Winter 1-1983

Modal Control of an Unstable Periodic Orbit

William Wiesel

William Shelton
USAF Retired

Recommended Citation

Wiesel, W. & Shelton, W. (1983). "Modal Control of an Unstable Periodic Orbit." Journal of the Astronautical Sciences, 31(1), 63-76.

The Journal of the Astronautical Sciences, Vol. XXXI, No. 1, pp. 63–76, January-March 1983

Modal Control of an Unstable Periodic Orbit

W. Wiesel[1] and W. Shelton[2]

Abstract

We apply Floquet theory to the problem of designing a control system for a satellite in an unstable periodic orbit. Expansion about a periodic orbit produces a time periodic linear system, which is augmented by a time periodic control term. We show that this can be done such that a) the application of control produces only inertial accelerations, b) positive real Poincaré exponents are shifted into the left half-plane, and c) the shift of the exponent is linear with control gain. We apply these developments to an unstable orbit near the Earth-Moon L_3 point perturbed by the Sun. Finally, we show that the control theory can be extended to include first order perturbations about the periodic orbit without increase in control cost.

Introduction

Control of a satellite following a nominal trajectory is a topic which arises in several areas of celestial mechanics. For example, stationkeeping costs are a major limiting factor on the lifetime of a synchronous satellite. Farquhar [1] has considered the application of control theory to satellites near the Earth-Moon libration points, and one satellite, ISEE-3, orbits the libration point between the Earth and Sun; Farquhar [2], Farquhar *et al.* [3]. Due to the relatively benign nature of this latter orbit, ISEE-3 is able to stationkeep using impulsive maneuvers every few months. Stationkeeping near an unstable periodic orbit has also been studied by Breakwell *et al.* [4], who developed a continuous feedback controller that stabilizes the orbit.

In this paper we introduce a new formulation of the periodic orbit control problem. Often, a periodic orbit can be found in a dynamical system which does not admit equilibrium points. This is important, since it enables more dynamics to be incorporated *ab initio,* reducing overall control costs. We introduce Floquet modal variables in the

[1]Department of Aeronautics and Astronautics, Air Force Institute of Technology, Wright-Patterson AFB, OH 45433.
[2]Johnson Space Center, Houston, TX 77058.

vicinity of the periodic orbit, and apply control only to suppress deviations from the orbit which grow with time. In common with Breakwell *et al.,* we do not attempt to suppress *all* deviations from the reference orbit. Thus, our controller does not expend energy to eliminate sinusoidal or decaying errors, further reducing control costs. Finally, we show that it is possible to incorporate some perturbations into the controlled system without further increase in control costs. The ideas developed here will be applied to an object in a periodic orbit about the Earth-Moon L_3 point, but the theory may be applied to any unstable periodic orbit.

Dynamics

In an earlier work, Wiesel [5], a model for the dynamics of a massless particle in the Earth-Moon-Sun system was developed. The geometry of this system is shown in Fig. 1, and the Hamiltonian is given by

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + yp_x - xp_y + n_\odot^{1/3} m_\odot (1 + m_\odot)^{-2/3} p_x \sin(1 - n_\odot)t$$
$$+ p_y \cos(1 - n_\odot)t - (1 - \mu)/r_{s\oplus} - \mu/r_{s☾} - m_\odot/r_{s\odot} \tag{1}$$

We have adopted the standard units and conventions of the restricted three body problem. In the form above, the rectangular coordinates refer to a frame rotating at the average lunar rate, while the momenta are *inertial* velocity components. We shall find this latter fact very useful in discussing the control theory.

For the current discussion, we shall use this model with the Earth and Moon in a periodic orbit about their center of mass, consistent with the circular orbit about the Sun. This produces a time periodic dynamical system with a period of one synodic month. This model thus includes the Sun and "variational" terms in the lunar orbit from the outset. The major effects neglected are the "Keplerian" component of lunar eccentricity, lunar inclination, and the solar eccentricity. These can be incorporated into the dynamics (1) (see [5]), and we discuss their incorporation into the controller in a later section.

Figure 2 shows the particular periodic orbit we shall study in this paper. It was constructed in [5], and can be thought of as a forced oscillation about the L_3 (opposite to the Moon) equilibrium point. It is unstable, and must be controlled if it is to be used as a reference trajectory.

Introduction of Modal Variables

Expanding the equations of motion from (1) about the periodic orbit, and grouping them into state vector form, we find the time periodic variational equations

