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Statistical Inference to Evaluate and Compare the Performance of Correlated Multi-State Classification Systems

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Statistical Inference to Evaluate and Compare Correlated Multi-State Classification Systems

DISSERTATION

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STATISTICAL INFERENCE TO EVALUATE AND COMPARE CORRELATED MULTI-STATE CLASSIFICATION SYSTEMS

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Abstract

The current emphasis on including correlation when comparing diagnostic test performance is quite important, however, there are cases in which correlation effects may be negligible with respect to inference. This proposed work examines the impact of including correlation between classification systems with continuous features by comparing the optimal performance of two diagnostic tests with multiple outcomes as well as providing inference for a sequence of tests. We define the optimal point using Bayes Cost, a metric that sums the weighted misclassifications within a diagnostic test using a cost/benefit structure. Through simulation, we quantify the impact of correlation on standard errors comparing two tests and evaluate the resulting errors with respect to CI coverage and width under varying diagnostic test accuracy, sample size, cost/benefit structures, parametric assumptions and correlation levels. When formulas are required for better inference to include correlation, we provide updated computational techniques that properly extend the Delta and Generalized method. Additionally, ,to date, no methods have been applied to quantify the performance of a sequence of tests. Therefore, the inference methods derived in this work are extended to sequenced tests where feature correlation is unavoidable and must be accounted for when developing inference on tests.

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First and foremost, I would like to thank my wonderful wife. Without her love and support during this time, my completion of this dissertation would have been impossible. I would also like to thank my children for being somewhat understanding when I needed to work a little longer than they thought I should. Finally, I would also like to think my amazing advisor, Dr. Schubert Kabban. She guided me faithfully, and sometimes painfully, through the research process.

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STATISTICAL INFERENCE TO EVALUATE AND COMPARE CORRELATED MULTI-STATE CLASSIFICATION SYSTEMS

I. Introduction

Decision making under uncertainty is a hallmark of statistical thought in the 21^{st} century. Whenever we make decisions, we take input information, classify it into a set of outcomes and develop a cost/benefit for each of those outcomes. This process represents a classification system which is weighted by some sort of desirability. Classification systems are present almost every time a decision is made, whether an autonomous vehicle is plotting a route or a doctor is diagnosing an illness. When a decision is important and the classification system is imperfect, statistics and best practices can be used to compare classification systems and quantify performance so as to choose the most appropriate classification method available.

The difficulty in making decisions based on imperfect information is uncertainty. Statistics can support this decision process by making inferences, based on reasonable assumptions, that can be used to quantify the uncertainty inherent in the data in order to determine the better course of action. Specifically, when deciding on a diagnostic test to use in a clinical setting, for instance, comparisons of tests require accurate inference adjusting the estimate that compares the two tests by a quantifiable measure of uncertainty, or standard error.

One of the simplifying assumptions often made when comparing classification systems is independence, though in many settings, statistical independence is rarely possible. Correlation can manifest with the selection of features to be measured and compared on the same experimental units. For example, the triglyceride measure and total cholesterol measures are highly correlated. If two tests assessing the likelihood of developing diabetes (especially if we consider multiple "risk" levels) are developed on the same set of subjects, one using triglycerides and the other using total cholesterol, it is likely that any inference from the tests could not assume statistical independence on these features. Whereas assumed independence may be more appropriate, if each test is conducted on different randomly sampled subjects, though higher triglycerides would be expected to be associated with higher total cholesterol. However, designs using repeated measures on the same subjects are useful in removing inter-subject variability, and therefore, at times preferred.

It is, therefore, important to appropriately estimate and adjust comparisons of tests for the correlation that exists between them. The risk in not doing so produces overly conservative statistical inference such that new tests or classification systems which are inherently more accurate may not be discovered because the error surrounding the comparison of the two tests was not appropriately quantified. In contrast, there exists at times minimal correlation, seemingly negligible, or correlation that is non-tractable. In such cases, it may still be possible to create appropriate inference, which is not too conservative, especially when test data already exists and new samples of paired data are infeasible. Therefore, this work addresses two research questions:

1. How robust are methodologies against correlation in the evaluation or comparison of classification systems?

2. What is the appropriate adjustment to methodological approaches to account for correlation when required?

Finally, a diagnostic test (or classification system) is expensive, takes a long time, or is otherwise difficult to perform. To improve overall detection in these cases, a screening test may be used to determine subjects that require continued testing for classification. These sequenced tests involve the same population subjects and correlation in this setting is unavoidable. Further, correlation in the sequential setting can be estimated utilizing similar methodological approaches to the correlation in the comparison of systems. To date, statistical inferential methods have not been applied to the performance of sequential systems.

Thus, this dissertation aims to improve the classification system selection and performance quantification involving three or more classes in a paired setting and develop inferential methods for performance of sequential tests. Specifically, this research addresses the two research questions by quantifying the need for accounting for correlation in a comparison of the performance of classification systems, quantifying the uncertainty present in the comparison of paired classification systems, and developing inferential methods on sequences of classification systems. These aims translate to the following research goals:

1. Determine the level of correlation at which inference around the comparison between classification systems is adversely impacted (Overestimates or Misspecifies Coverage).

2. Develop correct inferential procedures to account for correlation between features in a comparison of classification system.

3. Develop inferential procedures to account for correlation in sequence of subsystems.

The results of this work produces the following contributions to providing inference in the comparison and evaluation of correlated multi-state classification systems:

1. Under specific scenarios, ignoring correlation on inferential methods is shown to be a valid approach.

2. Current methods are extended to compare generalized multi-state systems (any k) with correlated features.

3. The first statistical inferential methods for sequential systems with respect to accuracy are developed.

These contributions enables researchers and decision makers to have more accurate and cost-effective solutions for comparing and quantifying uncertainty in classification systems.

II. Background

2.1 Classification Systems

A classification system (G) is a functional map that transforms k partitions of a set of events, $\mathbf{E} = \{(e_1, ..., e_k\}$ to k distinct elements of a label set $\mathbf{L} = \{l_1, ..., l_k\}$. Each of these partitions represent a class. This labeling of the events, or data, into distinct classes occurs because of a set of features, $\mathbf{F} = \{f_1, ..., f_m)\}$, that are used as a part of a functional map that transforms the partitioned event set into the label set. As such, we assume that there is specific parameter, or vector of parameters, $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, that, when varied with respect to each feature, affects the outcome class label assigned by the classification system. Thus, for every $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, there is a classification system $(\mathbf{A}_{\boldsymbol{\theta}})$, and the set of these systems, $\mathbb{A} = \{\mathbf{A}_{\boldsymbol{\theta}}, \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$ is called a classification system family (CSF) [33].

Table 2.1. Two-class contingency table

		TRUTH	
		Positive	Negative
CLASSIFICATION	"Positive"	True Positive	False Positive
	"Negative"	False Negative	True Negative

Consider the two-class contingency table presented in Table 2.1 [4] which contains four states of information with respect to the two truth states of positive and negative. Two of these states are correct classification: true positive (often referred to as sensitivity when the number of true positives is divided by the total number of positives in the truth data) and true negative (referred to as specificity when the number of true negatives is divided by the total number of negatives in the truth data). The other two outcomes are misclassifications: false positive and false negative. These misclassifications are the result of the classification system assigning an incorrect predictive label to the respective truth event.

Table 2.2. Three-class contingency table where each entry, $i|j$, represents a classification system assigning label i to an event whose truth class is j

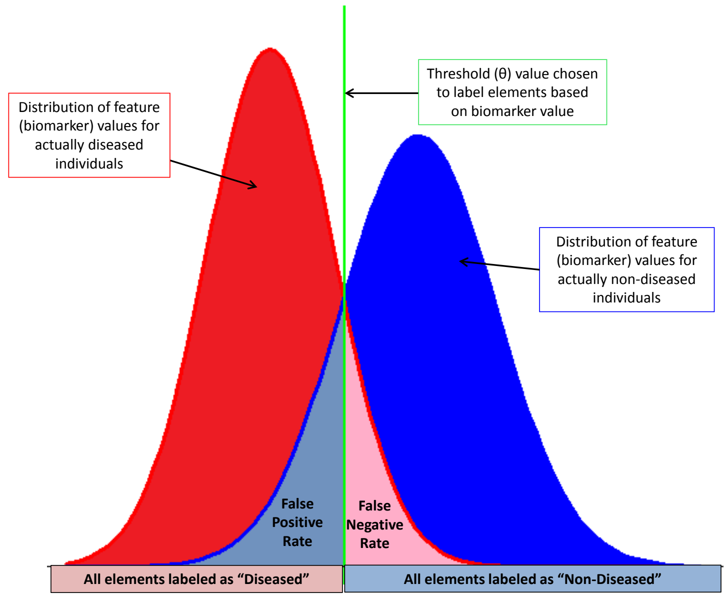
		TRUTH		
		CLASS 1	CLASS 2	CLASS 3
CLASSIFICATION	"CLASS 1"	1|1	1|2	1|3
	"CLASS 2"	2|1	2|2	2|3
	"CLASS 3"	3|1	3|2	3|3

Similarly, any classification system with k labels, which is designed to map events into k classes, contains k^2 states of information. Consider the three-class system included in Table 2.2 [4]. Here, green blocks on the diagonal represent the correct classifications ($k = 3$ correct classifications), and the remaining six blocks represent possible misclassifications ($k^2 - k$ incorrect classifications). When there are more than two classes, the terms "true positive" and "false negative" lose meaning, therefore, the notation $i|j$ is used where j represents the truth label and i represents the classification system label.

2.2 Optimal Points

Often, the features within a classification system are measured using continuous data which naturally occurs with variation, and which, when partitioned on a thresh-

Figure 2.1. Two-class continuous with threshold



old, generates the outcome class label for the classification system. Figure 2.1 [4] represents such a phenomenon for a two class system and the resulting classification as identified by Table 2.1 [4]. It should be clear to see that there is a compromise between correct (or incorrect) classifications as the threshold θ moves from left to right. The point at which θ results in the best classification performance for the CSF is referred to as the optimal point.

There are many ways to compute optimal points [29]. Although many methods are based upon the notion of accurate classification, in this work, we refer to the optimal point as the threshold settings of the classification system that maximizes classification accuracy. We define this as the "best performance" of the CSF.

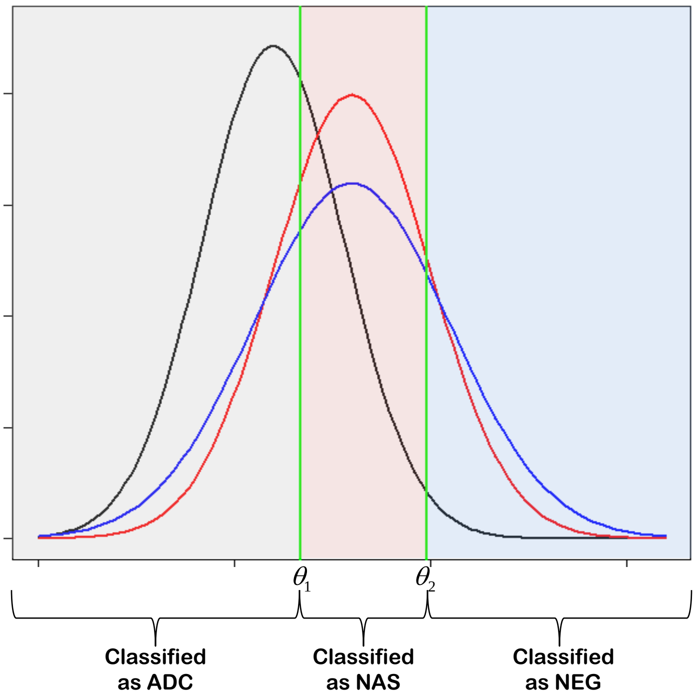
Ideally, the definition for best performance of a classification system should have some flexibility to allow for adjustments based on the requirements of the classification system. For example, if the importance of identifying all diseased individuals far outweighs the importance of mis-identifying non-diseased individuals, this may suggest a much different value of θ than when the two outcomes are equally important. Extensive work in the literature suggests that these costs (weighted importance on mis-classification rates) should be taken into account when evaluating optimal thresholds for a CSF [1, 18, 24, 33, 35, 36, 37, 38]. Similarly, thresholds can also be affected by the prevalence of each class. For example, a much larger sample of non-diseased individuals could shift the optimal threshold, and should be included when determining optimal threshold levels [7, 24]. Finally, when both prevalence and weighted misclassification costs are incorporated simultaneously into the definition of "best" performance, the optimal point of the CSF may change [13, 33, 38].

The optimal point for a k-class CSF often corresponds to $k-1$ threshold values. This occurs because separating a continuous set of numbers into k (ordinal) partitions requires a minimum of $k-1$ break points. For example, as described in Batterton [4] and reproduced here, in order to classify subjects into three categories (HIV negative (NEG), HIV positive non-symptomatic (NAS), and HIV-positive with AIDS dementia complex (ADC)), two threshold values ($\theta_1 < \theta_2$) on a biomarker (NAA/Cr) were used [25]. If a subject's NAA/Cr level was below θ_1 they were classified as ADC, if the subject's NAA/Cr level was between θ_1 and θ_2 they were classified as NAS, and finally if the subjects NAA/Cr level was greater than θ_2 they were classified as NEG [25] (see Figure 2.2 [4]). The values for θ_1 and θ_2 are chosen to maximize the performance of the classification system [29]. Typically, this is accomplished by collapsing the classification results into an univariate metric and optimizing the value of the associated metric with respect to the threshold of interest. Two such examples of such a metric that are the basis for this research are discussed in the Section 2.3.

2.3 Metrics for Optimal Points

For the purposes of this research, two metrics for determining the optimal point of a classification system are considered: the Youden Index and Bayes Cost. Both

Figure 2.2. Three Class Classifications for HIV Example.



of these metrics may account for the cost associated with misclassification and class prevalences and defines for each set of thresholds considered, the classification system accuracy. Each of these metrics will be described next.

2.3.1 The Youden Index.

The Youden Index (J), first introduced by W.J. Youden in 1950, is a method for rating classification systems with two classes [42]. Literature has shown J to be a useful metric in evaluating the performance of classification systems as a function of correct classifications [14, 28, 30, 42]. In the two-class system, J is defined as the maximum sum of sensitivity plus specificity (minus one) out of all possible choices of parameters, $\theta \in \Theta$, for the CSF:

$$J = \max_{\theta \in \Theta} \left[\text{sensitivity}(\theta) + \text{specificity}(\theta) - 1\right]. \tag{2.1}$$

The performance of a classification system that is worse than a pure guess (sensitivity and specificity both less than 0.5), is considered poor. Therefore, when considering only CSFs with both specificity and sensitivity bounded by [0.5,1], systems which perform better than chance have $J \in [0, 1]$ [42].

An important extension to the Youden Index is to a k-class systems. The extended J is defined as the summation of k correct classification probabilities (no longer specificity and sensitivity) [25, 26], and is expressed by:

$$J = \max_{\theta \in \Theta} \left[\sum_{\substack{i=1 \\ i=j}}^{k} \sum_{j=1}^{k} P_{i|j}(\theta) \right], \tag{2.2}$$

where $P_{i|j}(\theta)$ represents the probability associated with assigning class i to an event whose truth class is j.

