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Courtney A. Schmitt

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Courtney A. Schmitt, 2d Lt, USAF

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AIR FORCE INSTITUTE OF TECHNOLOGY

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Courtney A. Schmitt, B.S.
2d Lt, USAF

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Courtney A. Schmitt, B.S.
2d Lt, USAF

Committee Membership:

Dr. Matthew Fickus
Chairman

Dr. Dursun Bulutoglu
Member

Dr. Mark E. Oxley
Member

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Abstract

An equiangular tight frame (ETF) is a sequence of equal-norm vectors in a Euclidean space whose coherence achieves equality in the Welch bound, and thus yields an optimal packing in a projective space. A regular simplex is a simple type of ETF in which the number of vectors is one more than the dimension of the underlying space. More sophisticated examples include harmonic ETFs, which are formed by restricting the characters of a finite abelian group to a difference set. Recently, it was shown that some harmonic ETFs are themselves comprised of regular simplices. In this thesis, we continue the investigation into these special harmonic ETFs. We begin by characterizing when the subspaces spanned by the ETF's regular simplices form an equi-isoclinic tight fusion frame, which is a type of optimal packing in a Grassmannian space. It turns out that such ETFs yield complex circulant conference matrices; this is remarkable since real examples of such matrices are known to not exist. We further show that some of these ETFs yield mutually unbiased simplices, which are a natural generalization of the quantum-information-theoretic concept of mutually unbiased bases. Finally, we provide infinite families of ETFs that have all of these properties.

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COMPRISED OF REGULAR SIMPLICES

I. Introduction

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$ and let $\mathbb{H}$ be a D-dimensional Hilbert space over $\mathbb{F}$ whose inner product is conjugate-linear in the first argument. Further, let $\mathcal{N}$ be a general N-element indexing set. A sequence of nonzero vectors $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ in $\mathbb{H}$ has *coherence*

$$\operatorname{coh}(\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}) := \max_{n\neq n'} \tfrac{|\langle\boldsymbol{\varphi}_n,\boldsymbol{\varphi}_{n'}\rangle|}{\|\boldsymbol{\varphi}_n\|\|\boldsymbol{\varphi}_{n'}\|}. \tag{1}$$

A well-known lower bound on the coherence of a sequence of unit norm vectors is the *Welch bound* [31]

$$\sqrt{\tfrac{N-D}{D(N-1)}} \leq \max_{n\neq n'} |\langle\boldsymbol{\varphi}_n,\boldsymbol{\varphi}_{n'}\rangle| \tag{2}$$

and equality is achieved in this bound if and only if the vectors form an *equiangular tight frame (ETF)* for $\mathbb{H}$ [27]. In particular, the lines spanned by the vectors of an ETF have the property that the minimum angle between any pair of them is as large as possible, and so they are an optimal packing of points in projective space. Due to this optimality, ETFs have applications in areas such as compressed sensing [1, 2], quantum information theory [23, 33], and algebraic coding theory [18], however, constructing ETFs is not an easy task [11].

Letting $\operatorname{ETF}(D,N)$ denote an ETF of N vectors in a D-dimensional Hilbert space $\mathbb{H}$, an $\operatorname{ETF}(D,D)$ is equivalent to an orthonormal basis for $\mathbb{H}$ and so exists for any positive integer D. An $\operatorname{ETF}(1,N)$ is equivalent to N unimodular scalars and so exists for any positive integer N. We call an $\operatorname{ETF}(S,S+1)$ a *(regular S-)simplex* for $\mathbb{H}$ and these are known to exist for any positive integer S. Outside of these cases, all known infinite families of ETFs come from combinatorial designs. For example, real ETFs are equivalent to a certain type of strongly regular graph [17, 21, 24, 30]. Another class of constructions exist when the *redundancy* of the ETF, namely the quantity $\frac{N}{D}$, is close to two. In this case, ETFs can be related to conference matrices, Hadamard matrices, Paley tournaments, and Gauss sums [17, 22, 26, 27]. Less restrictive constructions are *Steiner ETFs* and *Harmonic ETFs*. Steiner ETFs arise from using a balanced incomplete block design to strategically arrange several regular simplices with respect to each other [13, 15]. Harmonic ETFs arise from difference sets in finite abelian groups [9, 27, 29, 32].

Steiner ETFs are made up of regular simplices by design in the sense that their vectors can be partitioned into subsequences of vectors that are regular simplices for their spans. It was shown in [18] that harmonic ETFs constructed from *McFarland* difference sets are a unitary transformation of a Steiner ETF and so they too are comprised of simplices. Further, it has been shown that the complement of both *twin prime power* and some *Singer* difference sets also have this property [10]. It was shown that these ETFs, while being an optimal packing of points in projective space, further have the property that the subspaces spanned by the simplices are a type of optimal packing in Grassmannian space known as an *equi-chordal tight fusion frame (ECTFF)* and so achieve the *simplex bound* of [7].

In this thesis we will further investigate known harmonic ETFs comprised of regular simplices and relate them to other structures. In Chapter II, we will establish notation and give already known results that will relate to later topics. Chapter III will further develop the properties of difference sets that result in harmonic ETFs comprised of simplices as well as show that a subset of these ETFs have the property that the subspaces spanned by the simplices produce a type of ECTFF known as an *equi-isoclinic tight fusion frame (EITFF)*. Further, we will see that this subset of ETFs also gives constructions for complex circulant conference matrices. In Chapter IV we further restrict the class of harmonic ETFs and show that this restricted class will produce collections of *mutually unbiased simplices (MUS)*, a simplex-based generalization of *mutually unbiased bases*. Finally, we will classify three known types of difference sets according to the properties needed for the construction of EITFFs, circulant conference matrices, and MUSs.

II. Background

For any $z \in \mathbb{F}$, let z^* denote the complex conjugate of z. Given any N-element indexing set $\mathcal{N}$, let $\mathbb{F}^{\mathcal{N}} := \{\mathbf{y} : \mathcal{N} \to \mathbb{F}\}$ be the Hilbert space equipped with the inner product $\langle \mathbf{y}_1, \mathbf{y}_2 \rangle := \sum_{n\in\mathcal{N}} [\mathbf{y}_1(n)]^* \mathbf{y}_2(n)$. Further, for an M-element indexing set $\mathcal{M}$ let $\mathbb{F}^{\mathcal{M}\times\mathcal{N}} := \{\mathbf{A} : \mathcal{M} \times \mathcal{N} \to \mathbb{F}\}$ be the space of all matrices whose rows and columns are indexed by $\mathcal{M}$ and $\mathcal{N}$, respectively, equipped with the Frobenius (Hilbert-Schmidt) inner product, $\langle \mathbf{A}_1, \mathbf{A}_2 \rangle_{\text{Fro}} := \text{Tr}(\mathbf{A}_1^* \mathbf{A}_2)$. Any such matrix represents a linear operator from $\mathbb{F}^{\mathcal{N}}$ to $\mathbb{F}^{\mathcal{M}}$.

The *synthesis operator* of a finite sequence of vectors $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ in a D-dimensional Hilbert space $\mathbb{H}$ is $\boldsymbol{\Phi} : \mathbb{F}^{\mathcal{N}} \to \mathbb{H}$, $\boldsymbol{\Phi}\mathbf{x} := \sum_{n\in\mathcal{N}} \mathbf{x}(n)\boldsymbol{\varphi}_n$. Its adjoint is the *analysis operator*, $\boldsymbol{\Phi}^* : \mathbb{H} \to \mathbb{F}^{\mathcal{N}}$, $(\boldsymbol{\Phi}^*\mathbf{y})(n) = \langle \boldsymbol{\varphi}_n, \mathbf{y} \rangle$. When $\mathcal{N} = [N] := \{1, ..., N\}$ and $\mathbb{H} = \mathbb{F}^D$, $\boldsymbol{\Phi}$ can be thought of as a $D \times N$ matrix whose nth column is $\boldsymbol{\varphi}_n$ and $\boldsymbol{\Phi}^*$ is the conjugate transpose of this matrix. Composing these operators yields the *frame operator* $\boldsymbol{\Phi}\boldsymbol{\Phi}^* : \mathbb{H} \to \mathbb{H}$, $\boldsymbol{\Phi}\boldsymbol{\Phi}^*\mathbf{y} = \sum_{n\in\mathcal{N}} \langle \boldsymbol{\varphi}_n, \mathbf{y} \rangle \boldsymbol{\varphi}_n$ and the $\mathcal{N} \times \mathcal{N}$ *Gram matrix* $\boldsymbol{\Phi}^*\boldsymbol{\Phi} : \mathbb{F}^{\mathcal{N}} \to \mathbb{F}^{\mathcal{N}}$ whose (n, n')th entry is $(\boldsymbol{\Phi}^*\boldsymbol{\Phi})(n, n') = \langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'} \rangle$. We also view each vector as its own synthesis operator $\boldsymbol{\varphi}_n : \mathbb{F} \to \mathbb{H}$, $\boldsymbol{\varphi}_n(x) = x\boldsymbol{\varphi}_n$. Its adjoint is the linear functional $\boldsymbol{\varphi}_n^* : \mathbb{H} \to \mathbb{F}$, $\boldsymbol{\varphi}_n^*\mathbf{y} = \langle \boldsymbol{\varphi}_n, \mathbf{y} \rangle$. Under this notation the frame operator of $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is $\boldsymbol{\Phi}\boldsymbol{\Phi}^* = \sum_{n\in\mathcal{N}} \boldsymbol{\varphi}_n \boldsymbol{\varphi}_n^*$.

In the special case where $\boldsymbol{\Phi}\boldsymbol{\Phi}^* = A\mathbf{I}$ for some $A > 0$, we say $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is an *(A-)tight frame* for $\mathbb{H}$. When the vectors $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ are regarded as members of some (larger) Hilbert space $\mathbb{K}$ which contains $\mathbb{H} = \text{span}\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ as a (proper) subspace, we say that $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is a *tight frame for its span*; elsewhere in the literature, such sequences are sometimes called "tight frame sequences." Here the analysis operator $\boldsymbol{\Phi}^* : \mathbb{H} \to \mathbb{F}^{\mathcal{N}}$ extends to an operator $\boldsymbol{\Phi}^* : \mathbb{K} \to \mathbb{F}^{\mathcal{N}}$ and $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is a tight frame for its span precisely when $\boldsymbol{\Phi}\boldsymbol{\Phi}^*\mathbf{y} = A\mathbf{y}$ for all $\mathbf{y} \in \mathbb{H} = C(\boldsymbol{\Phi})$. As shown in [12], this is equivalent to having either $\boldsymbol{\Phi}\boldsymbol{\Phi}^*\boldsymbol{\Phi} = A\boldsymbol{\Phi}$, $(\boldsymbol{\Phi}\boldsymbol{\Phi}^*)^2 = A\boldsymbol{\Phi}\boldsymbol{\Phi}^*$, or $(\boldsymbol{\Phi}^*\boldsymbol{\Phi})^2 = A\boldsymbol{\Phi}^*\boldsymbol{\Phi}$.

A *Naimark complement* of an A-tight frame $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ for a D-dimensional Hilbert space $\mathbb{H}$ is any sequence $\{\boldsymbol{\psi}_n\}_{n\in\mathcal{N}}$ in a space $\mathbb{K}$ with synthesis operator $\boldsymbol{\Psi}$ such that $\boldsymbol{\Phi}^*\boldsymbol{\Phi}+\boldsymbol{\Psi}^*\boldsymbol{\Psi}=A\mathbf{I}$. Since $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is an A-tight frame for $\mathbb{H}$, $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ has eigenvalues A and 0 with multiplicities D and $N-D$, respectively. Consequently $\boldsymbol{\Psi}^*\boldsymbol{\Psi}=A\mathbf{I}-\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ has eigenvalues A and 0 with multiplicities $N-D$ and D, respectively, meaning that $\{\boldsymbol{\psi}_n\}_{n\in\mathcal{N}}$ is a tight frame for its $(N-D)$-dimensional span. Being defined in terms of Gram matrices, Naimark complements are only unique up to unitary transformations. They exist whenever $N>D$: when $\mathbb{H}=\mathbb{F}^D$, $\mathcal{N}=[N]$, and $\boldsymbol{\Phi}$ is regarded as a $D\times N$ matrix, a natural way to construct a Naimark complement $\{\boldsymbol{\psi}_n\}_{n\in\mathcal{N}}$ is as the columns of any $(N-D)\times N$ matrix $\boldsymbol{\Psi}$ whose rows, together with the rows of $\boldsymbol{\Phi}$, form an equal-norm orthogonal basis for $\mathbb{F}^{\mathcal{N}}$.

2.1 Equi-Chordal and Equi-Isoclinic Tight Fusion Frames

Let $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ be M-dimensional subspaces of a D-dimensional Hilbert space $\mathbb{H}$. For each $n\in\mathcal{N}$ let $\boldsymbol{\Phi}_n$ be the synthesis operator of an orthonormal basis $\{\boldsymbol{\varphi}_{n,m}\}_{m\in\mathcal{M}}$ for $\mathcal{U}_n$ so that $\boldsymbol{\Phi}_n^*\boldsymbol{\Phi}_n=\mathbf{I}$. Here $\mathbf{P}_n=\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^*$ is the orthogonal projection operator onto $\mathcal{U}_n$. We can also consider the synthesis operator $\boldsymbol{\Phi}$ of the concatenation (union) $\{\boldsymbol{\varphi}_{n,m}\}_{n\in\mathcal{N},m\in\mathcal{M}}$ of all of these orthonormal bases. In the special case where $\mathbb{H}=\mathbb{F}^D$, $\mathcal{N}=[N]$, and $\mathcal{M}=[M]$ the operator $\boldsymbol{\Phi}$ can be regarded as a $1\times N$ block matrix whose nth block is the $D\times M$ matrix $\boldsymbol{\Phi}_n$, i.e.

$$\boldsymbol{\Phi}=\begin{bmatrix}\boldsymbol{\Phi}_1 & \cdots & \boldsymbol{\Phi}_N\end{bmatrix},\qquad \boldsymbol{\Phi}_n=\begin{bmatrix}\boldsymbol{\varphi}_{n,1} & \cdots & \boldsymbol{\varphi}_{n,M}\end{bmatrix}.$$

Because of this, in general we regard the $(\mathcal{N}\times\mathcal{M})\times(\mathcal{N}\times\mathcal{M})$ Gram matrix $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ as an $\mathcal{N}\times\mathcal{N}$ block matrix whose (n,n')th block is the $\mathcal{M}\times\mathcal{M}$ *cross-Gram matrix*, $\boldsymbol{\Phi}_n^*\boldsymbol{\Phi}_{n'}$. The *fusion frame operator* of $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is defined to be the sum of the orthogonal projection operators onto these subspaces and equates to the frame operator of $\{\boldsymbol{\varphi}_{n,m}\}_{n\in\mathcal{N},m\in\mathcal{M}}$ since

$$\boldsymbol{\Phi}\boldsymbol{\Phi}^* = \sum_{n\in\mathcal{N}}\sum_{m\in\mathcal{M}}\boldsymbol{\varphi}_{n,m}\boldsymbol{\varphi}_{n,m}^* = \sum_{n\in\mathcal{N}}\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^* = \sum_{n\in\mathcal{N}}\mathbf{P}_n.$$

The subspaces $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ are said to form a *tight fusion frame (TFF)* for $\mathbb{H}$ if there exists some scalar $A>0$ such that $\boldsymbol{\Phi}\boldsymbol{\Phi}^* = A\mathbf{I}$. In this case the tight frame constant is necessarily $A=\frac{MN}{D}$ since $AD = \mathrm{Tr}\,(A\mathbf{I}) = \mathrm{Tr}\,(\boldsymbol{\Phi}\boldsymbol{\Phi}^*) = \sum_{n\in\mathcal{N}}\mathrm{Tr}\,(\mathbf{P}_n) = MN$. Moreover, in general for any M-dimensional subspaces $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ of $\mathbb{H}$,

$$0\leq \Big\|\sum_{n\in\mathcal{N}}\mathbf{P}_n - \tfrac{MN}{D}\mathbf{I}\Big\|_{\mathrm{Fro}}^2 = \mathrm{Tr}\Big[(\sum_{n\in\mathcal{N}}\mathbf{P}_n - \tfrac{MN}{D}\mathbf{I})^2\Big] = \sum_{n\in\mathcal{N}}\sum_{\substack{n'\in\mathcal{N}\\ n'\neq n}}\mathrm{Tr}(\mathbf{P}_n\mathbf{P}_{n'}) - MN(\tfrac{MN}{D}-1)$$

and equality is achieved if and only if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is a TFF for $\mathbb{H}$. To continue, note that

$$\sum_{n\in\mathcal{N}}\sum_{\substack{n'\in\mathcal{N}\\ n'\neq n}}\mathrm{Tr}(\mathbf{P}_n\mathbf{P}_{n'}) \leq N(N-1)\max_{n\neq n'}\mathrm{Tr}(\mathbf{P}_n\mathbf{P}_{n'}) = N(N-1)\max_{n\neq n'}\|\boldsymbol{\Phi}_n^*\boldsymbol{\Phi}_{n'}\|_{\mathrm{Fro}}^2$$

and here equality is achieved if and only if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ are *equi-chordal*, i.e. if and only if $\mathrm{Tr}(\mathbf{P}_n\mathbf{P}_{n'})$ is constant over all $n\neq n'$. Combining these two inequalities we see that

$$\sqrt{\tfrac{M(MN-D)}{D(N-1)}} \leq \max_{n\neq n'}\|\boldsymbol{\Phi}_n^*\boldsymbol{\Phi}_{n'}\|_{\mathrm{Fro}} \tag{3}$$

where equality is achieved if and only if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an *equi-chordal tight fusion frame* (ECTFF) for $\mathbb{H}$. That is, a tight fusion frame for $\mathbb{H}$ with the property that the *chordal distance* between any two subspaces $\mathcal{U}_n$ and $\mathcal{U}_{n'}$ defined as

$$\mathrm{dist}_{\mathrm{c}}(\mathcal{U}_n,\mathcal{U}_{n'}) = \tfrac{1}{\sqrt{2}}\|\mathbf{P}_n-\mathbf{P}_{n'}\|_{\mathrm{Fro}} = \sqrt{M-\|\boldsymbol{\Phi}_n^*\boldsymbol{\Phi}_{n'}\|_{\mathrm{Fro}}^2}$$

is constant over all $n \neq n'$. From (3), we see that any ECTFF yields an optimal packing of subspaces with respect to the chordal distance, that is, an arrangement of N subspaces of $\mathbb{H}$, each of dimension M, whose minimum pairwise-chordal distance is maximal, satisfying the so-called simplex bound of [7].