$$\dot{\vec{\delta x}} = A(t)\vec{\delta x} \tag{2}$$

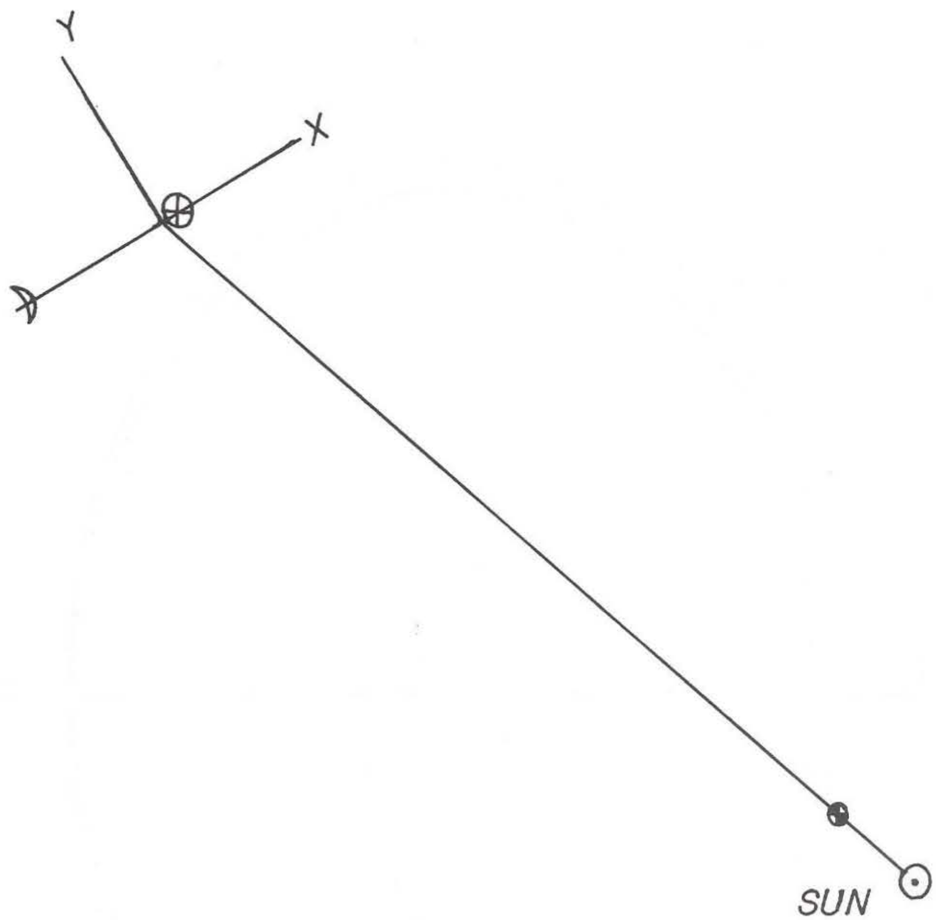

FIG. 1. Geometry of the Earth-Moon-Sun System.

The development of the periodic A matrix is contained in the Appendix. Floquet's theorem states that the state transition matrix $\Phi(t, 0)$ of (2) can be decomposed as

$$\Phi(t, 0) = \Lambda(t) e^{Jt} \Lambda^{-1}(t = 0) \tag{3}$$

where $\Lambda(t)$ is a time periodic matrix, and J is a constant matrix whose entries are Poincaré exponents. Direct substitution of (3) into (2) yields

$$\dot{\Lambda}(t) = A(t)\, \Lambda(t) - \Lambda(t) J \tag{4}$$

Initial conditions for this equation are supplied by the eigenvectors of the monodromy matrix $\Phi(\tau, 0)$, where τ is the period. In order to keep Λ real, it is assembled in column vectors which consist of either the real eigenvector conjugate to a real Poincaré exponent, or the real and imaginary parts of a complex eigenvector conjugate to a pair of