Another useful extension to J using properties discussed in Section 2.2 is to include the presence of a cost and/or prevalence multiplier to the sum of correct classifications. Such an extension for the 2-class case is referred to as the Generalized Youden Index (GYI) [31, 36, 18, 23, 35, 25, 26]. The equation for the GYI is given below:

$$GYI = \max_{\theta \in \Theta} \left[\text{sensitivity}(\theta) + m \cdot \text{specificity}(\theta) - 1 \right], \tag{2.3}$$

where m is a weight function of the prevalence of class 1 (p_1) and the cost benefit ratio (using true positive, true negative, false positive, and false negative costs respectively $(C_{TP}, C_{FP}, C_{TN}, C_{FN})$) given by $m = [(1-p_1)/p_1] \times [(C_{FP}-C_{TN})/(C_{FN}-C_{TP})]$. This weight function takes into account misclassification costs associated with false positive or false negative results in addition to costs associated with correct classifications as well as the prevalence of classes 1 and 2 [3].

A review of the literature demonstrates the variations discussed on J for classification performance which includes incorporating prevalence (with equal weight) in a two-class system [31, 36], incorporating prevalence and cost-benefit weights in a two-class system [18, 23, 35], and incorporating prevalence and cost to a k-class system [25, 26]. The equation for the GYI extension to k-classes as produced by Nakas [26], is given below:

$$J_{k;(1,2,\ldots,k-1)} = \sum_{i=1}^{k-1} J_{2;(i,i+1)} = J_{k-1;(1,2,\ldots,k-2)} + J_{2;(k-1,k)}, \tag{2.4}$$

where J_k is the sum of the Youden Indices for the adjacent classes with weight function m fixed to 1 in each comparison J_k [32]. A limitation of this extension to k classes is that it only considers the pairwise sums of correct classifications cp and therefore does not consider misclassification in a k class system as distinct. This important distinction is rendered moot when the costs of misclassifications and prevalences for each target class are equal between classes.

2.3.2 Bayes Cost.

While the Youden Index maximizes the correct classification rate, the goal of Bayes Cost (BC) as a metric of performance, is to minimize the misclassifications. In a two-class framework, these goals are equivalent [4, 5, 28, 37]. When more than two classes exist, or classes have unequal misclassification costs or prevalence, the equivalence does not hold because it is no longer feasible to associate directly the benefit of making the correct decision and the cost associated with making class-specific incorrect decisions [5, 33, 37]. When a metric is required that accounts for the information in these k^2-k misclassification probabilities, such as in the case of unequal costs or prevalence, BC loses no information about the classification system [33].

BC is defined as the minimum sum of the weighted misclassification probabilities across all possible parameters, $\theta \in \Theta$, and is given by:

$$BC = \min_{\theta \in \Theta} \left[\sum_{\substack{i=1 \\ i \neq j}}^{k} \sum_{j=1}^{k} c_{i|j} p_j P_{i|j}(\theta) \right], \tag{2.5}$$

where $c_{i|j}$ is the cost of assigning class i to an element from truth class j and p_j is the prevalence of class j. The use of BC allows for any cost/benefit and prevalence structure to be considered when determining the performance of a classification system [4]. As can be seen in Equation 2.5, BC extends to any number of k classes, and when $k = 2$ can be written as:

$$BC = \min_{\theta \in \Theta} c_{1|2} p_2 P_{1|2} + c_{2|1} p_1 P_{2|1}, \tag{2.6}$$

which has been shown to be equivalent to the two-class Youden Index with equal costs and weights [4, 37].

2.4 Parametric methods for Inference using Confidence Intervals

It is important to characterize uncertainty when estimating the metrics used to determine optimal points when they are constructed from data. Typically, this is done by developing confidence intervals (CI) around the metrics used to define the optimal point (Youden Index, BC, etc.) as well as the thresholds which correspond to the optimal point [5, 18, 28, 29, 30]. This section will outline the research and methods of developing inference using confidence intervals on performance metrics of classification systems.

2.4.1 Confidence Intervals.

A confidence interval on a random sample $\mathbf{X}$ can be characterized by $L(\mathbf{X})$ and $U(\mathbf{X})$ where, for some function of θ, $\tau(\theta)$, $P[L(\mathbf{X}) \leq \tau(\theta) \leq U(\mathbf{X})] = 1 - \alpha$. Then $L(\mathbf{X})$ and $U(\mathbf{X})$ forms a confidence interval with a confidence coefficient of $1-\alpha$ [8, 15].

CI performance is typically measured with coverage probability, a computed proportion across many trials that calculates the number of times that the parameter of interest resides within the interval, and width, measured as the distance between $L(\mathbf{X})$ and $U(\mathbf{X})$. The best CI is the smallest interval that guarantees a $1-\alpha$ coverage probability; however, often width is sacrificed to assure coverage is maintained. Besides resampling methods, there are two main methods of CI construction for J and BC, the Delta method and Generalized methods. Each of these are described in the following sections.

2.4.2 Delta Method (Large Samples).

For large samples, asymptotic normality on the feature space can allow for the use of the Delta method to construct variance estimates for simple Wald-based inference. The Delta method is given by the following theorem.

> Suppose that $\widehat{\boldsymbol{\theta}}$ is Asymptotic-Normal$_k(\boldsymbol{\theta}, b_n^2\boldsymbol{\Sigma})$ with $b_n \to 0$ and that g is a real-valued function with partial derivatives existing in a neighborhood of $\boldsymbol{\theta}$ and continuous at $\boldsymbol{\theta}$ with $g'(\boldsymbol{\theta}) = \partial g(\boldsymbol{\theta})/\partial\boldsymbol{\theta}$ not identically zero. Then as $n \to \infty$
>
> $$g(\widehat{\boldsymbol{\theta}}) \text{ is Asymptotic-Normal}_k[g(\boldsymbol{\theta}), b_n^2 g'(\boldsymbol{\theta})\boldsymbol{\Sigma}g'(\boldsymbol{\theta})^T].$$

The Multivariate Delta method described in Theorem 2.4.2 [6] allows for the creation of confidence intervals around the Bayes Cost metric and the associated optimal points by observing that asymptotically, BC is normally distributed and $\hat{\theta}_m^* \sim N(\hat{\theta}_m, Var(\hat{\theta}_m))$. The variance of the BC metric, with the assumption of independence in the feature space, is given in Equation 2.7 and the derivation of the partial derivatives in the function can be found in Batterton [4]:

$$Var\left(\widehat{BC}\right) \approx \sum_{j=1}^{k}\left[\left(\frac{\partial BC}{\partial \mu_j}\right)^2 Var(\widehat{\mu}_j) + \left(\frac{\partial BC}{\partial \sigma_j}\right)^2 Var(\widehat{\sigma}_j)\right]. \tag{2.7}$$

Confidence intervals are then calculated using the expected value of BC and variance from Equation 2.7 along with the appropriate standard normal probability (Z).

2.4.3 Generalized Method.

When the distribution of the features used for classification is normal, or suitably transformed to normal, the Generalized method for developing inference can be used. The strategy for developing Generalized Confidence Intervals (GCI) around threshold values, J and BC, involves developing generalized pivotal quantities for μ_j and σ_j, $j = 1, ..., k$, and using these pivotal quantities to construct the confidence interval. We define these general pivotal quantities (GPQs), as in Lai [20]:

$$R_{\mu_j} = \bar{x}_j - t_j \frac{S_j}{\sqrt{n_j}} \tag{2.8}$$

and

$$R_{\sigma_j} = \sqrt{\frac{(n_j - 1)S_j^2}{V_j}}, \tag{2.9}$$

where

$$t_j = \frac{\bar{X}_j - \mu_j}{S_j/\sqrt{n_j}} \tag{2.10}$$

and

$$V_j = \frac{(n_j - 1)S_j^2}{\sigma_j^2} \tag{2.11}$$

with the sample mean ($\bar{x}_j$) and standard deviation (S_j) from the j^{th} class, $t_j \sim t_{(n_j-1)}$, a t-distributed random variable, and $V_j \sim \chi^2_{(n_j-1)}$, a chi-square random variable. The distribution of X in this case is assumed normal, therefore this method is appropriate for large or small samples as long as the assumption of normality holds.

As described by Batterton [4], in order to construct the GCI around BC, Monte Carlo simulation is used to create a large number (K) of random draws from t_j and V_j for each of the k classes. These K sets of t_j and V_j values are then used in the Equations 2.8-2.11 to determine K values of R_{μ_j} and R_{σ_j}, $j = 1 \ldots k$. Then the GPQ for BC is found using the normal distribution parameters' and optimal thresholds' GPQs where

$$\begin{aligned} R_{BC} = &\sum_{j=2}^{k} c_{1|j} p_j \Phi\left(\frac{R_{\theta_1^*} - R_{\mu_j}}{R_{\sigma_j}}\right) \\ &+ \sum_{\substack{i=2 \\ i \neq j}}^{k-1} \sum_{j=1}^{k} c_{i|j} p_j \left[\Phi\left(\frac{R_{\theta^*_{m=i}} - R_{\mu_j}}{R_{\sigma_j}}\right) - \Phi\left(\frac{R_{\theta^*_{m=i-1}} - R_{\mu_j}}{R_{\sigma_j}}\right)\right] \\ &+ \sum_{j=1}^{k-1} c_{k|j} p_j \Phi\left(\frac{R_{\mu_j} - R_{\theta^*_{k-1}}}{R_{\sigma_j}}\right). \end{aligned} \tag{2.12}$$

The minimization is inherent in the use of optimal thresholds, θ_m^*, the $k-1$ optimal thresholds' GPQs ($R_{\theta_m^*}$) are found numerically for each of the K sets of R_{μ_j} and R_{σ_j} values (this requires K numerical minimizations of Equation 2.12).

2.4.4 Parametric Confidence Interval Inference on BC and J.

This section will summarize the application of those methods for inference under specific (normal) distributional assumptions on the underlying features used for classification. We start with inferences on a single test.

A large body of research exists on confidence intervals around optimal points and the thresholds that correspond to the optimal point [18, 28, 33, 30]. All of these require distributional assumptions on the parameters, typically normal. Jund [18] developed a Delta method for inference on J and optimal thresholds assuming independent samples of normal biomarkers that performed well in large samples, but struggled with small sample sizes. Perkins [28] developed a Delta method for variance of the 3-class Youden Index that performed consistently better than bootstrapping by accounting for measurement error, again, limited by performance on small sample sizes. Schisterman [30], compared performance of Delta method and Bootstrapping methods on J and optimal thresholds for 2-class Normal and Gamma distributed biomarkers, finding that the Delta method presented a reasonable compromise for maintaining coverage without excessive width for sample sizes larger than 50. Skaltsa [37] and Batterton [5] have performed inference on BC including optimal points. Performance of the Delta, Bootstrap, and Generlized methodologies on BC are similar to the counterparts utilizing J as the metric.

2.4.5 Parametric Confidence Interval on Test Comparisons.

In addition to inference on a single test, the Delta method has been used in recent literature for the comparison of tests using Youden Index and Bayes Cost [4, 5, 36, 37], all under the assumption of independence between features on compared tests. The Delta method generally guarantees coverage probability when considering large samples where the test metric will approach normality. As a result, when applying this method to smaller sample sizes, the width of the interval is naturally made wider to compensate for potential violations of assumptions and coverage is not guaranteed. Literature shows that the Delta method performs well at relatively small samples, as low as 20 subjects per class, while the Generalized method performs well at even lower sample sizes.

Whenever considering the difference between two statistical tests, the correlation between features on compared tests, either incidental based on sampling or the result of non-independent populations being tested, can have significant impacts on inference. Zhou includes this feature correlation in a paired Youden Index using bootstrap methods to create CIs on 2 class problems with sample sizes ranging from 20-200, resulting in intervals that maintained coverage [43] with comparable widths that would be expected using Delta or Generalized methods for CI construction. A few authors have developed methods to account for correlation in paired tests using the Youden Index [16, 40, 41]. They attempt to extend the Generalized method and Delta method to account for correlation between feature distributions between classes; however the distributional assumptions around the sampling distributions of the multivariate normal do not match the requirements of the Generalized method. Specifically, Yin utilized a multivariate normal distribution for the sampling of feature mean values. However, as this research shows, this decision is inappropriate and therefore led to confidence bounds that significantly exceed coverage in most of the scenarios presented [40, 41]. Additionally, in the Delta method extension provided by Yin only J is considered, without taking into account cost or prevalence, and performs similarly to the independent assumption counterparts [40, 41].

Because none of these methods correctly extend the Generalized method to k-class problems and no current research allows for a complex class specific cost/benefit structure when correlation is present, this research seeks to incorporate feature correlation in the the calculation of confidence intervals around BC in k-class problems.

2.4.6 Nonparametric Confidence Intervals.

Collapsing a classification system to the number of outcomes with respect to truth allows for it to be modeled without distributional assumptions. Similarly to parametric confidence intervals, work has been done on non-parametric confidence intervals [4, 22]. Batterton [4] developed fiducial intervals that require an assumption of independence between classes, but performs well on small sample-sizes using the BC metric. Luo [22] used Gaussian Kernel Smoothing to develop non-parametric confidence intervals, and used bootstrapping on the resulting density functions using J.

Some recent research has been focused on the inclusion of correlation on the Youden Index using non-parametric methods. Chen [9] used the Delta method to find an approximate estimate of variance as provided in Equation 2.13. This resulted in better coverage for the confidence intervals for samples ranging from 20-100 (per class). Chen's work was limited to computation for J in two class classification systems and did not extend either to k-class classification systems nor a class-specific cost structure as allowed in BC. Using the notation previously described, the expected values of TP, FP, TN, FN, the updated variance for J,

$$Var(J) = \left(\frac{TN}{FP+TN}\right)^2 \frac{TP \cdot FN}{(TP+FN)^3} + \left(\frac{TP}{FN+TP}\right)^2 \frac{TN \cdot FP}{(TN+FP)^3}, \tag{2.13}$$

is found by using a Delta method approximation. This research did not extend to the k class classification system, and, as previously stated, the Youden Index does not account for the cost of class specific misclassification errors (in k classes). Chen's method provided better coverage for two-class problems with correlation by sacrificing width. The method was identical to previous methods, with $Var(J)$ substituted as given in Equation 2.13 [10]. This method maintains coverage of the CI by adjusting the variance around the estimated value, and subsequently, the width of the CI. It has been shown to perform better in datasets that include correlated biomarkers [9]. As previously stated, the limitation of this work is that it only extended to the 2-class case. The research presented in this dissertation will strictly focus on the parametric approaches to inference on classification systems.

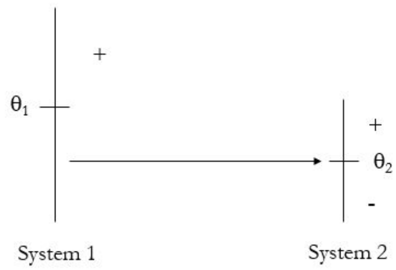
2.5 Sequencing Classification Systems

The work presented thus far has been concerned with inference on a single classification system, or a comparison of classification systems. Also of interest in this research is inference when sequencing of classification systems for diagnostic classification purposes [2, 3, 39, 34, 27], as sequences also exhibit inherent correlation and currently no inferential methods on J (GYI) or BC for a sequence of classification systems exists. We motivate and present classification system sequences through the application of sequences of diagnostic tests, yet recognize these concepts are also applicable to any generic sequence of classification systems. The use of screening tests and multiple tests in a diagnostic setting is common practice, and the need to examine strategies that maximize diagnostic accuracy in sequential testing is important. Using a strategy for sequencing tests can result in a reduction in cost for the overall testing procedures by eliminating the need for secondary tests, this is in contrast to a linear combination of tests (like a Bayes Network), where each subject is required to receive each test. Developing diagnostic results from a sequence of tests involves setting a strategy for selecting the test sequence.

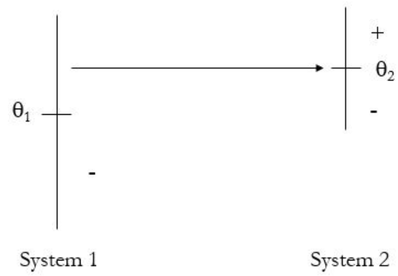
2.5.1 Strategies for Sequencing Classification Systems.