Letting $\|\mathbf{A}\|_2$ be the standard (induced) 2-norm of $\mathbf{A}$, namely its largest singular value, then $\frac{M(MN-D)}{D(N-1)} \leq \max_{n\neq n'} \|\mathbf{\Phi}_n^*\mathbf{\Phi}_{n'}\|_{\mathrm{Fro}}^2 \leq M\|\mathbf{\Phi}_n^*\mathbf{\Phi}_{n'}\|_2^2$ and so we also have

$$\sqrt{\tfrac{MN-D}{D(N-1)}} \leq \max_{n\neq n'} \|\mathbf{\Phi}_n^*\mathbf{\Phi}_{n'}\|_2. \tag{4}$$

Here equality is achieved if and only if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an *equi-isoclinic tight fusion frame* (EITFF) for $\mathbb{H}$, that is, an ECTFF for $\mathbb{H}$ such that for all $n \neq n'$, $\mathbf{\Phi}_n^*\mathbf{\Phi}_{n'}$ has constant singular values. This occurs if and only if there exists some $\sigma \geq 0$ such that $\mathbf{\Phi}_{n'}^*\mathbf{P}_n\mathbf{\Phi}_{n'} = \sigma^2\mathbf{I}$ for all $n \neq n'$. Conjugating by $\mathbf{\Phi}_{n'}$ gives $\mathbf{P}_{n'}\mathbf{P}_n\mathbf{P}_{n'} = \mathbf{\Phi}_{n'}\mathbf{\Phi}_{n'}^*\mathbf{P}_n\mathbf{\Phi}_{n'}\mathbf{\Phi}_{n'}^* = \sigma^2\mathbf{P}_{n'}$. Conversely, conjugating this by $\mathbf{\Phi}_{n'}^*$ yields $\mathbf{\Phi}_{n'}^*\mathbf{P}_n\mathbf{\Phi}_{n'} = \sigma^2\mathbf{I}$. To summarize, $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an EITFF for $\mathbb{H}$ if and only if it is a TFF for $\mathbb{H}$ and there exists $\sigma \geq 0$ such that $\mathbf{P}_{n'}\mathbf{P}_n\mathbf{P}_{n'} = \sigma^2\mathbf{P}_{n'}$ for all $n \neq n'$.

In the case that the subspaces $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ are of dimension one, i.e. $M = 1$, ECTFFs and EITFFs reduce to *equiangular tight frames* (ETFs): choosing a unit norm vector $\boldsymbol{\varphi}_n$ from each of the subspaces produces $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$. In this case, (3) and (4) both reduce to the Welch bound (2), where equality is achieved if and only if $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is an ETF for $\mathbb{H}$, that is $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is a tight frame for $\mathbb{H}$ and $|\langle\boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'}\rangle|$ is constant over all $n \neq n'$ making the vectors *equiangular*. This implies $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ has minimal coherence (1). The Naimark complement of an ETF is itself an ETF since $\mathbf{\Psi}^*\mathbf{\Psi} = A\mathbf{I} - \mathbf{\Phi}^*\mathbf{\Phi}$ and so $\|\boldsymbol{\psi}_n\|^2 = A - \|\boldsymbol{\varphi}_n\|^2$, implying that $\{\boldsymbol{\psi}_n\}_{n\in\mathcal{N}}$ is equal norm, and $\langle\boldsymbol{\psi}_n, \boldsymbol{\psi}_{n'}\rangle = -\langle\boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'}\rangle$, implying that $\{\boldsymbol{\psi}_n\}_{n\in\mathcal{N}}$ is equiangular.

An $\mathrm{ETF}(S, S+1)$ is called a *regular S-simplex*. Given any integer S, a regular S-simplex always exists as it is a Naimark complement of any sequence of $S+1$

unimodular scalars in $\mathbb{F}$. In light of the Welch bound (2), any $S+1$ linearly dependent unit vectors with coherence $\frac{1}{S}$ necessarily form a regular simplex for their span.

2.2 Harmonic Equiangular Tight Frames and Difference Sets

A *character* on a finite abelian group $\mathcal{G}$ is a homomorphism $\gamma : \mathcal{G} \to \mathbb{T}$ where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The set of all characters of $\mathcal{G}$, denoted $\hat{\mathcal{G}}$, is called the *(Pontryagin) dual* of $\mathcal{G}$ and is itself a group under entrywise multiplication. It is well known that $\hat{\mathcal{G}}$ is isomorphic to $\mathcal{G}$ and that $\{\gamma\}_{\gamma\in\hat{\mathcal{G}}}$ is an equal-norm orthogonal basis for $\mathbb{C}^{\mathcal{G}}$. Therefore, its synthesis operator $\mathbf{F} : \mathbb{C}^{\hat{\mathcal{G}}} \to \mathbb{C}^{\mathcal{G}}$ is invertible with $\mathbf{F}^{-1} = \frac{1}{G}\mathbf{F}^*$. The operator $\mathbf{F}$ is usually regarded as the $(\mathcal{G} \times \hat{\mathcal{G}})$-indexed *character table* of $\mathcal{G}$ whose (g, γ)th entry is $\mathbf{F}(g, \gamma) := \gamma(g)$. Its adjoint, $\mathbf{F}^* : \mathbb{C}^{\mathcal{G}} \to \mathbb{C}^{\hat{\mathcal{G}}}$, $(\mathbf{F}^*\boldsymbol{\varphi})(\gamma) = \langle \gamma, \boldsymbol{\varphi}\rangle$, is the *discrete Fourier transform* (DFT) on $\mathcal{G}$.

For any $\boldsymbol{\varphi}, \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \in \mathbb{C}^{\mathcal{G}}$, we let $(\boldsymbol{\varphi}_1 * \boldsymbol{\varphi}_2)(g) = \sum_{g'\in\mathcal{G}} \boldsymbol{\varphi}_1(g')\boldsymbol{\varphi}_2(g - g')$ denote the *convolution* of $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ and $\tilde{\boldsymbol{\varphi}}(g) = [\boldsymbol{\varphi}(-g)]^*$ denote the *involution* of $\boldsymbol{\varphi}$. Convolution and involution correspond to pointwise multiplication and conjugation, respectively, in the Fourier domain. That is, $[\mathbf{F}^*(\boldsymbol{\varphi}_1 * \boldsymbol{\varphi}_2)](\gamma) = [(\mathbf{F}^*\boldsymbol{\varphi}_1)(\gamma)][(\mathbf{F}^*\boldsymbol{\varphi}_2)(\gamma)]$ and $(\mathbf{F}^*\tilde{\boldsymbol{\varphi}})(\gamma) = [(\mathbf{F}^*\boldsymbol{\varphi})(\gamma)]^*$ for all $\gamma \in \hat{\mathcal{G}}$. Consequently, the *autocorrelation*, $\boldsymbol{\varphi} * \tilde{\boldsymbol{\varphi}}$, of $\boldsymbol{\varphi}$ corresponds to the pointwise modulus squared $|\mathbf{F}^*\boldsymbol{\varphi}|^2$ in the Fourier domain.

Since the rows and columns of the character table $\mathbf{F}$ of $\mathcal{G}$ are equal-norm orthogonal, given any $\mathcal{D} \subseteq \mathcal{G}$ we can restrict each character $\gamma \in \hat{\mathcal{G}}$ to $\mathcal{D}$, that is, we can regard $\gamma \in \mathbb{C}^{\mathcal{D}}$, and the rows of the resulting $(\mathcal{D} \times \hat{\mathcal{G}})$-indexed submatrix of $\mathbf{F}$ are still equal-norm orthogonal. Therefore its columns—the restricted characters—form a tight frame for $\mathbb{C}^{\mathcal{D}}$. Sometimes these restricted characters further form an ETF for $\mathbb{C}^{\mathcal{D}}$. For any $\mathcal{D} \subseteq \mathcal{G}$, the inner product of any two characters of $\mathcal{G}$ restricted to $\mathcal{D}$ is

$$\langle \gamma', \gamma\rangle_{\mathcal{D}} := \langle \gamma'\chi_{\mathcal{D}}, \gamma\chi_{\mathcal{D}}\rangle = \sum_{g\in\mathcal{D}}((\gamma')^{-1}\gamma)(g) = \langle \gamma'\gamma^{-1}, \chi_{\mathcal{D}}\rangle = (\mathbf{F}^*\chi_{\mathcal{D}})(\gamma'\gamma^{-1}). \quad (5)$$

As such, $\{\gamma\}_{\gamma\in\hat{\mathcal{G}}}$ is an ETF for $\mathbb{C}^{\mathcal{D}}$ if and only if $|(\mathbf{F}^*\chi_{\mathcal{D}})(\gamma)|$ is constant over all $\gamma \neq \mathbf{1}$. As we now discuss, this happens precisely when $\mathcal{D}$ is a *difference set* for $\mathcal{G}$.

A subset $\mathcal{D}$ of a finite abelian group $\mathcal{G}$ is called a difference set for $\mathcal{G}$ if the cardinality of $\{(d,d') \in \mathcal{G} : g = d - d'\}$ is constant over all nonzero $g \in \mathcal{G}$. Conceptually, $\mathcal{D}$ is a difference set for $\mathcal{G}$ if every nonzero element of $\mathcal{G}$ occurs the same number of times in the *difference table* for $\mathcal{D}$. For example, $\{1, 2, 4\}$ is a difference set for $\mathbb{Z}_7$ since its difference table is

−	1	2	4
1	0	6	5
2	1	0	4
4	3	2	0

.

Letting D be the cardinality of $\mathcal{D}$ and G be the order of $\mathcal{G}$, since there are $D(D-1)$ nonzero entries in such a table and $G-1$ nonzero elements of $\mathcal{G}$, each one must appear

$$\Lambda_{\mathcal{D}} := \tfrac{D(D-1)}{G-1} \tag{6}$$

times, i.e. for all nonzero $g \in \mathcal{G}$, $\#\{(d,d') \in \mathcal{G} : g = d - d'\} = \Lambda_{\mathcal{D}}$.

Any subset $\mathcal{D}$ of $\mathcal{G}$ can also be classified as a difference set based on the *autocorrelation,* $\chi_{\mathcal{D}} * \tilde{\chi}_{\mathcal{D}}$, of $\chi_{\mathcal{D}}$. In particular

$$(\chi_{\mathcal{D}} * \tilde{\chi}_{\mathcal{D}})(g) = \sum_{g'\in\mathcal{G}} \chi_{\mathcal{D}}(g')\chi_{\mathcal{D}}(g'-g) = \#[\mathcal{D} \cap (g+\mathcal{D})] = \#\{(d,d') \in \mathcal{G} : g = d - d'\}.$$

In summary, $\mathcal{D}$ is a difference set for $\mathcal{G}$ if and only if $\chi_{\mathcal{D}} * \tilde{\chi}_{\mathcal{D}} = (D - \Lambda_{\mathcal{D}})\boldsymbol{\delta}_0 + \Lambda_{\mathcal{D}}\mathbf{1}$. Equivalently, taking Fourier transforms, $\mathcal{D}$ is a difference set for $\mathcal{G}$ if and only if

$$|\mathbf{F}^*\chi_{\mathcal{D}}|^2 = \mathbf{F}^*(\chi_{\mathcal{D}} * \tilde{\chi}_{\mathcal{D}}) = \mathbf{F}^*[(D - \Lambda_{\mathcal{D}})\boldsymbol{\delta}_0 + \Lambda_{\mathcal{D}}\mathbf{1}] = (D - \Lambda_{\mathcal{D}})\mathbf{1} + G\Lambda_{\mathcal{D}}\boldsymbol{\delta}_1$$

since $\mathbf{F}^*\boldsymbol{\delta}_0 = \mathbf{1}$ and $\mathbf{F}^*\mathbf{1} = G\boldsymbol{\delta}_1$. Specifically, for all $\gamma \neq \mathbf{1}$ we have that

$$|(\mathbf{F}^*\chi_{\mathcal{D}})(\gamma)|^2 = D - \Lambda_{\mathcal{D}} = D - \tfrac{D(D-1)}{G-1} = D^2\left(\tfrac{G-D}{D(G-1)}\right) = \tfrac{D^2}{S^2}. \tag{7}$$

where $S := \sqrt{\frac{D(G-1)}{G-D}}$ is the reciprocal Welch Bound (2). Since this value is constant for all $\gamma \in \hat{\mathcal{G}}$, $\gamma \neq \mathbf{1}$, by (5) restricting the characters of $\mathcal{G}$ to any subset $\mathcal{D}$ of $\mathcal{G}$ forms an ETF for $\mathbb{C}^{\mathcal{D}}$ if and only if $\mathcal{D}$ is a difference set for $\mathcal{G}$. An ETF constructed in this manner is called a *harmonic ETF*. Here we note that the complement of any difference set is also a difference set and the corresponding harmonic ETFs are Naimark complements. Also any shift or automorphism of $\mathcal{G}$ applied to a difference set is again a difference set.

For any subgroup $\mathcal{H}$ of $\mathcal{G}$, its *annihilator* is $\mathcal{H}^\perp = \{\gamma \in \hat{\mathcal{G}} : \gamma(h) = 1,\ \forall h \in \mathcal{H}\}$. It is well known that $\mathcal{H}^\perp$ is a subgroup of $\hat{\mathcal{G}}$, and moreover that $\boldsymbol{\varphi} : \mathcal{H}^\perp \to (\mathcal{G}/\mathcal{H})^\wedge$, $[\boldsymbol{\varphi}(\gamma)](\overline{g}) := \gamma(g)$ is a well-defined isomorphism. Here and in later chapters we let $\overline{g}$ represent the coset $g + \mathcal{H}$ and $\overline{\gamma}$ represent the coset $\gamma\mathcal{H}^\perp$. We will denote the order of $\mathcal{G}$ by G and the order of $\mathcal{H}$ by H and so the order of $\mathcal{H}^\perp$ is $\frac{G}{H}$. Throughout the later results we will make use of the *Poisson summation formula*: if $\mathcal{H}$ is any subgroup of $\mathcal{G}$ of order H then $\mathbf{F}\chi_{\mathcal{H}^\perp} = \frac{G}{H}\chi_{\mathcal{H}}$, or equivalently, taking the DFT, $\mathbf{F}^*\chi_{\mathcal{H}} = H\chi_{\mathcal{H}^\perp}$. To see this, note that for all $\gamma' \in \mathcal{H}^\perp$ and $g \in \mathcal{G}$,

$$[\gamma'(g) - 1]\sum_{\gamma \in \mathcal{H}^\perp} \gamma(g) = \sum_{\gamma \in \mathcal{H}^\perp} (\gamma'\gamma)(g) - \sum_{\gamma \in \mathcal{H}^\perp} \gamma(g) = 0.$$

Then in the case that $g \notin \mathcal{H}$, there exists $\gamma' \in \mathcal{H}^\perp$ such that $\gamma'(g) \neq 1$ and so we must have that $\sum_{\gamma \in \mathcal{H}^\perp} \gamma(g) = 0$. In the case that $g \in \mathcal{H}$, $\sum_{\gamma \in \mathcal{H}^\perp} \gamma(g) = \sum_{\gamma \in \mathcal{H}^\perp} 1 = \frac{G}{H}$.

2.3 Difference Set Constructions

Many of the results in later sections will require a difference set that has certain properties. In particular, we will relate these results to three known types of difference set constructions, *Singer* difference sets, *McFarland* difference sets, and *twin prime power (TPP)* difference sets. Here we give the background for constructing these difference sets.

2.3.1 Singer Difference Sets

In general, for any prime power Q let $\mathbb{F}_Q$ be the finite field of order Q and let $\mathbb{F}_Q^\times$ be the multiplicative group formed by the nonzero elements of $\mathbb{F}_Q$. For any integer $J \geq 2$ the corresponding Singer difference set is

$$\mathcal{D} = \{\overline{\beta} \in \mathbb{F}_{Q^J}^\times / \mathbb{F}_Q^\times : \operatorname{tr}(\beta) = \beta + \beta^Q + \beta^{Q^2} + \cdots + \beta^{Q^{J-1}} = 0\}.$$

Here $\mathcal{D}$ is a difference set in the quotient group $\mathcal{G} = \mathbb{F}_{Q^J}^\times / \mathbb{F}_Q^\times$ and so the harmonic ETF is an ETF$(\frac{Q^{J-1}-1}{Q-1}, \frac{Q^J-1}{Q-1})$.