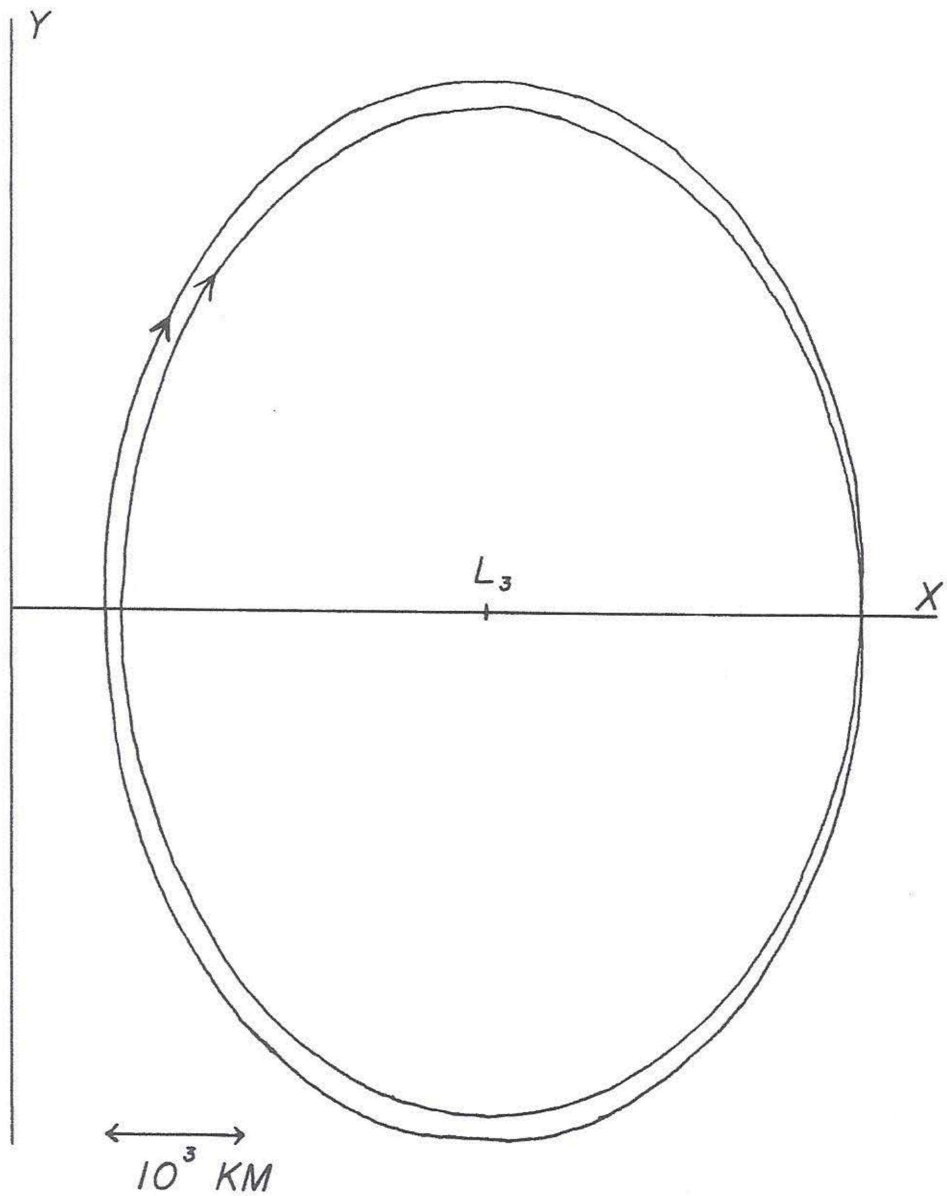

FIG. 2. The Unstable L_3 Periodic Orbit.

imaginary Poincaré exponents. For our L_3 orbit, which has 2 imaginary pairs and one real pair of Poincaré exponents, J assumes the form

$$J=\begin{bmatrix} 0 & +|\omega_1| & & & & \\ -|\omega_1| & 0 & & & & \\ & & \omega_2 & 0 & & \\ & & 0 & -\omega_2 & & \\ & & & & 0 & +|\omega_3| \\ & & & & -|\omega_3| & 0 \end{bmatrix} \tag{5}$$

which is also real. Table I gives the ω_i values.

The construction of this solution to (2) begins with the numerical integration of

$$\dot{\Phi}=A(t)\Phi$$

for one period, to obtain the monodromy matrix $\Phi(\tau)$. Since $\Lambda(t)$ is periodic, equation (3) implies that the eigenvectors of $\Phi(\tau)$ are $\Lambda(0)$, while its eigenvalues λ_i are related to the Poincaré exponents by

$$\omega_i = \frac{l}{\tau}\ln(\lambda_i)$$

Thus, a slight modification to the standard eigenvector/eigenvalue problem furnishes us with the $\underline{J}$ matrix and the value of the Λ matrix at $t = 0$.

Since $\Lambda(t)$ is periodic, we need to obtain it over the interval $(0, \tau)$ to complete the solution. This is performed by numerically integrating (4) over one period, starting from the initial conditions obtained in solving the eigenvector problem. In order to conveniently deal with functions obtained by numerical processes, we suggest the use of harmonic analysis, Brouwer and Clemence [6]. By this technique, a tabular function may be easily reduced to its Fourier series coefficients. Having obtained J and $\Lambda(t)$, the solution (3) to the variational equations (2) is completed. We shall also require $\Lambda^{-1}(t)$, which can be obtained by inverting $\Lambda(t)$ for many values of t over one period and applying harmonic analysis again. Equation (3) implies that $\Lambda(t)$ is never singular.

We now introduce Floquet modal variables $\vec{\eta}$ by

$$\vec{\eta}=\Lambda^{-1}(t)\delta\vec{x} \tag{6}$$

TABLE 1. Poincaré Exponents and Periods/*e*-Folding Times for the L_3 Periodic Orbit

Mode	ω_i	τ_i
1	0.0777384*i*	351.458 days; periodic
2	0.1789305	24.302 days; unstable
3	0.0841342*i*	324.739 days; periodic

Substituting into (2) and using (4), the variational equations (2) reduce to the constant coefficient system

$$\dot{\vec{\eta}} = J\vec{\eta} \tag{7}$$

Thus, the effect of the periodic linear transformation (6) is to reduce the system to an uncoupled, linear constant coefficient problem. For our L_3 orbit (where J has the form given in (5)) the flow assumes the character of a center when projected on either the (η_1, η_2) or (η_5, η_6) planes, and becomes a saddle point when projected on the (η_3, η_4) plane.

Figure 3 is an example of a projection on to the (η_3, η_4) plane. Trajectories near the periodic orbit were integrated using the full nonlinear equations of motion from (1). The $\delta\vec{x}$ vector was then computed, and converted to $\vec{\eta}$ variables for output. The success of the transformation to modal variables is confirmed by the appearance of a perfect saddle point structure near the origin. Figure 3 also emphasizes the local character of the modal transformation, since elsewhere on the figure the trajectories are time dependent and nonlinear. It is the unstable behavior of the η_3 coordinate which must be altered if this orbit is to be stabilized.