Three sequential strategies for classification systems using continuous features and assuming a 2-class outcome, e.g. “positive” or “negative”, are prevalent in the literature [2, 3, 34, 39]. Because these sequences have been discussed in terms of medical literature, we describe the 2-class outcome as positive (for disease) or negative (for non-disease).

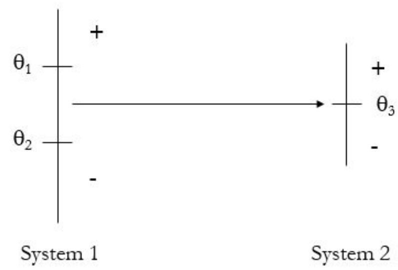
In a Believe the Positive strategy (BP) for a sequence of tests, the testing is concluded when a positive outcome is assigned or the number of tests has been reached in which case the result of the last test is assumed. In a Believe the Negative strategy



(a) Believe the Positive



(b) Believe the Negative



(c) Believe the Extreme

Figure 2.3. Strategies for Sequencing Systems

(BN) for a sequence of tests, the testing is concluded when a negative outcome is assigned or the number of tests has been reached in which case the result from the last test is assumed. In a Believe the Extreme strategy, a threshold is set to conclude a positive outcome, a threshold is set to conclude a negative outcome, and those values that do not meet either prescribed threshold are subjected to subsequent testing until a positive or negative result is reached. Figure 2.3 is a visual depiction of each strategy. For this research, only BP and BN strategies are considered, as the BE strategy has been shown to converge to either the BP or BN strategy [3].

Formulas for the false positive and true positive rates of these sequential testing strategies are readily available [2]. For a 2-class, 2-system sequence of systems, let $F_{X_{i,j}}$ where $i \in \{1, 2\}$ and $j \in \{N, D\}$ denote the cumulative distribution function of classification system i for those with a disease, D, and those without the disease, N. Let $F_{X_{1D},X_{2D}}$ and $F_{X_{1N},X_{2N}}$ denote the joint CDFs of the disease and non-disease distributions, respectively, for the system. Then, the formulas for the False Positive Rate (FPR) and True Positive Rate (TPR) of the BP and BN strategy are given by:

$$\begin{aligned} P_{FP}^{BP}(\theta^{BP}) =& 1 - P_{TN}^{BP} = 1 - P[(X_{1,N} < \theta_1) \cap (X_{2,N} < \theta_2)|-] \\ =& 1 - F_{X_{1,N},X_{2,N}}\left(\theta_1^{BP}, \theta_2^{BP}\right) \end{aligned} \tag{2.14}$$

$$\begin{aligned} P_{TP}^{BP}(\theta^{BP}) =& 1 - P[(X_{1,D} < \theta_1) \cap (X_{2,D} < \theta_2)|-] \\ =& 1 - F_{X_{1,D},X_{2,D}}\left(\theta_1^{BP}, \theta_2^{BP}\right) \end{aligned} \tag{2.15}$$

$$P_{FP}^{BN}(\theta^{BN}) = 1 - F_{X_{1,N}}\left(\theta_1^{BN}\right) - F_{X_{2,D}}\left(\theta_2^{BN}\right) + F_{X_{1,N},X_{2,N}}\left(\theta_1^{BN}, \theta_2^{BN}\right) \tag{2.16}$$

$$P_{TP}^{BN}(\theta^{BN}) = 1 - F_{X_{1,D}}\left(\theta_1^{BN}\right) - F_{X_{2,D}}\left(\theta_2^{BN}\right) + F_{X_{1,D},X_{2,D}}\left(\theta_1^{BN}, \theta_2^{BN}\right) \tag{2.17}$$

where θ_i corresponds to the threshold values of test $i = 1$ or 2 respectively, superscript BP or BN denotes the formula for the BP or BN strategy depicted in Figure 2.3, and the notation "$|-$" implies the probability conditioned on the negative, or non-disease, state.

2.5.2 Correlation in Sequenced Tests.

When considering strategy K ($K =$BP or BN) and the equations given in 2.17, we can explicitly state the GYI as:

$$GYI = \max_{\underset{\sim}{\boldsymbol{\theta}} \in \Theta} \left(TPR^K(\boldsymbol{\theta}) - mFPR^K(\boldsymbol{\theta})\right) \tag{2.18}$$

where $m = [(1 - p_1)/p_1] \times [(C_{FP} - C_{TN})/(C_{FN} - C_{TP})]$ as defined for Equation 2.3.

Because the equations for P_{FP}^K and P_{TP}^K include evatuations of joint distributions at fixed values of θ, correlation is included as part of the calculations of these values [2, 3, 17, 34, 39]. Correlation is unavoidable due to the use of the same subjects throughout the sequence (as subsets of the total population). The expectation in sequential classification systems is that the correlation is relatively large ($\rho \geq 0.5$); therefore, any methodological approach must be able to function in the presence of significant correlation.

2.6 Conclusion

Feature correlation, arising from the use of paired samples in testing, is important to consider when using methods to examine the inference in comparing tests or in a sequence of tests. Correlation has been largely ignored for a long time in research involving comparisons of classification systems. Sometimes low correlation may not have a large effect on inference. No recommendations exist for when correlation must be accounted for when comparing tests. When equations must be adjusted to account for correlation, methods exist for J [16, 40, 41, 43], but none exist for BC when a complex cost/prevalence structure is present. A clear gap in literature is that the Generalized method has not been correctly extended to k-class problems. The use of BC in such calculations, which has not been researched with feature correlation in consideration, can allow for a more complex cost structure among misclassification errors. Further, likely due to the number of parameters required for estimation and the ability to track correlation through the sequence, no inferential methods exist for sequence classification systems where correlation is inherent and likely large. This work seeks to address these gaps as given in the goals provided in Chapter I.

III. The Impact of Correlation on Classification Systems

3.1 Introduction

This chapter addresses the first research goal to determine the level of correlation at which inference around the comparison between classification systems is adversely impacted via misspecification of coverage. This goal is achieved by examining the effect of correlation on inference assuming independent samples rather than dependent (paired) samples in comparisons of classification systems. While the particular example used in this chapter is diagnostic tests, the application extends to all classification system families. The purpose of this chapter in achieveing the first research goal is twofold. The first purpose is to quantify the extent of overestimation and misspecification of coverage in CIs when correlation is ignored; and second, to provide guidance on the level of test correlation for which inference remains relatively robust under an assumption of independence in comparisons of tests. Under the latter purpose, recommendations on scenarios which do not require modification of existing inferential methods is offered so as to take advantage of the numerous tests and methods that currently exist to create CIs on the differences between tests using both the Youden Index (for the 2-class case) and Bayes Cost (for k-classes).

3.2 Methods

The metric chosen for comparison of the accuracy of diagnostic tests is Bayes Cost as defined in Equation 2.5. This reseach utilizes the three main parametric statistical methods used to develop inference in order to compare diagnostic tests, the Delta method, Generalized method and Bootstrapping. The Delta method and Generalized method for a single test can be found in Section 2.4.2 and Section 2.4.3, respectively. Under the assumption of independence, the comparison of diagnostic tests using the Delta method is found in Section 3.2.1, the Generalized method in Section 3.2.2 and a summary of the Bootstrap method is provided in Section 3.2.3.

3.2.1 Delta Method.

Confidence intervals for the difference between two independent test BC values (test 1 and 2) are calculated using the expected value of the performance metrics and variances from Equation 2.7 for each of the tests along with the appropriate standard Normal probability (Z). This process is shown in Equation 3.1.

$$\left(\widehat{BC}_1 - \widehat{BC}_2\right) \pm z_{\alpha/2}\sqrt{Var(\widehat{BC}_1) + Var(\widehat{BC}_2)} \tag{3.1}$$

.

Notice that under the assumption of independence, the variance of the difference in BC values contains no covariance, that is, covariance of BC_1 and BC_2 is assumed to be zero.

3.2.2 Generalized Method.

The vectors for R_{BC_1} and R_{BC_2} are developed independently utilizing the methodology for a single test as presented in Section 2.4.3. The resulting vectors can be subtracted from one another to create a CI for the difference of BC values of size α using the $\frac{\alpha}{2} \cdot 100\%$ and $(1 - \frac{\alpha}{2}) \cdot 100\%$ quantiles from the $R_{BC_1} - R_{BC_2}$ values. Figure 3.1 shows the algorithmic steps for computing a CI around the difference of BC values under the assumption of independence.

3.2.3 Bootstrapping.

The bootstrap method can be used to create CIs for both large and small data samples when parametric assumptions are not met. Several authors have used boot-

- Calculate $\bar{x}_j$ and s_j,
- For each $\bar{x}_j$, generate g monte carlo samples from a t distribution,
- Generate pivotal quantites for R_{μ_j} from $\bar{x}_j$, s_j and the generated t distributions,
- For each s_j, generate g monte carlo samples from a χ^2 distribution,
- Generate pivotal quantities for R_{σ_j} from s_j,
- Solve for R_{BC_1} and R_{BC_2} using the required pivotal quantites,
- Determine the required $\alpha/2$ quantiles from the vector $R_{BC} = R_{BC_1} - R_{BC_2}$ for the CI.

Figure 3.1. Algorithmic steps to compute the multivariate Generalized method including correlation between tests.

strap methods to create CIs for diagnostics tests and compare them to other, less robust, methods [4, 5, 43]. Because of its versatility, bootstrapping provides a good point of comparison.

One boostrap CI assumes asymptotic normality of the parameter estimate. This is accomplished by estimating the variance of the parameter estimate and creating a Wald-based confidence interval from the bootstrap samples. This method is generally known as an asymptotic normal (AN) bootstrap [11]. In addition to AN, the basic percentile (BP) and bias corrected and accelerated (BCa) methods were considered for comparison as well, and are perhaps the more common boostrap methods [12]. Further, because we seek to determine the impact of assuming independence when comparing tests, the boostrap methods employed assumed independence when generating the bootstrap samples. Finally, in accordance with the recommended parameters, a bootstrap sample of size 2000 for each test was used for creating the CIs [11].

3.3 Simulation

To quantify the effects of correlation when assuming independent samples, a simulation was performed where Normal variates representing the distributions for each class were created by varying the degree of correlation, classification accuracy, sample size, cost structure, feature distribution, and number of classes. We generate date with correlation between features used in each test and utilize methodologies presented in Section 3.2 to develop CIs. Six levels of class-specific correlation between tests were considered: $\rho = -0.3, 0, 0.1, 0.3, 0.5, 0.9$, these values were chosen as they span those utilized in previous works on correlated features [9]. Three test accuracy settings were created, given in Table 3.1 with the corresponding Bayes Cost (BC) and Youden Index (J) values. Two comparisons considered tests with the same accuracy, "Good, Good", "Fair, Fair", and two comparisons of tests considered those with differing accuracy, "Good, Fair", "Good, Poor". These were chosen to represent a wide range of possible and feasible diagnostic test accuracies. To generate the correlated data, data was drawn from a multivariate Normal using the appropriate marginals from Table 3.1 along with the associated correlation value for features between each test. The values listed for BC and J are under the assumption of equal cost and prevalence.

Four class-specific sample sizes from small to large ($n_j = 10, 20, 50, 100$) were examined in each of 2-class or 3-class scenarios. In the case of 2-classes, two cost structures ($c_{i|j}$), where i is the assigned class and j is the truth class, of equal (C_1) and unequal (C_2) weights were examined. All prevalence is assumed to be equal for testing. Therefore, when using C_1, no weights are placed on the $P_{i|j}$ values in BC. That is, $c_{i|j}p_j$ is equal to 1 for every class. Under C_2, a weighting structure placing higher emphasis on specific misclassifications is employed.

2-class system		
	Normal	Gamma
Test Accuracy	Class 1, Class 2	Class 1, Class 2
Good (BC=0.134, J=0.87)	$N(0,1), N(3,1)$	$G(1,1.3), G(7,1.64)$
Fair (BC=0.317, J=0.68)	$N(0,1), N(2,1)$	$G(1,1.3), G(5,1.85)$
Poor (BC=0.617, J=0.38)	$N(0,1), N(1,1)$	$G(1,1.3), G(3,1.15)$
3-class system		
	Normal	Gamma
Test Accuracy	Class 1, Class 2, Class 3	Class 1, Class 2, Class 3
Good (BC=0.267, J=2.73)	$N(-3,1), N(0,1), N(3,1)$	$G(1,1.3), G(2.3,3.7), G(5,13.70)$
Fair (BC=0.635, J=2.37)	$N(-2,1), N(0,1), N(2,1)$	$G(1,1.3), G(2,1.5), G(5,5.34)$
Poor (BC=1.234, J=1.77)	$N(-1,1), N(0,1), N(1,1)$	$G(1,1.3), G(2,1.5), G(3,1.74)$

Table 3.1. Class specifications for test accuracy .

$$C_1 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}.$$

In the case of 3-classes, equal (C_1) and unequal (C_2) weights were also considered as given by:

$$C_1 = \begin{bmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} 0 & 1 & 4 \\ 2 & 0 & 2 \\ 5 & 3 & 0 \end{bmatrix}.$$

Similar to the 2-class case, $P_j = 1/3$ for all 3 classes producing equal weights on the probabilities ($P_{i|j}$) when using C_1 and unequal weights when using C_2.

Finally, distribution misspecification was also examined by using Gamma distributed parameters that matched the BC values for the Normal distributions (Table 3.1). The Gamma variates were generated by applying an inverse distribution function on CDF values of Normally distributed random variates. For each scenario (a combination of correlation, accuracy, sample size, feature distribution, and class size), CIs were developed assuming independence for the inference using each of the previously described methods: Delta, Generalized and three forms of the Bootstrap (AN, BP, BCa). Combined, these factor levels resulted in 3840 scenarios for the simulation (6 correlations, 4 accuracies, 4 sample sizes, 2 cost structures, 2 class sizes, 5 methods, 2 distributional assumptions). CIs were generated 10,000 times in each of these scenarios, with $\alpha = 0.05$ and both coverage and width were recorded for comparison.

3.4 Simulation Results

Using the results of the simulation, effects of correlation, testing accuracy, cost, sample size and distribtuion misspecification were estimated for each of the methodological approaches comparing diagnostic tests when igonoring correlation. Results exemplifying findings from the simulation are provided in the text below, and all simulation results are tabulated in Appendix A.

Table 3.2 and Figure 3.2 demonstrate the coverage for comparisons of "Fair" accuracy tests based on Normally distributed features in the 2-class case with equal costs. In comparing these results, coverage is about what is expected (95%) for correlation up to 0.1 for the Delta and Generalized methods. When the correlation between diagnostic tests is $\geq$ 0.3, coverage becomes quite conservative and when the correlation between tests is negative (-0.3), the coverage is lower than expected. The Bootstrap methods perform the poorest and only reaches about 0.95 coverage for correlation levels of 0 or 0.1 when the sample size is 50 or greater with possibly the exception of the BP bootstrap. In general, though, coverage is maintained for positive correlation, albeit quite conservative when correlation is 0.3 or higher. The accuracy had relatively little effect on the the coverage of the Delta, Generalized, or BP bootstrap methods. Because the BP bootstrap performed better than the BCa or AN bootstrap, subsequent results focus only on the BP bootstrap; all bootstrap results can be found in Appendix A.

