{N} = 1$ supersymmetry breaking with the tensor multiplet as the Goldstone multiplet [15, 16]. The antisymmetric gauge field can be dualized into a scalar, by applying the appropriate Legendre transformation. The resulting model is manifestly U(1) invariant and it describes a 3-brane in six dimensions.

Other examples of U(1) duality invariant models of the scalar and antisymmetric tensor in four dimensions can be obtained by considering the bosonic sector of the self-dual tensor multiplet systems we discussed in sect. 7. It is worth noting that not all U(1) duality invariant models of the scalar and antisymmetric tensor admit a supersymmetric extension: the two fields have to appear in the action in the combination ψ as defined in (8.25). This is in contrast with what we found in self-dual nonlinear electrodynamics.

Using the results of [18, 19], the construction just described can be generalized to derive $\mathrm{U}(n)$ duality invariant models of n gauge $(p-1)$-forms and n gauge $(d-p-1)$-forms in four dimensions. The Lagrangian is

$$L = -\frac{1}{2\,p!}\,\mathrm{tr}\,(\chi + \bar{\chi})\ , \tag{8.26}$$

where the complex $n \times n$ matrix χ is a function of U^i and $\tilde{V}^i$ which satisfies the nonlinear constraint

$$\chi^{ij} + \frac{1}{2}\,\chi^{ik}\bar{\chi}^{jk} \ = \ \frac{1}{2}\,(U^i + \mathrm{i}\,\tilde{V}^i)\cdot(U^j + \mathrm{i}\,\tilde{V}^j)\ . \tag{8.27}$$

9 $\mathcal{N} = 2$ duality rotations

The construction of sect. 4 admits a natural generalization to $\mathcal{N} = 2$ supersymmetry [20], although here much less explicit results have been obtained so far. We will discuss the case of one single Abelian gauge multiplet only, the generalization to an arbitrary number being straightforward.

We will work in $\mathcal{N} = 2$ global superspace $\mathbb{R}^{4|8}$ parametrized by $\mathcal{Z}^A = (x^a, \theta^\alpha_i, \bar{\theta}^i_{\dot\alpha})$, where $i = \underline{1}, \underline{2}$. The flat covariant derivatives $\mathcal{D}_A = (\partial_a, \mathcal{D}^i_\alpha, \bar{\mathcal{D}}^{\dot\alpha}_i)$ satisfy the algebra

$$\{\, \mathcal{D}^i_\alpha, \mathcal{D}^j_\beta \,\} = \{\, \bar{\mathcal{D}}_{\dot\alpha i}, \bar{\mathcal{D}}_{\dot\beta j} \,\} = 0\ , \qquad \{\, \mathcal{D}^i_\alpha, \bar{\mathcal{D}}_{\dot\alpha j} \,\} = -2\,\mathrm{i}\,\delta^i_j\,(\sigma^a)_{\alpha\dot\alpha}\,\partial_a\ . \tag{9.1}$$

Throughout this section, we will use the notation:

$$\begin{aligned} \mathcal{D}^{ij} &\equiv \mathcal{D}^{\alpha(i}\mathcal{D}^{j)}_\alpha = \mathcal{D}^{\alpha i}\mathcal{D}^j_\alpha\ , \qquad & \bar{\mathcal{D}}^{ij} &\equiv \bar{\mathcal{D}}^{(i}_{\dot\alpha}\bar{\mathcal{D}}^{j)\,\dot\alpha} = \bar{\mathcal{D}}^i_{\dot\alpha}\bar{\mathcal{D}}^{j\,\dot\alpha} \\ \mathcal{D}^4 &\equiv \frac{1}{16}(\mathcal{D}^{\underline{1}})^2\,(\mathcal{D}^{\underline{2}})^2\ , \qquad & \bar{\mathcal{D}}^4 &\equiv \frac{1}{16}(\bar{\mathcal{D}}_{\underline{1}})^2\,(\bar{\mathcal{D}}_{\underline{2}})^2\ . \end{aligned} \tag{9.2}$$

