q-alg/9602020

The dual (p, q)-Alexander-Conway Hopf algebras and the associated universal $\mathcal{T}$-matrix

R. Chakrabarti† and R. Jagannathan‡§

† Department of Theoretical Physics, University of Madras, Guindy Campus, Madras-600025, INDIA

‡ The Institute of Mathematical Sciences, C.I.T. Campus, Tharamani, Madras-600113, INDIA

Abstract: The dually conjugate Hopf algebras $Fun_{p,q}(R)$ and $U_{p,q}(R)$ associated with the two-parametric (p, q)-Alexander-Conway solution (R) of the Yang-Baxter equation are studied. Using the Hopf duality construction, the full Hopf structure of the quasitriangular enveloping algebra $U_{p,q}(R)$ is extracted. The universal $\mathcal{T}$-matrix for $Fun_{p,q}(R)$ is derived. While expressing an arbitrary group element of the quantum group characterized by the noncommuting parameters in a representation independent way, the $\mathcal{T}$-matrix generalizes the familiar exponential relation between a Lie group and its Lie algebra. The universal $\mathcal{R}$-matrix and the FRT matrix generators, $L^{(\pm)}$, for $U_{p,q}(R)$ are derived from the $\mathcal{T}$-matrix.

1. Introduction

The quantum Yang-Baxter equation (QYBE) admits nonstandard solutions [1]-[5] characterizing quasitriangular Hopf algebras which are not deformations of classical algebras. The nonstandard quantum algebras associated with the Alexander-Conway solution of the QYBE has been studied ([1]-[5]) and its two-parametric generalization has been obtained [6, 7]. These algebras have the interesting property that, by the technique of superization [4], they can be associated with the graded quantized universal enveloping algebra (QUEA) $U_{\langle q\rangle}(gl(1|1))$, where $\langle q\rangle$ represents the set of pertinent deformation parameters. Interestingly, the quantum group realization of the Alexander polynomial was obtained [8, 9] from the algebra $U_q(gl(1|1))$ while yielding a free fermion model for the invariant. In addition, for the nongeneric values of the deformation parameters, the nonstandard R-matrices may be engendered using a coloured, generalized boson realization [5, 7] of the universal $\mathcal{R}$-matrices of the corresponding standard ungraded QUEA $U_{\langle q\rangle}(gl(2))$.

§E-mail: jagan@imsc.ernet.in

The FRT-construction [10] associates to any solution R of the QYBE a quantum matrix pseudogroup defined by the transfer matrix T. The elements of T generate the function algebra $Fun_{\langle q\rangle}(R)$. The QUEA $U_{\langle q\rangle}(R)$, dually conjugate to $Fun_{\langle q\rangle}(R)$ in the Hopf sense, may now be obtained [4] if $Fun_{\langle q\rangle}(R)$ contains a group-like, 'quantum determinent' type, element and some suitable ansatz for the matrix generators $L^{\pm}$ of $U_{\langle q\rangle}(R)$ exists. The significance of a key notion capping the Hopf duality structure [10] was recently highlighted by Frønsdal and Galindo [11] in the context of $Fun_{p,q}(GL(2))$ and $U_{p,q}(gl(2))$: In a representation independent way, they derived a closed expression for the dual form

$$\mathcal{T} = \sum e^A E_A = \mathcal{T}_{e,E}\,, \tag{1.1}$$

called the universal $\mathcal{T}$-matrix, and established it as the quantum group generalization of the familiar exponential map obtaining in the case of classical groups (see also [12, 13]). Here, the sets $\left\{e^A\right\}$ and $\{E_A\}$ are the respective basis elements of the dual Hopf algebras $\mathcal{A} = Fun_{\langle q\rangle}(R)$ and $\mathcal{U} = U_{\langle q\rangle}(R)$, satisfying the relation

$$\left\langle e^A\,,\, E_B\right\rangle = \delta^A_B\,, \tag{1.2}$$

where $\langle\,,\,\rangle$ is a doubly nondegenerate bilinear form. Conversely, as in the classical case, an infinitesimal analysis of the quantum group elements would lead to the quantum algebra, as has been demonstrated by Finkelstein [14] in the example of $Fun_q(GL(2))$. When the algebras $Fun_{\langle q\rangle}(R)$ and $U_{\langle q\rangle}(R)$ are finitely generated, a closed expression of the $\mathcal{T}$-matrix may be obtained in terms of the two sets of generators. The $\mathcal{T}$-matrix expresses a representation-free realization of a quantum group element depending on the noncommuting group parameters. As pointed out in ([11]-[13]), the main usefulness of the $\mathcal{T}$-matrix derive from the fact that the transition matrices of the integrable models appear, upon specialization, in passing from the structure to the representations.

The technique adopted in ([11]-[13]) may be summarized as follows. The structure relations embodying the duality between the conjugate Hopf algebras may be expressed succinctly in terms of the $\mathcal{T}$-matrix as

$$\begin{aligned}
\mathcal{T}_{e,E}\mathcal{T}_{e',E} &= \mathcal{T}_{\Delta(e),E}\,, \quad \mathcal{T}_{e,E}\mathcal{T}_{e,E'} = \mathcal{T}_{e,\Delta(E)}\,,\\
\mathcal{T}_{\epsilon(e),E} &= \mathbb{1}\,, \quad \mathcal{T}_{e,\epsilon(E)} = \mathbb{1}\,,\\
\mathcal{T}_{S(e),E} &= \mathcal{T}^{-1}\,, \quad \mathcal{T}_{e,S(E)} = \mathcal{T}^{-1}\,,
\end{aligned} \tag{1.3}$$

where e and e' (E and E') refer to two identical copies of $Fun_{\langle q\rangle}(R)$ ($U_{\langle q\rangle}(R)$) and Δ, ϵ and S denote, respectively, the coproduct, counit and the antipode maps.