All test comparisons in the 2-class Normal feature case were not noticeably affected

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.927	0.704	0.950	0.731	0.870	0.639	0.907	0.634	0.852	0.642
	0	0.946	0.704	0.962	0.732	0.892	0.633	0.929	0.629	0.879	0.635
	0.1	0.951	0.703	0.967	0.731	0.897	0.631	0.934	0.627	0.883	0.632
	0.3	0.971	0.703	0.981	0.730	0.932	0.638	0.956	0.633	0.914	0.640
	0.5	0.987	0.700	0.990	0.729	0.952	0.628	0.974	0.624	0.941	0.630
	0.9	1.000	0.696	1.000	0.726	0.999	0.622	1.000	0.620	0.997	0.622
20	-0.3	0.923	0.509	0.936	0.516	0.896	0.481	0.913	0.480	0.891	0.482
	0	0.949	0.509	0.957	0.516	0.925	0.481	0.939	0.480	0.918	0.482
	0.1	0.954	0.508	0.962	0.515	0.934	0.479	0.947	0.479	0.928	0.481
	0.3	0.969	0.508	0.976	0.515	0.950	0.479	0.962	0.479	0.944	0.480
	0.5	0.988	0.508	0.991	0.515	0.974	0.479	0.982	0.479	0.969	0.480
	0.9	1.000	0.506	1.000	0.514	1.000	0.477	1.000	0.477	1.000	0.477
50	-0.3	0.928	0.326	0.933	0.327	0.918	0.318	0.925	0.319	0.914	0.319
	0	0.949	0.326	0.952	0.327	0.937	0.318	0.945	0.318	0.935	0.319
	0.1	0.953	0.326	0.957	0.327	0.945	0.317	0.951	0.318	0.943	0.318
	0.3	0.971	0.326	0.974	0.327	0.966	0.318	0.969	0.318	0.964	0.319
	0.5	0.989	0.325	0.990	0.326	0.984	0.317	0.987	0.318	0.982	0.318
	0.9	1.000	0.325	1.000	0.326	1.000	0.317	1.000	0.317	1.000	0.317
100	-0.3	0.929	0.231	0.932	0.232	0.924	0.229	0.927	0.229	0.923	0.229
	0	0.950	0.232	0.952	0.232	0.946	0.229	0.949	0.229	0.945	0.229
	0.1	0.956	0.231	0.957	0.232	0.951	0.229	0.954	0.229	0.951	0.229
	0.3	0.971	0.231	0.972	0.232	0.969	0.228	0.970	0.229	0.968	0.229
	0.5	0.989	0.231	0.989	0.231	0.987	0.228	0.988	0.229	0.987	0.229
	0.9	1.000	0.231	1.000	0.232	1.000	0.229	1.000	0.229	1.000	0.229

Table 3.2. Coverage (Cov) and Length (Len) for comparisons of "Fair, Fair" 2-Class tests with Equal Cost

by an unequal cost structure, with the exception of the Delta method. Figure 3.3 shows the coverage of the Delta method for the Normal features in the 2-class case for both cost structures and across several test accuracies. In general, unlike the equal cost comparisons, coverage was not maintained for small samples in the Delta method when costs were not equal (C_2), with minimal exception. Only with large sample size ($n_j \geq 50$) is coverage maintained (about 95%) for positive correlation up to 0.10 and conservative (coverage $> 96\%$) for positive correlation ≥ 0.30. Further, for the Delta method overall, the effects of differing test accuracies was minimal with respect to the patterns of achieving or maintaining coverage (Figure 3.3). Cost had

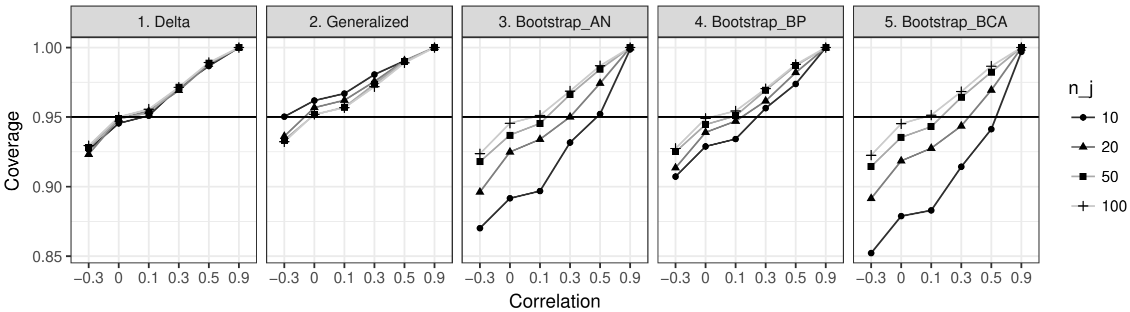


Figure 3.2. Coverage for comparisons of "Fair, Fair" 2-Class tests with Equal Cost

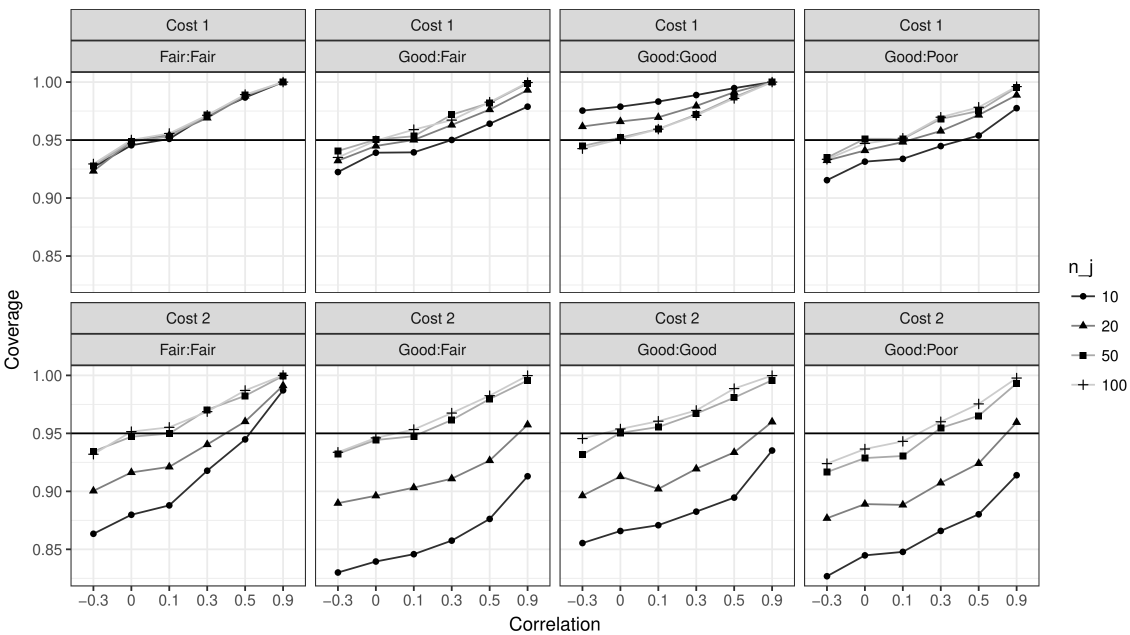


Figure 3.3. Delta method coverage for 2-class Normal biomarkers

similar impact on the performance of the Bootstrap. The Generalized method did not have an appreciable difference in coverage between the cost scenarios.

Interestingly, in the 2-class test comparison, when the features are distributed as Gamma instead of Normal, comparisons of tests with equal accuracies maintain coverage at about 95% across all levels of correlation, being conservative on coverage only for very highly positive correlation, $\rho \geq 0.5$ (Figure 3.4). However, when the accuracies of the tests were disparate (e.g. "Good" vs "Fair" or "Good" vs "Poor"), coverage could not be achieved for any methods, sample size or correlation level (Figure 3.4). The "Good" vs "Poor" comparison had higher coverage, yet, in general, could not achieve 95% coverage, nor could the unequal cost comparisons (see Appendix A). In each scenario, the inference on a single test with misspecification is not good; however, when identical tests are presented the methods create a relatively good confidence interval around the incorrect point. In a single test, this produces a very poor result; however, since the center and size of the CI produced is consistent, the resulting comparisons of tests is often surprisingly accurate.

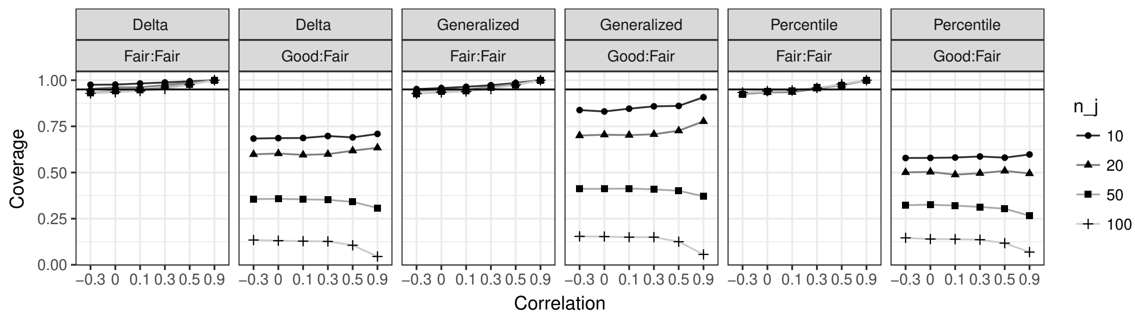


Figure 3.4. Coverage for 2-class Gamma Biomarkers under "Good, Good" and "Good, Fair" accuracies with equal cost

Results for the 3-class comparisons were similar when comparing the effects of estimation method on coverage, though class-specific sample size had a much larger impact on coverage for the Bootstrap and Delta methods (Figure 3.5) likely due to the additional samples included because of the additional (third) class. Coverage is maintained at 95% with samples of 50 or larger (100 for bootstrap) for all methods when correlation was greater than or equal to zero. Coverage is conservative for correlation $\rho \geq 0.3$ and too low for $\rho \leq -0.3$. Increased accuracy of the comparison test caused little change in the resulting coverage.

Finally, feature distribution had a larger impact on the coverage in the 3-class case (Figure 3.6). Comparing Figure 3.4 to Figure 3.6, we see similar, and relatively good coverage in the "Fair" vs "Fair" test comparison, yet test comparisons continued to struggle to reach coverage in comparisons of tests with different accuracies.

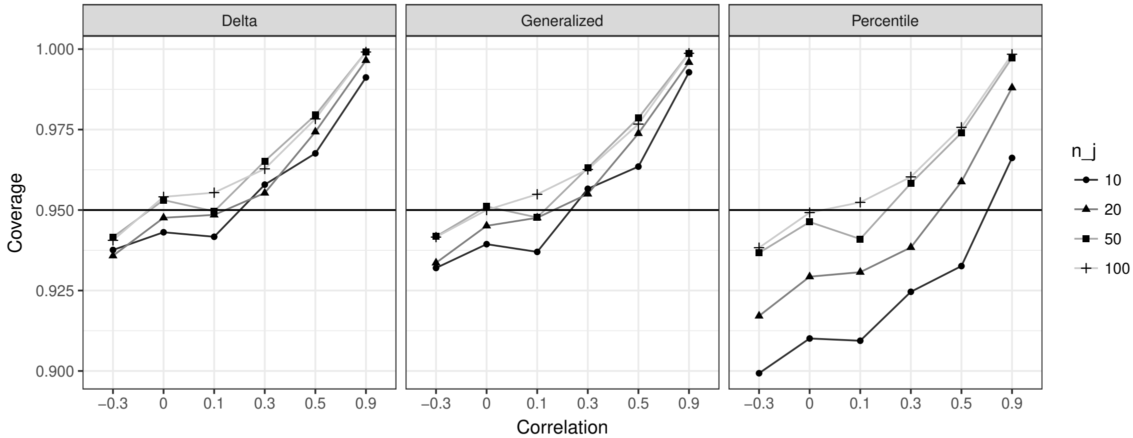


Figure 3.5. Coverage for 3-class Normal Biomarkers under "Good, Good" accuracies with equal cost

CI length for all scenarios are not compared as the method-specific CI length does not vary by correlation level (by design, standard errors are the same). However, there is a small correlation effect on length present in small samples for all methods. This is due to the generation within the simulation of class-specific data. In small samples with high correlation, outliers in a specific class are exceedingly rare because they would have a large effect on the correlation, causing standard error on the CI to be lower, on average. As sample size increases, this effect is less. Additionally, because the Generalized method has an additional step of generating values from a t-distribution, which has heavier tails than a Normal distribution, this effect is less pronounced in those results.

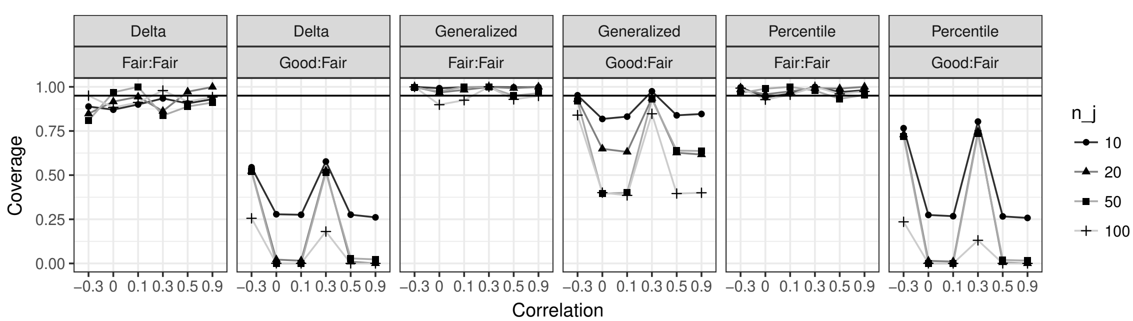


Figure 3.6. Coverage for 3-class Gamma Biomarkers under "Good, Good" and "Good, Fair" accuracies with equal cost

3.5 Discussion

Correlation within the samples when comparing diagnostic tests can cause decision makers to make incorrect or overly conservative inference, regardless of the inferential method utilized to make a conclusion. The most concerning scenario is when correlation between features is negative as this results in CIs with coverage below the confidence level. Minimal correlation ($0 \leq \rho \leq 0.1$) can be considered negligible with respect to its impact on CI coverage for most of the methods; however, the choice of method should be made with other considerations in mind. Sample size, cost structure, and misspecification of the parameter distributions can still have significant impact on the resulting inference. In small samples, when Normal features are being studied, the Generalized method provides more consistent coverage, regardless of the cost structure; this finding is consistent with other author's results [5, 20].

In situations with larger correlation ($\rho \geq 0.3$), inference will be overly conservative. However, any of the methods may still be applied with the understanding that intervals will be wider than necessary which may still provide useful inference in some cases. When this is not adequate, updated inferential methods that can account for correlation are required in order to maintain a reasonable coverage without overestimation of width for CIs.

When small samples ($n_j \leq 20$) are presented with Normal features and minimal correlation levels ($0 \leq \rho \leq 0.1$), the Generalized method outperforms the Delta and Bootstrap methods. With larger samples ($n_j \geq 50$) or larger correlation levels ($\rho \geq 0.3$), the Delta method, with its ease of calculation, is preferred when features are Normal. When the features are not Normally distributed, none of the methods described here provide reliable coverage for 2-sided CI in the 2- or 3-class case.

With respect to coverage, it is inferred that accounting for correlation would provide a 95% CI by reducing the standard error and associated CI length. In such cases, this work demonstrates the importance of considering correlation and providing some practical recommendations for when the use of more complex inference methods that require the estimation of correlation is needed. Overall, our recommendation is to include reasonable use of methods assuming independence when comparing diagnostic tests in which feature distributions are very weakly and positively correlated ($0 \leq \rho \leq 0.1$). For other scenarios, especially for any level of negative correlation and moderate or higher positive correlation, methods that appropriately adjust standard error for correlation are recommended.