An integral over the full superspace (with the measure $\mathrm{d}^{12}\mathcal{Z} = \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathrm{d}^4\bar{\theta}$) can be reduce to one over the chiral subspace (with the measure $\mathrm{d}^8\mathcal{Z} = \mathrm{d}^4x\,\mathrm{d}^4\theta$) or over the antichiral subspace ($\mathrm{d}^8\bar{\mathcal{Z}} = \mathrm{d}^4x\,\mathrm{d}^4\bar{\theta}$):

$$\int \mathrm{d}^{12}\mathcal{Z}\ \mathcal{L}(\mathcal{Z}) \ = \ \int \mathrm{d}^8\mathcal{Z}\ \mathcal{D}^4\mathcal{L}(\mathcal{Z}) \ = \ \int \mathrm{d}^8\bar{\mathcal{Z}}\ \bar{\mathcal{D}}^4\mathcal{L}(\mathcal{Z})\ . \tag{9.3}$$

The discussion in this section is completely analogous to the one presented in the first part of sect. 4. We will thus be brief. Let $\mathcal{S}[\mathcal{W}, \bar{\mathcal{W}}]$ be the action describing the dynamics of a single $\mathcal{N} = 2$ vector multiplet. The (anti) chiral superfield strengths $\bar{\mathcal{W}}$ and $\mathcal{W}$ satisfy the Bianchi identity [41]

$$\mathcal{D}^{ij}\,\mathcal{W} \ = \ \bar{\mathcal{D}}^{ij}\,\bar{\mathcal{W}}\ . \tag{9.4}$$

The general solution of the Bianchi identity [42],

$$\mathcal{W} = \bar{\mathcal{D}}^4 \mathcal{D}^{ij} V_{ij} , \qquad \bar{\mathcal{W}} = \mathcal{D}^4 \bar{\mathcal{D}}^{ij} V_{ij} \tag{9.5}$$

is in terms of a real unconstrained prepotential $V_{(ij)}$.

Suppose that $\mathcal{S}[\mathcal{W}, \bar{\mathcal{W}}]$ can be unambiguously defined as a functional of *unconstrained* (anti) chiral superfields $\bar{\mathcal{W}}$ and $\mathcal{W}$. Then, one can define (anti) chiral superfields $\bar{\mathcal{M}}$ and $\mathcal{M}$ as

$$\mathrm{i}\, \mathcal{M} \equiv 4 \, \frac{\delta}{\delta \mathcal{W}} \, \mathcal{S}[\mathcal{W}, \bar{\mathcal{W}}] , \qquad -\mathrm{i}\, \bar{\mathcal{M}} \equiv 4 \, \frac{\delta}{\delta \bar{\mathcal{W}}} \, \mathcal{S}[\mathcal{W}, \bar{\mathcal{W}}] \tag{9.6}$$

in terms of which the equations of motion are

$$\mathcal{D}^{ij} \, \mathcal{M} \;=\; \bar{\mathcal{D}}^{ij} \, \bar{\mathcal{M}} \, . \tag{9.7}$$

Again, since the Bianchi identity (9.4) and the equation of motion (9.7) have the same functional form, one can consider infinitesimal U(1) duality transformations

$$\delta \mathcal{W} \;=\; \lambda \, \mathcal{M} \, , \qquad \delta \mathcal{M} \;=\; - \, \lambda \, \mathcal{W} \, . \tag{9.8}$$

The analysis of Appendix A leads to

$$\begin{aligned} \delta \mathcal{S} &= -\frac{\mathrm{i}}{8} \, \lambda \int \mathrm{d}^8 \mathcal{Z} \, \Big(\mathcal{W}^2 - \mathcal{M}^2 \Big) + \frac{\mathrm{i}}{8} \, \lambda \int \mathrm{d}^8 \bar{\mathcal{Z}} \, \Big(\bar{\mathcal{W}}^2 - \bar{\mathcal{M}}^2 \Big) \\ &= \frac{\mathrm{i}}{4} \, \lambda \int \mathrm{d}^8 \mathcal{Z} \, \mathcal{M}^2 - \frac{\mathrm{i}}{4} \, \lambda \int \mathrm{d}^8 \bar{\mathcal{Z}} \, \bar{\mathcal{M}}^2 \end{aligned} \tag{9.9}$$