Example 2.3.1. Let $Q = 2$ and $J = 4$ giving that $\mathcal{G} = \mathbb{F}_{16}^\times / \mathbb{F}_2^\times \cong \mathbb{Z}_{15}$. To find the 7-element Singer difference set we use the fact that x^4+x+1 is a primitive polynomial over $\mathbb{F}_2$ [16] and so the multiplicative group of

$$\mathbb{F}_{16} = \{a + b\alpha + c\alpha^2 + d\alpha^3 : a, b, c, d \in \mathbb{F}_2,\ \alpha^4 + \alpha + 1 = 0\}$$

is generated by α, that is, $\mathbb{F}_{16}^\times = \langle\alpha\rangle$. Then the hyperplane $\{\beta \in \mathbb{F}_{16} : 0 = \operatorname{tr}(\beta) = \beta + \beta^2 + \beta^4 + \beta^8\}$ is the set $\{0\} \cup \{\alpha^j : j = 0, 1, 2, 4, 5, 8, 10\}$. To see this let $\mathbf{A}$ be the companion matrix of the primitive polynomial $x^4 + x + 1$ over $\mathbb{F}_2$, that is

$$\mathbf{A} = \begin{bmatrix} 0\,0\,0\,1 \\ 1\,0\,0\,1 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \end{bmatrix}.$$

Then the field trace, $\mathrm{tr}(\alpha^j)$, is equivalent to the matrix trace, $\mathrm{Tr}(\mathbf{A}^j)$, for all $j \geq 0$ where all arithmetic is done modulo 2. Finally, removing the zero element of $\mathbb{F}_{16}$ and identifying the remaining elements modulo $\mathbb{F}_2^\times$ gives the Singer difference set $\{0, 1, 2, 4, 5, 8, 10\}$. To see that this is a difference set, its difference table is given by:

−	0	1	2	4	5	8	10
0	0	14	13	11	10	7	5
1	1	0	14	12	11	8	6
2	2	1	0	13	12	9	7
4	4	3	2	0	14	11	9
5	5	4	3	1	0	12	10
8	8	7	6	4	3	0	13
10	10	9	8	6	5	2	0

,

and every nonzero element of $\mathbb{Z}_{15}$ appears exactly $\Lambda_{\mathcal{D}} = \frac{7(7-1)}{15-1} = 3$ times.

2.3.2 McFarland Difference Sets

For any prime power Q and any positive integer $J \geq 2$ let $\{\mathcal{U}_i : i = 1, \cdots, \frac{Q^J-1}{Q-1}\}$ enumerate the distinct hyperplanes of $\mathbb{F}_Q^J$. Further, let $\mathcal{K} = \{k_i : i = 0, \cdots, \frac{Q^J-1}{Q-1}\}$ be any finite abelian group of order $\frac{Q^J-1}{Q-1} + 1$ whose nonzero elements enumerate the hyperplanes of $\mathbb{F}_Q^J$. Then the corresponding McFarland difference set for the group

$\mathcal{G} = \mathcal{K} \times \mathbb{F}_Q^J$ is given by

$$\mathcal{D} = \{(k, u) \in \mathcal{G} : k = k_i, u \in \mathcal{U}_i \text{ for some } i = 1, \cdots, \tfrac{Q^J-1}{Q-1}\}.$$

Here the harmonic ETF is an ETF$(Q^{J-1}(\frac{Q^J-1}{Q-1}), Q^J(\frac{Q^J-1}{Q-1}+1))$. This leads to having the reciprocal Welch bound $S = \frac{Q^J-1}{Q-1}$ and $\Lambda_{\mathcal{D}} = \frac{Q^{J-1}(Q^{J-1}-1)}{Q-1}$.

Example 2.3.2. Let $Q = 2$ and $J = 2$. Here the distinct hyperplanes of $\mathbb{F}_2^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ are given by $\{\mathcal{U}_i : i = 1, 2, 3\}$ where $\mathcal{U}_1 = \{00, 01\}$, $\mathcal{U}_2 = \{00, 10\}$, and $\mathcal{U}_3 = \{00, 11\}$. Further let $\mathcal{K} = \mathbb{F}_2^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then the McFarland difference set in the group $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given by

$$\mathcal{D} = \{1000, 1001, 0100, 0110, 1100, 1111\}.$$

One way to verify this is a difference set is by its difference table where $\Lambda_{\mathcal{D}} = 2$:

−	1000	1001	0100	0110	1100	1111
1000	0000	0001	1100	1110	0100	0111
1001	0001	0000	1101	1111	0101	0110
0100	1100	1101	0000	0010	1000	1011
0110	1110	1111	0010	0000	1010	1001
1100	0100	0101	1000	1010	0000	0011
1111	0111	0110	1011	1001	0011	0000

.

2.3.3 Twin Prime Power Difference Sets

Given any odd prime power Q such that $Q+2$ is also a prime power, the corresponding TPP difference set in the group $\mathcal{G} = \mathbb{F}_Q \times \mathbb{F}_{Q+2}$ is given by

$$\mathcal{D} = \{(x,y) \in \mathbb{F}_Q \times \mathbb{F}_{Q+2} : (x,y) \in S_Q \times S_{Q+2} \text{ or } (x,y) \in N_Q \times N_{Q+2} \text{ or } y = 0\}.$$

Here N_Q denotes all nonsquares in $\mathbb{F}_Q$ and S_Q denotes all nonzero squares in $\mathbb{F}_Q$. Note that the number of elements in N_Q and S_Q are both $\frac{Q-1}{2}$. Therefore, the corresponding harmonic difference set has parameters $\text{ETF}(\frac{Q^2+2Q-1}{2}, Q(Q+2))$.

Example 2.3.3. Let $Q = 3$ so that $\mathcal{G} = \mathbb{F}_3 \times \mathbb{F}_5$. Since $N_3 = \{2\}$ and $S_3 = \{1\}$, while $N_5 = \{2,3\}$ and $S_5 = \{1,4\}$, the TPP difference set in $\mathcal{G}$ is

$$\mathcal{D} = \{(1,1),(1,4),(2,2),(2,3),(0,0),(1,0),(2,0)\}.$$

This difference set results in an $\text{ETF}(7,15)$ which is equivalent to the $\text{ETF}(7,15)$ produced from the difference set in Example 2.3.1 and so will share all the same properties. This equivalence to a Singer difference set is not common to all TPP difference sets.

2.4 Quantum Information Theory Problem

A lot of the work in this field is motivated by the following quantum information theory problem: Design unit vectors $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ in $\mathbb{H}$, a D-dimensional Hilbert space, so that

$$\text{every self-adjoint operator } \mathbf{A} : \mathbb{H} \to \mathbb{H} \text{ can be recovered from } \{\boldsymbol{\varphi}_n^* \mathbf{A} \boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}. \quad (8)$$

Here, $\{\mathbf{A} : \mathbb{H} \to \mathbb{H} \mid \mathbf{A}^* = \mathbf{A}\}$ is a real Hilbert space under the Frobenius (Hilbert-Schmidt) inner product $\langle \mathbf{A}, \mathbf{B} \rangle_{\text{Fro}} = \text{Tr}(\mathbf{AB}) = \sum_{d=1}^{D} \langle \mathbf{u}_d, \mathbf{AB}\mathbf{u}_d \rangle$, where $\{\mathbf{u}_d\}_{d=1}^{D}$ is any orthonormal basis for $\mathbb{H}$. As such, $\{\boldsymbol{\varphi}_n^* \mathbf{A} \boldsymbol{\varphi}_n\}_{n \in \mathcal{N}}$ equates to measurements of the form $\boldsymbol{\varphi}_n^* \mathbf{A} \boldsymbol{\varphi}_n = \text{Tr}(\boldsymbol{\varphi}_n^* \mathbf{A} \boldsymbol{\varphi}_n) = \text{Tr}(\boldsymbol{\varphi}_n \boldsymbol{\varphi}_n^* \mathbf{A}) = \langle \mathbf{P}_n, \mathbf{A} \rangle_{\text{Fro}}$, where $\mathbf{P}_n = \boldsymbol{\varphi}_n \boldsymbol{\varphi}_n^*$ is the rank-one orthogonal projection onto the line spanned by $\boldsymbol{\varphi}_n$. Thus, satisfying (8) reduces to having $\{\mathbf{P}_n\}_{n \in \mathcal{N}}$ span the space $\{\mathbf{A} : \mathbb{H} \to \mathbb{H} \mid \mathbf{A} = \mathbf{A}^*\}$. This space has dimension D^2 when the underlying field $\mathbb{F}$ is $\mathbb{C}$ and has dimension $\binom{D+1}{2}$ when $\mathbb{F}$ is $\mathbb{R}$. Meanwhile, the dimension of $\text{span}(\{\mathbf{P}_n\}_{n \in \mathcal{N}})$ is the same as the rank of its Gram matrix; since

$$\langle \mathbf{P}_n, \mathbf{P}_{n'} \rangle_{\text{Fro}} = \text{Tr}(\mathbf{P}_n \mathbf{P}_{n'}) = \text{Tr}(\boldsymbol{\varphi}_n \boldsymbol{\varphi}_{n'}^* \boldsymbol{\varphi}_{n'} \boldsymbol{\varphi}_n^*) = \langle \boldsymbol{\varphi}_{n'}, \boldsymbol{\varphi}_n \rangle \langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'} \rangle = |\langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'} \rangle|^2, \tag{9}$$

the Gram matrix of $\{\mathbf{P}_n\}_{n \in \mathcal{N}}$ is the pointwise modulus squared $|\boldsymbol{\Phi}^* \boldsymbol{\Phi}|^2$ of the Gram matrix $\boldsymbol{\Phi}^* \boldsymbol{\Phi}$ of $\{\boldsymbol{\varphi}_n\}_{n \in \mathcal{N}}$. In summary, we see that for any $\{\boldsymbol{\varphi}_n\}_{n \in \mathcal{N}}$ in $\mathbb{H}$,

$$\text{rank}(|\boldsymbol{\Phi}^* \boldsymbol{\Phi}|^2) \leq \begin{cases} D^2, & \mathbb{F} = \mathbb{C}, \\ \binom{D+1}{2}, & \mathbb{F} = \mathbb{R}, \end{cases} \tag{10}$$

and further, $\{\boldsymbol{\varphi}_n\}_{n \in \mathcal{N}}$ achieves equality in (10) if and only if it satisfies property (8).

2.4.1 Mutually Unbiased Bases

One way to try to achieve (8) is to construct *mutually unbiased bases* (MUBs) for $\mathbb{H}$. To elaborate, for all $v = 1, ..., V$, let $\{\mathbf{u}_{v,d}\}_{d=1}^{D}$ be an orthonormal basis for $\mathbb{H}$ and let $\mathbf{U}_v$ be its (unitary) synthesis operator. The union $\{\mathbf{u}_{v,d}\}_{v=1,\, d=1}^{V \quad D}$ of these bases is a tight frame for $\mathbb{H}$: denoting its synthesis operator by $\mathbf{U}$ we have $\mathbf{U}\mathbf{U}^* = \sum_{v=1}^{V} \mathbf{U}_v \mathbf{U}_v^* = V\mathbf{I}$ and so $V^2 D = \|\mathbf{U}\mathbf{U}^*\|_{\text{Fro}}^2 = \|\mathbf{U}^*\mathbf{U}\|_{\text{Fro}}^2$. Then expanding

$\|\mathbf{U}^*\mathbf{U}\|_{\mathrm{Fro}}^2$ and bounding this quantity using the coherence of $\{\mathbf{u}_{v,d}\}_{v=1,\, d=1}^{V\;\;\, D}$ gives

$$V^2D = VD + \sum_{v=1}^{V}\sum_{\substack{v'=1\\ v'\neq v}}^{V}\sum_{d=1}^{D}\sum_{d'=1}^{D}|\langle\mathbf{u}_{v,d},\mathbf{u}_{v',d'}\rangle|^2 \leq VD + V(V-1)D^2\mathrm{coh}(\{\mathbf{u}_{v,d}\}_{v=1,\, d=1}^{V\;\;\, D}).$$

Therefore, the coherence of any union of orthonormal bases for $\mathbb{H}$ is bounded below by $\frac{1}{\sqrt{D}}$ and achieves equality here if and only if $|\langle\mathbf{u}_{v,d},\mathbf{u}_{v',d'}\rangle| = \frac{1}{\sqrt{D}}$ for all $v \neq v'$, that is, if and only if these bases are *mutually unbiased*.

Letting $\mathbf{J}_A$ be an all-ones matrix of size $A \times A$, if $\{\mathbf{u}_{v,d}\}_{v=1,\, d=1}^{V\;\;\, D}$ are MUBs for $\mathbb{H}$ with synthesis operator $\mathbf{U}$ then $|\mathbf{U}^*\mathbf{U}|^2 = \mathbf{I}_{VD} + \frac{1}{D}[(\mathbf{J}_V - \mathbf{I}_V)\otimes\mathbf{J}_D]$. Thus, $|\mathbf{U}^*\mathbf{U}|^2$ has eigenvalues V, 1, and 0 with multiplicities 1, $V(D-1)$, and $V-1$, respectively, and so $\mathrm{rank}(|\mathbf{U}^*\mathbf{U}|^2) = V(D-1)+1$. Therefore, by (10) when $\mathbb{F} = \mathbb{C}$, $V(D-1)+1 \leq D^2$, i.e. $V \leq D+1$; when $\mathbb{F} = \mathbb{R}$, $V(D-1)+1 \leq \binom{D+1}{2}$, i.e. $V \leq \frac{D+2}{2}$. When equality is achieved we say the MUBs are *maximal* meaning $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}} = \{\mathbf{u}_{v,d}\}_{v=1,\, d=1}^{V\;\;\, D}$ achieves equality in (10) and so satisfies (8). Maximal MUBs are known to exist when the dimension D is a prime power, however, their existence is open for dimensions as small as $D = 6$ for example [4].

2.4.2 Gerzon's Bound

Note that when $\mathbb{F} = \mathbb{C}$ maximal MUBs consist of $D(D+1)$ vectors. However, (10) suggests that (8) can be satisfied with as few as D^2 vectors. Ideally, these vectors should have small coherence. Therefore, if possible, we would like to construct D^2 vectors in $\mathbb{H}$ that form an ETF for $\mathbb{H}$. By a similar argument when $\mathbb{F} = \mathbb{R}$, if possible, we would like to construct $\binom{D+1}{2}$ vectors in $\mathbb{H}$ that form an ETF for $\mathbb{H}$. As we now explain, *Gerzon's bound* states that such ETFs are *maximal* in the sense that they consist of the largest possible number of equiangular vectors in $\mathbb{H}$.

Indeed, when $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is equiangular, but not collinear, there exists $W \in [0, 1)$, such that for all $n \neq n'$, $|\langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'} \rangle| = W$. Then by (9), $\langle \mathbf{P}_n, \mathbf{P}_{n'} \rangle_{\text{Fro}} = |\langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'} \rangle|^2 = W^2 \neq 1$. Therefore, the Gram matrix of $\{\mathbf{P}_n\}_{n \in \mathcal{N}}$ is $|\boldsymbol{\Phi}^* \boldsymbol{\Phi}|^2 = (1 - W^2)\mathbf{I} + W^2 \mathbf{J}$ which is positive definite and so $\text{rank}(|\boldsymbol{\Phi}^* \boldsymbol{\Phi}|^2) = N$. Thus by (10), $N \leq D^2$ when $\mathbb{F} = \mathbb{C}$ and $N \leq \binom{D+1}{2}$ when $\mathbb{F} = \mathbb{R}$.

For a maximal ETF to be real, D must be two less than an odd square, i.e. $D \in \{3, 7, 23, 47, ...\}$ [11]. It is known that maximal real ETFs exist for D being 3, 7, or 28 and do not exist for $D = 47$ [3, 11]. Complex maximal ETFs are also known as *symmetric, informationally complete, positive operator-valued measures* (SIC-POVMs). *Zauner's conjecture*, which remains open, is that SIC-POVMs exist for any dimension D [33]. SIC-POVMs have been proven to exist for D being 1-24, 28, 30, 31, 35, 37, 39, 43, 48, 124, and 323 [14, 19], while many other dimensions have numerical constructions.

2.4.3 Orthoplex Bound

When N exceeds Gerzon's bound, equiangularity is not attainable and so neither is the Welch bound (2). In this case, a new bound for minimal coherence can be derived. For unit norm vectors $\{\boldsymbol{\varphi}_n\}_{n \in \mathcal{N}}$, the corresponding projections $\{\mathbf{P}_n\}_{n \in \mathcal{N}}$ can be transformed to be traceless and therefore lie in the orthogonal complement of $\mathbf{I}$. Normalizing these gives the self-adjoint operators

$$\hat{\mathbf{P}}_n = \sqrt{\tfrac{D}{D-1}}(\mathbf{P}_n - \tfrac{1}{D}\mathbf{I}).$$

Here, $\langle \hat{\mathbf{P}}_n, \hat{\mathbf{P}}_{n'} \rangle_{\text{Fro}} = \frac{D}{D-1}\left(|\langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'} \rangle|^2 - \frac{1}{D}\right)$ since

$$\langle \mathbf{P}_n - \tfrac{1}{D}\mathbf{I}, \mathbf{P}_{n'} - \tfrac{1}{D}\mathbf{I} \rangle_{\text{Fro}} = \text{Tr}\left[(\mathbf{P}_n - \tfrac{1}{D}\mathbf{I})(\mathbf{P}_{n'} - \tfrac{1}{D}\mathbf{I})\right] = |\langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_{n'} \rangle|^2 - \tfrac{1}{D}. \quad (11)$$

Given any $D+2$ vectors in a real D-dimensional Hilbert space, one can show that at least two of these vectors have a nonnegative inner product with each other [6]. Since $\{\hat{\mathbf{P}}_n\}_{n\in\mathcal{N}}$ lie in $\mathbf{I}^\perp$, which is a subspace of $\{\mathbf{A}:\mathbb{H}\to\mathbb{H} \mid \mathbf{A}=\mathbf{A}^*\}$ of codimension one, we thus have that $0\leq\langle\hat{\mathbf{P}}_n,\hat{\mathbf{P}}_{n'}\rangle$ for some $n\neq n'$, provided $N>D^2$ when $\mathbb{F}=\mathbb{C}$, or $N>\binom{D+1}{2}$ when $\mathbb{F}=\mathbb{R}$. From (11) we have $0\leq\max\limits_{n\neq n'}\frac{D}{D-1}(|\langle\boldsymbol{\varphi}_n,\boldsymbol{\varphi}_{n'}\rangle|^2 - \frac{1}{D})$. Rearranging gives the *orthoplex bound* which takes over as the lower bound on coherence from the Welch bound when $N>D^2$ or $N>\binom{D+1}{2}$ for $\mathbb{F}=\mathbb{C}$ or $\mathbb{F}=\mathbb{R}$, respectively:

$$\tfrac{1}{\sqrt{D}}\leq\max_{n\neq n'}|\langle\boldsymbol{\varphi}_n,\boldsymbol{\varphi}_{n'}\rangle|.$$

In particular, maximal MUBs achieve the orthoplex bound.