Control Theory

We return to the variational equations in physical variables, and augment them with the usual control term

$$\delta\dot{\vec{x}} = A(t)\delta\vec{x} + B(t)\vec{u}(t) \tag{8}$$

At this stage we apply the physical constraint that a satellite can only produce control accelerations with respect to inertial space. Since the momenta in the Hamiltonian (1) are the inertial velocity components resolved on the rotating axes, this constraint implies that control terms can only occur in momenta (p_x, p_y) equations of motion. We elect to control only the η_3 coordinate, implying we need only consider scalar u, and take $B(t)$ as the constant vector

$$\vec{B}^T = (0, 1, 0, 1, 0) \tag{9}$$

This choice corresponds to 2 fixed thrusters operating at a 45° angle to the Earth-Moon line. It is otherwise arbitrary.

Transforming (8) to the modal variables, we find

$$\dot{\vec{\eta}} = J\eta + \Lambda^{-1}(t)\vec{B}u \tag{10}$$

which is again a time periodic system. If we choose to feed back the unstable η_3 coordinate

$$u = k\eta_3 \tag{11}$$

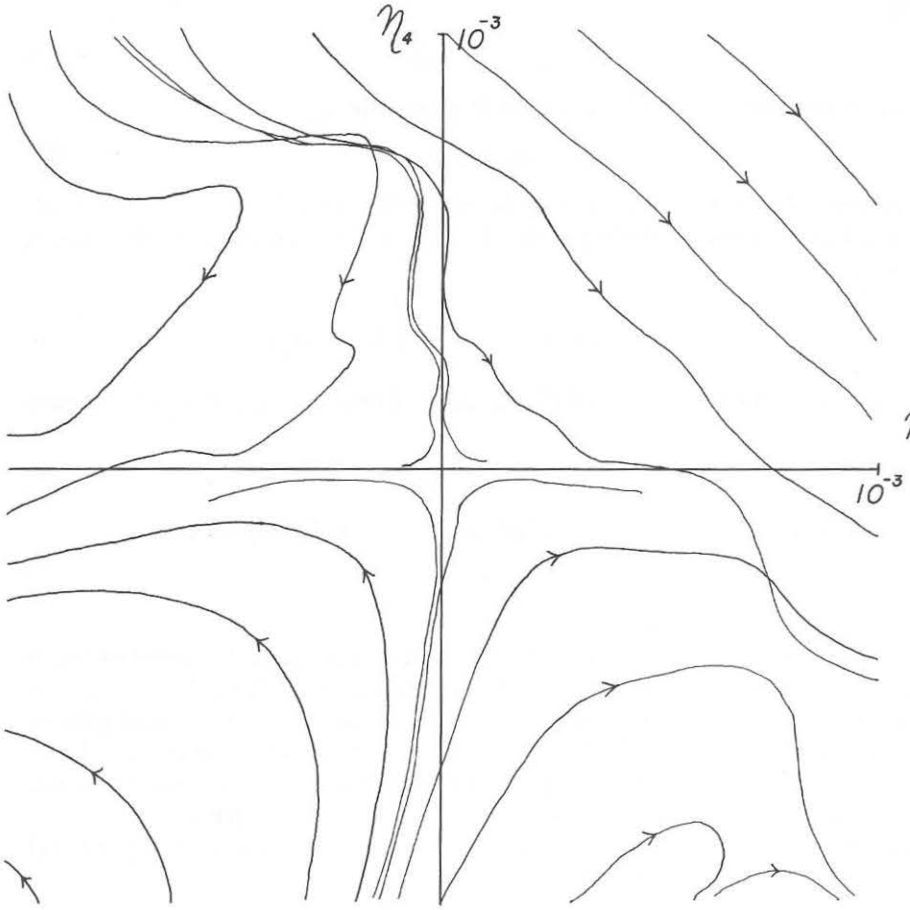

FIG. 3. Phase Portrait of the Uncontrolled System.

where k is a gain, we are led to another Floquet problem of particularly simple form

$$\dot{\vec{\eta}} = \begin{bmatrix} 0 & +|\omega_1| & 0 & 0 & 0 & 0 \\ -|\omega_1| & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_2 + k\vec{\lambda}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +|\omega_3| \\ 0 & 0 & 0 & 0 & -|\omega_3| & 0 \end{bmatrix} \vec{\eta}, \tag{12}$$

where

$$\vec{\lambda}(t) = \Lambda^{-1}(t)\vec{B} \tag{13}$$

Of particular interest is the new equation of motion for η_3

$$\dot{\eta}_3 = [\omega_2 + k\lambda_3(t)]\eta_3 \tag{14}$$

Equation (14) can be solved by using an integrating factor. From (13), $\vec{\lambda}(t)$ is available as a Fourier series. Separating λ_3 into its constant and sinusoidal parts, the solution to (14) is

$$\eta_3(t) = \eta_3(0)e^{(\omega_2 + k\lambda_{3c})t}\exp\left\{\int_0^t k\lambda_{3p}(t)\,dt\right\} \tag{15}$$

Since the last exponential is a periodic function of time, the new Poincaré exponent becomes

$$\omega_2' = \omega_2 + k\lambda_{3c} \tag{16}$$

which yields the rule for gain calculation, and the controllability condition is

$$\lambda_{3c} \neq 0 \tag{17}$$

This condition is satisfied for our L_3 orbit.