The theory is thus duality invariant provided the following reality condition is satisfied:

$$\int \mathrm{d}^8 \mathcal{Z} \, \Big(\mathcal{W}^2 + \mathcal{M}^2 \Big) \;=\; \int \mathrm{d}^8 \bar{\mathcal{Z}} \, \Big(\bar{\mathcal{W}}^2 + \bar{\mathcal{M}}^2 \Big) . \tag{9.10}$$

Here $\mathcal{M}$ and $\bar{\mathcal{M}}$ are defined as in (9.6), and $\mathcal{W}$ and $\bar{\mathcal{W}}$ should be considered as *unconstrained* chiral and antichiral superfields, respectively. Eq. (9.10) serves as our master functional equation ($\mathcal{N} = 2$ self-duality equation) to determine duality invariant models of the $\mathcal{N} = 2$ vector multiplet.

We remark that, as in the $\mathcal{N} = 0, 1$ cases, the action itself is not duality invariant, but

$$\delta \left(\mathcal{S} - \frac{\mathrm{i}}{8} \int \mathrm{d}^8 \mathcal{Z} \, \mathcal{M} \mathcal{W} + \frac{\mathrm{i}}{8} \int \mathrm{d}^8 \bar{\mathcal{Z}} \, \bar{\mathcal{M}} \bar{\mathcal{W}} \right) \;=\; 0 \, . \tag{9.11}$$

The invariance of the latter functional under a finite U(1) duality rotation by $\pi/2$, is equivalent to the self-duality of $\mathcal{S}$ under Legendre transformation,

$$\mathcal{S}[\mathcal{W}, \bar{\mathcal{W}}] - \frac{\mathrm{i}}{4} \int \mathrm{d}^8 \mathcal{Z} \, \mathcal{W} \mathcal{W}_{\mathrm{D}} + \frac{\mathrm{i}}{4} \int \mathrm{d}^8 \bar{\mathcal{Z}} \, \bar{\mathcal{W}} \bar{\mathcal{W}}_{\mathrm{D}} \;=\; \mathcal{S}[\mathcal{W}_{\mathrm{D}}, \bar{\mathcal{W}}_{\mathrm{D}}] \, , \tag{9.12}$$

where $\mathcal{W}_{\rm D}$ is the dual chiral field strength,

$$\mathcal{W}_{\rm D} = \bar{\mathcal{D}}^4 \mathcal{D}_{ij} \, V_{\rm D}{}^{ij} \ , \tag{9.13}$$

with $V_{\rm D}{}^{ij}$ a real unconstrained prepotential.

Apart from the $\mathcal{N} = 2$ Maxwell action

$$\mathcal{S}_{\rm free} = \frac{1}{8} \int {\rm d}^8 \mathcal{Z} \, \mathcal{W}^2 + \frac{1}{8} \int {\rm d}^8 \bar{\mathcal{Z}} \, \bar{\mathcal{W}}^2 \ , \tag{9.14}$$

only one other solution of (9.10) is known [21]:

$$\mathcal{S} \ = \ \frac{1}{4} \int {\rm d}^8 \mathcal{Z} \, \mathcal{X} + \frac{1}{4} \int {\rm d}^8 \bar{\mathcal{Z}} \, \bar{\mathcal{X}} \ , \tag{9.15}$$

where the chiral superfield $\mathcal{X}$ is a functional of $\mathcal{W}$ and $\bar{\mathcal{W}}$ defined via the constraint

$$\mathcal{X} \ = \ \mathcal{X} \, \bar{\mathcal{D}}^4 \bar{\mathcal{X}} \ + \ \frac{1}{2} \mathcal{W}^2 \ . \tag{9.16}$$