III. Harmonic ETFs comprised of regular simplices

3.1 Fine Difference Sets

In [10] it was shown that certain harmonic ETFs are disjoint unions of simplices. To elaborate, let $\mathcal{D}$ be a D-element difference set for a finite abelian group $\mathcal{G}$ of order G and let $\{\boldsymbol{\varphi}_\gamma\}_{\gamma\in\hat{\mathcal{G}}} = \{\gamma\chi_{\mathcal{D}}\}$ be the corresponding harmonic ETF for $\mathbb{C}^{\mathcal{D}}$. If there exists a subgroup $\mathcal{H}$ of $\mathcal{G}$ of order $H := \frac{G}{S+1}$ where $S := \sqrt{\frac{D(G-1)}{G-D}}$ is the reciprocal Welch bound (2) such that $\mathcal{D}\cap\mathcal{H} = \emptyset$, then $\{\boldsymbol{\varphi}_\gamma\}_{\gamma\in\hat{\mathcal{G}}}$ is a disjoint union of H regular S-simplices. These simplices correspond to the cosets of $\mathcal{H}$, that is, for all $\gamma\in\hat{\mathcal{G}}$, the $S+1$ vectors $\{\boldsymbol{\varphi}_{\gamma'}\}_{\gamma'\in\gamma\mathcal{H}^\perp}$ form a simplex for their span and satisfy $\sum_{\gamma'\in\gamma\mathcal{H}^\perp}\boldsymbol{\varphi}_{\gamma'} = 0$. Note that S must necessarily be an integer. The results presented in this and the next chapter will rely upon having a difference set that satisfies these properties, therefore, we give these difference sets a name:

Definition 3.1.1. A difference set $\mathcal{D}$ of cardinality D in a finite abelian group $\mathcal{G}$ of order G is *fine* if there exists a subgroup $\mathcal{H}$ of $\mathcal{G}$ with $H := \#(\mathcal{H}) = \frac{G}{S+1}$ where $S = \sqrt{\frac{D(G-1)}{G-D}}$ and such that $\mathcal{D}\cap\mathcal{H} = \emptyset$.

We call these differences sets fine because as the following theorem will show they are disjoint from the largest subgroup $\mathcal{H}$ possible. Further, we show that all nontrivial shifts of $\mathcal{D}$ intersect with $\mathcal{H}$ in the same number of points. To show this it helps to establish the following notation:

$$\mathcal{D}_g = (\mathcal{D}-g)\cap\mathcal{H}, \forall g\in\mathcal{G}. \tag{12}$$

Theorem 3.1.2. *Let $\mathcal{D}$ be a D-element difference set for a finite abelian group $\mathcal{G}$ of order G. Then if $\mathcal{H}$ is a subgroup of $\mathcal{G}$ of order H that is disjoint from $\mathcal{D}$, $H\le\frac{G}{S+1}$ where $S := \sqrt{\frac{D(G-1)}{G-D}}$. Moreover, the following are equivalent:*

(i) $H = \frac{G}{S+1}$,

(ii) $(\mathbf{F}^*\chi_{\mathcal{D}})(\gamma) = -\frac{D}{S}$ *for all* $\gamma \in \mathcal{H}^\perp$, $\gamma \neq \mathbf{1}$,

(iii) $\#(\mathcal{D}_g) = \frac{D}{S}$ *for all* $g \notin \mathcal{H}$.

Proof. Since $\mathcal{H} \cap \mathcal{D} = \emptyset$, we have $0 = \langle \chi_{\mathcal{H}}, \chi_{\mathcal{D}} \rangle = \frac{1}{G}\langle \mathbf{F}^*\chi_{\mathcal{H}}, \mathbf{F}^*\chi_{\mathcal{D}} \rangle$. Then by the Poisson summation formula,

$$0 = \tfrac{H}{G}\langle \chi_{\mathcal{H}^\perp}, \mathbf{F}^*\chi_{\mathcal{D}} \rangle = \tfrac{H}{G}\sum_{\gamma \in \mathcal{H}^\perp} (\mathbf{F}^*\chi_{\mathcal{D}})(\gamma) = \tfrac{HD}{G} + \tfrac{H}{G}\sum_{\substack{\gamma \in \mathcal{H}^\perp \\ \gamma \neq \mathbf{1}}} (\mathbf{F}^*\chi_{\mathcal{D}})(\gamma).$$

Multiplying by $\frac{GS}{DH}$ and subtracting the summation we have that

$$S = -\sum_{\substack{\gamma \in \mathcal{H}^\perp \\ \gamma \neq \mathbf{1}}} \tfrac{S}{D}(\mathbf{F}^*\chi_{\mathcal{D}})(\gamma). \tag{13}$$

Taking the modulus of this equation and applying (7) gives that $H \leq \frac{G}{S+1}$:

$$S = |S| = |-\sum_{\substack{\gamma \in \mathcal{H}^\perp \\ \gamma \neq \mathbf{1}}} \tfrac{S}{D}(\mathbf{F}^*\chi_{\mathcal{D}})(\gamma)| \leq \sum_{\substack{\gamma \in \mathcal{H}^\perp \\ \gamma \neq \mathbf{1}}} \tfrac{S}{D}|(\mathbf{F}^*\chi_{\mathcal{D}})(\gamma)| = (\tfrac{G}{H} - 1).$$

(i⇔ii) Here since $\#(\mathcal{H}^\perp) = \frac{G}{H}$, (13) is a summation of $\frac{G}{H} - 1$ terms each of which has modulus one by (7). If $H = \frac{G}{S+1}$, then this summation has S unimodular terms and can only equal S if $(\mathbf{F}^*\chi_{\mathcal{D}})(\gamma) = -\frac{D}{S}$ for all $\gamma \in \mathcal{H}^\perp$, $\gamma \neq \mathbf{1}$. Conversely, if $(\mathbf{F}^*\chi_{\mathcal{D}})(\gamma) = -\frac{D}{S}$ for all $\gamma \in \mathcal{H}^\perp$, $\gamma \neq \mathbf{1}$ then (13) gives that $H = \frac{G}{S+1}$.

(i⇔iii) Let $S' := \frac{G}{H} - 1$ and let $\{g_s\}_{s=1}^{S'}$ be coset representatives of the nontrivial cosets of $\mathcal{H}$. For each $s = 1, ..., S'$ recall that $\mathcal{D}_{g_s} := (\mathcal{D} - g_s) \cap \mathcal{H}$ and let $D_{g_s} := \#(\mathcal{D}_{g_s})$. Since $\mathcal{D} \cap \mathcal{H} = \emptyset$, $\sum_{s=1}^{S'} D_{g_s} = D$ and for any $d, d' \in \mathcal{D}$, $d - d' \in \mathcal{H}$ if and only if there exists s such that $d, d' \in \mathcal{D}_{g_s}$. Then the number of entries of the difference table of $\mathcal{D}$

that are nonzero elements of $\mathcal{H}$ is

$$\#\{(d,d') \in \mathcal{D} \times \mathcal{D} : d - d' \in \mathcal{H} \setminus \{0\}\} = \sum_{s=1}^{S'} (D_{g_s}^2 - D_{g_s}) = -D + \sum_{s=1}^{S'} D_{g_s}^2.$$

Since $\mathcal{D}$ is a difference set for $\mathcal{G}$, $\#\{(d,d') \in \mathcal{D} \times \mathcal{D} : d - d' = g\} = \frac{D(D-1)}{G-1}$ for any nonzero $g \in \mathcal{G}$. As such, the number of nonzero members of $\mathcal{H}$ in the difference table of $\mathcal{D}$ is equivalently $\#\{(d,d') \in \mathcal{D} \times \mathcal{D} : d - d' \in \mathcal{H} \setminus \{0\}\} = \frac{D(D-1)(H-1)}{G-1}$. Therefore, $\frac{D(D-1)(H-1)}{G-1} = -D + \sum_{s=1}^{S'} D_{g_s}^2$. Thus

$$\frac{(D-1)(H-1)}{G-1} = -1 + \frac{1}{D} \sum_{s=1}^{S'} [(D_{g_s} - \frac{D}{S'}) + \frac{D}{S'}]^2 = \frac{D}{S'} - 1 + \frac{1}{D} \sum_{s=1}^{S'} (D_{g_s} - \frac{D}{S'})^2 \geq \frac{D}{S'} - 1 \quad (14)$$

where equality holds if and only if $D_{g_s} = \frac{D}{S'}$ for all $s = 1, ..., S'$. Solving this inequality for H gives $H \geq \frac{1}{D-1}(G-1)(\frac{D}{S'} - 1) + 1 = \frac{1}{D-1}[\frac{D(G-1)}{S'} - (G-D)]$. Multiplying by $(S'+1)(D-1)$ yields

$$G(D-1) \geq \frac{D(G-1)(S'+1)}{S'} - (G-D)(S'+1) = D(G-1) + \frac{D(G-1)}{S'} - (G-D)(S'+1).$$

Subtracting $G(D-1)$ from both sides we have

$$0 \geq (G-D) + \frac{D(G-1)}{S'} - (G-D)(S'+1) = \frac{D(G-1)}{S'} - S'(G-D).$$

Finally this gives that $(S')^2 \geq \frac{D(G-1)}{G-D} = S^2$. Since equality held in (14) if and only if $D_{g_s} = \frac{D}{S'}$ for all $s = 1, ..., S'$ and each step since has been reversible, we have that $S' = S$ if and only if $D_{g_s} = \frac{D}{S}$ for all $s = 1, ..., S$. Finally, $\frac{G}{H} - 1 = S' \geq S$ and so $H \leq \frac{G}{S+1}$ where equality holds if and only if $D_{g_s} = \frac{D}{S}$ for all $s = 1, ..., S$. □

This shows that for a fine difference set S must necessarily divide D since $\#(\mathcal{D}_g)$ must be an integer. Further, when $\mathcal{D}$ is a fine difference set, using (6) we know that

$D - \Lambda_{\mathcal{D}} = D - \frac{D(D-1)}{G-1} = \frac{D(G-D)}{G-1} = \frac{D^2}{S^2}$. Therefore, we have that $\frac{D}{S} = \sqrt{D - \Lambda_{\mathcal{D}}}$ and so $\sqrt{D - \Lambda_{\mathcal{D}}}$ must also be an integer.

Example 3.1.3. Recall from Example 2.3.1 that when $Q = 2$ and $J = 4$ the corresponding Singer difference set is $\{0, 1, 2, 4, 5, 8\}$ in the group $\mathcal{G} = \mathbb{Z}_{15}$. Here we have that $S = \frac{7}{2}$ and so the corresponding harmonic ETF cannot be a disjoint union of simplices. However, its complement is $\mathcal{D} = \{3, 6, 7, 11, 12, 13, 14\}$ and corresponds to an ETF$(8, 15)$ which has the inverse Welch bound $S = 4$. Further, this difference set avoids a subgroup of $\mathcal{G}$ of order $H = \frac{G}{S+1} = 3$, namely the subgroup $\mathcal{H} = \{0, 5, 10\}$. Therefore, $\mathcal{D}$ is fine and the ETF$(8, 15)$ is a disjoint union of 3 regular 4-simplices.

Further, since the cosets of $\mathcal{H}$ partition $\mathcal{G}$, we can partition $\mathcal{D}$ into its intersections with these cosets giving

$$\mathcal{D} = \emptyset \cap \{6, 11\} \cap \{7, 12\} \cap \{13, 3\} \cap \{9, 14\}.$$

For this reason we will order $\mathcal{D}$ as $\mathcal{D} = \{6, 11, 7, 12, 13, 3, 9, 14\}$ from here forward. To develop certain results later we will need to express this partition of $\mathcal{D}$ instead as subsets of $\mathcal{H}$. From Theorem 3.1.2, for all $g \in \mathcal{G}$, each $\mathcal{D}_g$ is a subset of $\mathcal{H}$ of cardinality $\frac{D}{S} = 2$. These sets are $\mathcal{D}_0 = \mathcal{D}_5 = \mathcal{D}_{10} = \emptyset$, $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_4 = \mathcal{D}_8 = \{5, 10\}$, $\mathcal{D}_6 = \mathcal{D}_7 = \mathcal{D}_9 = \mathcal{D}_{13} = \{0, 5\}$, and $\mathcal{D}_3 = \mathcal{D}_{11} = \mathcal{D}_{12} = \mathcal{D}_{14} = \{0, 10\}$.

To construct the ETF$(8, 15)$ we extract the rows corresponding to $\mathcal{D}$ of the 15×15 character table for $\mathbb{Z}_{15}$ indexed by $\mathcal{G} \times \hat{\mathcal{G}}$ and scale it so that each vector is unit norm. Here we let $\omega = e^{\frac{2\pi i}{15}}$ and represent $\hat{\mathcal{G}}$ as $\mathbb{Z}_{15}$, identifying $n \in \mathbb{Z}_{15}$ with the character

$\gamma(m) = \omega^{mn}$. Therefore, the columns of the following matrix form an ETF$(8, 15)$:

$$\mathbf{\Phi} = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & \omega^6 & \omega^{12} & \omega^3 & \omega^9 & 1 & \omega^6 & \omega^{12} & \omega^3 & \omega^9 & 1 & \omega^6 & \omega^{12} & \omega^3 & \omega^9 \\ 1 & \omega^{11} & \omega^7 & \omega^3 & \omega^{14} & \omega^{10} & \omega^6 & \omega^2 & \omega^{13} & \omega^9 & \omega^5 & \omega^1 & \omega^{12} & \omega^8 & \omega^4 \\ 1 & \omega^7 & \omega^{14} & \omega^6 & \omega^{13} & \omega^5 & \omega^{12} & \omega^4 & \omega^{11} & \omega^3 & \omega^{10} & \omega^2 & \omega^9 & \omega^1 & \omega^8 \\ 1 & \omega^{12} & \omega^9 & \omega^6 & \omega^3 & 1 & \omega^{12} & \omega^9 & \omega^6 & \omega^3 & 1 & \omega^{12} & \omega^9 & \omega^6 & \omega^3 \\ 1 & \omega^{13} & \omega^{11} & \omega^9 & \omega^7 & \omega^5 & \omega^3 & \omega^1 & \omega^{14} & \omega^{12} & \omega^{10} & \omega^8 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} \\ 1 & \omega^9 & \omega^3 & \omega^{12} & \omega^6 & 1 & \omega^9 & \omega^3 & \omega^{12} & \omega^6 & 1 & \omega^9 & \omega^3 & \omega^{12} & \omega^6 \\ 1 & \omega^{14} & \omega^{13} & \omega^{12} & \omega^{11} & \omega^{10} & \omega^9 & \omega^8 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{bmatrix}.$$

Moreover, since $\mathcal{D}$ is fine, we know that any subset of 5 vectors from this ETF corresponding to a coset of $\mathcal{H}^\perp = \{0, 3, 6, 9, 12\} \subseteq \hat{\mathcal{G}}$ form a simplex for their span. Choosing to index these simplices by $\mathcal{H}^\perp$, $1 + \mathcal{H}^\perp$ and $2 + \mathcal{H}^\perp$ gives the following three simplices:

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_0 & \mathbf{\Phi}_1 & \mathbf{\Phi}_2 \end{bmatrix} = \tfrac{1}{\sqrt{8}} \left[\begin{array}{ccccc|ccccc|ccccc} 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^6 & \omega^9 & \omega^{12} & 1 & \omega^3 & \omega^{12} & 1 & \omega^3 & \omega^6 & \omega^9 \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{11} & \omega^{14} & \omega^2 & \omega^5 & \omega^8 & \omega^7 & \omega^{10} & \omega^{13} & \omega & \omega^4 \\ 1 & \omega^6 & \omega^{12} & \omega^3 & \omega^9 & \omega^7 & \omega^{13} & \omega^4 & \omega^{10} & \omega & \omega^{14} & \omega^5 & \omega^{11} & \omega^2 & \omega^8 \\ 1 & \omega^6 & \omega^{12} & \omega^3 & \omega^9 & \omega^{12} & \omega^3 & \omega^9 & 1 & \omega^6 & \omega^9 & 1 & \omega^6 & \omega^{12} & \omega^3 \\ 1 & \omega^9 & \omega^3 & \omega^{12} & \omega^6 & \omega^{13} & \omega^7 & \omega & \omega^{10} & \omega^4 & \omega^{11} & \omega^5 & \omega^{14} & \omega^8 & \omega^2 \\ 1 & \omega^9 & \omega^3 & \omega^{12} & \omega^6 & \omega^3 & \omega^{12} & \omega^6 & 1 & \omega^9 & \omega^6 & 1 & \omega^9 & \omega^3 & \omega^{12} \\ 1 & \omega^{12} & \omega^9 & \omega^6 & \omega^3 & \omega^9 & \omega^6 & \omega^3 & 1 & \omega^{12} & \omega^3 & 1 & \omega^{12} & \omega^9 & \omega^6 \\ 1 & \omega^{12} & \omega^9 & \omega^6 & \omega^3 & \omega^{14} & \omega^{11} & \omega^8 & \omega^5 & \omega^2 & \omega^{13} & \omega^{10} & \omega^7 & \omega^4 & \omega \end{array}\right].$$

3.2 A Construction of EITFFs

In [10] it was shown that the complements of appropriately shifted Singer difference sets with J even, all McFarland difference sets, and the complements of TPP difference sets are all fine. Further it was shown that if an ETF for $\mathbb{H}$ is a disjoint union of simplices, then the subspaces spanned by those simplices form an ECTFF for $\mathbb{H}$. We now establish some notation to show that for fine difference sets with certain additional properties, these ECTFFs for $\mathbb{H}$ are further EITFFs for $\mathbb{H}$.