IV. Accounting for Correlation in Classification System Comparisons

4.1 Introduction

This chapter addresses the second research goal to develop correct inferential methods to account for correlation in comparisons of classification system performance. Specifically, in this chapter methodologies for the Delta method and Generalized inference are developed that properly characterize the uncertainty between correlated classification systems in order to compare the performance of classification systems using CIs.

The extension of current methodologies is burdensome. Estimation of the variance within the Delta method requires more assumptions and parameter estimations, increasing the requirement for samples to reach asymptotic behavior. There is no true extension for the Generalized method into the multivariate case that meets the criteria for pivotal quantities. Although other work has attempted such an extension [41, 40], they mistakenly extend the Generalized method inappropriately for multivariate distributions. However, by relaxing assumptions of independence among pivotal quantities, we are able to extend the Generalized method appropriately to incorporate a correlation estimation in the creation of CIs that perform at the expected level of coverage.

4.2 Methods to Compare Two Classification Systems with Correlation

Two main parametric methods, using the BC metric provided in Equation 2.5, were developed in order to compare classification systems: the Delta method developed in Section 4.2.1 and the Generalized method developed in Section 4.2.2.

4.2.1 Delta Method.

With Normal, suitably transformable to Normal or asymptotically Normal features presented, the Delta method is suitable to construct variance estimates for simple Wald-based inference. The Wald-based confidence interval for the difference of two BC values is given by:

$$E(BC_1 - BC_2) \pm z_{0.975}\sqrt{Var(BC_1 - BC_2)}). \tag{4.1}$$

Since a normal feature is assumed, the standard subtraction of two correlated, Normally distributed random variables can be used to develop mean and variance estimates. The formula

$$Var(BC_1 - BC_2) = Var(BC_1) + Var(BC_2) - 2Cov(BC_1, BC_2), \tag{4.2}$$

is used for the variance present in the difference of the performance between the two classification systems. The process of calculating the covariance in Equation 4.2 follows a natural correlation analysis. Specifically, the marginal and conditional distributions of the BC value for each test are Normal. Due to the normality of the features, the conditional distribution of features in the second test can be represented as a linear combination of those in first test. Conveniently, this result allows for the application of the law of total variance, which can be used to develop the following formulas.

$$Cov(\mu_1, \mu_2) = \rho\sigma_1\sigma_2. \tag{4.3}$$

$$Cov(\sigma_1^2, \sigma_2^2) = \rho^2\sigma_1\sigma_2. \tag{4.4}$$

The Delta method calculations can be extended using Kline's approximation for covariance in the Delta method [19], such that:

$$\begin{aligned} Cov(BC_1, BC_2) = &\sum_i \left(\frac{\partial BC_i}{\partial \mu_{1,i}}\right)\left(\frac{\partial BC_i}{\partial \mu_{2,i}}\right) Cov(\mu_{1,i}, \mu_{2,i}) \\ &+ \sum_i \left(\frac{\partial BC_i}{\partial \sigma_{1,i}}\right)\left(\frac{\partial BC_i}{\partial \sigma_{2,i}}\right) Cov(\sigma^2_{1,i}, \sigma^2_{2,i}), \end{aligned} \tag{4.5}$$

where 1,2 represent the first and second test respectively from the point estimate of $BC_1 - BC_2$.

The partial derivatives and estimation of all required parameters for the application of this methodology can be accomplished as in previous work [5] and are reproduced in Appendix B of this document.

4.2.2 Generalized Method.

The methodological approach for the Genearlized method is repeated here as it was presented in Section 2.4.3 specifically to demonstrate how this must be modified to include correlation. Recall, the Generalized method, in the univariate normal case, relies on Generalized pivotal quantities to develop sampling distributions for parameters μ and σ. For k classes, in a single system the pivotal quantities can be defined as:

$$R_{\mu_j} = \overline{x}_j - t_j \frac{S_j}{\sqrt{n_j}} \tag{4.6}$$

and

$$R_{\sigma_j} = \sqrt{\frac{(n_j - 1)S_j^2}{V_j}} \tag{4.7}$$

where

$$t_j = \frac{\overline{X}_j - \mu_j}{S_j/\sqrt{n_j}} \tag{4.8}$$

and

$$V_j = \frac{(n_j - 1)S_j^2}{\sigma_j^2}, \tag{4.9}$$

The sample mean ($\overline{x}_j$) and standard deviation (S_j) are from the j^{th} class, $t_j \sim t(n_j - 1)$, a t-distribution random variable with $n_j - 1$ degrees of freedom, and $V_j \sim \chi^2_{n_j-1}$, a chi-square random variable with $n_j - 1$ degrees of freedom. In order to construct the GCI around BC, monte carlo simulation is used to create a large number (K) of random draws from t_j and V_j for each of the k classes. These K sets of t_j and V_j values are then used in Equations 4.6 and 4.7 to determine K values of R_{μ_j} and R_{σ_j}, $j = 1 \ldots k$.

These parameters are used to calculate K values of R_{BC} utilizing Equation 2.12 and a confidence interval is built from the quantiles of the resulting K values of the R_{BC} vector.

To achieve the goal of extending the method, the marginal distribution requirements may be met with the multivariate counterparts for the univariate distributions, the multivariate-t and the Wishart distribution. Drawing the t-random variable from a multivariate t-distribution with a scale matrix equivalent to that of the correlation matrix in our original sample meets pivotal quantity and correlation requirements for R_μ. Specifically, we draw:

$$(t_1, t_2) \sim MVT\left(\mathbf{P}\frac{n-1}{n-3}\right) \tag{4.10}$$

where $\mathbf{P}$ represents the size correlation matrix for the features, by class, where the i, j element represents the correlation between system features within class. Then following the standard procedure for for the Generalized method defines

$$R_{\mu_i} = \mu_i - t_i\frac{\sigma_i}{n_i} \tag{4.11}$$

for each class.

The covariance matrix for a multivariate Normal random sample is distributed as a Wishart distribtuion,

$$(n-1)S \sim Wishart(n-1, \mathbf{\Sigma}). \tag{4.12}$$

With $\mathbf{\Sigma}$ representing the covariance matrices of each feature between tests. The vector of matrices R_{Σ}, an extenstion of Equation 4.7, can be directly simulated using the Wishart distribution. R_{σ_i} can be constructed by taking the appropriate diagonal element from the matrices within R_{Σ}, the samples from this Wishart distribution.

Using these new pivotal quantities, R_{BC} can be solved as given in Equation 2.12 for each instance of R_{μ} and R_{σ}. Confidence intervals can be built on the quantiles of these distributions, or when comparing multiple tests, the $R_{BC_1} - R_{BC_2}$ vector quantiles.

A CI for BC utilizing the Generalized method is then developed by utilizing the algorithm in Figure 4.1.

- Calculate $\bar{\mathbf{x}}$ and $\mathbf{S}$,
- Generate g monte carlo samples from a multivariate t distribution using the dispersion matrix from $\mathbf{S}$,
- Generate pivotal quantites for R_{μ} from $\bar{\mathbf{x}}$ and the generated t distribution,
- Generate g monte carlo samples from a multivariate Wishart distribution using $\mathbf{S}$, the diagonal elements are the pivotal quantites,
- Solve for R_{BC_1} and R_{BC_2} using the required pivotal quantites,
- Determine the required $\alpha/2$ quantiles from the vector $R_{BC} = R_{BC_1} - R_{BC_2}$ for the CI.

Figure 4.1. Algorithmic steps to compute the multivariate Generalized method for including correlation between tests

4.2.3 Simulation.

In order to examine the extended Delta method and multivariate Generalized method, a simulation was performed where normal variates representing the distribution for each class were created with varying degrees of correlation, classification accuracy, sample size, cost structure, biomarker distribution, and number of classes. Six levels of class-specific correlation between tests were considered: $\rho = -0.3, 0, 0.1, 0.3, 0.5, 0.9$. Again, these values span the range of typical correlation values examined in applications of diagnostic test comparisons. Three test accuracy settings were created, given in Table 4.1 with the corresponding BC and Youden Index values. Two comparisons considered tests with the same accuracy, "Good, Good", "Fair, Fair", and two comparisons of tests considered those with differing accuracy, "Good, Fair", "Good, Poor". These were chosen to represent a wide range of possible and feasible diagnostic test accuracies.

2-class system		
	Normal	Gamma
Test Accuracy	Class 1, Class 2	Class 1, Class 2
Good (BC=0.134, J=0.87)	$N(0,1), N(3,1)$	$G(1,1.3), G(7,1.64)$
Fair (BC=0.317, J=0.68)	$N(0,1), N(2,1)$	$G(1,1.3), G(5,1.85)$
Poor (BC=0.617, J=0.38)	$N(0,1), N(1,1)$	$G(1,1.3), G(3,1.15)$
3-class system		
	Normal	Gamma
Test Accuracy	Class 1, Class 2, Class 3	Class 1, Class 2, Class 3
Good (BC=0.267, J=2.73)	$N(-3,1), N(0,1), N(3,1)$	$G(1,1.3), G(2.3,3.7), G(5,13.70)$
Fair (BC=0.635, J=2.37)	$N(-2,1), N(0,1), N(2,1)$	$G(1,1.3), G(2,1.5), G(5,5.34)$
Poor (BC=1.234, J=1.77)	$N(-1,1), N(0,1), N(1,1)$	$G(1,1.3), G(2,1.5), G(3,1.74)$

Table 4.1. Class specifications for test accuracy .

Four class-specific sample sizes from small to large ($n_j = 10, 20, 50, 100$) were examined in each of 2-class or 3-class scenarios. In the case of 2-classes, two cost structures $c_{i|j}$, where i represents the assigned class and j represents the truth class, of equal (C_1) and unequal (C_2) weights were examined. Equal prevalence of all classes is assumed in all scenarios.

$$C_1 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \qquad \text{and} \qquad C_2 = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}.$$

In the case of 3-classes, equal (C_1) and unequal (C_2) weights were also considered as given by:

$$C_1 = \begin{bmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix} \qquad \text{and} \qquad C_2 = \begin{bmatrix} 0 & 1 & 4 \\ 2 & 0 & 2 \\ 5 & 3 & 0 \end{bmatrix}.$$

Finally, distribution misspecification was also examined by using Gamma distributed parameters that matched the BC and J values for the Normal distributions in Table 4.1.

For each scenario, CIs were developed to compare values using each of the previously described methods: Delta, Generalized and three forms of the Bootstrap (AN, BP, BCa). Combined, these factor levels resulted in 3840 scenarios for the simulation (6 correlations, 4 accuracies, 4 sample sizes, 2 cost structures, 2 class sizes, 5 methods, 2 distributional assumptions). CIs were generated 10,000 times in each of these scenarios, with $\alpha = 0.05$ and both coverage and width were recorded for comparison.

4.3 Results

Using the results from the data simulation, correlation, testing accuracy, number of classes, cost, sample size and distribution misspecification effects were estimated for each of the methodological approaches comparing the diagnostic tests with correlated biomarkers. The text below provides results exemplifying findings from the simulation. Full results are tabulated in Appendix C.

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
100	-0.3	0.947	0.250	0.948	0.249	0.942	0.247	0.944	0.247	0.941	0.247
	0	0.950	0.231	0.946	0.230	0.945	0.228	0.948	0.229	0.946	0.229
	0.1	0.947	0.223	0.947	0.222	0.941	0.220	0.945	0.221	0.941	0.221
	0.3	0.947	0.203	0.951	0.202	0.940	0.200	0.942	0.201	0.937	0.201
	0.5	0.948	0.177	0.950	0.176	0.945	0.174	0.946	0.175	0.942	0.175
	0.9	0.954	0.084	0.952	0.083	0.948	0.083	0.949	0.083	0.943	0.083
50	-0.3	0.947	0.350	0.950	0.348	0.937	0.343	0.944	0.344	0.937	0.344
	0	0.942	0.325	0.953	0.322	0.932	0.318	0.936	0.318	0.929	0.318
	0.1	0.943	0.313	0.946	0.310	0.932	0.306	0.938	0.306	0.929	0.306
	0.3	0.951	0.286	0.951	0.283	0.939	0.278	0.945	0.279	0.935	0.279
	0.5	0.951	0.250	0.948	0.247	0.940	0.243	0.945	0.244	0.936	0.244
	0.9	0.954	0.119	0.953	0.117	0.942	0.115	0.945	0.116	0.934	0.116
20	-0.3	0.945	0.543	0.951	0.535	0.918	0.518	0.935	0.517	0.913	0.519
	0	0.944	0.503	0.947	0.493	0.919	0.478	0.934	0.478	0.912	0.480
	0.1	0.946	0.486	0.948	0.476	0.921	0.462	0.934	0.461	0.911	0.464
	0.3	0.949	0.444	0.951	0.434	0.920	0.420	0.935	0.420	0.906	0.422
	0.5	0.950	0.390	0.947	0.380	0.917	0.368	0.931	0.368	0.902	0.370
	0.9	0.954	0.189	0.948	0.184	0.929	0.178	0.936	0.178	0.902	0.180
10	-0.3	0.931	0.742	0.944	0.728	0.888	0.687	0.915	0.682	0.868	0.693
	0	0.940	0.689	0.946	0.672	0.888	0.634	0.919	0.630	0.864	0.641
	0.1	0.943	0.666	0.949	0.648	0.894	0.611	0.921	0.607	0.867	0.617
	0.3	0.943	0.611	0.949	0.593	0.883	0.560	0.910	0.556	0.855	0.568
	0.5	0.947	0.540	0.951	0.522	0.885	0.493	0.911	0.489	0.845	0.501
	0.9	0.961	0.268	0.954	0.258	0.909	0.250	0.921	0.245	0.849	0.255

Table 4.2. Coverage (Cov) and Length (Len) of the “Fair, Fair” accuracy test with 2 classes and normal biomarkers with equal cost

Table 4.2 and Figure 4.2 show the coverage and mean length for comparisons of “Fair” accuracy tests on normally distributed biomarkers with equal costs in the 2-class case. Comparing the methods used to construct the CI resulted in a good coverage for the difference in BC values with the Generalized method, regardless of the correlation level. In general, the delta method, too, provided good coverage with the exception of small samples size ($n_j = 10$) at a lower correlations ($\rho \leq 0.3$). The Generalized and Delta methods had similar lengths, although the generalized CI was generally slightly tighter than the Delta method. The Bootstrap methods generally perform the poorest, only reaching the desired accuracy levels at the largest sample

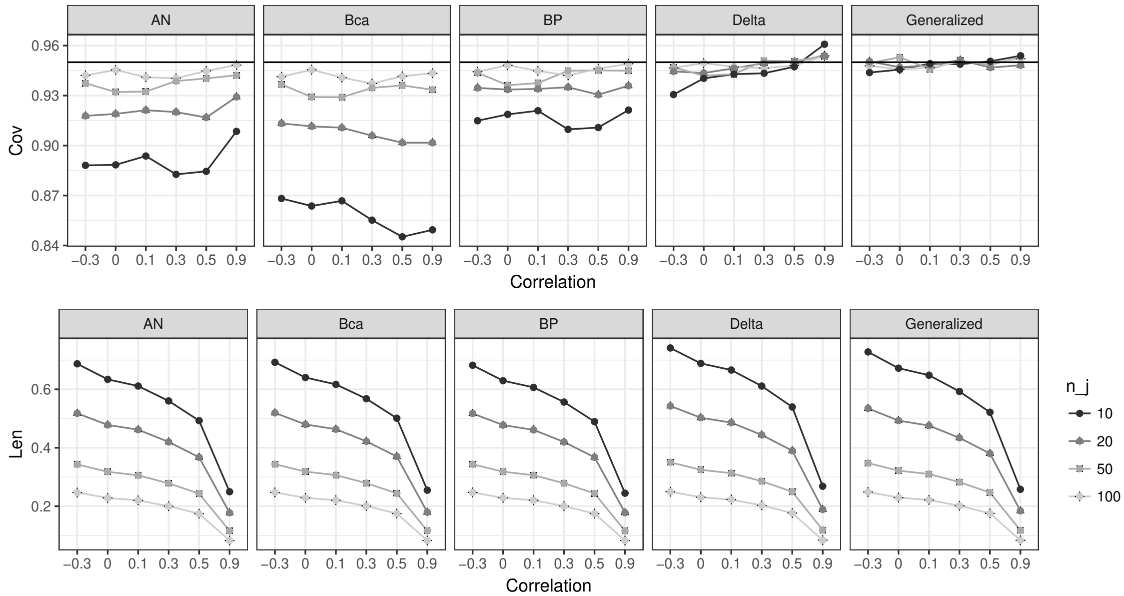


Figure 4.2. Coverage (Cov) and Length (Len) of the "Fair, Fair" accuracy test with 2 classes and normal biomarkers with equal cost