Using the known Hopf structure of $Fun_{\langle q\rangle}(R)$, the dual Hopf structure of the QUEA $U_{\langle q\rangle}(R)$ may be read from (1.3). When both the Hopf algebras are finitely generated, the basis elements $\left\{e^A\right\}$ and $\{E_A\}$ may be expressed as ordered monomials in the respective sets of generators. A closed expression of the dual form may now be explicitly obtained in terms of the two sets of generators.

Using the q-exponentials, an explicit expression for the $\mathcal{T}$-matrix corresponding to the dual Hopf algebras $Fun_{p,q}(GL(2))$ and $U_{p,q}(gl(2))$ was first obtained in [11]. The dual forms for the standard quantum $gl(n)$ and the twisted quantum $gl(n)$ were considered in [12] and [13], respectively. Following the Frønsdal-Galindo approach, Bonechi *et al.* [15] derived the $\mathcal{T}$-matrices for some inhomogeneous quantum groups. Morozov and Vinet [16] constructed these generalized exponential maps for all standard simple quantum groups with a single deformation parameter. The dual super-Hopf algebras $Fun_{p,q}(GL(1|1))$ and $U_{p,q}(gl(1|1))$ were studied in [17] using the above approach. In another developement, the (p,q)-generalization of the Wigner d-functions have been obtained [18] using the finite dimensional representations of the $\mathcal{T}$-matrix for the algebra $Fun_{p,q}(GL(2))$. With a view to provide further concrete examples of the universal $\mathcal{T}$-matrix we consider here the Hopf duality structure of the (p,q)-Alexander-Conway algebras and obtain the associated universal $\mathcal{T}$-matrix.

The universal $\mathcal{R}$-matrix for the (p,q)-Alexander-Conway algebra is known [7] and we show here that it may be obtained from the universal $\mathcal{T}$-matrix via a homomorphic map $\phi\ :Fun_{p,q}(R)\longrightarrow U_{p,q}(R)$ using the relation

$$(\mathrm{id}\otimes\phi)\mathcal{T}=\mathcal{R}\,. \qquad (1.4)$$

Such a map was first used in [13] in the context of quantum $gl(n)$. It is then obvious that in view of the link between the universal $\mathcal{R}$-matrix and the FRT matrix generators $L^{(\pm)}$ [10] it should be possible to obtain $L^{(\pm)}$ directly from the $\mathcal{T}$-matrix. Such a procedure has already been demonstrated in [19] in the case of $U_q(sl(2))$. We shall exhibit here a similar derivation of $L^{(\pm)}$-matrices for $U_{p,q}(R)$ from the $\mathcal{T}$-matrix.

2. Hopf structure of $Fun_{p,q}(R)$

We study the Hopf algebra associated with the two-parametric nonstandard solution [6, 7] of the QYBE, namely,

$$R=\begin{pmatrix} Q & 0 & 0 & 0\\ 0 & \lambda^{-1} & s & 0\\ 0 & 0 & \lambda & 0\\ 0 & 0 & 0 & -Q^{-1}\end{pmatrix}, \qquad s=Q-Q^{-1}\,. \qquad (2.1)$$

The defining relation of the quantum inverse scattering method [10],

$$\sum_{m,n=1,2} R_{im,jn} T_{mk} T_{nl} = \sum_{m,n=1,2} T_{jn} T_{im} R_{mk,nl} \,, \tag{2.2}$$

with the R-matrix given by (2.1), describes a transfer matrix

$$T = [t_{ij}] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{2.3}$$

whose variable elements obey the braiding relations

$$\begin{aligned} ab &= p^{-1} ba \,, \quad ac = q^{-1} ca \,, \quad db = -p^{-1} bd \,, \quad dc = -q^{-1} cd \,, \\ p^{-1} bc &= q^{-1} cb \,, \quad ad - da = (p^{-1} - q) bc \,, \quad b^2 = 0 \,, \quad c^2 = 0 \,, \end{aligned} \tag{2.4}$$

where

$$p = \lambda Q \,, \qquad q = \lambda^{-1} Q \,. \tag{2.5}$$

If the diagonal elements of T, a and d, are invertible, the elements $\{a, b, c, d, a^{-1}, d^{-1}\}$ generate a Hopf algebra $Fun_{p,q}(R)$ whose coalgebraic structure readily follows. The coproduct, counit and the antipode maps are, respectively, given by

$$\begin{aligned} \Delta(T) &= T \dot{\otimes} T \,, \\ \Delta(a^{-1}) &= a^{-1} \otimes a^{-1} - a^{-1} b a^{-1} \otimes a^{-1} c a^{-1} \,, \\ \Delta(d^{-1}) &= d^{-1} \otimes d^{-1} - d^{-1} c d^{-1} \otimes d^{-1} b d^{-1} \,, && (2.6) \\ \epsilon(T) &= \mathbb{1} && (2.7) \\ S(T) &= T^{-1} \,, \quad S(a^{-1}) = a - b d^{-1} c \,, \quad S(d^{-1}) = d - c a^{-1} b \,, && (2.8) \end{aligned}$$

where

$$T^{-1} = \begin{pmatrix} a^{-1} + a^{-1} b d^{-1} c a^{-1} & -a^{-1} b d^{-1} \\ -d^{-1} c a^{-1} & d^{-1} + d^{-1} c a^{-1} b d^{-1} \end{pmatrix} \tag{2.9}$$

and $\dot{\otimes}$ denotes the tensor product coupled with matrix multiplication. Despite the appearance of the relations ($b^2 = 0$, $c^2 = 0$) in (2.4), suggestive of a superalgebraic structure, the Hopf algebra $Fun_{p,q}(R)$ is bosonic as it follows the tensor product rule

$$(\Gamma_1 \otimes \Gamma_2)(\Gamma_3 \otimes \Gamma_4) = \Gamma_1 \Gamma_3 \otimes \Gamma_2 \Gamma_4 \,, \quad \forall \;\; \Gamma \in Fun_{p,q}(R) \,. \tag{2.10}$$