Following [20], let us prove that this system provides a solution of the self-duality equation (9.10). Under an infinitesimal variation of $\mathcal{W}$ only, we have

$$\begin{aligned} \delta_{\mathcal{W}} \mathcal{X} &= \delta_{\mathcal{W}} \mathcal{X} \, \bar{\mathcal{D}}^4 \bar{\mathcal{X}} + \mathcal{X} \, \bar{\mathcal{D}}^4 \delta_{\mathcal{W}} \bar{\mathcal{X}} + \mathcal{W} \, \delta \mathcal{W} \ , \\ \delta_{\mathcal{W}} \bar{\mathcal{X}} &= \delta_{\mathcal{W}} \bar{\mathcal{X}} \, \mathcal{D}^4 \mathcal{X} + \bar{\mathcal{X}} \, \mathcal{D}^4 \delta_{\mathcal{W}} \mathcal{X} \ . \end{aligned} \tag{9.17}$$

From these relations one gets

$$\delta_{\mathcal{W}} \mathcal{X} = \frac{1}{1 - \mathcal{Q}} \left[\frac{\mathcal{W} \, \delta \mathcal{W}}{1 - \bar{\mathcal{D}}^4 \bar{\mathcal{X}}} \right] \ , \qquad \delta_{\mathcal{W}} \bar{\mathcal{X}} = \frac{\bar{\mathcal{X}}}{1 - \mathcal{D}^4 \mathcal{X}} \, \mathcal{D}^4 \delta_{\mathcal{W}} \mathcal{X} \ , \tag{9.18}$$

where

$$\begin{aligned} \mathcal{Q} &= \mathcal{P} \, \bar{\mathcal{P}} \ , & \bar{\mathcal{Q}} &= \bar{\mathcal{P}} \, \mathcal{P} \ , \\ \mathcal{P} &= \frac{\mathcal{X}}{1 - \bar{\mathcal{D}}^4 \bar{\mathcal{X}}} \, \bar{\mathcal{D}}^4 \ , & \bar{\mathcal{P}} &= \frac{\bar{\mathcal{X}}}{1 - \mathcal{D}^4 \mathcal{X}} \, \mathcal{D}^4 \ . \end{aligned} \tag{9.19}$$

With these results, it is easy to compute $\mathcal{M}$:

$${\rm i} \, \mathcal{M} = \frac{\mathcal{W}}{1 - \bar{\mathcal{D}}^4 \bar{\mathcal{X}}} \left\{ 1 + \bar{\mathcal{D}}^4 \, \bar{\mathcal{P}} \, \frac{1}{1 - \mathcal{Q}} \, \frac{\mathcal{X}}{1 - \bar{\mathcal{D}}^4 \bar{\mathcal{X}}} + \bar{\mathcal{D}}^4 \, \frac{1}{1 - \bar{\mathcal{Q}}} \, \frac{\bar{\mathcal{X}}}{1 - \mathcal{D}^4 \mathcal{X}} \right\} \ . \tag{9.20}$$

Now, a short calculation gives

$${\rm Im} \int {\rm d}^8 \mathcal{Z} \, \left\{ \mathcal{M}^2 + 2 \, \frac{1}{1 - \mathcal{Q}} \, \frac{\mathcal{X}}{1 - \bar{\mathcal{D}}^4 \bar{\mathcal{X}}} \right\} \ = \ 0 \ . \tag{9.21}$$

On the other hand, the constraint (9.16) implies

$$\int \mathrm{d}^8\mathcal{Z}\,\mathcal{X} - \int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{X}} = \frac{1}{2}\int \mathrm{d}^8\mathcal{Z}\,\mathcal{W}^2 - \frac{1}{2}\int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{W}}^2\ , \tag{9.22}$$

and hence

$$\frac{\delta}{\delta\mathcal{W}}\left\{\int \mathrm{d}^8\mathcal{Z}\,\mathcal{X} - \int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{X}}\right\} = \mathcal{W}\ . \tag{9.23}$$

