Gravitational fields of rotating disks and black holes

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Abstract

The two known exact solutions of Einstein's field equations describing rotating objects of physical significance – a black hole and a rigidly rotating disk of dust – are discussed using a single mathematical framework related to Jacobi's inversion problem. Both solutions can be represented in such a form that they differ in the choice of a complex parameter and a real solution of the axisymmetric Laplace equation only.

A recently found family of solutions describing *differentially* rotating disks of dust fits into the same scheme.

1 Introduction

Infinitesimally thin disks and black holes can be treated by means of the vacuum Einstein equations. In both cases boundary value problems are to be solved, cf. the contribution by G. Neugebauer [1].

In the stationary and axially symmetric case the vacuum Einstein equations are equivalent to the Ernst equation

$$(\Re f)\triangle f = (\nabla f)^2 \tag{1}$$

with $\triangle = \frac{\partial^2}{\partial\varrho^2} + \frac{1}{\varrho}\frac{\partial}{\partial\varrho} + \frac{\partial^2}{\zeta^2}$ and $\nabla = (\frac{\partial}{\partial\varrho}, \frac{\partial}{\partial\zeta})$, where ϱ and ζ are cylindrical (Weyl-) coordinates. (The ζ-axis represents the axis of symmetry.) The full metric can be calculated from the complex Ernst potential $f(\varrho, \zeta)$.

By means of soliton-theoretical techniques it was possible to solve the problem of a rigidly rotating disk of dust in terms of ultraelliptic functions [2, 3]. The mathematical structure of this solution allowed a generalization to a class of solutions related to Jacobi's inversion problem in the general (hyperelliptic) case [4]. These solutions turned out to be closely related to finite-gap solutions of the Ernst equation [5, 6, 7, 8].

In this paper I will discuss a subclass of solutions related to the ultraelliptic case of Jacobi's inversion problem. They contain the Kerr solution describing a rotating black hole, the above mentioned solution found by Neugebauer and Meinel [2] describing a rigidly rotating disk of dust, and a three-parameter family of solutions recently found by Ansorg and Meinel [9] describing differentially rotating disks of dust. In this formulation, the solutions differ in the choice of a complex parameter and a real solution of the axisymmetric

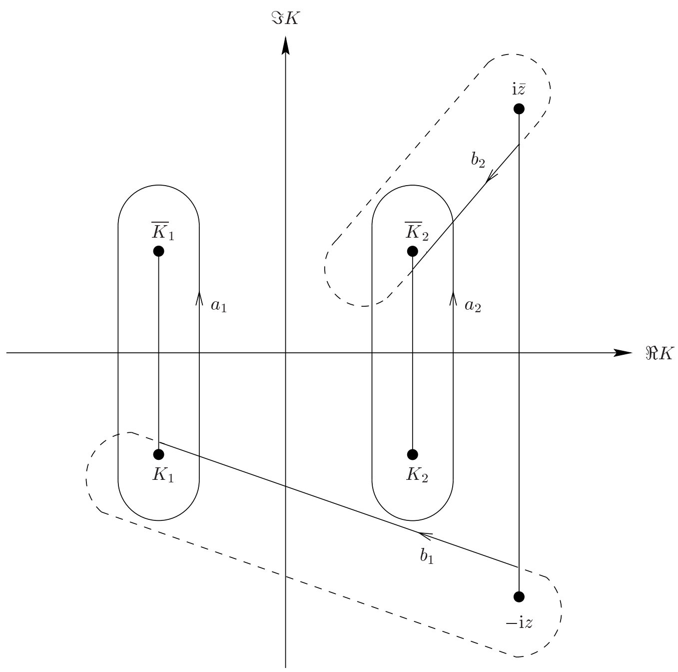

Figure 1: The a- and b-periods. Dotted lines are on the lower sheet of the Riemann surface.

Laplace equation only. This will provide further insight into a certain parameter limit ("ultrarelativistic limit") where the disk solutions coincide with the extreme Kerr solution.