In the Hopf algebra $Fun_{p,q}(R)$ an invertible group-like element D exists:

$$D = a d^{-1} - b d^{-1} c d^{-1} \,, \qquad D^{-1} = d a^{-1} - b a^{-1} c a^{-1} \,. \tag{2.11}$$

Using (2.4), the commutation relations for D follow:

$$[D,a]=0\,,\quad [D,d\,]=0\,,\quad \{D,b\}=0\,,\quad \{D,c\}=0\,. \tag{2.12}$$

The induced coalgebra maps for D are

$$\Delta(D)=D\otimes D\,,\quad \epsilon(D)=1\,,\quad S(D)=D^{-1}\,, \tag{2.13}$$

as obtained from the relations (2.6-2.9).

A Gauss decomposition of the T-matrix (2.3) as

$$T=\begin{pmatrix}1 & 0\\ \zeta & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & \hat{d}\end{pmatrix}\begin{pmatrix}1 & \xi\\ 0 & 1\end{pmatrix} \tag{2.14}$$

introduces new variables related to the old ones by

$$b=a\xi\,,\qquad c=\zeta a\,,\qquad d=\zeta a\xi+\hat{d}\,, \tag{2.15}$$

where $\hat{d}$ is an invertible element. The element D and its inverse now read

$$D=a\hat{d}^{-1}\,,\qquad D^{-1}=\hat{d}a^{-1}\,. \tag{2.16}$$

The algebra (2.4) assumes the form

$$\begin{aligned} a\xi &= p^{-1}\xi a\,,\quad a\zeta=q^{-1}\zeta a\,,\quad \hat{d}\xi=-p^{-1}\xi\hat{d}\,,\quad \hat{d}\zeta=-q^{-1}\zeta\hat{d}\,,\\ [a,\hat{d}\,] &= 0\,,\quad [\xi,\zeta]=0\,,\quad \xi^2=0\,,\quad \zeta^2=0\,. \end{aligned} \tag{2.17}$$

The coalgebra maps (2.6-2.9) are rewritten as

$$\begin{aligned} \Delta(a) &= a\otimes a+a\xi\otimes\zeta a\,,\qquad \Delta(\xi)=\mathbb{1}\otimes\xi+\xi\otimes D^{-1}\,,\\ \Delta(\zeta) &= \zeta\otimes\mathbb{1}+D^{-1}\otimes\zeta\,,\qquad \Delta(\hat{d})=\hat{d}\otimes\hat{d}-\hat{d}\xi\otimes\zeta\hat{d}\,, \end{aligned} \tag{2.18}$$

$$\epsilon(a) = 1\,,\quad \epsilon(\xi)=0\,,\quad \epsilon(\zeta)=0\,,\quad \epsilon(\hat{d})=1\,, \tag{2.19}$$

$$\begin{aligned} S(a) &= a^{-1}+\xi a^{-1}D\zeta\,,\qquad S(\xi)=-\xi D\,,\\ S(\zeta) &= -D\zeta\,,\qquad S(\hat{d})=\hat{d}^{-1}-\xi\hat{d}^{-1}D\zeta\,. \end{aligned} \tag{2.20}$$

Assuming that the algebra can be augmented with the logarithms of a and $\hat{d}$, we use the map

$$a=\mathrm{e}^{x}\,,\qquad \hat{d}=\mathrm{e}^{\hat{x}}\,, \tag{2.21}$$

and a reparametrization

$$p=\mathrm{e}^{-\omega}\,,\qquad q=\mathrm{e}^{-\nu} \tag{2.22}$$

to convert (2.17) to the algebraic structure

$$\begin{aligned}
[x, \xi] &= \omega\xi\,, \quad [x, \zeta] = \nu\zeta\,, \quad [\hat{x}, \xi] = \hat{\omega}\xi\,, \quad [\hat{x}, \zeta] = \hat{\nu}\zeta\,, \\
& \qquad\qquad \hat{\omega} = \omega + \mathrm{i}\pi\,, \quad \hat{\nu} = \nu + \mathrm{i}\pi \\
[x, \hat{x}] &= 0\,, \quad [\xi, \zeta] = 0\,, \quad \xi^2 = 0\,, \quad \zeta^2 = 0\,,
\end{aligned} \tag{2.23}$$

where the choice of identical phases for $\hat{\omega}$ and $\hat{\nu}$ is necessitated by the requirement that the single deformation parameter limit ($p = q$) exists and, via a duality construction, yields the q-Alexander-Conway algebra ([1]-[5]). Following the technique of Ref. [11], the coalgebra maps for (2.23) are seen to be