Let $\{\boldsymbol{\varphi}_\gamma\}_{\gamma\in\hat{\mathcal{G}}}$ be a harmonic ETF arising from the fine difference set $\mathcal{D}$. Then the synthesis operator of $\{\boldsymbol{\varphi}_\gamma\}_{\gamma\in\hat{\mathcal{G}}}$, after scaling so that each vector has unit norm, is

$$\mathbf{\Phi} \in \mathbb{C}^{\mathcal{D}\times\hat{\mathcal{G}}}, \qquad \mathbf{\Phi}(d,\gamma) = \tfrac{1}{\sqrt{D}}\gamma(d).$$

Further we know this harmonic ETF is a disjoint union of simplices corresponding to the cosets of $\mathcal{H}^\perp$. That is for all $\gamma \in \hat{\mathcal{G}}$, $\{\boldsymbol{\varphi}_{\gamma'}\}_{\gamma'\in\gamma\mathcal{H}^\perp}$ is a regular simplex for its span and we define $\mathbf{\Phi}_\gamma$ to be its synthesis operator:

$$\mathbf{\Phi}_\gamma \in \mathbb{C}^{\mathcal{D}\times\mathcal{H}^\perp}, \qquad \mathbf{\Phi}_\gamma(d,\gamma') = \tfrac{1}{\sqrt{D}}(\gamma\gamma')(d). \tag{15}$$

By this definition, $\mathbf{\Phi}_\gamma$ is dependent on the choice of coset representative. However, if $\overline{\gamma} = \overline{\gamma}'$, then $\mathbf{\Phi}_\gamma = \mathbf{\Phi}_{\gamma'}\mathbf{T}^{\gamma(\gamma')^{-1}}$ where $\mathbf{T}^\gamma$ is the "translation by γ operator over $\mathcal{H}^\perp$" defined by $\mathbf{T}^\gamma(\gamma_1,\gamma_2) = 1$ if and only if $\gamma_1\gamma_2^{-1} = \gamma$ for all $\gamma,\gamma_1,\gamma_2 \in \mathcal{H}^\perp$. Because of this relationship between $\mathbf{\Phi}_\gamma$ and $\mathbf{\Phi}_{\gamma'}$ when γ and γ' are in the same coset of $\mathcal{H}^\perp$ we know that the column spaces of $\mathbf{\Phi}_\gamma$ and $\mathbf{\Phi}_{\gamma'}$ are equal. In Theorem 3.2.1 we will verify that for all $\gamma \in \hat{\mathcal{G}}$, $\mathbf{\Phi}_\gamma$ is indeed a regular S-simplex in a D-dimensional Hilbert space by showing it is an isometry of the canonical regular S-simplex in an $(S+1)$-dimensional Hilbert space.

The synthesis operator of this canonical regular S-simplex is

$$\mathbf{\Psi} \in \mathbb{C}^{(\mathcal{G}/\mathcal{H})\backslash\{\overline{0}\}\times\mathcal{H}^{\perp}}, \qquad \mathbf{\Psi}(\overline{g},\gamma) = \tfrac{1}{\sqrt{S}}\gamma(g). \tag{16}$$

Intuitively, $\mathbf{\Psi}$ is a regular S-simplex since the character table of $\mathcal{G}/\mathcal{H}$ is the $(\mathcal{G}/\mathcal{H}) \times \mathcal{H}^{\perp}$ indexed matrix whose $(\overline{g},\gamma)$th entry is $\gamma(g)$. Then $\mathbf{\Psi}$ is a Naimark complement of the first row of this matrix, which is all ones, and so $\mathbf{\Psi}$ is a regular S-simplex.

To computationally verify this, first note that $\mathbf{\Psi}$ is well-defined: If $\overline{g} = \overline{g}'$, then $g - g' \in \mathcal{H}$. Therefore, for any $\gamma \in \mathcal{H}^{\perp}$, $\gamma(g)\gamma^{-1}(g') = \gamma(g-g') = 1$. Multiplying by $\frac{1}{\sqrt{S}}\gamma(g')$ then gives that $\mathbf{\Psi}(\overline{g},\gamma) = \mathbf{\Psi}(\overline{g}',\gamma)$. To continue, recall that $(\mathcal{G}/\mathcal{H})^\wedge \cong \mathcal{H}^{\perp}$ and so $\#(\mathcal{H}^{\perp}) = S+1$. Then $\mathbf{\Psi}$ is a regular $S-$simplex since

$$(\mathbf{\Psi}^*\mathbf{\Psi})(\gamma,\gamma') = \sum_{\substack{\overline{g}\in\mathcal{G}/\mathcal{H}\\ \overline{g}\neq\overline{0}}} \mathbf{\Psi}^*(\gamma,\overline{g})\mathbf{\Psi}(\overline{g},\gamma') = \sum_{\substack{\overline{g}\in\mathcal{G}/\mathcal{H}\\ \overline{g}\neq\overline{0}}} \tfrac{1}{S}[\gamma(g)]^*\gamma'(g)$$

and applying $[\gamma(g)]^* = \gamma^{-1}(g)$ together with the Poisson summation formula gives

$$(\mathbf{\Psi}^*\mathbf{\Psi})(\gamma,\gamma') = \sum_{\substack{\overline{g}\in\mathcal{G}/\mathcal{H}\\ \overline{g}\neq\overline{0}}} \tfrac{1}{S}(\gamma^{-1}\gamma')(g) = \begin{cases} 1, & \gamma = \gamma', \\ -\frac{1}{S}, & \gamma \neq \gamma'. \end{cases}$$

For all $\gamma \in \hat{\mathcal{G}}$ we would like $\mathbf{\Phi}_\gamma = \mathbf{E}_\gamma\mathbf{\Psi}$ where $\mathbf{E}_\gamma \in \mathbb{C}^{\mathcal{D}\times(\mathcal{G}/\mathcal{H})\backslash\{\overline{0}\}}$ is an isometry. Applying $\mathbf{\Psi}^*$ to the right gives $\mathbf{\Phi}_\gamma\mathbf{\Psi}^* = \frac{S+1}{S}\mathbf{E}_\gamma$. Applying (15) and (16) we have

$$\mathbf{E}_\gamma(d,\overline{g}) = \tfrac{S}{S+1}\sum_{\gamma'\in\mathcal{H}^{\perp}} \mathbf{\Phi}_\gamma(d,\gamma')\mathbf{\Psi}^*(\gamma',\overline{g}) = \tfrac{\sqrt{S}\gamma(d)}{\sqrt{D}(S+1)}\sum_{\gamma'\in\mathcal{H}^{\perp}} \gamma'(d-g).$$

Then by the Poisson summation formula we find that in order for $\mathbf{E}_\gamma$ to satisfy

$\mathbf{\Phi}_\gamma = \mathbf{E}_\gamma \mathbf{\Psi}$ we must have

$$\mathbf{E}_\gamma(d,\overline{g}) = \tfrac{\sqrt{S}\gamma(d)}{\sqrt{D}(S+1)} \begin{cases} S+1, & d-g\in\mathcal{H}, \\ 0, & d-g\notin\mathcal{H} \end{cases} = \sqrt{\tfrac{S}{D}} \begin{cases} \gamma(d), & \overline{d}=\overline{g}, \\ 0, & \overline{d}\neq\overline{g}. \end{cases}$$

With this established notation, we now prove, for all $\gamma \in \hat{\mathcal{G}}$, $\mathbf{E}_\gamma$ is an isometry and so $\mathbf{\Phi}_\gamma$ is a regular S-simplex. Further these isometries have the property that their cross-Gram matrices are diagonal and the subspaces of $\mathbb{C}^{\mathcal{D}}$ spanned by their column spaces, which are the same subspaces spanned by the simplices of the ETF, form an ECTFF for $\mathbb{C}^{\mathcal{D}}$. Note that from [10] we already know that these subspaces form an ECTFF for $\mathbb{C}^{\mathcal{D}}$, however, our proof of this fact is different from the one given in [10]. Further, we show that for certain fine difference sets these ECTFFs also form an EITFF for $\mathbb{C}^{\mathcal{D}}$.

Theorem 3.2.1. *Let $\mathcal{D}$ be a fine difference set in a finite abelian group $\mathcal{G}$ as given by Definition 3.1.1 and let $\mathbf{\Phi}_\gamma$ and $\mathbf{\Psi}$ be defined by* (15) *and* (16)*, respectively. For any $\gamma \in \hat{\mathcal{G}}$, let*

$$\mathbf{E}_\gamma \in \mathbb{C}^{\mathcal{D}\times\mathcal{G}/\mathcal{H}\backslash\{\overline{0}\}}, \qquad \mathbf{E}_\gamma(d,\overline{g}) := \tfrac{\sqrt{S}}{\sqrt{D}} \begin{cases} \gamma(d), & \overline{d}=\overline{g}, \\ 0, & \overline{d}\neq\overline{g}. \end{cases} \tag{17}$$

Then

(a) $\mathbf{E}_\gamma$ is an isometry, i.e. $\mathbf{E}_\gamma^*\mathbf{E}_\gamma = \mathbf{I}$, and $\mathbf{\Phi}_\gamma = \mathbf{E}_\gamma\mathbf{\Psi}$.

(b) $\mathbf{E}_\gamma^*\mathbf{E}_{\gamma'}$ is a diagonal matrix with

$$(\mathbf{E}_\gamma^*\mathbf{E}_{\gamma'})(\overline{g},\overline{g}) = \tfrac{S}{D}\sum_{\substack{d\in\mathcal{D}\\ \overline{d}=\overline{g}}}(\gamma^{-1}\gamma')(d). \tag{18}$$

(c) If $\mathcal{U}_\gamma := C(\mathbf{\Phi}_\gamma) = C(\mathbf{E}_\gamma)$ is the subspace spanned by both $\mathbf{E}_\gamma$ and $\mathbf{\Phi}_\gamma$ then

$\{\mathcal{U}_{\overline{\gamma}}\}_{\overline{\gamma}\in\hat{\mathcal{G}}/\mathcal{H}^\perp}$ *form an ECTFF for* $\mathbb{C}^{\mathcal{D}}$.

(d) $\{\mathcal{U}_{\overline{\gamma}}\}_{\overline{\gamma}\in\hat{\mathcal{G}}/\mathcal{H}^\perp}$ *form an EITFF for* $\mathbb{C}^{\mathcal{D}}$ *if and only if* $\mathcal{D}_g$ (12) *is a difference set for* $\mathcal{H}$ *for all* $g \in \mathcal{G}$.

(e) If $\mathcal{D}_g$ (12) *is a difference set for* $\mathcal{H}$ *for all* $g \in \mathcal{G}$ *then* $|(\mathbf{E}_\gamma^*\mathbf{E}_{\gamma'})(\overline{g},\overline{g})|^2 = \frac{1}{S}$ *for all* $\overline{g} \in \mathcal{G}/\mathcal{H}$, $\overline{g} \neq \overline{0}$.

Proof. For all $\gamma, \gamma' \in \hat{\mathcal{G}}$ we have

$$(\mathbf{E}_\gamma^*\mathbf{E}_{\gamma'})(\overline{g},\overline{g}') = \sum_{d\in\mathcal{D}} \mathbf{E}_\gamma^*(\overline{g},d)\mathbf{E}_{\gamma'}(d,\overline{g}') = \tfrac{S}{D}\begin{cases} \sum\limits_{\substack{d\in\mathcal{D}\\ \overline{d}=\overline{g}}} (\gamma^{-1}\gamma')(d), & \overline{g} = \overline{g}', \\ 0, & \text{else}, \end{cases}$$

giving (b). Since $\mathcal{D}$ is fine, by Theorem 3.1.2, $\#(\mathcal{D}_g) = \frac{D}{S}$ for all $g \notin \mathcal{H}$ and so

$$(\mathbf{E}_\gamma^*\mathbf{E}_{\gamma})(\overline{g},\overline{g}') = \tfrac{S}{D}\left\{\begin{matrix} \sum\limits_{\substack{d\in\mathcal{D}\\ \overline{d}=\overline{g}}} 1, & \overline{g} = \overline{g}', \\ 0, & \text{else}, \end{matrix}\right\} = \mathbf{I}(\overline{g},\overline{g}').$$

Applying $\mathbf{E}_\gamma$ to $\mathbf{\Psi}$ gives (a) since

$$(\mathbf{E}_\gamma\mathbf{\Psi})(d,\gamma') = \sum_{\substack{\overline{g}\in\mathcal{G}/\mathcal{H}\\ \overline{g}\neq\overline{0}}} \mathbf{E}_\gamma(d,\overline{g})\mathbf{\Psi}(\overline{g},\gamma') = \tfrac{1}{\sqrt{D}}\gamma(g)\gamma'(g) = \mathbf{\Phi}_\gamma(d,\gamma').$$

For any $d \in \mathcal{D}$, $\overline{d} = \overline{g}$ if and only if $d - g \in \mathcal{H}$ or equivalently $d - g \in \mathcal{D}_g$ which happens precisely when $d \in g + \mathcal{D}_g$. Applying this gives that

$$\#\{(d,d') \in \mathcal{D}\times\mathcal{D} : g' = d'-d, \overline{d} = \overline{d}' = \overline{g}\} = \#\{(d,d') \in (g+\mathcal{D}_g)\times(g+\mathcal{D}_g) : g' = d'-d\}.$$

Therefore,

$$\begin{aligned}\#\{(d,d')\in\mathcal{D}\times\mathcal{D} : g' = d'-d, \overline{d} = \overline{d}' = \overline{g}\} &= \#\{d\in g+\mathcal{D}_g : d\in(g-g')+\mathcal{D}_g\}\\ &= \#\left[\mathcal{D}_g\cap(\mathcal{D}_g-g')\right]\\ &= (\chi_{\mathcal{D}_g}*\tilde{\chi}_{\mathcal{D}_g})(-g').\end{aligned}$$

Taking the modulus squared of (18) and applying this equality gives

$$\begin{aligned}|(\mathbf{E}_\gamma^*\mathbf{E}_{\gamma'})(\overline{g},\overline{g})|^2 &= \tfrac{S^2}{D^2}\sum_{\substack{d\in\mathcal{D}\\ \overline{d}=\overline{g}}}\sum_{\substack{d'\in\mathcal{D}\\ \overline{d}'=\overline{g}}}(\gamma^{-1}\gamma')(d'-d) &(19)\\ &= \tfrac{S^2}{D^2}\sum_{g'\in\mathcal{G}}(\gamma^{-1}\gamma')(g')(\chi_{\mathcal{D}_g}*\tilde{\chi}_{\mathcal{D}_g})(-g')\\ &= \tfrac{S^2}{D^2}[\mathbf{F}^*(\chi_{\mathcal{D}_g}*\tilde{\chi}_{\mathcal{D}_g})](\gamma^{-1}\gamma') &(20)\\ &= \tfrac{S^2}{D^2}|(\mathbf{F}^*\chi_{\mathcal{D}_g})(\gamma^{-1}\gamma')|^2.\end{aligned}$$

Then the singular values of the cross-Grams are $|(\mathbf{E}_\gamma^*\mathbf{E}_{\gamma'})(\overline{g},\overline{g})| = \frac{S}{D}|(\mathbf{F}^*\chi_{\mathcal{D}_g})(\gamma^{-1}\gamma')|$. These singular values are constant if and only if $\mathcal{D}_g$ is a difference set for $\mathcal{H}$. This proves (e).

To prove (c) first note that

$$\begin{aligned}\sum_{\substack{\overline{g}\in\mathcal{G}/\mathcal{H}\\ \overline{g}\neq\overline{0}}}(\chi_{\mathcal{D}_g}*\tilde{\chi}_{\mathcal{D}_g})(-g') &= \sum_{\substack{\overline{g}\in\mathcal{G}/\mathcal{H}\\ \overline{g}\neq\overline{0}}}\#\{(d,d')\in\mathcal{D}\times\mathcal{D} : g'=d'-d, \overline{d}=\overline{d}'=\overline{g}\}\\ &= \#\Big(\bigcup_{\substack{\overline{g}\in\mathcal{G}/\mathcal{H}\\ \overline{g}\neq\overline{0}}}\{(d,d')\in\mathcal{D}\times\mathcal{D} : g'=d'-d, \overline{d}=\overline{d}'=\overline{g}\}\Big).\end{aligned}$$

Since this is a disjoint union and $\mathcal{D}\cap\mathcal{H}=\emptyset$ we have

$$\begin{aligned}\sum_{\substack{\overline{g}\in\mathcal{G}/\mathcal{H}\\ \overline{g}\neq\overline{0}}}(\chi_{\mathcal{D}_g}*\tilde{\chi}_{\mathcal{D}_g})(-g') &= \#\{(d,d')\in\mathcal{D}\times\mathcal{D}: g'=d'-d, \overline{d}=\overline{d}'\}\\ &= \#\{(d,d')\in\mathcal{D}\times\mathcal{D}: g'=d'-d\in\mathcal{H}\}\\ &= [(D-\Lambda_{\mathcal{D}})\boldsymbol{\delta}_0+G\Lambda_{\mathcal{D}}\chi_{\mathcal{H}}](g)\end{aligned}$$

where $\Lambda_{\mathcal{D}}$ is as defined by (6). Therefore, applying this to (20) gives

$$\|\mathbf{E}_\gamma^*\mathbf{E}_{\gamma'}\|_{\text{Fro}}^2=\sum_{\substack{\overline{g}\in\mathcal{G}/\mathcal{H}\\ \overline{g}\neq\overline{0}}}\tfrac{S^2}{D^2}[\mathbf{F}^*(\chi_{\mathcal{D}_g}*\tilde{\chi}_{\mathcal{D}_g})](\gamma^{-1}\gamma')=\tfrac{S^2}{D^2}[(D-\Lambda_{\mathcal{D}})\mathbf{1}+G\Lambda_{\mathcal{D}}\chi_{\mathcal{H}^\perp}](\gamma^{-1}\gamma')$$

which is constant over all $\gamma^{-1}\gamma'\notin\mathcal{H}^\perp$ thus proving (c).