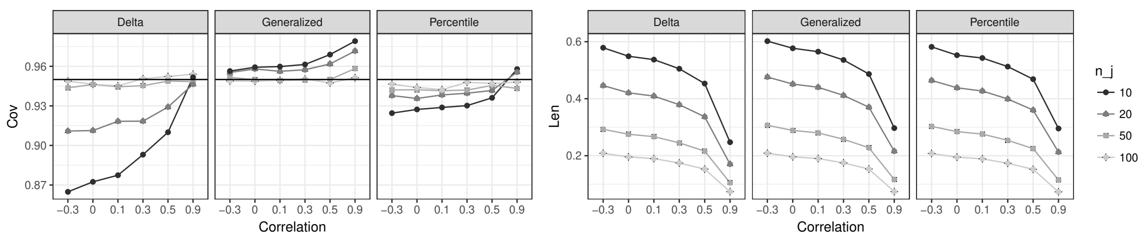


Figure 4.3. Coverage and Length of the Generalized, Delta, BP methods on "Fair, Fair" test comparisons with unequal cost structure.

size examined ($n_j = 100$) for BCa and BP. Of the Bootstrap methods examined, BP performed better than BCa or AN when considering coverage and in general met the expected 95% coverage for $n_j = 100$ and for $\rho \geq 0.3$ for $n_j = 50$. Therefore, subsequent results will compare only the results from the BP bootstrap, although all bootstrap results can be found in the Appendix C. Of note is the drastic decrease in CI width across all methods and sample sizes as correlation is increased (Figure 4.2). This is attributable to the growing covariance at higher correlations, and subsequent lower standard error. Differences in test accuracies did not have an effect on the coverage of these methods except that for good tests and small sample sizes $n_j < 50$, Delta coverage exceeds 95%.

With the introduction of a complex cost structure, a difference in coverage can be seen in each of the methods. Figure 4.3 shows the resulting coverage and length of CIs when comparing the Fair test accuracies with unequal cost. The Generalized method exceeds coverage when $n_j \leq 20$ in all scenarios, although maintains about 95% coverage for $n_j > 20$. In larger sample sizes ($n_j \geq 50$) and with some correlation present ($\rho \geq 0.3$) the Delta method maintains coverage. The bootstrap requires a large sample ($n_j = 100$) to maintain the required coverage. All methods demonstrate an increasing trend in coverage as correlation becomes higher. In addition, the Generalized and Delta methods maintain approximately the same width for CIs.

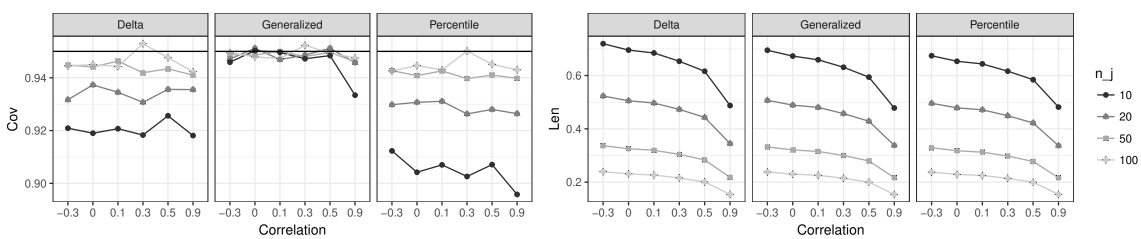


Figure 4.4. Coverage and Length of the Generalized, Delta and Basic Percentile methods on 3-class, "Fair, Fair" test comparisons with unequal cost structure.

Figure 4.4 shows the results for a 3-class case for the Fair, Fair test comparisons of normally distributed biomarkers with an unequal cost structure. The Generalized method maintains coverage in most scenarios, with the exception of small sample sizes exhibiting extreme correlation ($n_j = 10, \rho = 0.9$). In general, neither the Delta method nor BP Bootstrap achieve coverage in any of the scenarios; however, they both approach the 95% threshold with relatively large sample sizes ($n_j \geq 50$). The upward trend in coverage with increasing correlation that showed in the 2-class case is not present in the 3-class case. With more parameters of estimation from the inclusion of a third class, the effect of an higher correlation in a specific pairwise set of samples is minimized. Compared with the 2-class case, the effect of sample size in each of the methods is more pronounced in the 3-class case. The requirement of estimation on the numerous parameters required to accomplish each of the methods causes an increase in the required number of samples to maintain coverage. Length remains approximately the same for each of the CI methodologies.

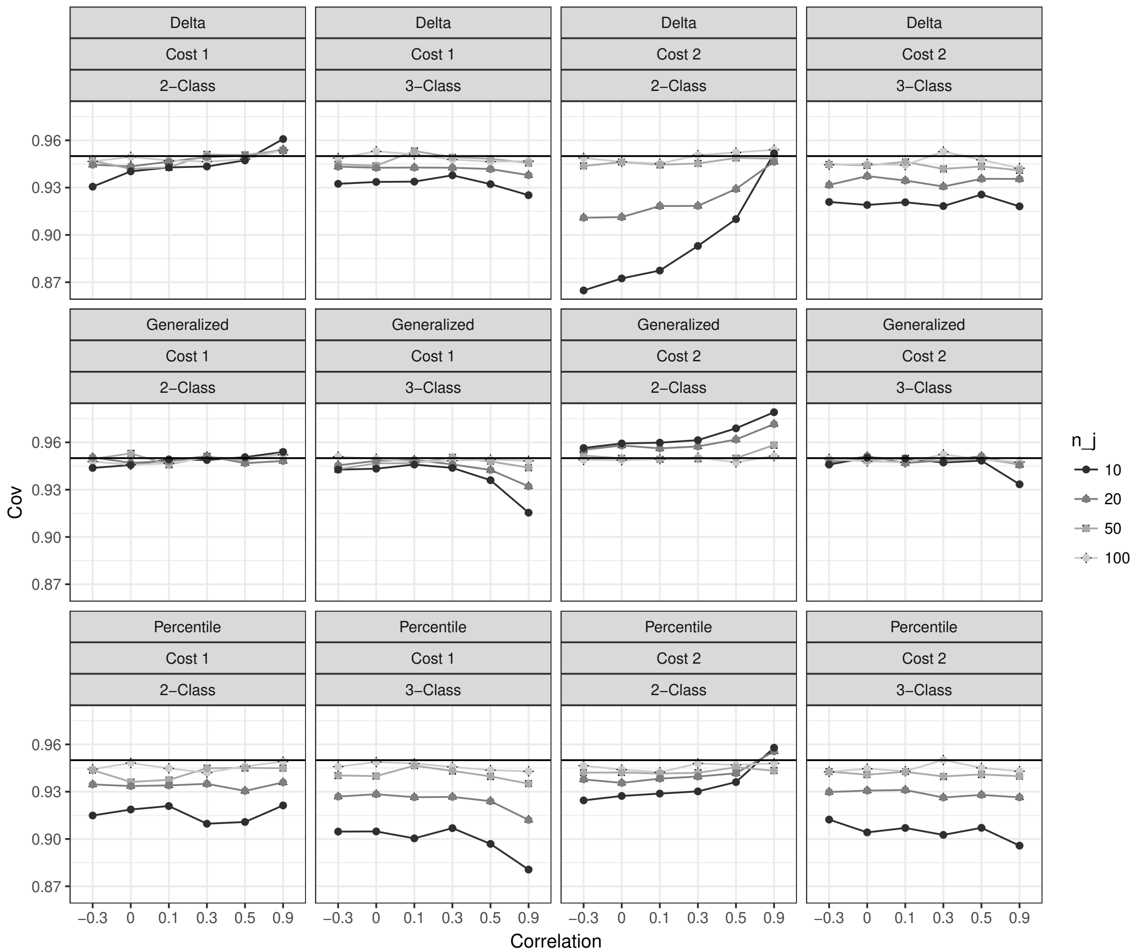


Figure 4.5. Coverage on Fair, Fair test comparisons with both cost and class structures.

In the 3-class case, cost had a smaller impact on coverage. Figure 4.5 shows both scenarios in the 2-class and 3-class case. The coverage in the 3-class case does not drop as significantly as that in the 2-class case, and this effect is constant amongst all of the methods.

None of the testing methodologies performed well when presented with a set of

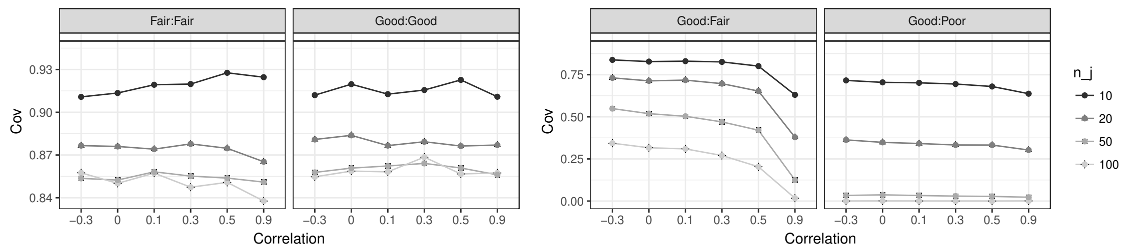


Figure 4.6. Coverage of the Generalized Method for 2-class equal cost with Gamma distributed biomarkers.

Gamma distributed biomarkers (Table 4.3). The individual test, as shown by Batterton [4], does not maintain coverage in these scenarios. As a result, comparisons of tests were not reliable in maintaining coverage. A notable exception to this outcome when tests are equal, the methods will at times maintain or exceed coverage (e.g. Generalized and BP for 3-class in Table 4.3).

For instance, Table 4.3 shows a comparison of the results from the 2-class and 3-class Gamma misspecification. Here, the BP Bootstrap has the most consistent coverage, with the Delta method providing good coverage in the 2-class problem and the Generalized and BP methods maintain coverage across all sample sizes for the 3-class problem.

4.4 Discussion

While accounting for correlation is burdensome computationally, Chapter III showed that assuming independence would result in overestimating the width of a confidence interval. Adjusting for correlation, in certain scenarios, can maintain effective coverage at a reasonable length of the CI. This chapter has fully extended the common parametric methods to account for correlation in multiple testing scenarios, with reasonable coverage and minimal width.

In situations with moderate correlation levels ($\rho \geq 0.3$), care should be made to account for correlation using these equations. When in-class samples are small

n_j	ρ	2-class Coverages Delta	Gen	BP	3-class Coverages— Delta	Gen	BP
10	-0.3	0.973	0.911	0.930	0.822	0.953	0.969
	0	0.974	0.914	0.927	0.830	0.962	0.977
	0.1	0.977	0.919	0.930	0.825	0.966	0.979
	0.3	0.974	0.920	0.929	0.839	0.970	0.982
	0.5	0.975	0.928	0.928	0.838	0.978	0.987
	0.9	0.977	0.925	0.931	0.871	0.993	0.990
20	-0.3	0.960	0.877	0.931	0.896	0.964	0.956
	0	0.955	0.876	0.927	0.893	0.966	0.961
	0.1	0.954	0.874	0.929	0.891	0.972	0.959
	0.3	0.954	0.878	0.929	0.896	0.976	0.964
	0.5	0.954	0.875	0.927	0.898	0.983	0.967
	0.9	0.955	0.865	0.923	0.899	0.994	0.973
50	-0.3	0.947	0.854	0.942	0.916	0.987	0.949
	0	0.938	0.852	0.933	0.915	0.988	0.954
	0.1	0.939	0.858	0.937	0.912	0.990	0.951
	0.3	0.939	0.855	0.938	0.908	0.990	0.953
	0.5	0.939	0.854	0.938	0.902	0.989	0.953
	0.9	0.936	0.851	0.939	0.902	0.971	0.958
100	-0.3	0.940	0.858	0.943	0.912	0.981	0.947
	0	0.935	0.850	0.937	0.910	0.973	0.953
	0.1	0.933	0.857	0.936	0.905	0.971	0.947
	0.3	0.931	0.847	0.940	0.910	0.963	0.953
	0.5	0.926	0.851	0.934	0.904	0.957	0.955
	0.9	0.929	0.838	0.942	0.891	0.921	0.956

Table 4.3. Coverages for the Gamma distributed "Fair, Fair" test using all three methods for 2-class and 3-class problems.

($n_j \leq 20$), the Generalized method performs superior to other methods accross all methods, Delta and Generalized methods appropriately extended performs better than the Bootstrap. Comparing these results to those presented by Yin [40], the Delta method performs similarly, but the Generalized method, as presented in this chapter, maintains coverage at a much more reasonable rate, whereas Yin's Generalized method is overly conservative, exceeding 97% coverage in most scenarios. While length of these methods and those cannot be directly compared due to choice of parameters, any overestimation of coverage would be the result of overly wide confidence intervals.

Finally, all of these methods are numerical in nature, and extreme correlation ($\rho \geq 0.9$) can present numerical issues due to the nature of the methodologies used. In a practical sense, we suggest a correlation greater than 0.9 could make the estimation of a CI of the comparison of tests somewhat unreliable, specifically, as $\rho \to 1$ the tests become identical as the features are perfectly linearly correlated.

V. Inference on Sequential Systems

5.1 Introduction

This chapter addresses the third research goal to develop inferential procedures to account for correlation in sequences of classification systems. A brief motivation for sequential classification systems utilizing the typical language associted with medical testing follows. Several techniques exist that improve the overall diagnostic process. Sequential testing combines multiple diagnostic tests sequentially in order to classify subjects in one of two groups of the classification system. Sequential methods are generally used to increase accuracy or reduce operational cost in a diagnostic setting, where operational cost is akin to the number of tests required in order to classify a subject [2, 3, 27, 34, 38]. This is in contrast to a linear combination of tests, like a Bayes Network, where each subject is required to receive each test.

While sequential testing can achieve the goals of reducing costs and increasing overall diagnostic accuracy, there are no current methods for developing inference on sequential systema for a metric, such as Bayes Cost (BC) or Youden Index (J). The use of sequential testing is common practice, however, and the need to examine strategies that maximize diagnostic accuracy in sequential testing is important. Thereby, methods for inference and to compare diagnostic accuracy of sequences is important.

5.1.1 Sequenced Tests.

We alter the notation of the formulas in Equation 2.17 as we need it for readability to develop the CI around BC. Therefore, we restate Equation 2.17 as it given in Equation 5.1.