$$\begin{aligned}
\Delta(x) &= x \otimes \mathbb{1} + \mathbb{1} \otimes x + \frac{\nu + \omega}{\mathrm{e}^{\nu} - \mathrm{e}^{-\omega}} \xi \otimes \zeta\,, \\
\Delta(\xi) &= \mathbb{1} \otimes \xi + \xi \otimes \mathrm{e}^{\hat{x} - x}\,, \\
\Delta(\zeta) &= \zeta \otimes \mathbb{1} + \mathrm{e}^{\hat{x} - x} \otimes \zeta\,, \\
\Delta(\hat{x}) &= \hat{x} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{x} + \frac{\nu + \omega + \mathrm{i}2\pi}{\mathrm{e}^{\nu} - \mathrm{e}^{-\omega}} \xi \otimes \zeta\,, && (2.24) \\
\epsilon(x) &= \epsilon(\xi) = \epsilon(\zeta) = \epsilon(\hat{x}) = 0\,, && (2.25) \\
S(x) &= -x + \frac{\nu + \omega}{\mathrm{e}^{\nu} - \mathrm{e}^{-\omega}} \xi \mathrm{e}^{x - \hat{x}} \zeta\,, \qquad S(\xi) = -\xi \mathrm{e}^{x - \hat{x}}\,, \\
S(\zeta) &= -\mathrm{e}^{x - \hat{x}} \zeta\,, \qquad S(\hat{x}) = -\hat{x} + \frac{\nu + \omega + \mathrm{i}2\pi}{\mathrm{e}^{\nu} - \mathrm{e}^{-\omega}} \xi \mathrm{e}^{x - \hat{x}} \zeta\,. && (2.26)
\end{aligned}$$

The above Hopf structure suggests that $Fun_{p,q}(R)$ may be embedded in the enveloping algebra of $(x, \hat{x}, \xi, \zeta)$ satisfying the algebraic relations (2.23) and endowed with a noncocommutative coproduct rule. This enveloping algebra is dual to the Hopf algebra $U_{p,q}(R)$.

All the Hopf algebra axioms for the coalgebra maps (2.24-2.26) can be explicitly proved. To this end, the following identities are to be noted. If X and $\tilde{X}$ are elements satisfying

$$[x, \xi] = c_{\xi}\xi\,, \quad [X, \zeta] = c_{\zeta}\zeta\,, \quad [X, \tilde{X}] = 0 \tag{2.27}$$

and C is an arbitrary c-number, we have

$$\begin{aligned}
\exp(X \otimes \mathbb{1} + \mathbb{1} \otimes X + C\xi \otimes \zeta) &= \left(\mathrm{e}^{X} \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes \mathbb{1} + \tilde{C}\xi \otimes \zeta\right)\left(\mathbb{1} \otimes \mathrm{e}^{X}\right) && (2.28) \\
\exp\left(-X + \xi\tilde{X}\zeta\right) &= \mathrm{e}^{X} + \Lambda\xi\mathrm{e}^{-X}\tilde{X}\zeta && (2.29)
\end{aligned}$$

where $\Lambda = \frac{\mathrm{e}^{c_{\zeta}} - \mathrm{e}^{-c_{\xi}}}{c_{\zeta} + c_{\xi}}$, $\tilde{C} = C\Lambda$. From the above identities and (2.23), it follows that the one parametric set of elements

$$D(\theta) = \mathrm{e}^{\theta(x - \hat{x})} \tag{2.30}$$

satisfies a group-like property

$$\Delta(D(\theta)) = D(\theta) \otimes D(\theta)\,, \qquad S(D(\theta)) = D(\theta)^{-1} \tag{2.31}$$

for any $\theta \in \mathbb{Z}$. It is interesting to note that the elements $\{D(\theta)|\theta \in \mathbb{Z}\}$ form a discrete group that cannot be embedded in a continuous one-parameter group.

Before proceeding to derive the structure of $U_{p,q}(R)$ and construct the universal $\mathcal{T}$-matrix, let us note an interesting application of the characterization of the elements of the T-matrix (2.3) through the algebraic algebraic structure (2.23). Let $\left\{\left(a_1\,,\,a_1^\dagger\right)\,,\,\left(a_2\,,\,a_2^\dagger\right)\right\}$ and $\left\{\left(b_1\,,\,b_1^\dagger\right)\,,\,\left(b_2\,,\,b_2^\dagger\right)\right\}$ be fermion operators corresponding to two disparate, commuting, Fermi fields; they may be the basic operators from which a pair of 2-nd order para-Fermi operators can be obtained through the well known Green's ansatz [20], or they can be considered as belonging to two Fermi fields of different colours. Such commuting Fermi fields can also be constructed from a single Fermi field [21]. The algebra of these (a,b)-operators is

$$\begin{aligned}
\left\{a_i, a_j^\dagger\right\} &= \delta_{ij}\,, \quad \{a_i, a_j\} = 0\,, \quad i,j = 1,2\,, \\
\left\{b_i, b_j^\dagger\right\} &= \delta_{ij}\,, \quad \{b_i, b_j\} = 0\,, \quad i,j = 1,2\,, \\
[a_i, b_j] &= 0\,, \quad \left[a_i, b_j^\dagger\right] = 0\,, \quad i,j = 1,2\,.
\end{aligned} \tag{2.32}$$

Using the Bogoliubov transformation, we define

$$\begin{aligned}
\xi &= (\cos\alpha)a_1 + (\sin\alpha)a_2^\dagger \quad \left(\text{ or } \ (\cos\alpha)a_2 - (\sin\alpha)a_1^\dagger\ \right)\,, \\
\zeta &= (\cos\beta)b_1 + (\sin\beta)b_2^\dagger \quad \left(\text{ or } \ (\cos\beta)b_2 - (\sin\beta)b_1^\dagger\ \right)\,.
\end{aligned} \tag{2.33}$$

Then, we can take

$$x = \omega\xi^\dagger\xi + \nu\zeta^\dagger\zeta + f(\alpha,\beta)\,, \qquad \hat{x} = \hat{\omega}\xi^\dagger\xi + \hat{\nu}\zeta^\dagger\zeta + g(\alpha,\beta)\,, \tag{2.34}$$

where $f(\alpha,\beta)$ and $g(\alpha,\beta)$ are two arbitrary c-number functions of α and β. Now, it is seen that, using (2.15), (2.22) and (2.32)-(2.34), it is possible to realize the variable group element (2.3) parametrized by the classical variables α and β. A similar realization of $GL_{p,q}(1|1)$ in terms of a fermion field was obtained in [17] following [18] in which $GL_{p,q}(2)$ was realized in terms of a boson field.