The latter relation can be shown to be equivalent to

$$\frac{1}{1-\mathcal{Q}}\,\frac{\mathcal{X}}{1-\bar{\mathcal{D}}^4\bar{\mathcal{X}}} = \mathcal{P}\,\frac{1}{1-\bar{\mathcal{Q}}}\,\frac{\bar{\mathcal{X}}}{1-\mathcal{D}^4\mathcal{X}} + \mathcal{X}\ . \tag{9.24}$$

Using this result in eq. (9.21), we arrive at the relation

$$\int \mathrm{d}^8\mathcal{Z}\,\mathcal{M}^2 - \int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{M}}^2 = -2\int \mathrm{d}^8\mathcal{Z}\,\mathcal{X} + 2\int \mathrm{d}^8\bar{\mathcal{Z}}\,\bar{\mathcal{X}} \tag{9.25}$$

which is equivalent, due to (9.22), to (9.10).

The dynamical system (9.15), (9.16) was introduced in [21] as the $\mathcal{N}=2$ supersymmetric BI action (*c.f.* with the similar construction for the $\mathcal{N}=1$ super BI action we described in sect. 3). Such an interpretation is supported in part by the fact that the theory correctly reduces to the $\mathcal{N}=1$ BI in a special $\mathcal{N}=1$ limit; we now briefly discuss this issue.

Let us introduce the $\mathcal{N}=1$ components of the $\mathcal{N}=2$ vector multiplet. Given an $\mathcal{N}=2$ superfield U, its $\mathcal{N}=1$ projection is defined to be $U| = U(\mathcal{Z})|_{\theta_{\underline{2}}=\bar{\theta}^{\underline{2}}=0}$. The $\mathcal{N}=2$ vector multiplet contains two independent chiral $\mathcal{N}=1$ components

$$\mathcal{W}| = \sqrt{2}\,\Phi\ , \qquad \mathcal{D}^{\underline{2}}_{\alpha}\,\mathcal{W}| = 2\mathrm{i}\,W_\alpha\ , \qquad (\mathcal{D}^{\underline{2}})^2\mathcal{W}| = \sqrt{2}\,\bar{D}^2\bar{\Phi}\ . \tag{9.26}$$

Using in addition that

$$\int \mathrm{d}^8\mathcal{Z} = -\frac{1}{4}\int \mathrm{d}^6z\,(\mathcal{D}^{\underline{2}})^2\ , \qquad \int \mathrm{d}^{12}\mathcal{Z} = \frac{1}{16}\int \mathrm{d}^8z\,(\mathcal{D}^{\underline{2}})^2\,(\bar{\mathcal{D}}_{\underline{2}})^2\ , \tag{9.27}$$

the above definitions imply that the free $\mathcal{N}=2$ vector multiplet action (9.14) straightforwardly reduces to $\mathcal{N}=1$ superfields

$$\mathcal{S}_{\rm free} = \int \mathrm{d}^8z\,\bar{\Phi}\Phi + \frac{1}{4}\int \mathrm{d}^6z\,W^2 + \frac{1}{4}\int \mathrm{d}^6\bar{z}\,\bar{W}^2\ . \tag{9.28}$$

If one switches off Φ,

$$\Phi = 0 \qquad \Longrightarrow \qquad (\mathcal{D}^{\underline{2}})^2\mathcal{W}| = 0\ , \tag{9.29}$$

one readily observes that the theory (9.15), (9.16) reduces to the $\mathcal{N}=1$ BI theory (5.2), (5.3). However, it was shown in [20] that there exist infinitely many manifestly $\mathcal{N}=2$ supersymmetric models possessing this very property. Of course, the specific feature of the system (9.15), (9.16) is its invariance under U(1) duality rotations, and the requirement of self-duality severely restricts the class of possible models. But it turns out that even the latter requirement is not sufficient to uniquely fix the $\mathcal{N}=2$ supersymmetric BI action.