The calculation of the optimal point for sequential tests using these strategies is straightforward and readily available [2]. Let $F_{X_{i,j}}$ where $i \in \{a, b\}$ and $j \in \{1, 2\}$ denote a cumulative distribution function (CDF). Here, a small, but significant notational change is made for this chapter for simplicity of reading the derivations. The disease classes j, formerly 1 and 2, have been replaced with a and b. Let $F_{X_{a,1},X_{a,2}}$ and $F_{X_{b,1},X_{b,2}}$ denote the joint CDFs for the system. Then, the formulas for the false positive rate (P_{FP}) and the true positive rate (P_{TP}) of each strategy are found in Equation 5.1.

$$\begin{aligned}
P_{FP}^{BP}(\theta^{BP}) =& 1 - F_{X_{a,1},X_{a,2}}\left(\theta_1^{BP}, \theta_2^{BP}\right) \\
P_{TP}^{BP}(\theta^{BP}) =& 1 - F_{X_{b,1},X_{b,2}}\left(\theta_1^{BP}, \theta_2^{BP}\right) \\
P_{FP}^{BN}(\theta^{BN}) =& 1 - F_{X_{a,1}}\left(\theta_1^{BN}\right) - F_{X_{a,2}}\left(\theta_2^{BN}\right) + F_{X_{a,1},X_{a,2}}\left(\theta_1^{BN}, \theta_2^{BN}\right) \\
P_{TP}^{BN}(\theta^{BN}) =& 1 - F_{X_{b,1}}\left(\theta_1^{BN}\right) - F_{X_{b,2}}\left(\theta_2^{BN}\right) + F_{X_{b,1},X_{b,2}}\left(\theta_1^{BN}, \theta_2^{BN}\right)
\end{aligned} \tag{5.1}$$

where θ_i corresponds to the threshold values, superscripts BP and BN refer to the respective strategy.

5.1.2 Bayes Cost.

The performance metric, BC, minimizes misclassification errors. It allows for the application of a cost associated with specific misclassifications as well as differing class prevalences. BC in the 2-class case is defined here, by Equation 5.2.

$$BC = \min_{\theta \in \Theta} c_{b|a} p_a \left(1 - P_{TP}(\theta)\right) + c_{a|b} p_b \left(P_{FP}(\theta)\right), \tag{5.2}$$

where $c_{a|b}$ is the cost of assigning a class a element from truth class b, p_b is the prevalence of class b, and P_{TP} and P_{FP} are the true and false positive rates for the classifier at each parameter setting, θ. Depending on the strategy chosen for a specific sequence of tests, P_{TP} and P_{FP} are substituted in Equation 5.2 with the specific P_{TP} and P_{FP}, as computed from equations given in 5.1, associated with that particular strategy.

5.2 Methods for Inference on Sequenced Tests

Two methods of inference are considered for developing CIs on sequenced diagnostic tests; the Delta method and the Generalized method. Each of these methods has a set of assumptions that are required to make inferences utilizing that method. In addition, theory for each of these methods have been extended in the previous chapter to be able to handle the correlation induced by performing a testing sequence.

5.2.1 Delta Method.

With Normal, suitably transformable to Normal or asymptotically Normal features presented, the Delta method is suitable to construct variance estimates for simple Wald-based inference. The Wald-based confidence interval for the BC of a sequence is given in Equation 5.3

$$E(BC) \pm z_{0.975}\sqrt{Var(BC)}. \tag{5.3}$$

The value of $E(BC)$ is the optimal BC value determined by applying Equation 5.2 to the sample. The $Var(BC)$ requires more calculation. An application of the method developed by Klein [19] may be used to derive an estimate of the variance utilizing the partial derivatives of the BC equation under the assumption that the total variance of BC is small.

The chain rule for higher order functions is required for the calculation and is provided in Equation 5.4. Define $f = \Phi(z_1, z_2)$ where $z_1 = \frac{\theta_1 - \mu_{a,1}}{\sigma_{a,1}}$ and $z_2 = \frac{\theta_2 - \mu_{a,2}}{\sigma_{a,2}}$.

$$\frac{\partial f}{\partial \mu_{a,1}} = \frac{\partial f}{\partial z_1}\frac{\partial z_1}{\partial \mu_{a,1}} + \frac{\partial f}{\partial z_2}\frac{\partial z_2}{\partial \mu_{a,1}}. \tag{5.4}$$

Taking the derivative of $\Phi(z_1, z_2)$ w.r.t. each of its parameters yields the following:

$$\begin{aligned}
\frac{\partial\Phi(z_1,z_2)}{\partial z_1} =& \frac{\partial}{\partial z_1}\int_{-\infty}^{z_1}\int_{-\infty}^{z_2}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp^{-\frac{u^2-\rho uv+v^2}{2\pi(1-\rho^2)}}\partial v\partial u \\
=& \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\int_{-\infty}^{z_2}\exp^{-\frac{z_1^2-2\rho z_1 v+v^2}{2(1-\rho^2)}}\partial v \\
=& \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\int_{-\infty}^{z_2}\exp^{-\frac{(v-\rho z_1)^2+z_1^2(1-\rho^2)}{2(1-\rho^2)}}\partial v \\
=& \frac{1}{\sqrt{2\pi}}\exp^{-\frac{z_1^2}{2}}\int_{-\infty}^{z_2}\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\exp^{-\frac{(v-\rho z_1)^2}{2(1-\rho^2)}}\partial v. \\
\Rightarrow \frac{\partial\Phi(z_1,z_2)}{\partial z_1} =& \phi(z_1)\Phi\left(z_2;\rho z_1,\sqrt{1-\rho^2}\right). \qquad (5.5)
\end{aligned}$$

Similarly,

$$\frac{\partial\Phi(z_1,z_2)}{\partial z_2} = \phi(z_2)\Phi\left(z_1;\rho z_2,\sqrt{1-\rho^2}\right). \qquad (5.6)$$

Using these results, we can evaluate the partial derivative of $\mu_{a,1}$ as follows:

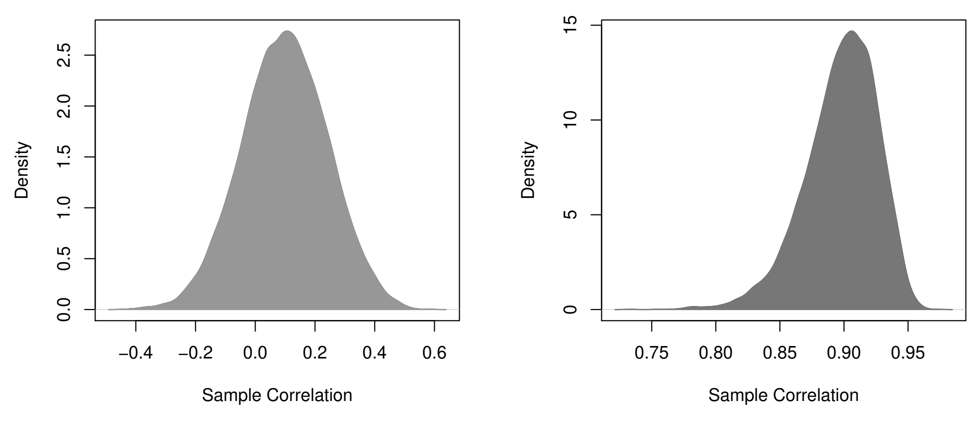
$$
\begin{aligned}
\frac{\partial \Phi\left[z_{a,1}, z_{a,2}\right]}{\partial \mu_{a,1}} =&\phi(z_{a,1})\Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2}\right)\frac{\partial z_{a,1}}{\partial \mu_{a,1}} \\
&+ \phi(z_{a,2})\Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right)\frac{\partial z_{a,2}}{\partial \mu_{a,1}} \\
=&\phi(z_{a,1})\Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2}\right)\frac{\partial}{\partial \mu_{a,1}}\left(\frac{\theta_1-\mu_{a,1}}{\sigma_{a,1}}\right)\sigma_{a,1}^{-1} \\
&+ \phi(z_{a,2})\Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right)\frac{\partial \theta_2}{\partial \mu_{a,1}}\sigma_{a,2}^{-1} \\
=&\phi(z_{a,1})\Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2}\right)\left(\frac{\partial \theta_1}{\partial \mu_{a,1}}-1\right)\sigma_{a,1}^{-1} \\
&+ \phi(z_{a,2})\Phi\left(z_{a,1}\rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right)\frac{\partial \theta_2}{\partial \mu_{a,1}}\sigma_{a,2}^{-1}. \\
\frac{\partial \Phi\left[z_{b,1}, z_{b,2}\right]}{\partial \mu_{a,1}} =&\phi(z_{b,1})\Phi\left(z_{b,2}; \rho_b z_{b,1}, \sqrt{1-\rho_b^2}\right)\frac{\partial \theta_1}{\partial \mu_{a,1}}\sigma_{b,1}^{-1} \\
&+ \phi(z_{b,1})\Phi\left(z_{b,2}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2}\right)\frac{\partial \theta_2}{\partial \mu_{a,1}}\sigma_{b,2}^{-1}.
\end{aligned} \tag{5.7}
$$

Utilizing these partial derivatives, the formula for variance of BC is given in Equation 5.8.

$$
\begin{aligned}
Var(BC) =& \sum_i\sum_j\left(\frac{\partial BC}{\partial \mu_{i,j}}\right)^2 Var(\mu_{i,j}) + \sum_i\sum_j\left(\frac{\partial BC}{\partial \sigma_{i,j}}\right)^2 Var(\sigma_{i,j}) \\
&+ \sum_i\left(\prod_j \frac{\partial BC}{\partial \rho_{i,j}}\right) Var(\rho_j) + \sum_i\left(\prod_j \frac{\partial BC}{\partial \mu_{i,j}}\right) Cov(\mu_{i,1}, \mu_{i,2}) \\
&+ \sum_i\left(\prod_j \frac{\partial BC}{\partial \sigma_{i,j}}\right) Cov(\sigma_{i,1}, \sigma_{i,2}).
\end{aligned} \tag{5.8}
$$

The application of Equation 5.8 requires estimation of parameters as given in Equations 5.9-5.13. Equation 5.11 is an asymptotic variance provided by Lehman and Casella [21]. The assumption of variance of ρ requires asymptotic Normality of the parameter, however, as we see in Figure 5.1, the distributional assumption of asymptotic Normality is not robust at extreme correlation levels. This may affect performance (coverage or length) of the Delta method at extreme correlation levels. This was not an issue in Chapter IV when the standard error for the difference in the performance of two correlated systems was dreived, as only a point estimate of ρ was utilized.

$$
\begin{aligned}
Var(\mu_{i,j}) =& \frac{\sigma_{i,j}^2}{n_j},\ \forall i \in \{a,b\}, j \in \{1,2\}, && (5.9)\\
Var(\sigma_{i,j}^2) =& \frac{\sigma_{i,j}^2}{2(n_j-1)},\ \forall i \in \{a,b\}, j \in \{1,2\}, && (5.10)\\
Var(\rho_i) =& \left(1-\rho_i^2\right)^2,\ \forall i \in \{a,b\}, && (5.11)\\
Cov(\mu_{i,1},\mu_{i,2}) =& \hat{\rho}\sqrt{Var(\mu_{i,1})Var(\mu_{i,1})},\ \forall i \in \{a,b\}, && (5.12)\\
Cov(\sigma_{i,1},\sigma_{i,2}) =& \hat{\rho}^2\sqrt{Var(\sigma_{i,1})Var(\sigma_{i,1})},\ \forall i \in \{a,b\}. && (5.13)
\end{aligned}
$$



(a) Sample Correlation when ρ=0.1 (b) Sample Correlation when ρ=0.9

Figure 5.1. Distributional Comparisons of ρ

The partial derivatives and estimation of all required parameters for the application of this methodology for BP and BN strategies are provided in Appendix D.

With all of the partial derivatives in place, it is now possible to develop the variance required for the Wald-based CI developed by the Delta method by using Equation 5.14

$$Var(BC) = Var\left[c_{b|a}p_a\left(1 - P_{TP}(\theta)\right) + c_{a|b}p_b\left(P_{FP}(\theta)\right)\right], \tag{5.14}$$

substituting the appropriate P_{TP} and P_{FP} equations for the BP and BN strategies. For a two-test system, both BP and BN require estimates for 10 parameters. The BN strategy, however, requires the estimates in multiple terms within both the P_{TP} and P_{FP} equations.

5.2.2 Generalized Method.

The Generalized method relies on pivotal quantities that match a set of required distribution assumptions in the features associated with the BC value. When $\boldsymbol{Y} \sim N(0, \boldsymbol{\Sigma})$ and $U \sim \chi^2_\nu$ then

$$\boldsymbol{\mu} + \boldsymbol{Y}\sqrt{\nu/U} \sim t(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \tag{5.15}$$

is a multivariate t-distributed random variable. With $\boldsymbol{\Sigma}$ being a dispersion matrix $\nu/(\nu-2)$ times the Normal covariance matrix.

Similarly, the multivariate sampling distribution for a covariance matrix of a Normal distribution is

$$(n-1)\boldsymbol{S} \sim Wishart(n-1, \boldsymbol{\Sigma}). \tag{5.16}$$

The Generalized method is then developed, using these pivotal quantities. Define:

$$R_{\mu_j} = t(\bar{\mathbf{x}}_j, \boldsymbol{\Sigma}_j, \nu) \tag{5.17}$$

with $\boldsymbol{\mu}_j$, the mean vector, $\mathbf{x_j}$, the observed mean vector for class j, $\boldsymbol{\Sigma}_j$, the dispersion (scale) matrix for the j^{th} class, and ν as the degrees of freedom. Next, define

$$R_{\Sigma_j} = W(\nu - 1, \boldsymbol{S}_j)/(\nu - 1) \tag{5.18}$$

with $\boldsymbol{\Sigma}_j$ as the covariance matrix for the j^{th} class and $\boldsymbol{S}_j$ the observed covariance matrix for the j^{th} class. Then

$$R_{BC} = c_1 p_a \left(1 - F_1(\theta_1, \theta_2)\right) + c_2 p_b F_2(\theta_1, \theta_2). \tag{5.19}$$

For each k element of R_μ and R_Σ, a numerical solution for R_{BC}, θ_1 and θ_2 can be provided. Specifically,

$$\begin{aligned} R_{BC} = & c_{b|a} p_a \left(1 - \Phi\left[(R_{\theta^*} - R_{\mu_1})^T R_{\Sigma_1}^{-1} (R_{\theta^*} - R_{\mu_1})\right]\right) \\ & + c_{a|b} p_b \Phi\left[(R_{\theta^*} - R_{\mu_2})^T R_{\Sigma_2}^{-1} (R_{\theta^*} - R_{\mu_2})\right]. \end{aligned} \tag{5.20}$$

From this vector of R_{BC} values of k length, the confidence interval for BC can be accomplished with the $\alpha/2$ vector quantiles. The Generalized algorithm consists of the approach found in Figure 5.2

5.2.3 Simulation.

Simulations were performed with normal variates representing the distributions for each class in each test in a sequence with varying degrees of correlation, classification accuracy, sample size and cost structure for each strategy, BP and BN. Four levels of correlation between class-specific distributions in test 1 and test 2 were considered, $\rho = 0, 0.2, 0.5, 0.8$. These correlation levels they are common in medical applications for sequenced testing. Three combinations of test accuracies were considered (Table 5.1) to represent sequences of two Good tests, two Poor tests and a Poor (Test

- Calculate $\bar{\mathbf{x}}$ and $\mathbf{S}$,
- Generate g monte carlo samples from a multivariate t distribution using the dispersion matrix from $\mathbf{S}$,
- Generate pivotal quantites for R_μ from $\bar{\mathbf{x}}$ and the generated t distribution,
- Generate g monte carlo samples from a multivariate Wishart distribution using $\mathbf{S}$, these are the pivotal quantites,
- Solve for R_{BC} using the required pivotal quantites,
- Determine the required $\alpha/2$ quantiles from the vector R_{BC} for the CI

Figure 5.2. Algorithmic steps to compute the multivariate Generalized method for sequential systems

1) and Good (Test 2) sequence of tests. The BC values for these scenarios were identical for both BP and BN strategies. Four class sample sizes were considered, $n_j = 50, 100, 250, 500$. Finally, two cost structures were considered, one with equal cost for misclassification, and one that weights false positive results twice as heavily as false negative results, which was performed on the "Poor, Good" test scenario.