2 Solutions of the Einstein equations related to Jacobi's inversion problem

In [4] it has been shown that

$$f = \exp\left(\int_{K_1}^{K_a} \frac{K^2 dK}{Z} + \int_{K_2}^{K_b} \frac{K^2 dK}{Z} - v_2\right) \tag{2}$$

with

$$Z = \sqrt{(K-K_1)(K-\overline{K}_1)(K-K_2)(K-\overline{K}_2)(K+\mathrm{i}z)(K-\mathrm{i}\bar{z})}, \tag{3}$$

a bar denoting complex conjugation, K_1 and K_2 being arbitrary complex parameters, and

$$z = \varrho + \mathrm{i}\zeta, \tag{4}$$

represents a solution of the Ernst equation (1) if K_a and K_b (and the integration paths) are determined from Jacobi's inversion problem

$$\int\limits_{K_1}^{K_a} \frac{dK}{Z} + \int\limits_{K_2}^{K_b} \frac{dK}{Z} = v_0, \quad \int\limits_{K_1}^{K_a} \frac{KdK}{Z} + \int\limits_{K_2}^{K_b} \frac{KdK}{Z} = v_1, \tag{5}$$

where v_0 is an arbitrary real solution of the (axisymmetric) Laplace equation $\triangle v_0 = 0$ and the real functions v_1 and v_2 satisfy the differential relations

$$\mathrm{i}v_{j,z} = \frac{1}{2}v_{j-1} + zv_{j-1,z}, \quad j = 1, 2. \tag{6}$$

(As a consequence, v_1 and v_2 are solutions of the Laplace equation as well.) The ultraelliptic functions $K_a(v_0, v_1)$ and $K_b(v_0, v_1)$ have four independent periods corresponding to the closed integrals in the two-sheeted Riemann surface related to (3) as indicated in Fig. 1. They are called a- and b-periods according to the integration contours a_1, a_2, b_1, and b_2.

Sometimes the following reformulation of Eqs. (2), (5) proves to be useful:

$$f = \exp\left(\int\limits_{K_b}^{K_a} \frac{K^2dK}{Z} - \tilde{v}_2\right), \quad \int\limits_{K_b}^{K_a} \frac{dK}{Z} = \tilde{v}_0, \quad \int\limits_{K_b}^{K_a} \frac{KdK}{Z} = \tilde{v}_1 \tag{7}$$

with

$$\tilde{v}_j = v_j - \int\limits_{K_1}^{K_2} \frac{K^jdK}{Z}, \quad j = 0, 1, 2. \tag{8}$$

Note that K_b is now on the other sheet of the Riemann surface. Introducing

$$\hat{v}_j = v_j - \Re \int\limits_{K_1}^{K_2} \frac{K^jdK}{Z}, \quad j = 0, 1, 2 \tag{9}$$

and using the obvious relation

$$\Im \int\limits_{K_1}^{K_2} \frac{K^jdK}{Z} = \frac{1}{4\mathrm{i}}\left(\oint\limits_{a_1} \frac{K^jdK}{Z} + \oint\limits_{a_2} \frac{K^jdK}{Z}\right) \tag{10}$$

we obtain

$$\tilde{v}_j = \hat{v}_j - \frac{1}{4}\left(\oint\limits_{a_1} \frac{K^jdK}{Z} + \oint\limits_{a_2} \frac{K^jdK}{Z}\right). \tag{11}$$

It can easily be verified that the real functions $\hat{v}_j$ are solutions of the Laplace equation and satisfy the same recursion relations (6) as v_j. Note that an asymptotically flat solution ($f \to 1$ at infinity) is obtained for $v_j \to 0$ (or $\hat{v}_j \to 0$) at infinity. This condition fixes the integration constants in (6).

In the next section I will discuss physically interesting examples. They differ in the choice of the potential function v_0 (or $\hat{v}_0$) and the parameter K_1. In all cases I assume

$$K_2 = -\overline{K}_1, \quad \Re K_1 \leq 0, \quad \Im K_1 \leq 0. \tag{12}$$

3 Examples

3.1 The rotating black hole

The Kerr solution is obtained for real K_1, i.e.

$$Z = (K - K_1)(K - K_2)\sqrt{(K + \mathrm{i}z)(K - \mathrm{i}\bar{z})}, \tag{13}$$

and

$$\hat{v}_0 = C\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \tag{14}$$

with a positive parameter C and

$$r_k = \sqrt{(K_k + \mathrm{i}z)(K_k - \mathrm{i}\bar{z})}, \quad r_k > 0;\ k = 1, 2. \tag{15}$$

From (6) we obtain

$$\hat{v}_j = C\left(\frac{K_1^j}{r_1} + \frac{K_2^j}{r_2}\right), \quad j = 1, 2. \tag{16}$$