3. Dual Hopf algebra $U_{p,q}(R)$ and the universal $\mathcal{T}$-matrix

The monomials $\{e^A|e^A = \zeta^{a_1}x^{a_2}\hat{x}^{a_3}\xi^{a_4},\ A = (a_1,a_2,a_3,a_4),\ a_1,a_4 = (0,1),\ a_2,a_3 = 0,1,2,\ldots\}$ constitute a basis for $\mathcal{A} = Fun_{p,q}(R)$ obeying the multiplication and the induced coalgebra maps

$$e^A e^B = \sum_C f_C^{AB} e^C , \tag{3.1}$$

$$\Delta(e^A) = \sum_{BC} h_{BC}^A e^B \otimes e^C , \quad \epsilon\left(e^A\right) = \delta_{\underline{0}}^A , \quad S\left(e^A\right) = \sum_B S_B^A e^B . \tag{3.2}$$

The unit element is obtained by choosing $A = \underline{0}$, where $\underline{0} = (0,0,0,0)$. The elements $\{E_A\}$, defined by (1.2), form a basis set for the algebra $\mathcal{U}$, dual to $\mathcal{A}$, namely, the QUEA $U_{p,q}(R)$. The duality construction (1.3) enforces the following Hopf structures for the basis set $\{E_A\}$:

$$E_A E_B = \sum_C h_{AB}^C E_C , \tag{3.3}$$

$$\Delta(E_A) = \sum_{BC} f_A^{BC} E_B \otimes E_C . \tag{3.4}$$

Using the algebra (2.23) the structure tensor f_C^{AB} is derived:

$$\begin{aligned} f_C^{AB} = \; & \bar{\delta}^{a_1 b_1} \bar{\delta}^{a_4 b_4} \delta_{c_1}^{a_1+b_1} \theta_{c_2}^{a_2+b_2} \theta_{c_3}^{a_3+b_3} \delta_{c_4}^{a_4+b_4} \\ & \times \sum_{kl} \begin{pmatrix} a_2 \\ k \end{pmatrix} \begin{pmatrix} b_2 \\ c_2 - k \end{pmatrix} \begin{pmatrix} a_3 \\ l \end{pmatrix} \begin{pmatrix} b_3 \\ c_3 - l \end{pmatrix} \\ & \times (\nu b_1)^{a_2-k} (-\omega a_4)^{b_2-c_2+k} ((\nu + \mathrm{i}\pi) b_1)^{a_3-l} (-(\omega + \mathrm{i}\pi) a_4)^{b_3-c_3+l} , \end{aligned} \tag{3.5}$$

where $\bar{\delta}^{ab} = \delta_0^a \delta_0^b + \delta_1^a \delta_0^b + \delta_0^a \delta_1^b$ and $\theta_b^a = 1\,(0)$ if $a \geq b$ ($\leq b$). The tensor h_{BC}^A is determined using the induced coproduct for the basis set $\{e^A\}$, namely,

$$\Delta(e^A) = \Delta(\zeta)^{a_1} \Delta(x)^{a_2} \Delta(\hat{x})^{a_3} \Delta(\xi)^{a_4} . \tag{3.6}$$

The following special cases, necessary for determining the dual algebraic structure, may be directly read from (3.6):

$$h_{B\underline{0}}^A = \delta_B^A , \qquad h_{\underline{0}B}^A = \delta_B^A , \tag{3.7a}$$

$$h_{0b_2b_300c_2c_30}^A = \delta_0^{a_1} \delta_{b_2+c_2}^{a_2} \delta_{b_3+c_3}^{a_3} \delta_0^{a_4} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} , \tag{3.7b}$$

$$h_{1000B}^A = \delta_1^{a_1} \delta_{b_1}^0 \prod_{i=2}^4 \delta_{b_i}^{a_i} , \qquad h_{B0001}^A = \left(\prod_{i=1}^3 \delta_{b_i}^{a_i} \right) \delta_1^{a_4} \delta_{b_4}^0 , \tag{3.7c}$$

$$h_{01001000}^A = \delta_1^{a_1} \delta_1^{a_2} \delta_0^{a_3} \delta_0^{a_4} - \delta_1^{a_1} \prod_{i=2}^4 \delta_0^{a_i} , \tag{3.7d}$$

$$h_{00101000}^A = \delta_1^{a_1} \delta_0^{a_2} \delta_1^{a_3} \delta_0^{a_4} + \delta_1^{a_1} \prod_{i=2}^4 \delta_0^{a_i} , \tag{3.7e}$$

$$h^A_{00010100} = \delta^{a_1}_0 \delta^{a_2}_1 \delta^{a_3}_0 \delta^{a_4}_1 - \left(\prod_{i=1}^{3} \delta^{a_i}_0\right) \delta^{a_4}_1 , \tag{3.7f}$$

$$h^A_{00010010} = \delta^{a_1}_0 \delta^{a_2}_0 \delta^{a_3}_1 \delta^{a_4}_1 + \left(\prod_{i=1}^{3} \delta^{a_i}_0\right) \delta^{a_4}_1 , \tag{3.7g}$$

$$h^A_{00011000} = \delta^{a_1}_1 \delta^{a_2}_0 \delta^{a_3}_0 \delta^{a_4}_1 + \delta^{a_1}_0 \delta^{a_4}_0 \Omega_{a_2,a_3} , \tag{3.7h}$$

where

$$\Omega_{j,k} = \frac{\nu^j(\nu + \mathrm{i}\pi)^k - (-\omega)^j(-\omega - \mathrm{i}\pi)^k}{\mathrm{e}^\nu - \mathrm{e}^{-\omega}} . \tag{3.8}$$