Now assume that $\mathcal{D}_g$ is a difference set for $\mathcal{H}$. Continuing from (19), we separate from the summation the case when $d'=d$ and in the remaining sum we apply the fact that $(\mathcal{D}-g)\cap\mathcal{H}$ is a nonempty difference set for $\mathcal{H}$ for all nonzero $g\in\mathcal{G}$. This gives (d) since

$$\begin{aligned}|(\mathbf{E}_\gamma^*\mathbf{E}_{\gamma'})(\overline{g},\overline{g}')|^2 &= \tfrac{S^2}{D^2}[\tfrac{D}{S}(\gamma^{-1}\gamma')(0)+\tfrac{1}{H-1}(\tfrac{D}{S})(\tfrac{D}{S}-1)\sum_{h\in\mathcal{H}\setminus\{0\}}(\gamma^{-1}\gamma')(h)]\\ &= \tfrac{S}{D}\{1+\tfrac{1}{H-1}(\tfrac{D}{S}-1)[-1+\sum_{h\in\mathcal{H}}(\gamma^{-1}\gamma')(h)]\}\\ &= \tfrac{S}{D}(1+\tfrac{1}{H-1})(\tfrac{D}{S}-1)\left\{\begin{array}{c}-1+H,\ \gamma^{-1}\gamma'\in\mathcal{H}^\perp,\\ -1,\quad \gamma^{-1}\gamma'\notin\mathcal{H}^\perp,\end{array}\right\}\\ &= \begin{cases}1,\ \gamma^{-1}\gamma'\in\mathcal{H}^\perp,\\ \tfrac{1}{S},\ \gamma^{-1}\gamma'\notin\mathcal{H}^\perp.\end{cases}\end{aligned}$$ □

We have just shown that for all $\gamma,\gamma'\in\hat{\mathcal{G}}$, $\mathbf{E}_\gamma^*\mathbf{E}_{\gamma'}$ is diagonal with constant modulus precisely when $\mathcal{D}_g$ is a difference set for $\mathcal{H}$. Recall that a necessary condition for a set $\mathcal{D}$ to be a difference set is that $\Lambda_{\mathcal{D}}$ (6) is an integer. Therefore, for $\mathcal{D}_g$ to be a difference set for $\mathcal{H}$ we must have that $\Lambda_{\mathcal{D}_g}$ is an integer. Recall from the proof of Theorem 3.1.2 that $\frac{D}{S}-1=\frac{(D-1)(H-1)}{G-1}$ whenever $\mathcal{D}$ is a fine difference set. Applying this gives

$$\Lambda_{\mathcal{D}_g}=\tfrac{1}{H-1}(\tfrac{D}{S})(\tfrac{D}{S}-1)=\tfrac{D}{S(H-1)}\big[\tfrac{(D-1)(H-1)}{G-1}\big]=\tfrac{\Lambda_{\mathcal{D}}}{S}. \tag{21}$$

Example 3.2.2. Recall from Examples 2.3.1 and 3.1.3 that $\mathcal{D}=\{6,11,7,12,13,3,9,14\}$ is a fine difference set for $\mathbb{Z}_{15}$ resulting in an ETF$(8,15)$ comprised of 3 regular 4-simplices. The canonical regular 4-simplex (16) and the distinct isometries (17) are

$$\mathbf{\Psi}=\tfrac{1}{2}\begin{bmatrix}1&\omega^3&\omega^6&\omega^9&\omega^{12}\\1&\omega^6&\omega^{12}&\omega^3&\omega^9\\1&\omega^9&\omega^3&\omega^{12}&\omega^6\\1&\omega^{12}&\omega^9&\omega^6&\omega^3\end{bmatrix},$$

$$\mathbf{E}_0=\tfrac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&0\\1&0&0&0\\0&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&0\\0&0&0&1\\0&0&0&1\end{bmatrix},\ \mathbf{E}_1=\tfrac{1}{\sqrt{2}}\begin{bmatrix}\omega^6&0&0&0\\\omega^{11}&0&0&0\\0&\omega^7&0&0\\0&\omega^{12}&0&0\\0&0&\omega^{13}&0\\0&0&\omega^3&0\\0&0&0&\omega^9\\0&0&0&\omega^{14}\end{bmatrix},\ \mathbf{E}_2=\tfrac{1}{\sqrt{2}}\begin{bmatrix}\omega^{12}&0&0&0\\\omega^7&0&0&0\\0&\omega^{14}&0&0\\0&\omega^9&0&0\\0&0&\omega^{11}&0\\0&0&\omega^6&0\\0&0&0&\omega^3\\0&0&0&\omega^{13}\end{bmatrix}.$$

By Theorem 3.2.1(c) the subspaces of $\mathbb{C}^{\mathcal{D}}$ given by the column spaces of $\mathbf{E}_0$, $\mathbf{E}_1$, and $\mathbf{E}_2$ form an ECTFF for $\mathbb{C}^{\mathcal{D}}$. However, recall that for all $g\in\mathcal{G}$, $\mathcal{D}_g$ is either $\emptyset$, $\{0,5\}$, $\{5,10\}$, or $\{0,10\}$, all of which are difference sets for $\mathcal{H}$, and so by Theorem 3.2.1(e) this ECTFF is further an EITFF for $\mathbb{C}^{\mathcal{D}}$. Further by Theorem 3.2.1(d) the cross-

Gram matrices of the isometries are diagonal and every diagonal entry has modulus $\frac{1}{\sqrt{S}} = \frac{1}{2}$:

$$\mathbf{E}_0^*\mathbf{E}_1 = -\frac{1}{2}\begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & \omega^8 & 0 \\ 0 & 0 & 0 & \omega^4 \end{bmatrix}, \mathbf{E}_0^*\mathbf{E}_2 = -\frac{1}{2}\begin{bmatrix} \omega^2 & 0 & 0 & 0 \\ 0 & \omega^4 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega^8 \end{bmatrix}, \mathbf{E}_1^*\mathbf{E}_2 = -\frac{1}{2}\begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & \omega^8 & 0 \\ 0 & 0 & 0 & \omega^4 \end{bmatrix}.$$

3.3 Circulant Conference Matrices

A complex conference matrix is an $N \times N$ matrix $\mathbf{A}$ with zeros on the diagonal and unimodular entries on the off-diagonal such that $\mathbf{A}\mathbf{A}^* = (N-1)\mathbf{I}$, that is any two columns of $\mathbf{A}$ are orthogonal. It is known that the only real circulant conference matrix is of order 2 [8, 25, 28]. To conclude this chapter we will construct complex circulant conference matrices. To elaborate, in the case that $\mathcal{D}$ is a fine difference set such that for all $g \in \mathcal{G}$, $\mathcal{D}_g$ is a difference set for $\mathcal{H}$, by Theorem 3.2.1 the cross-Gram matrices of the isometries are diagonal with constant modulus. As we now explain, appending the sequence of these diagonal entries with a zero yields the first column of a complex circulant conference matrix.

Theorem 3.3.1. *Let $\mathcal{D}$ be a fine difference set in a finite abelian group $\mathcal{G}$ as given in Definition 3.1.1 together with the property that for all $g \in \mathcal{G}$, $\mathcal{D}_g$ defined by* (12) *is a difference set for $\mathcal{H}$. Also, for any $\gamma \in \hat{\mathcal{G}}$, $\gamma \notin \mathcal{H}^\perp$ define*

$$\mathbf{x} \in \mathbb{C}^{\mathcal{G}/\mathcal{H}}, \qquad \mathbf{x}(\overline{g}) = \tfrac{S^{3/2}}{D} \sum_{\substack{d \in \mathcal{D} \\ \overline{d} = \overline{g}}} \gamma(d).$$

Then the translates of $\mathbf{x}$ *are orthogonal with* $|\mathbf{x}(\overline{g})| = \begin{cases} 0, \ \overline{g} = \overline{0}, \\ 1, \ \overline{g} \neq \overline{0}. \end{cases}$

Proof. It is clear that $\mathbf{x}(\overline{0}) = 0$ and by Theorem 3.2.1(d) we have $|\mathbf{x}(\overline{g})| = |\sqrt{S}(\mathbf{E}_\gamma^* \mathbf{E}_{\gamma'})(\overline{g}, \overline{g})| = 1$ for all $\overline{g} \neq \overline{0}$.

Now note, that by (22) and (23), for all $\gamma \in \hat{\mathcal{G}}$ and $\gamma', \gamma'' \in \mathcal{H}^\perp$ we have

$$(\mathbf{\Phi}_\mathbf{1}^* \mathbf{\Phi}_\gamma)(\gamma', \gamma'') = \sum_{d \in \mathcal{D}} \tfrac{1}{\sqrt{D}} [\gamma'(d)]^* \tfrac{1}{\sqrt{D}} (\gamma\gamma'')(d) = (\mathbf{\Phi}^* \mathbf{\Phi})(\gamma', \gamma\gamma'').$$

Therefore,

$$|(\mathbf{\Phi}_\mathbf{1}^* \mathbf{\Phi}_\gamma)(\gamma', \gamma'')| = |(\mathbf{\Phi}^* \mathbf{\Phi})(\gamma', \gamma\gamma'')| = \begin{cases} 1, \ \gamma' = \gamma\gamma'', \\ \frac{1}{S}, \ \gamma' \neq \gamma\gamma''. \end{cases}$$

This means that when $\gamma \notin \mathcal{H}^\perp$, $|(\mathbf{\Phi}_\mathbf{1}^* \mathbf{\Phi}_\gamma)(\gamma', \gamma'')| = \frac{1}{S}$. Now recall that for all $\gamma \in \hat{\mathcal{G}}$, $\mathbf{\Phi}_\gamma = \mathbf{E}_\gamma \mathbf{\Psi}$ where $\mathbf{E}_\gamma$ and $\mathbf{\Psi}$ are defined by (17) and (16), respectively. Using this and (18) we have that for all $\gamma' \in \mathcal{H}^\perp$,

$$\begin{aligned}(\mathbf{\Phi}_\mathbf{1}^* \mathbf{\Phi}_\gamma)(1, \gamma') &= (\mathbf{\Psi}^* \mathbf{E}_\mathbf{1}^* \mathbf{E}_\gamma \mathbf{\Psi})(1, \gamma') \\ &= \sum_{\substack{\overline{g} \in \mathcal{G}/\mathcal{H} \\ \overline{g} \neq \overline{0}}} \sum_{\substack{\overline{g}' \in \mathcal{G}/\mathcal{H} \\ \overline{g}' \neq \overline{0}}} \mathbf{\Psi}^*(\mathbf{1}, \overline{g}) (\mathbf{E}_1^* \mathbf{E}_\gamma)(\overline{g}\, \overline{g}') \mathbf{\Psi}(\overline{g}', \gamma') \\ &= \tfrac{1}{D} \sum_{\overline{g} \in \mathcal{G}/\mathcal{H}} \sum_{\substack{d \in \mathcal{D} \\ \overline{d} = \overline{g}}} \gamma(d) \gamma'(g).\end{aligned}$$

Let $\mathbf{F}$ be the character table of $\mathcal{G}/\mathcal{H}$ and so $\mathbf{F}^*$ is the DFT on $\mathcal{G}/\mathcal{H}$. This gives

$$(\mathbf{\Phi}_\mathbf{1}^* \mathbf{\Phi}_\gamma)(\mathbf{1}, \gamma') = \sum_{\overline{g} \in \mathcal{G}/\mathcal{H}} \mathbf{F}(\overline{g}, \gamma') [\tfrac{1}{S^{3/2}} \mathbf{x}(\overline{g})]^* = \Big[\sum_{\overline{g} \in \mathcal{G}/\mathcal{H}} \mathbf{F}^*(\gamma', \overline{g}) \tfrac{1}{S^{3/2}} \mathbf{x}(\overline{g}) \Big]^* = [(\mathbf{F}^*(\tfrac{1}{S^{3/2}\mathbf{x}}))(\gamma')]^*.$$

Now for all $\gamma \notin \mathcal{H}^\perp$, $\big|[\mathbf{F}^*(\frac{1}{S^{1/2}} \mathbf{x})](\gamma')\big| = |S(\mathbf{\Phi}_\mathbf{1}^* \mathbf{\Phi}_\gamma)(\mathbf{1}, \gamma')| = 1$ and so the translates of $\frac{1}{S^{1/2}} \mathbf{x}$ are orthonormal. Therefore, the translates of $\mathbf{x}$ are orthogonal. □

Example 3.3.2. Continuing Examples 2.3.1, 3.1.3, and 3.2.2 since the difference set $\mathcal{D} = \{6, 11, 7, 12, 13, 3, 9, 14\}$ for $\mathbb{Z}_{15}$ is fine and each $\mathcal{D}_g$ is a difference set for $\mathcal{H}$, the diagonal entries of the cross-Grams of the embedding operators (17) form a circulant conference matrix. In particular, using the diagonal entries of $\mathbf{E}_0^*\mathbf{E}_1$ and $\mathbf{E}_0^*\mathbf{E}_2$ (3.2.2) gives the following circulant conference matrices, respectively:

$$-\begin{bmatrix} 0 & \omega^4 & \omega^8 & \omega^2 & \omega \\ \omega & 0 & \omega^4 & \omega^8 & \omega^2 \\ \omega^2 & \omega & 0 & \omega^4 & \omega^8 \\ \omega^8 & \omega^2 & \omega & 0 & \omega^4 \\ \omega^4 & \omega^8 & \omega^2 & \omega & 0 \end{bmatrix}, \qquad -\begin{bmatrix} 0 & \omega^8 & \omega & \omega^4 & \omega^2 \\ \omega^2 & 0 & \omega^8 & \omega & \omega^4 \\ \omega^4 & \omega^2 & 0 & \omega^8 & \omega \\ \omega & \omega^4 & \omega^2 & 0 & \omega^8 \\ \omega^8 & \omega & \omega^4 & \omega^2 & 0 \end{bmatrix}.$$

Example 3.3.3. From Example 2.3.2, $\mathcal{D} = \{1000, 1001, 0100, 0110, 1100, 1111\}$ is a McFarland difference set for the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with $D = 6$, $G = 16$, and $S = \sqrt{\frac{D(G-1)}{G-D}} = 3$. Since $\mathcal{D}$ is disjoint from a subgroup of order $H = \frac{G}{S+1} = 4$, namely $\mathcal{H} = \{0\} \times \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{0000, 0010, 0001, 0011\}$, $\mathcal{D}$ is fine. Therefore, by Theorem 3.2.1, for all $g \in \mathcal{G}$, $g \notin \mathcal{H}$, $\#(\mathcal{D}_g) = \frac{D}{S} = 2$. Further, the cosets of $\mathcal{H}$ intersect $\mathcal{D}$ in 2 points and each of these sets is either $\{1000, 1001\}$, $\{0100, 0110\}$ or $\{1100, 1111\}$. However, not every $\mathcal{D}_g$ is a difference set for $\mathcal{H}$. For example, $\mathcal{D}_{1000} = \{0000, 0001\}$ is not a difference set for $\mathcal{H}$ since its difference table would only contain two nonzero elements of $\mathcal{H}$, however there are three total. As such, by Theorem 3.2.1 the subspaces spanned by the regular simplices that comprise the resulting harmonic ETF do not form an EITFF, however they do form an ECTFF for $\mathbb{C}^{\mathcal{D}}$.

Here, we identify $\hat{\mathcal{G}}$ with $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, regarding $n_1n_2n_3n_4$ as the character $g_1g_2g_3g_4 \mapsto (-1)^{g_1n_1+g_2n_2+g_3n_3+g_4n_4}$. In particular, $\mathcal{H}^\perp$ is identified with those $n_1n_2n_3n_4$ such that $(-1)^{n_3} = (-1)^{n_4} = 1$, namely $\{0000, 1000, 0100, 1100\}$. Extracting the 6 rows that correspond to $\mathcal{D}$ from the resulting 16×16 character table and then normalizing columns yields the synthesis operator of an ETF$(6, 16)$:

$$\mathbf{\Phi} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

where the columns are indexed by

$$\{0000, 1000, 0100, 1100, 0010, 1010, 0110, 1110,$$
$$0001, 1001, 0101, 1101, 0011, 1011, 0111, 1111\}.$$

Since $\mathcal{D}$ is fine, $\mathbf{\Phi}$ is the union of 4 regular 3-simplices indexed by the cosets of $\mathcal{H}^{\perp}$, namely $\mathbf{\Phi}_{0000}$, $\mathbf{\Phi}_{0010}$, $\mathbf{\Phi}_{0001}$ and $\mathbf{\Phi}_{0011}$ where for any $n_1 n_2 n_3 n_4$ in $\hat{\mathcal{G}}$,

$$\mathbf{\Phi}_{n_1 n_2 n_3 n_4} = \begin{bmatrix} \boldsymbol{\varphi}_{n_1 n_2 n_3 n_4} & \boldsymbol{\varphi}_{(n_1+1) n_2 n_3 n_4} & \boldsymbol{\varphi}_{n_1 (n_2+1) n_3 n_4} & \boldsymbol{\varphi}_{(n_1+1)(n_2+1) n_3 n_4} \end{bmatrix}.$$

The canonical regular 3-simplex (16) is given by

$$\mathbf{\Psi} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

We have that the isometries (17) that satisfy $\mathbf{\Phi}_{n_1n_2n_3n_4} = \mathbf{E}_{n_1n_2n_3n_4}\mathbf{\Psi}$ are

$$\mathbf{E}_{0000} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{E}_{0010} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\mathbf{E}_{0001} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{E}_{0011} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Further by Theorem 3.2.1 we know that the corresponding cross-Gram matrices are all diagonal. In particular, these cross-Gram matrices are all one of the following matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

These cross-Grams do not have a constant modulus on the diagonal since $\mathcal{D}_g$ is not a difference set for $\mathcal{H}$ for every $g \in \mathcal{G}$.