The a-periods in Eq. (11) can easily be calculated by means of the residues at the poles K_1 and K_2 [note that $K_2 = -K_1$ according to (12)]:

$$\oint\limits_{a_1} \frac{K^j dK}{Z} = \frac{\pi \mathrm{i} K_1^{j-1}}{r_1}, \quad \oint\limits_{a_2} \frac{K^j dK}{Z} = \frac{\pi \mathrm{i} K_2^{j-1}}{r_2}. \tag{17}$$

By a suitable combination of Eqs. (7) this leads to

$$f = \exp\left(\int\limits_{K_b}^{K_a} \frac{dK}{\sqrt{(K + \mathrm{i}z)(K - \mathrm{i}\bar{z})}}\right) \tag{18}$$

with

$$\int\limits_{K_b}^{K_a} \frac{dK}{(K - K_1)\sqrt{(K + \mathrm{i}z)(K - \mathrm{i}\bar{z})}} = \frac{1}{r_1}\left(2CK_1 - \frac{\pi \mathrm{i}}{2}\right), \tag{19}$$

$$\int\limits_{K_b}^{K_a} \frac{dK}{(K + K_1)\sqrt{(K + \mathrm{i}z)(K - \mathrm{i}\bar{z})}} = \frac{1}{r_2}\left(-2CK_1 - \frac{\pi \mathrm{i}}{2}\right). \tag{20}$$

These intergrals can elementarily be calculated with the final result

$$f = 1 - \frac{4M}{r_1 + r_2 + 2M + \mathrm{i}\frac{J}{\sqrt{M^4 - J^2}}(r_1 - r_2)}, \tag{21}$$

where the parameters M (mass) and J (angular momentum) are related to K_1 and C according to

$$M = K_1 \coth(2CK_1), \quad \frac{J}{\sqrt{M^4 - J^2}} = \frac{1}{\sinh(-2CK_1)}. \tag{22}$$

This is exactly the Ernst potential of the Kerr solution. Note that the extreme limit ($J = M^2$) is obtained for $K_1 \to 0$ (with $M = 1/2C$).

3.2 The rigidly rotating disk of dust

The solution describing the gravitational field of a rigidly rotating disk of dust (placed at $\zeta = 0$, $\varrho \le \varrho_0$) is obtained for [2]

$$K_1 = \varrho_0 \sqrt{\frac{\mathrm{i} - \mu}{\mu}}, \quad v_0 = \frac{1}{\pi \mathrm{i} \varrho_o^2} \int\limits_{-\mathrm{i}\varrho_o}^{\mathrm{i}\varrho_o} \frac{D(K) dK}{\sqrt{(K + \mathrm{i}z)(K - \mathrm{i}\bar{z})}} \tag{23}$$

(integration along the imaginary K-axis, $\Re\sqrt{(K + \mathrm{i}z)(K - \mathrm{i}\bar{z})} < 0$ for ϱ, ζ outside the disk) with

$$D(K) = \frac{\mu \ln\left(\sqrt{1 + \mu^2(1 + K^2/\varrho_0^2)^2} + \mu(1 + K^2/\varrho_0^2)\right)}{\sqrt{1 + \mu^2(1 + K^2/\varrho_0^2)^2}}. \tag{24}$$

From (6) we obtain

$$v_j = \frac{1}{\pi \mathrm{i} \varrho_o^2} \int\limits_{-\mathrm{i}\varrho_o}^{\mathrm{i}\varrho_o} \frac{D(K) K^j dK}{\sqrt{(K + \mathrm{i}z)(K - \mathrm{i}\bar{z})}}, \quad j = 1, 2. \tag{25}$$

Here μ is a real parameters. (The total mass M and the angular momentum J of the disk are functions of ϱ_0 and μ.) It turns out that the solution is regular for $0 \le \mu < \mu_0 = 4.62966\ldots$ where the limit $\mu \to \mu_0$, for finite M, leads to $\varrho_0 \to 0$. According to (23), this means $K_1 \to 0$ and it can be shown that

$$\hat{v}_0 \to \frac{1}{Mr}, \quad r = \sqrt{z\bar{z}} = \sqrt{\varrho^2 + \zeta^2}. \tag{26}$$

Comparing this with Eqs. (14), (15) [note that r_1 and r_2 approach r for $K_1 \to 0$], and the final remark of the previous subsection, we are led to the conclusion that the solution approaches exactly the extreme Kerr metric (for $r > 0$) in the limit $\mu \to \mu_0$. More details concerning this “ultrarelativistic” limit can be found in [10] and [11].