The antipode maps for the basis elements $\{e^A\}$ are obtained from (2.26) :

$$\begin{aligned} S(e^A) &= S(\xi)^{a_4} S(\hat{x})^{a_3} S(x)^{a_2} S(\zeta)^{a_1} \\ &= (-1)^{a_2+a_3} \zeta^{a_1} \left(\hat{x} + a_1(\nu + \mathrm{i}\pi) - a_4(\omega + \mathrm{i}\pi) - \frac{\nu + \omega + \mathrm{i}2\pi}{\mathrm{e}^\nu - \mathrm{e}^{-\omega}} \zeta \mathrm{e}^{x-\hat{x}} \xi \right)^{a_3} \\ &\quad \times \left(x + a_1\nu - a_4\omega - \frac{\nu + \omega}{\mathrm{e}^\nu - \mathrm{e}^{-\omega}} \zeta \mathrm{e}^{x-\hat{x}} \xi \right)^{a_2} \mathrm{e}^{(a_1+a_4)(x-\hat{x})} \xi^{a_4} . \end{aligned} \tag{3.9}$$

The second equality in (3.9) follows by using the commutation relations (2.23) and will be later used to compute the antipode maps for the dual basis elements.

Employing the duality property, we now extract the multiplication relations for the dual basis elements $\{E_A\}$. From (3.3) and (3.7a) the unit element follows directly:

$$E_A E_{\underline{0}} = E_A , \quad E_{\underline{0}} E_A = E_A \quad \longrightarrow \quad E_{\underline{0}} = 1 . \tag{3.10}$$

By choosing the generators of the dual algebra as

$$E_- = E_{1000} , \qquad H = E_{0100} , \qquad \hat{H} = E_{0010} , \qquad E_+ = E_{0001} , \tag{3.11}$$

and using the tensor structure (3.7b) and (3.7c) we express an arbitrary dual basis element as

$$E_A = (a_2! a_3!)^{-1} E_-^{a_1} H^{a_2} \hat{H}^{a_3} E_+^{a_4} . \tag{3.12}$$

Further use of the special values of the structure tensor h^A_{BC} in (3.7) now yields the commutation relations for the generators of the dual alglebra $U_{p,q}(R)$:

$$\begin{aligned} [H, E_\pm] &= \pm E_\pm , \quad [\hat{H}, E_\pm] = \mp E_\pm , \quad [H, \hat{H}] = 0 , \quad E^2_\pm = 0 , \\ [E_+, E_-] &= \frac{\mathrm{e}^{\nu(H+\hat{H})} g^{-1} - \mathrm{e}^{-\omega(H+\hat{H})} g}{\mathrm{e}^\nu - \mathrm{e}^{-\omega}} , \end{aligned} \tag{3.13}$$

where

$$g = \mathrm{e}^{-\mathrm{i}\pi\hat{H}} . \tag{3.14}$$

The algebra (3.13) may now be exploited to compute the general expression for the structure tensor h^A_{BC}, which reads

$$\begin{aligned} h^A_{BC} &= (-1)^{b_2+c_2-a_2}\bar{\delta}^{b_1c_1}\bar{\delta}^{b_4c_4}\delta^{b_1+c_1}_{a_1}\theta^{b_2+c_2}_{a_2}\theta^{b_3+c_3}_{a_3}\delta^{b_4+c_4}_{a_4} \\ &\times a_2!a_3!(b_2!b_3!c_2!c_3!)^{-1}\sum_{kl}\begin{pmatrix} b_2 \\ k \end{pmatrix}\begin{pmatrix} c_2 \\ a_2-k \end{pmatrix}\begin{pmatrix} b_3 \\ l \end{pmatrix}\begin{pmatrix} c_3 \\ a_3-l \end{pmatrix} \\ &\times c_1^{b_2+b_3-k-l}b_4^{c_2+c_3-a_2-a_3+k+l} + \delta^{a_1}_{b_1}\delta^{b_4}_1\delta^{c_1}_1\delta^{a_4}_{c_4}a_2!a_3! \\ &\times (b_2!c_2!b_3!c_3!(a_2-b_2-c_2)!(a_3-b_3-c_3)!)^{-1}\Omega_{a_2-b_2-c_2,a_3-b_3-c_3}\,. \end{aligned} \tag{3.15}$$

The coproduct rules for the generators of the dual algebra $U_{p,q}(R)$ are obtained from (3.4) and (3.5):

$$\begin{aligned} \Delta(H) &= H\otimes\mathbb{1}+\mathbb{1}\otimes H\,, \qquad \Delta(\hat{H}) = \hat{H}\otimes\mathbb{1}+\mathbb{1}\otimes\hat{H}\,, \\ \Delta(E_+) &= E_+\otimes \mathrm{e}^{-\omega(H+\hat{H})}g^{-1}+\mathbb{1}\otimes E_+\,, \quad \Delta(E_-) = E_-\otimes\mathbb{1}+\mathrm{e}^{\nu(H+\hat{H})}g\otimes E_-\,. \end{aligned} \tag{3.16}$$