The complete solution to (12) is obtainable by the methods we developed earlier. It should be mentioned that the control term does not alter any other Poincaré exponent in the controlled system. Also, should a periodic orbit have multiple real pairs of Poincaré exponents, the procedure used here can be reapplied to (12) after it is rediagonalized by a second modal transformation. Our technique is thus the Floquet analogue of the "pole placement" method used for constant coefficient systems.

Converting back to physical variables, the controller is implemented in the full nonlinear system by adding the term

$$k\vec{B}\eta_3 = k\vec{B}\{\Lambda^{-1}(t)[\vec{x} - \vec{x}_p(t)]\}_3 \tag{18}$$

to the equation of motion, where $\{\cdot\}_3$ represents the 3rd component of the vector in brackets. Here, $\vec{x}_p(t)$ is the periodic orbit, and $\vec{x}$ the full system state vector. It is still convenient to display results of trajectory integrations in the $\vec{\eta}$ variables, although these variables no longer completely decouple the system.

Figure 4 is a phase portrait for a gain $k = +0.3$, just above what is needed to stabilize the system. (A positive gain is needed because λ_{3c} is itself negative.) Some trajectories do indeed enter the origin, but the behavior over much of the (η_3, η_4) plane is complicated, and many trajectories depart to infinity.

The situation improves for somewhat higher gain. Figure 5 shows a phase portrait for $k = 0.5$. The pattern is now that of a nodal point—both exponents are real and negative. For a gain of 0.8 (Fig. 6) the nodal pattern appears again, and covers a larger area on the (η_3, η_4) plane. The region of validity of the linear theory thus increases with increasing gain. All trajectories shown in Figure 6 enter the origin. Control accelerations applied in these cases average about $10^{-5}\,g$ during the suppression of the

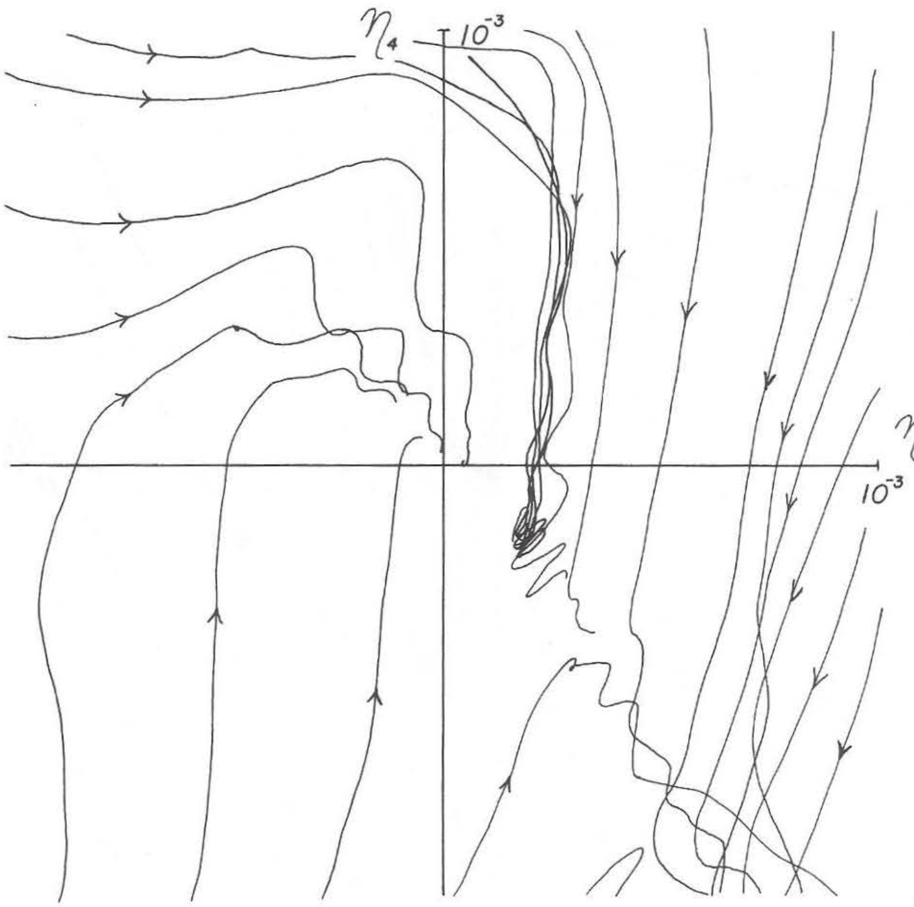

FIG. 4. Phase Portrait for a Gain of 0.3.

transient. As yet, we cannot cite a long term control cost, since once the satellite reaches the reference trajectory no further control is needed, ignoring outside perturbations.

The last two figures also show oscillations which occur along the "slow" axis of the nodal structure. These are not a result of the periodic nature of the dynamics, but are a nonlinear effect predicted by Poincaré [7]. The cause of the effect is shown schematically in Fig. 7. Over most of the plane we may neglect higher order terms, and if $k_1 \gg k_2$, y is the "fast" variable and quickly decays to a small value. However, near the x axis, higher order terms (proportional to x^2) dominate the linear term in $\dot{y}$. The x behavior is still linear, but the trajectory exhibits decaying oscillations transverse to the x axis as the origin is approached. Since the trajectory cannot depart too far from the x axis (or the linear term in $\dot{y}$ dominates), this interesting effect does not alter the overall stability of the controlled orbit.