3.3 Differentially rotating disks of dust

A three-parameter family of solutions describing differentially rotating disks of dust is obtained for [9]

$$K_1 = \varrho_0 X_1, \quad v_0 = \frac{1}{\pi \mathrm{i} \varrho_o^2} \int\limits_{-\mathrm{i}\varrho_o}^{\mathrm{i}\varrho_o} \frac{D(K) dK}{\sqrt{(K + \mathrm{i}z)(K - \mathrm{i}\bar{z})}}, \tag{27}$$

$$\overline{D(K)} = D(-K) = D(K), \ -\varrho_0 \le K/\mathrm{i} \le \varrho_0; \quad D(\pm \mathrm{i}\varrho_0) = 0, \tag{28}$$

where ϱ_0 is again the (Weyl-coordinate) radius of the disk, X_1 is an arbitrary complex parameter [we only assume $\Re X_1 \le 0$, $\Im X_1 \le 0$ according to (12)], and $D(K)$ is determined such that the following “dust condition” [12] is satisfied in the disk, i.e. for $\zeta = 0$, $\varrho \le \varrho_0$:

$$[\Im(A + B - 4\varrho AB)]^2 = 4\,\Im A\,\Im B, \quad A = \frac{f_{,z}}{f + \bar{f}}\ , \ B = \frac{\bar{f}_{,z}}{f + \bar{f}}. \tag{29}$$

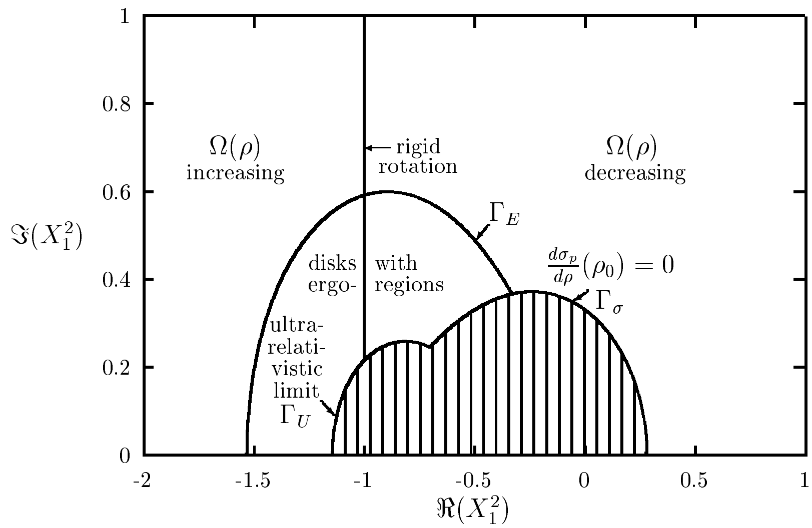

Figure 2: Differentially rotating disks of dust [9] in dependence on the complex parameter X_1^2. Regular solutions have been found outside the hatched region.

Note that for arbitrary $D(K)$ one obtains solutions which might be interpreted as disks consisting of two counter-rotating streams of particles moving on geodesics (i.e. two dust components), see [13]. The condition (29) guarantees that there is one stream of particles only[1]. The angular velocity $\Omega(\varrho)$ can be calculated afterwards.

The solution depends on the three parameters ϱ_0, $\Re X_1$, and $\Im X_1$. According to (23), the rigidly rotating disk of dust ($\Omega = \text{const.}$) is included for $\Re X_1^2 = -1$. The condition (29) leads to a complicated nonlinear integral equation for $D(K)$ which has been solved numerically to an extremely high accuracy, see [9]. [For $\Re X_1^2 = -1$ one obtains (24), of course.] A regular solution with positive surface mass-density has been found in the parameter region as indicated in Fig. 2. For X_1^2 approaching the curve Γ_U, $\varrho_0 \to 0$ follows for finite M, and the extreme Kerr solution is reached again. This confirms the conjecture formulated by Bardeen and Wagoner [10] that differential rotation will not change the ultrarelativistic limit. The curve Γ_σ is characterized by a vanishing derivative of the surface mass–density σ_p at the rim of the disk. (Normally only σ_p itself vanishes at the rim.) Γ_E divides the parameter-space into parts with and without ergoregions of the solutions. The angular velocity Ω is always a monotonic function of ϱ, increasing for $\Re X_1^2 < -1$ and decreasing for $\Re X_1^2 > -1$.

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[1] In this case the geodesic motion is a consequence of the Einstein equations. On the other hand, a formal superposition of two dust energy-momentum tensors does not lead automatically to a geodesic motion of the particles. Therefore, the physical interpretation [14] of a particular solution of this class is unsatisfactory.

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