The counit maps for the dual generators follow from (3.4) :

$$\epsilon(X) = 0\,, \qquad \forall\, X\in(H,\hat{H},E_\pm)\,. \tag{3.17}$$

Special values of the tensor $\left\{S^A_B\right\}$, obtained from (3.9), yields, via (3.4), the antipode maps for the dual generators :

$$\begin{aligned} S(H) &= -H\,, \qquad S(\hat{H}) = -\hat{H}\,, \\ S(E_+) &= g^{-1}\mathrm{e}^{\omega(H+\hat{H})}E_+\,, \qquad S(E_-) = E_-\mathrm{e}^{-\nu(H+\hat{H})}g\,. \end{aligned} \tag{3.18}$$

For the single deformation parameter case ($\nu=\omega$), the Hopf structure (3.13, 3.16, 3.17, and 3.18) for the dual algebra $U_{p,q}(R)$, after appropriate mappings, reduce to the results obtained in [4] using the FRT construction [10]. Following [4], we note that the element g has a group-like coalgebra structure

$$\Delta(g) = g\otimes g\,, \quad \epsilon(g)=\mathbb{1}\,, \quad S(g)=g^{-1} \tag{3.19}$$

and the element g^2 is central. By superization map [4] the super-Hopf algebra $U_{p,q}(gl(1|1))$ may be realized from the quotient algebra $U_{p,q}(R)/g^2-\mathbb{1}$. After a redefinition of the generators

$$Z=\frac{1}{2}(H+\hat{H})\,, \quad J=\frac{1}{2}(H-\hat{H})\,, \quad \chi_\pm = E_\pm Q^{\mp Z}\lambda^{-Z+\frac{1}{2}}\,, \tag{3.20}$$

the structure of the Hopf algebra $U_{p,q}(R)$ assumes the form

$$\begin{aligned}
[J,\chi_\pm] &= \pm\chi_\pm\,, \quad [\chi_+,\chi_-] = \frac{Q^{2Z}g - Q^{-2Z}g^{-1}}{Q-Q^{-1}}\,, \quad \chi_\pm^2 = 0\,, \\
[Z,X] &= 0\,, \qquad \forall\, X \in (J,\chi_\pm)\,, \qquad (3.21)\\
\Delta(Z) &= Z\otimes \mathbb{1} + \mathbb{1}\otimes Z\,, \qquad \Delta(J) = J\otimes\mathbb{1} + \mathbb{1}\otimes J\,, \\
\Delta(\chi_+) &= \chi_+ \otimes Q^Z\lambda^Z g + Q^{-Z}\lambda^{-Z}\otimes\chi_+\,, \\
\Delta(\chi_-) &= \chi_-\otimes Q^Z\lambda^{-Z} + Q^{-Z}\lambda^Z g^{-1}\otimes\chi_-\,, \qquad (3.22)\\
\epsilon(X) &= 0\,, \qquad \forall\, X\in (Z,J,\chi_\pm)\,, \qquad (3.23)\\
S(Z) &= -Z\,, \quad S(J) = -J\,, \quad S(\chi_+) = g^{-1}\chi_+\,, \quad S(\chi_-) = \chi_- g\,. \qquad (3.24)
\end{aligned}$$

In spite of the nonstandard commutation relations (3.21) the Hopf algebra $U_{p,q}(R)$ is bosonic as it obeys the direct product rule (2.10). The algebra $U_{p,q}(R)$ is quasi-triangular with the universal $\mathcal{R}$-matrix given by [7]

$$\mathcal{R} = (-1)^{(Z-J)\otimes(Z-J)} Q^{2(Z\otimes J + J\otimes Z)}\lambda^{2(Z\otimes J - J\otimes Z)} \mathrm{e}^{sQ^Z\lambda^Z\chi_+\otimes Q^{-Z}\lambda^Z\chi_-}\,. \qquad (3.25)$$

For later use, we note that, in terms of the generators (3.11),

$$\mathcal{R} = (-1)^{\hat{H}\otimes\hat{H}} Q^{\left(H\otimes H - \hat{H}\otimes\hat{H}\right)}\lambda^{-\left(H\otimes\hat{H} - \hat{H}\otimes H\right)}\mathrm{e}^{\lambda s E_+\otimes E_-} \qquad (3.26)$$

Finally, following the prescription (1.1), we now explicitly write down the universal $\mathcal{T}$-matrix of $Fun_{p,q}(R)$:

$$\mathcal{T} = \mathrm{e}^{\zeta E_-}\mathrm{e}^{xH + \hat{x}\hat{H}}\mathrm{e}^{\xi E_+}\,. \qquad (3.27)$$

It may be noted that in contrast to the case of $Fun_{p,q}(GL(2))$, discussed in [11], in the present case the q-exponentials do not appear in the expression for the $\mathcal{T}$-matrix as a consequence of the algebraic property $E_\pm^2 = 0$. Corresponding to the two-dimensional irreducible representation of the generators of $U_{p,q}(R)$ given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{H} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad (3.28)$$

the universal $\mathcal{T}$-matrix (3.27) reduces, as required, to the T-matrix (2.3).