The $\mathcal{N}=2$ supersymmetric BI action is expected to describe a single D3-brane in six dimensions

$$L_{\mathrm{D3-brane}} \;=\; 1 \;-\; \sqrt{-\det\left(\eta_{ab}+F_{ab}+\partial_a\bar{\varphi}\partial_b\varphi\right)}\;. \tag{9.30}$$

Here the complex transverse coordinates φ of the brane should, in general, be related to the scalars $\phi=\mathcal{W}|_{\theta=0}$ and the other components of the $\mathcal{N}=2$ vector multiplet by a nonlinear field redefinition (see, e.g. [43]). Since $L_{\mathrm{D3-brane}}$ is manifestly invariant under constant shifts of the transverse coordinates

$$\varphi(x) \;\longrightarrow\; \varphi(x)+\sigma\;, \tag{9.31}$$

the full supersymmetric theory must also be invariant under such transformations acting on $\mathcal{W}$ in a nonlinear way

$$\mathcal{W}(\mathcal{Z}) \;\longrightarrow\; \mathcal{W}(\mathcal{Z})+\sigma+\mathcal{O}(\mathcal{W},\bar{\mathcal{W}})\;. \tag{9.32}$$

Moreover, the $\mathcal{N}=2$ supersymmetric BI action is expected to provide a model for partial $\mathcal{N}=4\to\mathcal{N}=2$ supersymmetry breaking [44]. It means that the action should be invariant under nonlinear transformations

$$\mathcal{W}(\mathcal{Z}) \;\longrightarrow\; \mathcal{W}(\mathcal{Z})+\epsilon(\theta)+\mathcal{O}(\mathcal{W},\bar{\mathcal{W}})\;, \qquad \epsilon(\theta)=\sigma+\epsilon^{\alpha}_{i}\,\theta^{i}_{\alpha}\;, \tag{9.33}$$

with ϵ^{α}_{i} a constant spinor parameter. We now demonstrate that the system (9.15), (9.16) is not compatible even with the simpler transformations (9.32).

To start with, it is worth pointing out the following. When looking for nonlinear symmetry transformations (9.32) or (9.33), one might first try to duplicate the trick[10] which successfully worked in the case of the $\mathcal{N}=1$ supersymmetric BI action (see sect. 5). Namely, one can introduce the transformation of $\mathcal{X}$

$$\delta\mathcal{X}=\epsilon(\theta)\,\mathcal{W}\;, \qquad \bar{\mathcal{D}}^{i}_{\dot\alpha}\,\epsilon(\theta)=\mathcal{D}^{ij}\,\epsilon(\theta)=0\;, \tag{9.34}$$

which obviously leaves the action (9.15) invariant. But this variation of $\mathcal{X}$ must be induced by a variation of $\mathcal{W}$ consistent with the constraint (9.16). A direct analysis shows that

[10] This course was taken up in [45].

the variation $\delta\mathcal{W}$, that is derived in this way, does not satisfy the Bianchi identity (9.4). The difference from the $\mathcal{N}=1$ case is simple but crucial: the $\mathcal{N}=2$ vector multiplet does not possess any analogue of the property $W^3=0$, typical for the $\mathcal{N}=1$ vector multiplet.

We will use the following general Ansatz

$$\delta\mathcal{W} = \sigma + \sigma\,\bar{\mathcal{D}}^4\bar{\mathcal{Y}} + \bar{\sigma}\,\Box\mathcal{Y}\ , \qquad \bar{\mathcal{D}}^i_{\dot\alpha}\mathcal{Y} = 0 \tag{9.35}$$

for symmetry transformations (9.32). The variation is consistent with the Bianchi identity (9.4). The chiral superfield $\mathcal{Y}$ is some unknown functional of $\mathcal{W}$ and $\bar{\mathcal{W}}$. The precise form of $\mathcal{Y}$ as well as of the $\mathcal{N}=2$ supersymmetric BI action, $\mathcal{S}_{\rm BI}$, should be determined, order by order in perturbation theory, from three requirements: (i) the action is to be invariant under transformations (9.35); (ii) the action should solve the self-duality equation (9.10); (iii) to order $\mathcal{W}^4$, the action should have the form:

$$\mathcal{S}_{\rm BI} = \mathcal{S}_{\rm free} \ +\ \mathcal{S}_{\rm int}\ , \qquad \mathcal{S}_{\rm int} = \frac{1}{8}\int {\rm d}^{12}\mathcal{Z}\,\mathcal{W}^2\,\bar{\mathcal{W}}^2 \ +\ \mathcal{O}(\mathcal{W}^6)\ . \tag{9.36}$$

This reproduces the known F^4 terms in the BI action.[11] Direct calculation gives for $\mathcal{Y}$

$$\begin{aligned}\mathcal{Y} = &-\frac{1}{2}\,\mathcal{W}^2\Big\{1+\frac{1}{2}\,\bar{\mathcal{D}}^4\,\bar{\mathcal{W}}^2+\frac{1}{8}\,\bar{\mathcal{D}}^4\,(\bar{\mathcal{W}}^2\,\mathcal{D}^4\mathcal{W}^2)+\frac{1}{8}\,(\bar{\mathcal{D}}^4\,\bar{\mathcal{W}}^2)^2\Big\}\\ &-\frac{1}{36}\,\bar{\mathcal{D}}^4\,(\mathcal{W}^3\,\Box\bar{\mathcal{W}}^3) \ +\ \mathcal{O}(\mathcal{W}^8)\ ,\end{aligned} \tag{9.37}$$

while $\mathcal{S}_{\rm int}$ reads

$$\begin{aligned}\mathcal{S}_{\rm int} = &\frac{1}{8}\int {\rm d}^{12}\mathcal{Z}\,\mathcal{W}^2\,\bar{\mathcal{W}}^2\,\Big\{1+\frac{1}{2}\Big(\mathcal{D}^4\mathcal{W}^2+\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2\Big)\\ &+\frac{1}{4}\Big((\mathcal{D}^4\mathcal{W}^2)^2+(\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2)^2\Big)+\frac{3}{4}\,(\mathcal{D}^4\mathcal{W}^2)(\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2)\Big\}\\ &+\frac{1}{24}\int {\rm d}^{12}\mathcal{Z}\,\Big\{\frac{1}{3}\mathcal{W}^3\Box\bar{\mathcal{W}}^3+\frac{1}{2}(\mathcal{W}^3\Box\bar{\mathcal{W}}^3)\bar{\mathcal{D}}^4\bar{\mathcal{W}}^2+\frac{1}{2}(\bar{\mathcal{W}}^3\Box\mathcal{W}^3)\mathcal{D}^4\mathcal{W}^2+\frac{1}{48}\mathcal{W}^4\Box^2\bar{\mathcal{W}}^4\Big\}\\ &+\ \ \mathcal{O}(\mathcal{W}^{10})\ .\end{aligned} \tag{9.38}$$

The expression in the first two lines of (9.38) comes from the expansion of (9.15) in powers of $\mathcal{W}$ and its conjugate to the order indicated. As concerns the expression in the third line of (9.38), it is not present in the power series expansion of (9.15), but it is required for invariance under transformations (9.35). It is also worth noting that the expression in the first line of (9.37) coincides with the decomposition of $(-\mathcal{X})$ (9.16) to the given order.

[11]This is the only known superinvariant with this property.

Our conclusion is that the system (9.15), (9.16) cannot be identified with the correct $\mathcal{N} = 2$ supersymmetric D3-brane world-volume action, and the problem of constructing such an action is still open.