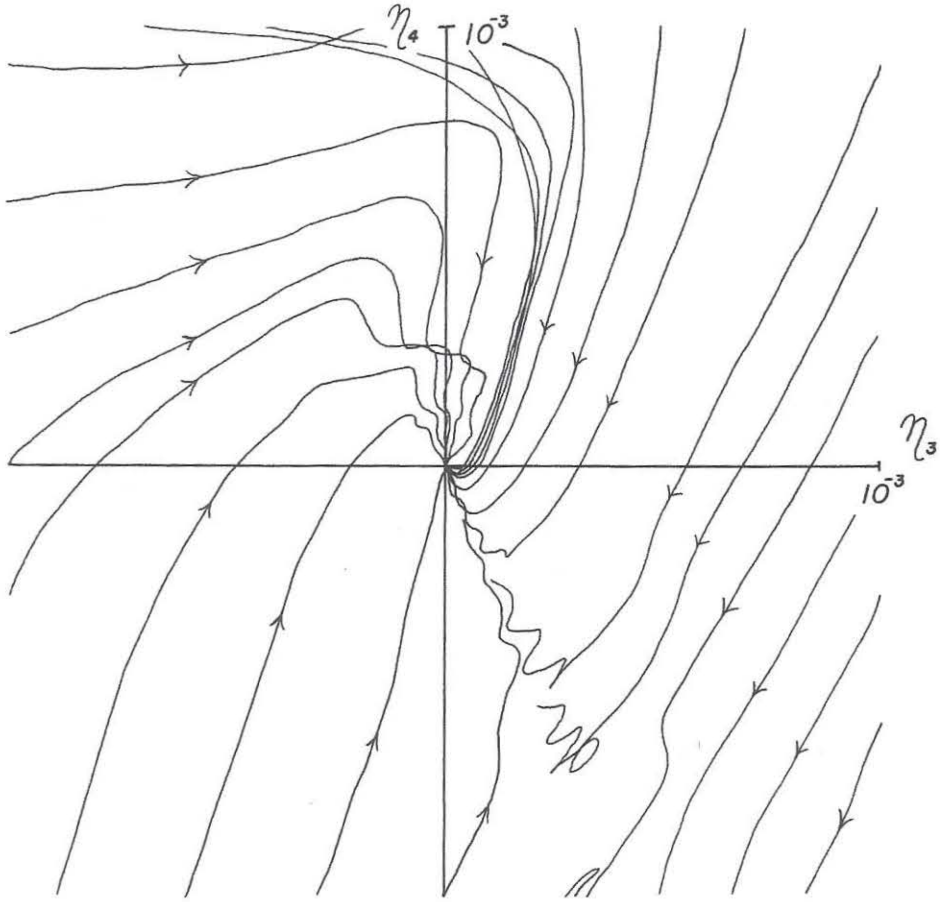

FIG. 5. Phase Portrait for a Gain of 0.5.

Perturbation Theory

We have seen that it is possible to use Floquet theory to design a control system for an unstable periodic orbit. If our L_3 satellite is delivered to the vicinity of the periodic orbit, the control system will drive the unstable modal variable η_3 to zero, while ignoring the stable oscillations in the (η_1, η_2) and (η_5, η_6) planes. Once the origin of the (η_3, η_4) plane is reached, the control system turns off, continuing to ignore small oscillations along inherently stable directions.

However, the above results hold only in the dynamics in which the periodic orbit itself exists. As mentioned in the first section, the free lunar eccentricity, the lunar inclination, and the eccentricity of the Sun have not yet been included. There are two possible approaches which might be taken to include these effects. The first approach would be to add damping to the two stable Floquet modes. This would move the pure imaginary Poincaré exponents into the left half plane, and the controller would