IV. Mutually Unbiased Simplices

4.1 Composite Difference Sets

To begin this chapter, we introduce an extension of the concept of MUBs to simplices:

Definition 4.1.1. Let $\mathbb{H}$ be an S-dimensional Hilbert space. A collection of H simplices $\{\boldsymbol{\varphi}_{h,s}\}_{h=1,\ s=1}^{H\quad S+1}$ for $\mathbb{H}$ are called *mutually unbiased simplices (MUSs)* for $\mathbb{H}$ if $|\langle\boldsymbol{\varphi}_{h,s},\boldsymbol{\varphi}_{h',s'}\rangle|$ is constant over all $s,s'=1,...,S+1$ and all $h\neq h'$.

To continue to develop the concept of MUSs there are certain properties we would like to know. Here we show that like MUBs, MUSs have coherence $\frac{1}{\sqrt{S}}$ (1). Further there is a natural upper bound on the number, H, of MUSs and when this bound is achieved we say the MUSs are maximal:

Theorem 4.1.2. *Let $\{\boldsymbol{\varphi}_{h,s}\}_{h=1,\ s=1}^{H\quad S+1}$ be a collection of H mutually unbiased simplices for an S-dimensional Hilbert space $\mathbb{H}$ over $\mathbb{F}$. Then $\operatorname{coh}(\{\boldsymbol{\varphi}_{h,s}\}_{h=1,\ s=1}^{H\quad S+1})=\frac{1}{\sqrt{S}}$ and $H\leq S-1$ when $\mathbb{F}=\mathbb{C}$ or $H\leq\frac{(S+2)(S-1)}{2S}$ when $\mathbb{F}=\mathbb{R}$.*

Proof. In the case that $h=h'$,

$$|\langle\boldsymbol{\varphi}_{h,s},\boldsymbol{\varphi}_{h',s'}\rangle|=\begin{cases}1,\ s=s',\\ \frac{1}{S},\ s\neq s'.\end{cases}$$

In the case $h\neq h'$, let $x:=|\langle\boldsymbol{\varphi}_{v,s},\boldsymbol{\varphi}_{d',s'}\rangle|$. Further let $\boldsymbol{\Phi}$ be the synthesis operator of $\{\boldsymbol{\varphi}_{h,s}\}_{h=1,\ s=1}^{H\quad S+1}$. Then

$$\frac{H^2(S+1)^2}{S}=\|\boldsymbol{\Phi}\boldsymbol{\Phi}^*\|_{\text{Fro}}^2=\|\boldsymbol{\Phi}^*\boldsymbol{\Phi}\|_{\text{Fro}}^2=(S+1)+\frac{1}{S^2}(S)(S+1)+V(V-1)(S+1)^2x^2$$

and solving gives that $x=\frac{1}{\sqrt{S}}$. Therefore, $\operatorname{coh}(\{\boldsymbol{\varphi}_{h,s}\}_{h=1,\ s=1}^{H\quad S+1})=\frac{1}{\sqrt{S}}$.

To compute the maximal number of MUSs, first note that

$$|\mathbf{\Phi}^*\mathbf{\Phi}|^2 = (1 - \tfrac{1}{S^2})\mathbf{I}_{H(S+1)} + \tfrac{1}{S}(\tfrac{1-S}{S}\mathbf{I}_H + \mathbf{J}_H) \otimes \mathbf{J}_{S+1}.$$

Therefore, $|\mathbf{\Phi}^*\mathbf{\Phi}|^2$ has eigenvalues 0, $\frac{H(S+1)}{S}$, and $\frac{S^2-1}{S^2}$ with multiplicities $H-1$, 1, and HS, respectively, and so $\operatorname{rank}(|\mathbf{\Phi}^*\mathbf{\Phi}|^2) = HS+1$. Therefore, by (10) when $\mathbb{F} = \mathbb{C}$, $HS+1 \leq S^2$, i.e. $H \leq S - \frac{1}{S}$; when $\mathbb{F} = \mathbb{R}$, $HS+1 \leq \binom{S+1}{2}$, i.e. $H \leq \frac{1}{2}(S+1) - \frac{1}{S}$. In both cases this upper bound on H is not an integer. Specifically in the case that $\mathbb{F} = \mathbb{C}$, we can simplify the bound to $H \leq S-1$. □

For the remainder of this section we will focus on the case where $\mathbb{F} = \mathbb{C}$. In this case, since the true upper bound $H \leq \frac{S^2-1}{S} = S - \frac{1}{S}$ is not attainable, maximal MUSs consisting of $S-1$ simplices cannot solve the quantum information theory problem (8) since equality is not achieved in (10). When the number of MUSs, $H = S-1$, $\operatorname{rank}(|\mathbf{\Phi}^*\mathbf{\Phi}|^2) = HS+1 = S^2 - S + 1$ and so the outer products of $\{\boldsymbol{\varphi}_{h,s}\}_{h=1,\, s=1}^{S-1\ \, S+1}$ only span an (S^2-S+1)-dimensional subspace of all self-adjoint operators. A natural candidate for such a space is the subspace of all $S \times S$ self-adjoint matrices whose diagonal entries are constant. In order to recover the true diagonal entries and so solve the quantum information theory problem (8) we must also include, with the $S-1$ simplices, an $S \times S$ identity matrix. This allows us to solve (8) with S^2+S-1 vectors, that is one less vector than maximal MUBs.

In order to give a construction of MUSs we will need to have a difference set with properties stronger than those of a fine difference set. To be precise we will need to have a fine difference set such that for all $g \in \mathcal{G}$, $g \notin \mathcal{H}$, each $\mathcal{D}_g$ is the shift of the same difference set for $\mathcal{H}$:

Definition 4.1.3. Let $\mathcal{D}$ be a fine difference set, c.f. Definition 3.1.1. $\mathcal{D}$ is called a *composite* difference set if for all $g \in \mathcal{G}$, $g \notin \mathcal{H}$, $\mathcal{D}_g := (\mathcal{D} - g) \cap \mathcal{H}$ is a shift of the same difference set in $\mathcal{H}$. That is, there exists a difference set $\mathcal{B}$ for $\mathcal{H}$ such that for all $g \in \mathcal{G}$, $g \notin \mathcal{H}$, there exists $h_g \in \mathcal{H}$ satisfying $\mathcal{D}_g = h_g + \mathcal{B}$.

From Theorem 3.2.1 when $\mathcal{D}$ is a difference set for $\mathcal{G}$ such that for all $g \in \mathcal{G}$, $\mathcal{D}_g$ is a difference set for $\mathbb{H}$, the subspaces of $\mathbb{C}^{\mathcal{D}}$ spanned by the simplices that make up the harmonic ETF form an EITFF. Therefore, letting $\mathbf{P}_\gamma = \mathbf{E}_\gamma \mathbf{E}_\gamma^*$ be the orthogonal projection operators onto these subspaces, we know that for all $\gamma, \gamma' \in \hat{\mathcal{G}}$, $\mathbf{P}_\gamma \mathbf{P}_{\gamma'} \mathbf{P}_\gamma = \sigma^2 \mathbf{P}_{\gamma'}$ for some $\sigma \geq 0$. This all still holds true when $\mathcal{D}$ is a composite difference set. However, as the following lemma shows we have a property that appears stronger then equi-chordality since for all $\gamma, \gamma', \gamma'' \in \hat{\mathcal{G}}$, $\mathbf{P}_\gamma \mathbf{P}_{\gamma'} \mathbf{P}_{\gamma''} = \kappa \mathbf{P}_\gamma \mathbf{P}_{\gamma''}$ for some constant κ:

Lemma 4.1.4. *Let $\mathcal{D}$ be a composite difference set for the group $\mathcal{G}$ as in Definition 4.1.3 and let $\mathbf{E}_\gamma$ be defined by* (17) *for all $\gamma \in \hat{\mathcal{G}}$. Then $\mathbf{E}_\gamma^* \mathbf{E}_{\mathbf{1}} \mathbf{E}_{\mathbf{1}}^* \mathbf{E}_{\gamma'} = \kappa \mathbf{E}_\gamma^* \mathbf{E}_{\gamma'}$ for some constant κ of modulus $\frac{1}{\sqrt{S}}$.*

Proof. By definition of $\mathbf{E}_\gamma$, for all $\overline{g} \in \mathcal{G}/\mathcal{H}$, $\overline{g} \neq \overline{0}$ we have

$$\begin{aligned}
\tfrac{D^3}{S^3}(\mathbf{E}_{\gamma'}^* \mathbf{E}_\gamma \mathbf{E}_\gamma^* \mathbf{E}_{\mathbf{1}} \mathbf{E}_{\mathbf{1}}^* \mathbf{E}_{\gamma'})(\overline{g}, \overline{g}) &= \sum_{\substack{d \in \mathcal{D} \\ \overline{d} = \overline{g}}} [(\gamma')^{-1}\gamma](d) \Big(\sum_{\substack{d' \in \mathcal{D} \\ \overline{d}' = \overline{g}}} \gamma^{-1}(d') \Big) \Big(\sum_{\substack{d' \in \mathcal{D} \\ \overline{d}' = \overline{g}}} \gamma'(d') \Big) \\
&= \sum_{\substack{d \in \mathcal{D} \\ \overline{d} = \overline{g}}} \Big(\sum_{\substack{d' \in \mathcal{D} \\ \overline{d}' = \overline{g}}} \gamma^{-1}(d' - d) \Big) \Big(\sum_{\substack{d' \in \mathcal{D} \\ \overline{d}' = \overline{g}}} \gamma'(d' - d) \Big).
\end{aligned}$$

To simplify this sum note that for all $\overline{g} \neq \overline{0}$, $\{d \in \mathcal{D} : \overline{d} = \overline{g}\} = \{d \in \mathcal{D} : d - g \in \mathcal{H}\}$. Now letting $h = d - g$ gives

$$\{d \in \mathcal{D} : \overline{d} = \overline{g}\} = \{h + g \in \mathcal{D} : h \in \mathcal{H}\} = g + \{h \in \mathcal{D} - g : h \in \mathcal{H}\} = g + \mathcal{D}_g.$$

Applying this gives

$$\tfrac{D^3}{S^3}(\mathbf{E}^*_{\gamma'}\mathbf{E}_{\gamma}\mathbf{E}^*_{\gamma}\mathbf{E}_{\mathbf{1}}\mathbf{E}^*_{\mathbf{1}}\mathbf{E}_{\gamma'})(\overline{g},\overline{g}) = \sum_{d\in g+\mathcal{D}_g}\Big(\sum_{d'\in g+\mathcal{D}_g}\gamma^{-1}(d'-d)\Big)\Big(\sum_{d'\in g+\mathcal{D}_g}\gamma'(d'-d)\Big).$$

Letting $d = h + g$ and $d' = h' + g$ gives

$$\begin{aligned}\tfrac{D^3}{S^3}(\mathbf{E}^*_{\gamma'}\mathbf{E}_{\gamma}\mathbf{E}^*_{\gamma}\mathbf{E}_{\mathbf{1}}\mathbf{E}^*_{\mathbf{1}}\mathbf{E}_{\gamma'})(\overline{g},\overline{g}) &= \sum_{h\in\mathcal{D}_g}\Big(\sum_{h'\in\mathcal{D}_g}\gamma^{-1}(h'-h)\Big)\Big(\sum_{h'\in\mathcal{D}_g}\gamma'(h'-h)\Big)\\ &= \sum_{h\in\mathcal{D}_g}[\gamma(\gamma')^{-1}](h)(\sum_{h'\in\mathcal{D}_g}\gamma^{-1}(h'))(\sum_{h'\in\mathcal{D}_g}\gamma'(h')).\end{aligned}$$

Since $\mathcal{D}$ is composite, we know that for all $g \notin \mathcal{H}$ there exists some difference set $\mathcal{B}$ for $\mathcal{H}$ such that $\mathcal{D}_g = h_g + \mathcal{B}$ for some $h_g \in \mathcal{H}$. Letting $h = h_g + b$ gives

$$\begin{aligned}\tfrac{D^3}{S^3}(\mathbf{E}^*_{\gamma'}\mathbf{E}_{\gamma}\mathbf{E}^*_{\gamma}\mathbf{E}_{\mathbf{1}}\mathbf{E}^*_{\mathbf{1}}\mathbf{E}_{\gamma'})(\overline{g},\overline{g}) &= \sum_{b\in\mathcal{B}}[\gamma(\gamma')^{-1}](h_g+b)(\sum_{b\in\mathcal{B}}\gamma^{-1}(h_g+b))(\sum_{b\in\mathcal{B}}\gamma'(h_g+b))\\ &= \sum_{b\in\mathcal{B}}[\gamma(\gamma')^{-1}](b)(\sum_{b\in\mathcal{B}}\gamma^{-1}(b))(\sum_{b\in\mathcal{B}}\gamma'(b))\\ &= (\mathbf{F}^*_{\mathcal{H}}\chi_{\mathcal{B}})(\gamma^{-1}\gamma')(\mathbf{F}^*\chi_{\mathcal{B}})(\gamma)(\mathbf{F}^*\chi_{\mathcal{B}})[(\gamma')^{-1}].\end{aligned}$$

Therefore, $\mathbf{E}^*_{\gamma}\mathbf{E}_{\mathbf{1}}\mathbf{E}^*_{\mathbf{1}}\mathbf{E}_{\gamma'} = \kappa_{\gamma,\gamma'}\mathbf{E}^*_{\gamma}\mathbf{E}_{\gamma'}$ where

$$\kappa_{\gamma,\gamma'} = \tfrac{S^4}{D^3}(\mathbf{F}^*_{\mathcal{H}}\chi_{\mathcal{B}})(\gamma^{-1}\gamma')(\mathbf{F}^*\chi_{\mathcal{B}})(\gamma)(\mathbf{F}^*\chi_{\mathcal{B}})[(\gamma')^{-1}].$$

Since $\mathcal{D}$ is composite, applying Theorem 3.1.2 and (21) we know that in the notation of Definition 4.1.3

$$|(\mathbf{F}^*_{\mathcal{H}}\chi_{\mathcal{B}})(\gamma)| = \tfrac{\#(\mathcal{B})}{S_{\mathcal{B}}} = \sqrt{\#(B)-\Lambda_{\mathcal{B}}} = \sqrt{\tfrac{D}{S}-\tfrac{\Lambda_{\mathcal{D}}}{S}} = \tfrac{D}{S^{3/2}}.$$

Therefore, $|\kappa_{\gamma,\gamma'}| = \frac{S^4}{D^3}(\frac{D}{S^{3/2}})^3 = \frac{1}{\sqrt{S}}$. □

Example 4.1.5. Recall that $\mathcal{D} = \{6, 11, 7, 12, 13, 3, 9, 14\}$ is a composite difference set for $\mathcal{G} = \mathbb{Z}_{15}$ and $\mathbf{E}_0^*\mathbf{E}_1$, $\mathbf{E}_0^*\mathbf{E}_2$, and $\mathbf{E}_1^*\mathbf{E}_2$ are given in Example 3.3.2. Using these cross-Gram matrices we see that as shown in Lemma 4.1.4, $\mathbf{E}_1^*\mathbf{E}_0\mathbf{E}_0^*\mathbf{E}_2$ is a constant multiple of $\mathbf{E}_1^*\mathbf{E}_2$:

$$\mathbf{E}_1^*\mathbf{E}_0\mathbf{E}_0^*\mathbf{E}_2 = (\mathbf{E}_0^*\mathbf{E}_1)^*(\mathbf{E}_0^*\mathbf{E}_2) = \tfrac{1}{4}\begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & \omega^8 & 0 \\ 0 & 0 & 0 & \omega^4 \end{bmatrix} = -\tfrac{1}{2}\mathbf{E}_1^*\mathbf{E}_2.$$

As we now show when you have a composite difference set, applying the cross-Grams of the isometries (17) to the canonical simplex (16) gives a set of MUSs.

Theorem 4.1.6. *Let $\mathcal{D}$ be a composite difference set, cf. Definition 4.1.3. Then $\mathbf{\Psi} \cup \{\sqrt{S}\mathbf{E}_{\mathbf{1}}^*\mathbf{E}_\gamma\mathbf{\Psi}\}_{\gamma\in\hat{\mathcal{G}}/\mathcal{H}^\perp}$ is an MUS for $\mathbb{H}$ where $\mathbf{\Psi}$ and $\mathbf{E}_\gamma$ are defined by* (16) *and* (17)*, respectively.*

Proof. First we consider inner products of vectors from $\sqrt{S}\mathbf{E}_{\mathbf{1}}^*\mathbf{E}_\gamma\mathbf{\Psi}$ for any $\gamma \in \hat{\mathcal{G}}$ with vectors from $\mathbf{\Psi}$. In particular, recalling that $\mathbf{E}_\gamma\mathbf{\Psi} = \mathbf{\Phi}_\gamma$ as given in (15), we have $|(\sqrt{S}\mathbf{\Psi}^*\mathbf{E}_{\mathbf{1}}^*\mathbf{E}_\gamma\mathbf{\Psi})(\gamma', \gamma'')| = |(\sqrt{S}\mathbf{\Phi}_{\mathbf{1}}^*\mathbf{\Phi}_\gamma)(\gamma', \gamma'')| = \frac{1}{\sqrt{S}}$.