A natural possibility to look for $\mathcal{N} = 2$ supersymmetric BI action, advocated in [21], is first to derive a manifestly $(1,0)$ supersymmetric BI action in six dimensions and, then to dimensionally reduce to four dimensions. By construction, the resulting four-dimensional model should be manifestly $\mathcal{N} = 2$ supersymmetric and invariant under constant shift transformations $\mathcal{W} \to \mathcal{W} + \sigma$, without any nonlinear terms. However, the problem of constructing the manifestly $(1,0)$ supersymmetric BI action in six dimensions is not simple. In $d = 6$ there exists an off-shell formulation for the $(1,0)$ vector multiplet [46]. But super-extensions of F^2, F^4 and F^6 terms, which appear in the decomposition of the $d = 6$ BI action, cannot be represented by integrals over $(1,0)$ superspace or its subspace. The super-extension of F^2 term was already derived in [46]. As to the super-extensions of F^4 and F^6 terms, candidates were proposed in [21]. Unfortunately, the proof of their invariance under $(1,0)$ supersymmetry transformations was based on the use of the identity (here we follow the $d = 6$ notation of [46]) $D_{\alpha\,(i}\left\{W^{[\beta}_{j}W^{\gamma]}_{k)}\right\} = 0$, which holds *on-shell* [46], and not *off-shell* as claimed in [21]. Therefore, the super-extensions of F^4 and F^6 terms proposed in [21] are not invariant under $(1,0)$ supersymmetry transformations. Thus the problem of constructing a manifestly $(1,0)$ supersymmetric BI action is six dimensions remains unsolved. If such an action exists, its dimensional reduction to $d = 4$ will be manifestly supersymmetric, but not all terms in the action can be represented as integrals over $\mathcal{N} = 2$ superspace or its supersymmetric subspaces.

Acknowledgements
We are grateful to Evgeny Ivanov, Dima Sorokin and Arkady Tseytlin for their interest in this project. We thank Paolo Aschieri for helpful discussions on duality rotations. Support from DFG-SFB-375, from GIF, the German-Israeli foundation for Scientific Research and from the EEC under TMR contract ERBFMRX-CT96-0045 is gratefully acknowledged. This work was also supported in part by the NATO collaborative research grant PST.CLG 974965, by the RFBR grant No. 99-02-16617, by the INTAS grant No. 96-0308 and by the DFG-RFBR grant No. 99-02-04022.

Appendix A Derivation of the self-duality equation

Eq. (2.9) is derived as follows. For an infinitesimal U(1) duality rotation, we have

$$\tilde{G}'_{ab}(F') = \tilde{G}_{ab}(F) - \lambda \tilde{F}_{ab} = \tilde{G}_{ab}(F) + 2\frac{\partial}{\partial F^{ab}} \left(-\frac{1}{4}\lambda\, F \cdot \tilde{F} \right) \ , \tag{A.1}$$

where we have used the infinitesimal version of eq.(2.8). At the same time, from the definition of $\tilde{G}'(F')$ it follows

$$\tilde{G}'(F') = 2\, \frac{\partial L(F')}{\partial F'} = 2 \left(\frac{\partial}{\partial F'}\, L(F) + \frac{\partial}{\partial F}\, \delta L \right) \ , \tag{A.2}$$

where

$$\delta L = L(F') - L(F) \ . \tag{A.3}$$

Using $F' = F + \lambda G$, one can express $\partial/\partial F'$ on the right-hand side of (A.2) via $\partial/\partial F$ with the result

$$\tilde{G}'_{ab}(F') = \tilde{G}_{ab}(F) + 2\frac{\partial}{\partial F^{ab}} \left(\delta L - \frac{1}{4}\lambda\, G \cdot \tilde{G} \right) \ . \tag{A.4}$$

Comparing eqs. (A.1) and (A.4) gives

$$\delta L = \frac{1}{4}\lambda \left(G \cdot \tilde{G} - F \cdot \tilde{F} \right) \ . \tag{A.5}$$

On the other hand, the Lagrangian can be varied directly to give

$$\delta L = \frac{\partial L}{\partial F^{ab}}\, \delta F^{ab} = \frac{1}{2}\lambda\, \tilde{G} \cdot G. \tag{A.6}$$

This is consistent with eq. (A.5) iff the self-duality equation (2.9) holds.

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