Let us now consider some applications of the $\mathcal{T}$-matrix (3.27). First, we use the relation (1.4) to reproduce the $\mathcal{R}$-matrix (3.26) from the $\mathcal{T}$-matrix (3.27). To this end, the required homomorphism is

$$\begin{aligned}
\phi(x) &= (\ln Q)H - (\ln\lambda)\hat{H}\,, \quad \phi(\hat{x}) = (\ln\lambda)H - (\ln Q - \mathrm{i}\pi)\hat{H}\,, \\
\phi(\xi) &= \lambda s E_-\,, \quad \phi(\zeta) = 0\,, \qquad (3.29)
\end{aligned}$$

which can be proved to satisfy the algebra (2.23) using the relations (3.13). The map (1.4) now gives the $\mathcal{R}$-matrix (3.26) from the corresponding $\mathcal{T}$-matrix (3.27). As noted by Frønsdal [13], in the context of quantum $gl(n)$, there should exist an alternative homomorphism, say $\tilde{\phi}\ : \mathcal{A} \longrightarrow \mathcal{U}$, such that $(\mathrm{id} \otimes \tilde{\phi})\mathcal{T}$ corresponds to the alternative form of the universal $\mathcal{R}$-matrix. We note that in the present case the alternative homomorphism is given by

$$\begin{aligned} \tilde{\phi}(x) &= -(\ln Q)H - (\ln\lambda)\hat{H}\,, \quad \tilde{\phi}(\hat{x}) = (\ln\lambda)H + (\ln Q - \mathrm{i}\pi)\hat{H}\,, \\ \tilde{\phi}(\xi) &= 0\,, \quad \tilde{\phi}(\zeta) = -\lambda s E_+\,. \end{aligned} \tag{3.30}$$

such that, as expected, we get

$$\begin{aligned} (\mathrm{id} \otimes \tilde{\phi})\mathcal{T} &= \tilde{\mathcal{R}} = \mathrm{e}^{-\lambda s E_- \otimes E_+} \lambda^{H \otimes \tilde{H} - \tilde{H} \otimes H} Q^{\tilde{H} \otimes \tilde{H} - H \otimes H} (-1)^{\tilde{H} \otimes \tilde{H}} \\ &= (\sigma(\mathcal{R}))^{-1} = \left(\mathcal{R}^{(+)}\right)^{-1}\,, \end{aligned} \tag{3.31}$$

where $\sigma(a \otimes b) = b \otimes a$.

As is well known [10], $L^{(\pm)}$, the FRT matrix generators of $\mathcal{U}$, can be obtained from the universal $\mathcal{R}$-matrix. Now, since the universal $\mathcal{R}$-matrix is related to the universal $\mathcal{T}$-matrix through a map it is obvious that we can get the $L^{(\pm)}$-matrices also directly from $\mathcal{T}$. Let us demonstrate this procedure in the present case following [19] where the $L^{(\pm)}$-matrices of $U_q(sl(2))$ have been derived directly from the corresponding universal $\mathcal{T}$-matrix. Using the representation (3.28) in (3.29) and (3.30) we get two representations of $\{x, \hat{x}, \xi, \zeta\}$ as follows :

$$\begin{aligned} \pi^+ \ : \quad & x = \begin{pmatrix} \ln Q & 0 \\ 0 & -\ln\lambda \end{pmatrix}, \quad \hat{x} = \begin{pmatrix} \ln\lambda & 0 \\ 0 & -(\ln Q) + \mathrm{i}\pi \end{pmatrix}, \\ & \xi = \begin{pmatrix} 0 & 0 \\ \lambda s & 0 \end{pmatrix}, \quad \zeta = 0\,, \end{aligned} \tag{3.32}$$

$$\begin{aligned} \pi^- \ : \quad & x = \begin{pmatrix} -\ln Q & 0 \\ 0 & -\ln\lambda \end{pmatrix}, \quad \hat{x} = \begin{pmatrix} \ln\lambda & 0 \\ 0 & (\ln Q) - \mathrm{i}\pi \end{pmatrix}, \\ & \xi = 0\,, \quad \zeta = \begin{pmatrix} 0 & -\lambda s \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.33}$$

Now, we obtain the required results :

$$(\pi^+ \otimes \mathbb{1})(\mathcal{T}) = L^{(+)} = \begin{pmatrix} \lambda^{\hat{H}} Q^H & 0 \\ s g^{-1} \lambda^{1-H} Q^{-\hat{H}} E_+ & g^{-1} \lambda^{-H} Q^{-\hat{H}} \end{pmatrix}, \tag{3.34}$$

$$(\pi^- \otimes \mathbb{1})(\mathcal{T}) = L^{(-)} = \begin{pmatrix} \lambda^{\hat{H}} Q^{-H} & -s E_- \lambda^{1-H} Q^{\hat{H}} g \\ 0 & \lambda^{-H} Q^{\hat{H}} g \end{pmatrix}. \tag{3.35}$$

It can be verified directly that these $L^{(\pm)}$-matrices generate the algebra $U_{p,q}(R)$ (3.13) in the FRT-approach [10].

4. Conclusion

We have studied the dually paired Hopf algebras $\mathcal{A} = Fun_{p,q}(R)$ and $\mathcal{U} = U_{p,q}(R)$ associated with the nonstandard R-matrix (2.1) involving two independent parameters. Using the technique developed by Frønsdal and Galindo [11]-[13] we have extracted the full Hopf structure of the algebra $U_{p,q}(R)$. A representation of the quantum group element, known as the universal $\mathcal{T}$-matrix, and associated with the corresponding dual form, is obtained and found to exhibit the suitably modified familiar exponential map relating the Lie group with the corresponding Lie algebra.

Following the construction in [16], the present derivation of the $\mathcal{T}$-matrix may be extended to general nonstandard algebras [22] related to the Hopf superalgebras $U_{\langle q\rangle}(gl(m|n))$. The corresponding nonstandard R-matrices are known [23] to underly the mathematical structure of the Park-Schultz (generalized six vertex) model and its associated quantum spin chains, which are of much current interest [24, 25].

Acknowledgement

We are thankful to the referee for useful comments.

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