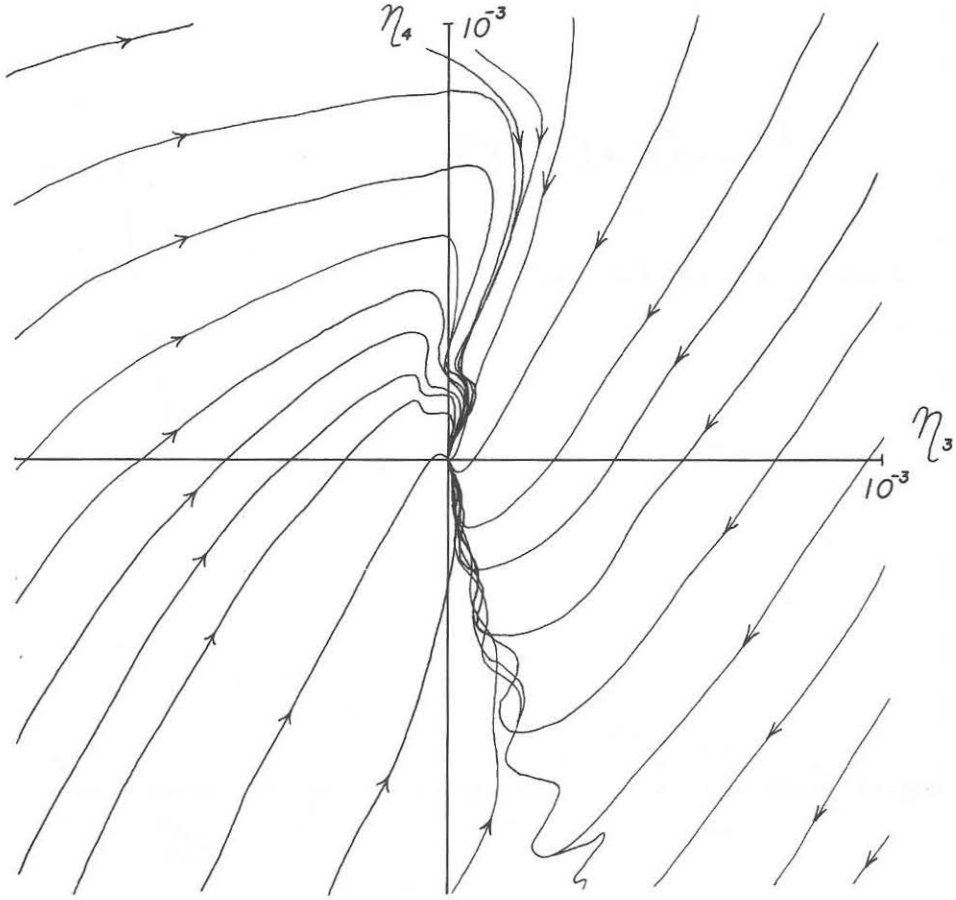

FIG. 6. Phase Portrait for a Gain of 0.8.

then suppress all oscillations about the periodic orbit, regardless of their source. The second, and more interesting option, is to continue to teach the controller to ignore any stable effects.

It appears that classical perturbation techniques can be extended to this case, since with the active controller operating, the periodic orbit becomes a stable reference solution. If we write the variational equations again, we may also include the lowest order perturbation terms to find:

$$\delta\dot{\vec{x}} = \{A(t) + k\vec{B}\Lambda_3^{-1}(t)\}\delta\vec{x} + \vec{P}(t) \tag{19}$$

Here, $\vec{P}(t)$ represents the perturbing effects evaluated on the periodic orbit, and is a function of time alone. If we apply the Floquet transform to modal variables for the *controlled* orbit, we find