Now we consider inner products between vectors from $\sqrt{S}\mathbf{E}_{\mathbf{1}}^*\mathbf{E}_\gamma\mathbf{\Psi}$ and vectors from $\sqrt{S}\mathbf{E}_{\mathbf{1}}^*\mathbf{E}_{\gamma'}\mathbf{\Psi}$ when $\overline{\gamma} \neq \overline{\gamma}'$. These are the entries of the matrix $S\mathbf{\Psi}^*\mathbf{E}_\gamma^*\mathbf{E}_{\mathbf{1}}\mathbf{E}_{\mathbf{1}}^*\mathbf{E}_{\gamma'}\mathbf{\Psi}$. By Lemma 4.1.4 we have that

$$|S\mathbf{\Psi}^*\mathbf{E}_\gamma^*\mathbf{E}_{\mathbf{1}}\mathbf{E}_{\mathbf{1}}^*\mathbf{E}_{\gamma'}\mathbf{\Psi}| = |S\kappa_{\gamma,\gamma'}\mathbf{\Psi}^*\mathbf{E}_\gamma^*\mathbf{E}_{\gamma'}\mathbf{\Psi}| = |S\kappa_{\gamma,\gamma'}\mathbf{\Phi}_\gamma^*\mathbf{\Phi}_{\gamma'}| = \tfrac{1}{\sqrt{S}}\mathbf{J}$$

and so $\mathbf{\Psi} \cup \{\sqrt{S}\mathbf{E}_{\mathbf{1}}^*\mathbf{E}_\gamma\mathbf{\Psi}\}_{\gamma\in\hat{\mathcal{G}}/\mathcal{H}^\perp}$ are MUSs for $\mathbb{H}$. □

Example 4.1.7. Revisiting the complement of a Singer difference last seen in Example 4.1.5 and the cross-Grams seen in Example 3.2.2 we can construct the following set of $S-1=3$ MUSs for the 4-dimensional Hilbert space $\mathbb{H}$:

$$\frac{1}{2}\left[\begin{array}{ccccc|ccccc|ccccc} 1 & \omega^{3} & \omega^{6} & \omega^{9} & \omega^{12} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{10} & \omega^{13} & \omega^{2} & \omega^{5} & \omega^{8} & \omega^{11} & \omega^{14} \\ 1 & \omega^{6} & \omega^{12} & \omega^{3} & \omega^{9} & \omega^{2} & \omega^{8} & \omega^{14} & \omega^{5} & \omega^{11} & \omega^{4} & \omega^{10} & \omega^{1} & \omega^{7} & \omega^{13} \\ 1 & \omega^{9} & \omega^{3} & \omega^{12} & \omega^{6} & \omega^{8} & \omega^{2} & \omega^{11} & \omega^{5} & \omega^{14} & \omega^{1} & \omega^{10} & \omega^{4} & \omega^{13} & \omega^{7} \\ 1 & \omega^{12} & \omega^{9} & \omega^{6} & \omega^{3} & \omega^{4} & \omega^{1} & \omega^{13} & \omega^{10} & \omega^{7} & \omega^{8} & \omega^{5} & \omega^{2} & \omega^{14} & \omega^{11} \end{array}\right].$$

4.2 Classifications of Known Difference Sets

4.2.1 Singer Difference Sets

From Example 3.1.3 we know that some complements of Singer difference sets are fine. In general for any even J the complement of a Singer difference set can be shifted so as to yield a fine difference set [10]. To see this let $\mathcal{D}$ be the complement of a Singer difference set. The harmonic ETF produced from $\mathcal{D}$ is an ETF$(Q^{J-1}, \frac{Q^J-1}{Q-1})$. Thus, the inverse Welch bound is $S = Q^{J/2}$ which is an integer for even J. For $\mathcal{D}$ to be fine, it would need to avoid a subgroup of order $H = \frac{G}{S+1} = \frac{Q^{J/2}-1}{Q-1}$. Letting $\mathcal{H} = \mathbb{F}_{Q^{J/2}}^\times/\mathbb{F}_Q^\times$ shows that $\mathcal{D}$ is fine when J is even. It is natural to ask whether all of these fine difference sets are also composite. As we now show, this is indeed the case since any Singer difference set, $\mathcal{D}$, has the property that $\mathcal{D}_g$ is a difference set for $\mathcal{H}$ for all $g \in \mathcal{G}$ and when $g \notin \mathcal{H}$ each $\mathcal{D}_g$ is a shift of the same difference set for $\mathcal{H}$. Since taking the set complement and shifting the set will not change these properties, they also hold for the complements of Singer difference sets that are fine.

Theorem 4.2.1. *If $\mathcal{D}$ is a Singer difference set for the group $\mathcal{G} = \mathbb{F}_{Q^J}^\times/\mathbb{F}_Q^\times$, then for all $g \in \mathcal{G}$, $\mathcal{D}_g$ is a difference set for $\mathcal{H}$ and for all $g \notin \mathcal{H}$, every $\mathcal{D}_g$ is a shift of the same difference set for $\mathcal{H}$. Moreover, if $\mathcal{G} = \langle\alpha\rangle$, then $(\alpha^j\mathcal{D}) \cap \mathcal{H} = \mathcal{H}$ precisely when $j = 0$ or $j = \frac{Q^{J/2}+1}{2}$ for Q being even or odd, respectively.*

Proof. Let α be a generator of $\mathcal{G}$. Recall that $\mathcal{D} = \{\bar{\beta} \in \mathcal{G} : \mathrm{tr}_{Q^J/Q}(\beta) = 0\}$ and

$\mathcal{H} = \langle [\alpha]^{Q^{J/2}+1} \rangle = \mathbb{F}_{Q^{J/2}}^{\times} / \mathbb{F}_{Q}^{\times} \subseteq \mathcal{D}$. For all $j = 0, ..., Q^{J/2}$, we define

$$\begin{aligned} C_j := \left(\bar{\alpha}^{-j}\mathcal{D}\right) \cap \mathcal{H} &= \{\bar{\beta} \in \mathcal{H} : \mathrm{tr}_{Q^J/Q}(\alpha^j \beta) = 0\} \\ &= \{\overline{\beta} \in \mathcal{H} : \mathrm{tr}_{Q^{J/2}/Q}[\mathrm{tr}_{Q^J/Q^{J/2}}(\alpha^j \beta)] = 0\}. \end{aligned}$$

to be the $\#(G/H) = Q^{J/2} + 1$ cosets of $\mathcal{D}$ intersected with $\mathcal{H}$. Since $\overline{\beta} \in \mathcal{H}$, $\beta \in \mathbb{F}_{Q^{J/2}}$ and so $\mathrm{tr}_{Q^J/Q^{J/2}}(\alpha^j \beta) = \beta\, \mathrm{tr}_{Q^J/Q^{J/2}}(\alpha^j)$. Therefore,

$$C_j = \{\overline{\beta} \in \mathcal{H} : \mathrm{tr}_{Q^{J/2}/Q}[\beta\, \mathrm{tr}_{Q^J/Q^{J/2}}(\alpha^j)] = 0\}.$$

First consider the case when $\mathrm{tr}_{Q^J/Q^{J/2}}(\alpha^j) = 0$. Then $C_j = \mathcal{H}$ which is a difference set for $\mathcal{H}$. Since

$$\mathrm{tr}_{Q^J/Q^{J/2}}(\alpha^j) = \alpha^j + \left(\alpha^j\right)^{Q^{J/2}} = \alpha^j(1 + \alpha^{j(Q^{J/2}-1)}),$$

we have that $\mathrm{tr}_{Q^J/Q^{J/2}}(\alpha^j) = 0$ if and only if

$$\alpha^{j(Q^{J/2}-1)} = -1 = \begin{cases} 1, & Q \text{ is even,} \\ \alpha^{\frac{Q^J-1}{2}}, & Q \text{ is odd.} \end{cases}$$

When Q is even this becomes $\mathrm{tr}_{Q^J/Q^{J/2}}(\alpha^j) = 0$ if and only if $j(Q^{J/2} - 1) \equiv 0 \mod (Q^J - 1)$ or equivalently $j \equiv 0 \mod (Q^{J/2} + 1)$. This implies that $j = 0$, i.e. that $C_0 = \mathcal{D} \cap \mathcal{H} = \mathcal{H}$ as expected. Alternatively, in the case that Q is odd this becomes, $\mathrm{tr}_{Q^J/Q^{J/2}}(\alpha^j) = 0$ if and only if $j(Q^{J/2} - 1) \equiv \frac{Q^J-1}{2} \mod (Q^J - 1)$ or equivalently $j \equiv \frac{Q^{J/2}+1}{2} \mod (Q^{J/2} + 1)$. This implies that $j = \frac{Q^{J/2}+1}{2}$.

In the case that $\mathrm{tr}_{Q^J/Q^{J/2}}(\alpha^j) \neq 0$, C_j is a Singer difference set with parameters Q and $\frac{J}{2}$. □

4.2.2 McFarland Difference Sets

Recall that a McFarland difference set $\mathcal{D}$ has parameters $D = Q^{J-1}(\frac{Q^J-1}{Q-1})$, $G = Q^J(\frac{Q^J-1}{Q-1}+1)$, $S = \frac{Q-1}{Q^J-1}$, and $\Lambda_{\mathcal{D}} = \frac{Q^{J-1}(Q^{J-1}-1)}{Q-1}$. It was shown in [10] that all McFarland difference sets are fine. To see this, note that we need $\mathcal{D}$ to be disjoint from a subset of $\mathcal{G}$ of order $H = \frac{G}{S+1} = Q^J$. Choosing $\mathcal{H} = \{0\} \times \mathbb{F}_{Q^J}$ then gives that $\mathcal{D}$ is fine. Again it is natural to ask if all McFarland difference sets are composite. From Example 3.3.3 we know that this is not the case.

In general given a McFarland difference set $\mathcal{D}$, $\mathcal{D}_g$ is not a difference set for $\mathcal{H}$, and so $\mathcal{D}$ is not composite since

$$\Lambda_{\mathcal{D}_g} = \frac{\Lambda_{\mathcal{D}}}{S} = \frac{Q^{J-1}(Q^{J-1}-1)}{Q^{J-1}-1} = \frac{Q^{J-2}[Q^J-1-(Q-1)]}{Q^J-1} = Q^{J-2} - \frac{Q^{J-2}(Q-1)}{Q^J-1} \notin \mathbb{Z}.$$

4.2.3 Twin Prime Power Difference Sets

Finally, we consider TPP difference sets. In [10] it was shown that the complement of any TPP difference set is fine. To see this let $\mathcal{D}$ be the complement of a TPP difference set in the group $\mathcal{G} = \mathbb{F}_Q \times \mathbb{F}_{Q+2}$. $\mathcal{D}$ is given by

$$\mathcal{D} = \{(x,y) \in \mathbb{F}_Q \times \mathbb{F}_{Q+2} : x = 0, y \neq 0 \text{ or } (x,y) \in S_Q \times N_{Q+2} \text{ or } (x,y) \in N_Q \times S_{Q+2}\}$$

and results in an ETF$(\frac{1}{2}(Q+1)^2, Q(Q+2))$. Further such a difference set has

$$\Lambda_{\mathcal{D}} = \frac{D(D-1)}{G-1} = \frac{1}{Q(Q+1)-1}[\tfrac{1}{2}(Q+1)^2][\tfrac{1}{2}(Q+1)^2 - 1] = \frac{(Q+1)^2}{4}$$

and an inverse Welch bound of $S = Q+1$. For $\mathcal{D}$ to be fine, it would need to avoid a subgroup $\mathcal{H}$ of $\mathcal{G}$ of order $H = Q$. Letting $\mathcal{H} = \mathbb{F}_Q \times \{0\}$ which is a subgroup of $\mathcal{G}$ disjoint from $\mathcal{D}$ shows that $\mathcal{D}$ is fine. To determine when the complements of TPP difference sets are composite, first note that

$$\Lambda_{\mathcal{D}_g} = \frac{\Lambda_{\mathcal{D}}}{S} = \frac{(Q+1)^2}{4(Q+1)} = \frac{Q+1}{4}. \tag{22}$$

Here we show that TPP difference sets are not compostite, but when $Q \equiv 3 \mod 4$ they do have the property that for all $g \in \mathcal{G}$, $\mathcal{D}_g$ is a difference set for $\mathcal{H}$.

Theorem 4.2.2. *Let $\mathcal{D}$ be the complement of the TPP difference set for the group $\mathcal{G} = \mathbb{F}_Q \times \mathbb{F}_{Q+2}$. If $Q \equiv 3 \mod 4$, then $\mathcal{D}$ has the property that for each $g \in \mathcal{G}$, $\mathcal{D}_g$ (12) is a difference set for $\mathcal{H}$. If $Q \equiv 1 \mod 4$ this property does not hold.*

Proof. First consider the case that $Q \equiv 1 \mod 4$. Then from (22) $\Lambda_{\mathcal{D}_g} = \frac{Q+1}{4} \notin \mathbb{Z}$ and so $\mathcal{D}_g$ cannot be a difference set for $\mathcal{H}$.

In the case that $Q \equiv 3 \mod 4$, take any $g_0 \notin \mathcal{H}$ so $g_0 = (x_0, y_0)$ where $y_0 \neq 0$. Then

$$g_0+\mathcal{D} = \{(x_0, y+y_0) : y \neq 0\} \cup \{(x_0+x, y_0+y) : (x, y) \in (S_Q \times N_{Q+2}) \cup (N_Q \times S_{Q+2})\}.$$

Therefore, intersecting this set with $\mathcal{H}$ gives

$$(g_0 + \mathcal{D}) \cap \mathcal{H} = \{(x_0, 0)\} \cup \{(x_0 + x, 0) : (x, -y_0) \in (S_Q \times N_{Q+2}) \cup (N_Q \times S_{Q+2})\}.$$

Note that since $-1 \in S_{Q+2}$, $y_0 \in N_{Q+2}$ if and only if $-y_0 \in N_{Q+2}$ and $y_0 \in S_{Q+2}$ if and only if $-y_0 \in S_{Q+2}$. Applying this result gives that

$$(g_0 + \mathcal{D}) \cap \mathcal{H} = \{(x_0, 0)\} \cup \{(x_0 + x, 0) : (x, y_0) \in (S_Q \times N_{Q+2}) \cup (N_Q \times S_{Q+2})\}.$$

In the case that $y_0 \in N_{Q+2}$, this becomes

$$(g_0 + \mathcal{D}) \cap \mathcal{H} = \{(x_0, 0)\} \cup \{(x_0 + x, 0) : x \in S_Q\} = (x_0, 0) + (S_Q \cup \{0\}) \times \{0\}.$$

This is a difference set in $\mathcal{H} = \mathbb{F}_Q \times \{0\}$ since $S_Q \cup \{0\} = N_Q^C = (-S_Q)^C$ is a difference set in $\mathbb{F}_Q$. In the case that $y_0 \in S_{Q+2}$, we have

$$(g_0 + \mathcal{D}) \cap \mathcal{H} = \{(x_0, 0)\} \cup \{(x_0 + x, 0) : x \in N_Q\} = (x_0, 0) + (N_Q \cup \{0\}) \times \{0\}.$$

This is a difference set in $\mathcal{H} = \mathbb{F}_Q \times \{0\}$ since $N_Q \cup \{0\} = S_Q^C$ is a difference set in $\mathbb{F}_Q$. $\mathcal{D}$ is not composite since, in general, N_Q is not a shift of S_Q. $\square$

V. Conclusion

Since TPP difference sets with $Q \equiv 3 \mod 4$ and appropriately shifted complements of Singer difference sets with J even satisfy that each $\mathcal{D}_g$ is a difference set for $\mathcal{H}$ we can apply Theorems 3.2.1 and 3.3.1 to construct EITFFs and complex circulant conference matrices, respectively. Moreover, since these complements of Singer difference sets are composite we can apply Theorem 4.1.6 to construct MUSs. The following theorem summarizes these results:

Theorem 5.0.1. *Let Q be any prime power. Then there exists an EITFF for $\mathbb{C}^D$ consisting of H subspaces each of dimension S and there exists a complex circulant conference matrix of size $(S+1) \times (S+1)$ whenever*

(a) $D = Q^{J-1}$, $H = \frac{Q^{J/2}-1}{Q-1}$, *and* $S = Q^{J/2}$ *for any even integer* $J \geq 4$.

(b) $D = \frac{1}{2}(Q+1)^2$, $H = Q$, *and* $S = Q+1$ *provided* $Q+2$ *is a odd prime power.*

Further, there exists an MUS for a $Q^{J/2}$-dimensional Hilbert space consisting of $\frac{Q^{J/2}-1}{Q-1}$ simplices for any even integer $J \geq 4$ and when $Q = 2$ these MUSs are maximal.

EITFFs of these sizes are known to exist [20], but it is still to be determined whether they are equivalent to the construction presented in this thesis. Some of these MUSs were previously obtained in [10], however, here we have shown that each of these is actually the first member of a distinct infinite family. Further, it is known that there exist sequences of vectors with the same cardinality and coherence of the maximal MUSs presented here [5]. In fact, in [5], the union of such a sequence and a standard basis is shown to meet the orthoplex bound. Whether these sequences of vectors are equivalent is yet to be determined. Complex circulant complex matrices of these sizes are seemingly unknown in the literature.

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