$$\dot{\vec{\eta}}' = J'\vec{\eta}' + \Lambda'^{-1}(t)\vec{P}(t) \tag{20}$$

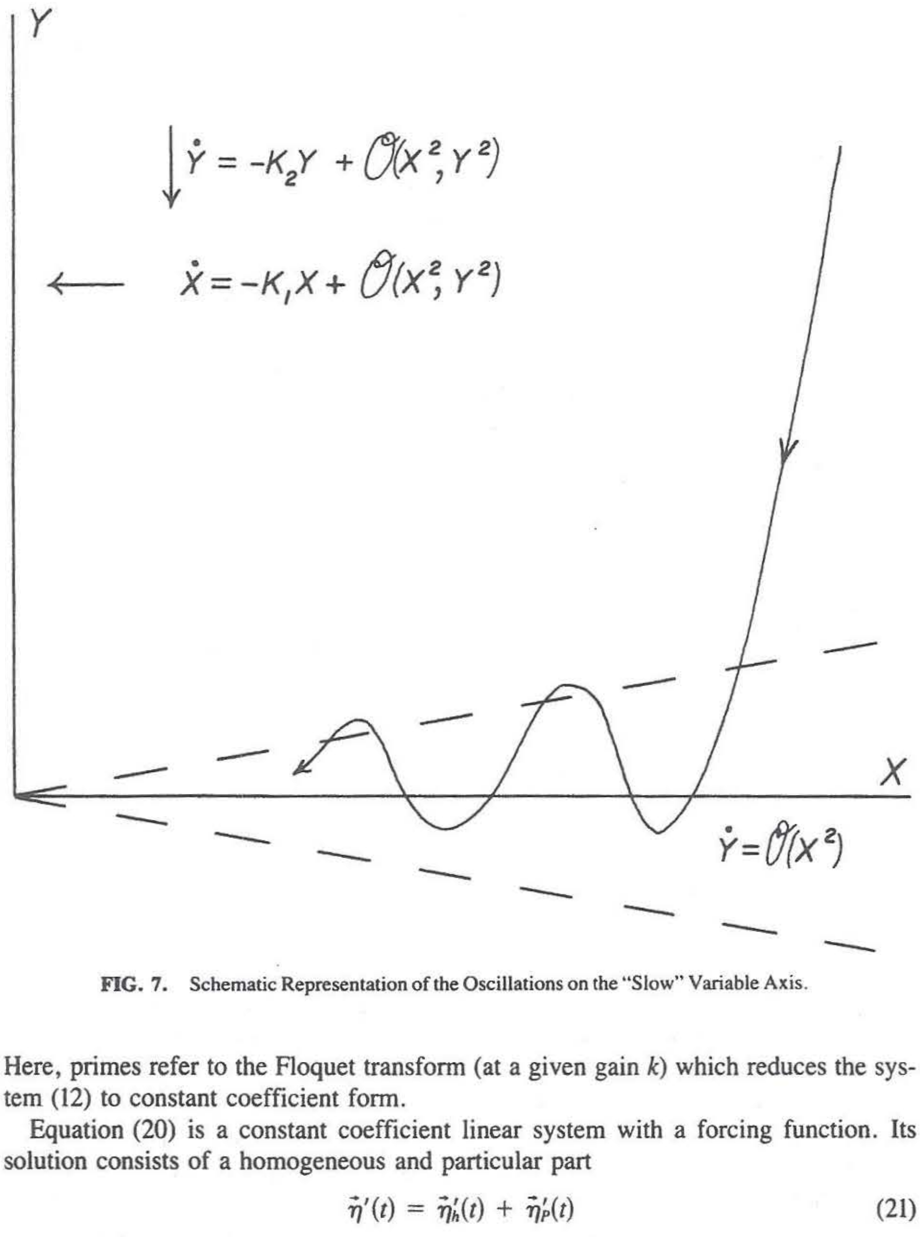

FIG. 7. Schematic Representation of the Oscillations on the "Slow" Variable Axis.

Here, primes refer to the Floquet transform (at a given gain k) which reduces the system (12) to constant coefficient form.

Equation (20) is a constant coefficient linear system with a forcing function. Its solution consists of a homogeneous and particular part

$$\vec{\eta}'(t) = \vec{\eta}_h'(t) + \vec{\eta}_P'(t) \tag{21}$$

The standard series techniques can be used to obtain $\vec{\eta}_P'(t)$ if they are generalized to allow series expansions of the form

$$\vec{\eta}_P'(t) = \sum_{\kappa} \vec{C}_{\kappa} \exp(j_1\omega_1 + j_2\omega_2 + \cdots) \tag{23}$$

where the ω_i can be either real or imaginary. If $\vec{\eta}'_P$ is well behaved (*e.g.*, free of resonances *and* positive real ω_i), then the controller can be taught to suppress only the "free" component of η_3 in the unprimed variables. This is a topic of current research.

Higher order perturbation terms in (19) would be of order δx times a small parameter $(e_{☾}, i_{☾}, e_{\odot})$, or of order δx^2. If the controller operates satisfactorily in the lowest order perturbed system, there is then every reason to believe that higher order terms would be negligible. We would again arrive at a control system with effectively zero long-term stationkeeping costs.

Conclusions

By using Floquet theory, we have found that the variational equations (2) can be solved. The Λ matrix obtained in this solution can be used to reduce the variational equations to a decoupled, constant-coefficient system. This reduction considerably simplifies the control problem, and in the case of a single unstable root the pole shift is linear with the gain. The long term control costs can approach zero if perturbation theory is used to include nonlinearities.

This paper addresses a problem which is very similar to the problem solved by Breakwell *et al.*, and we reach many of the same conclusions. However, our approaches differ substantially. While Breakwell *et al.* solved their problem using an optimal control approach entirely in physical state variables, our method centers around the modal decoupling transformation (6). Both methods have advantages, and since applications to orbital stationkeeping, satellite stability, and helicopter problems are only beginning, there is still much to learn.

Acknowledgment:

The authors wish to express their sincere appreciation to Dr. Robert Calico and Dr. James Silverthorn for many valuable discussions in control theory.

Appendix: Hamiltonian Variational Equations

The Hamilton equations of motion are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{A-1}$$

where q_i is the coordinate conjugate to the momentum p_i. They can be written in state vector form

$$\dot{\mathbf{x}} = Z\frac{\partial H}{\partial x} \tag{A-2}$$

where Z is a skew-symmetric matrix containing the off-diagonal ± 1 entries necessary to reproduce (A-1). Variational equations are used to study small deviations from a known trajectory. Expanding (A-2) about a given orbit, and retaining only first order terms, we find

$$\delta\dot{\mathbf{x}} = ZH_2(t)\delta\mathbf{x} \tag{A-3}$$

Here $\delta\mathbf{x}$ represents a small deviation, and H_2 is the symmetric matrix

$$[H_2]_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_j} \tag{A-4}$$

evaluated on the known solution. Comparing (A-3) to (2), we see

$$A(t) = ZH_2(t) \tag{A-5}$$

References

[1] FARQUHAR, R. W. *The Control and Use of Libration Point Satellites*, NASA TR R–346, 1970.

[2] FARQUHAR, R. W. "Limit Cycle Analysis of a Controlled Libration-Point Satellite," *J. Astronautical Sciences*, Vol. 17, 1970, pp. 267–291.

[3] FARQUHAR, R. W., MUHONEN, D., NEWMAN, C. and HEUBERGER "Trajectory and Orbital Maneuvers for the First Libration Point Satellite," *J. of Guidance and Control*, Vol. 3, 1981, pp. 544–549.

[4] BREAKWELL, J. V., KAMEL, A. and RATNER, M. J. "Stationkeeping for a Translunar Communication Station," *Celestial Mechanics*, Vol. 10, 1974, pp. 357–373.

[5] WIESEL, W. "The Restricted Earth-Moon-Sun Problem I: Dynamics and Libration Point Orbits," *Celestial Mechanics*, 1982, in press.

[6] BROWER, D. and CLEMENCE, G. M. *Methods of Celestial Mechanics*, Academic Press, 1961, page 108.

[7] POINCARÉ, H. *Les Methodes Nouvelles De la Mechanique Celeste*, 1899.