			BC at $\rho =$			
Accuracy	Test 1	Test 2	0	0.2	0.5	0.8
Poor, Poor	$N(0,1), N(2,1)$	$N(0,1), N(2,1)$	0.54	0.56	0.58	0.61
Poor, Good	$N(0,1), N(1,1)$	$N(0,1), N(2,1)$	0.31	0.32	0.32	0.32
Good, Good	$N(0,1), N(1,1)$	$N(0,1), N(1,1)$	0.22	0.25	0.28	0.30

Table 5.1. Testing accuracies considered in the simulation

For each scenario, CIs were developed utilizing the Delta method and the Generalized method for inference. Combined, these factor levels resulted in 128 scenarios for the simulation (4 correlation, 3 accuracies, 4 sample sizes, 2 strategies and an additional cost scenario for one accuracy). CIs were generated 2000 times in each of these scenarios with $\alpha = 0.05$ and both coverage and width were recorded for comparison.

5.3 Results

5.3.1 Results for BP Strategy.

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.935	0.056	0.5	1000			0.922	0.566
	500	0.946	0.084	0.933	0.079		500	0.95	0.089	0.912	0.08
	250	0.943	0.119	0.928	0.112		250	0.942	0.126	0.925	0.114
	100	0.939	0.187	0.926	0.178		100	0.945	0.198	0.932	0.184
	50	0.93	0.263	0.933	0.253		50	0.923	0.281	0.916	0.263
0.2	1000			0.920	0.056	0.8	1000			0.914	0.057
	500	0.954	0.087	0.936	0.08		500	0.946	0.091	0.912	0.082
	250	0.948	0.122	0.929	0.113		250	0.946	0.129	0.923	0.118
	100	0.935	0.192	0.927	0.18		100	0.943	0.204	0.934	0.19
	50	0.922	0.271	0.916	0.256		50	0.945	0.291	0.931	0.271

Table 5.2. Results for the BP strategy for a "Poor, Poor" test comparison at equal cost

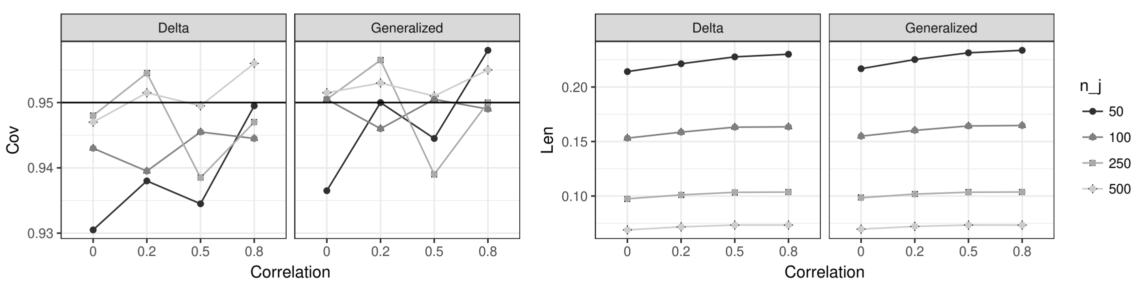


Figure 5.3. Coverange and Length for BP strategy for a "Poor, Good" test comparison at equal cost

Table 5.2 and Figure 5.3 show the results for the BP strategy in "Poor, Poor" and "Poor, Good" test comparison, respectively, at equal cost. The Generalized method achieves coverage for most correlation settings at a sample size of between $100 \leq n_j \leq 250$, while the Delta method fails to achieve coverage even at samples sizes of $n_j \geq 500$ in most testing scenarios. The Generalized method always has wider CI lengths when compared to the Delta method in all scenarios.

When presented with an unequal test accuracy (Table 5.3), there is a slight in-

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.946	0.064	0.5	1000			0.952	0.071
	500	0.951	0.093	0.947	0.091		500	0.950	0.100	0.948	0.100
	250	0.943	0.131	0.936	0.128		250	0.945	0.141	0.941	0.141
	100	0.949	0.206	0.939	0.201		100	0.943	0.222	0.941	0.221
	50	0.941	0.288	0.931	0.282		50	0.945	0.313	0.933	0.308
0.2	1000			0.944	0.068	0.8	1000			0.955	0.071
	500	0.948	0.097	0.945	0.096		500	0.955	0.100	0.957	0.100
	250	0.942	0.137	0.936	0.135		250	0.954	0.141	0.952	0.141
	100	0.945	0.215	0.942	0.212		100	0.950	0.223	0.945	0.222
	50	0.932	0.301	0.921	0.294		50	0.945	0.317	0.935	0.313

Table 5.3. Results for the BP strategy for a "Poor, Good" test comparison at unequal cost

crease in coverage with the Delta method meeting coverage for $n_j = 500$, and $n_j > 100$ for $\rho = 0.8$. When presented with an unequal cost scenario, in general, coverage is similar to the equal cost scenarios.

Overall, the Generalized method meets coverage faster than the Delta method at a cost of a slightly wider confidence interval. Full results can be found in Appendix E.

5.3.2 Results for BN Strategy.

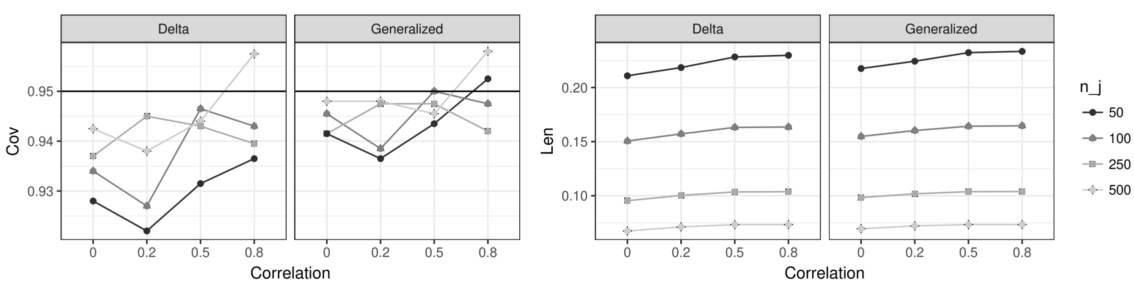


Figure 5.4. Coverange and Length for BP strategy for a "Poor, Good" test comparison at equal cost

Table 5.4 and Figure 5.4 show the analogous results for the BN strategy to those presented for the BP strategy in Table 5.2 and Figure 5.3. There is a slight drop in

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.935	0.056	0.5	1000			0.922	0.057
	500	0.952	0.084	0.935	0.079		500	0.950	0.089	0.927	0.081
	250	0.939	0.119	0.926	0.113		250	0.951	0.126	0.937	0.116
	100	0.937	0.187	0.932	0.181		100	0.933	0.198	0.927	0.190
	50	0.924	0.263	0.929	0.259		50	0.931	0.281	0.929	0.276
0.2	1000			0.921	0.056	0.8	1000			0.914	0.057
	500	0.947	0.087	0.922	0.080		500	0.940	0.091	0.921	0.084
	250	0.949	0.122	0.930	0.114		250	0.938	0.129	0.922	0.133
	100	0.941	0.192	0.932	0.183		100	0.938	0.204	0.934	0.202
	50	0.926	0.270	0.923	0.264		50	0.910	0.290	0.909	0.289

Table 5.4. Results for the BN strategy for a "Poor, Poor" test comparison at equal cost

coverage at $n_j \leq 100$ and $\rho \geq 0.8$ in both the Generalized and Delta methods. This drop in coverage can be attributed to the increase in the quantity of estimable parameters present in the functional forms of BC for the BN strategy. The Generalized method for BN meets coverage at a slightly larger sample size than BP ($n_j \geq 250$), and the Delta method does not meet coverage in most settings within this strategy.

5.4 Discussion

To support decision making under the circumstances where sequential testing is appropriate, accuracy in the method is important. For this reason, in the testing scenarios considered, the Generalized method is preferred. The Generalized method achieved coverage at smaller sample sizes and maintained a comparable width to CIs developed using the Delta method.

Sample size requirements were noticeably higher than in diagnostic test comparisons presented in Chapter IV, as correlation is an integral part of the functional form of BC derived from sequential tests, instead of being involved in just the comparison, the simultaneous estimable parameter requirement is much higher with sequential testing, requiring the estimation of ρ, which also affects covariance matrices (as opposed to only taking variance estimates in the previous method). In most practical situations, an increase of sample size requirements in discovering appropriate diagnostic settings and choosing the correct testing scenario may be preferable to a drastic increase in subjects requiring a particular diagnostic test without a screening test eliminating the requirement.

VI. Conclusions

The performance of classification systems at their optimal point is important to classification methods. Current methods of stratification, the Youden Index and Bayes Cost, allow for summarizing a classification system's performance at the optimal thresholds. In particular, the use of Bayes Cost minimizes the misclassification rates and has been shown to be a more flexible metric for characterizing performance due to the ability to impose a complex cost structure on the misclassifications. This classification is critical to compare and pick the best classification system, which requires new methods to be able to accomplish this.

Although estimation of BC is of interest, quantifying the uncertainty in the system performance is of great practical use. In such cases as when a new or varying test requires a comparison to the current "gold standard" test, inference on the difference between the tests is critical to decision methods. It is desirable in these scenarios to test the same group of people in order to avoid unintended variation and reduce overall sample sizes. This introduces a covariance of BC values into the system that must be identified and my require a methodological adjustment. This work has determined the impact of correlation on paired testing in order to characterize the situations that require a methodological adjustment. Additionally, methods have been developed that account for correlation in the system and adjust CIs accordingly.

Sequential testing involves multiple tests on the same group of people in order to avoid expensive or intrusive testing when possible. In sequential tests, the characterization of uncertainty is of great practical use. This work has developed inferential methods that allow for the creation of confidence intervals on BC values in sequential testing scenarios.

The results of this work produces the following contributions to providing inference in the comparison and evaluation of correlated multi-state classification systems:

1. Under specific scenarios, ignoring correlation on inferential methods is shown to be a valid approach.

2. Current methods are extended to compare generalized multi-state systems (any k) with correlated features.

3. The first statistical inferential methods for sequential systems with respect to accuracy are developed.

Future work may consider different distributional assumptions on the features of interest, allowing for inference without the requirement of transforming data. Also, this work assumed known prevalence structures for each class. Future work may consider a distribution on prevalence for situations where the prevalence of each class is not known. Finally, future work can expand sequential testing to include the Believe the Extreme strategy, as discussed in Section 2.5, account for testing scenarios that involve 3 or more sequential systems, or sequential testing with 3 or more classes.

Appendix A. Results from Naive Independence Assumption

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.957	0.668	0.963	0.756	0.860	0.553	0.929	0.545	0.841	0.555
	0	0.955	0.667	0.960	0.755	0.858	0.553	0.926	0.545	0.837	0.556
	0.1	0.965	0.668	0.967	0.756	0.872	0.554	0.936	0.546	0.854	0.555
	0.3	0.973	0.666	0.975	0.755	0.887	0.551	0.950	0.542	0.868	0.552
	0.5	0.987	0.665	0.987	0.754	0.922	0.551	0.967	0.543	0.901	0.552
	0.9	1.000	0.659	1.000	0.752	0.999	0.542	1.000	0.535	0.996	0.539
20	-0.3	0.952	0.474	0.955	0.507	0.901	0.423	0.935	0.422	0.890	0.424
	0	0.954	0.474	0.957	0.506	0.902	0.422	0.937	0.421	0.894	0.423
	0.1	0.952	0.474	0.955	0.506	0.901	0.422	0.937	0.421	0.893	0.424
	0.3	0.967	0.473	0.970	0.506	0.923	0.421	0.954	0.420	0.916	0.422
	0.5	0.983	0.473	0.983	0.506	0.947	0.420	0.970	0.420	0.941	0.422
	0.9	1.000	0.470	1.000	0.504	0.999	0.417	1.000	0.417	0.999	0.418
50	-0.3	0.952	0.300	0.953	0.309	0.931	0.286	0.945	0.286	0.927	0.286
	0	0.953	0.300	0.953	0.308	0.934	0.285	0.946	0.285	0.930	0.285
	0.1	0.953	0.300	0.955	0.309	0.934	0.285	0.947	0.286	0.932	0.286
	0.3	0.966	0.300	0.969	0.308	0.950	0.285	0.962	0.285	0.947	0.286
	0.5	0.982	0.300	0.982	0.309	0.968	0.285	0.977	0.285	0.965	0.286
	0.9	1.000	0.299	1.000	0.308	1.000	0.284	1.000	0.284	1.000	0.284
100	-0.3	0.952	0.212	0.951	0.215	0.942	0.207	0.948	0.207	0.939	0.207
	0	0.948	0.212	0.948	0.215	0.937	0.206	0.945	0.207	0.936	0.207
	0.1	0.953	0.212	0.953	0.215	0.943	0.207	0.950	0.207	0.941	0.207
	0.3	0.968	0.212	0.969	0.215	0.958	0.207	0.964	0.207	0.957	0.207
	0.5	0.983	0.212	0.983	0.215	0.978	0.206	0.981	0.207	0.976	0.207
	0.9	1.000	0.212	1.000	0.215	1.000	0.206	1.000	0.206	1.000	0.206

Table A.1. Results of Naive Independence Assumption with: Normal, 3-class, Good:Good, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.963	0.439	0.968	0.536	0.875	0.374	0.930	0.368	0.839	0.385
	0	0.965	0.437	0.965	0.534	0.877	0.372	0.932	0.366	0.840	0.383
	0.1	0.965	0.438	0.968	0.535	0.885	0.374	0.935	0.367	0.850	0.384
	0.3	0.977	0.438	0.978	0.535	0.900	0.373	0.949	0.367	0.866	0.382
	0.5	0.987	0.435	0.988	0.533	0.928	0.371	0.968	0.365	0.899	0.378
	0.9	0.999	0.434	1.000	0.533	0.997	0.366	1.000	0.362	0.995	0.366
20	-0.3	0.957	0.310	0.957	0.337	0.906	0.277	0.936	0.277	0.888	0.279
	0	0.959	0.311	0.959	0.338	0.908	0.279	0.938	0.278	0.893	0.281
	0.1	0.961	0.310	0.963	0.337	0.913	0.278	0.944	0.277	0.897	0.280
	0.3	0.971	0.309	0.971	0.337	0.933	0.277	0.956	0.277	0.917	0.279
	0.5	0.984	0.309	0.984	0.337	0.955	0.277	0.973	0.276	0.944	0.278
	0.9	1.000	0.308	1.000	0.336	1.000	0.275	1.000	0.275	1.000	0.275
50	-0.3	0.955	0.196	0.955	0.202	0.935	0.186	0.946	0.186	0.929	0.187
	0	0.953	0.196	0.954	0.202	0.931	0.186	0.945	0.187	0.927	0.187
	0.1	0.958	0.196	0.958	0.202	0.939	0.187	0.948	0.187	0.931	0.187
	0.3	0.966	0.196	0.966	0.202	0.950	0.187	0.959	0.187	0.944	0.187
	0.5	0.982	0.195	0.982	0.202	0.969	0.186	0.976	0.186	0.966	0.187
	0.9	1.000	0.196	1.000	0.202	1.000	0.186	1.000	0.186	1.000	0.187
100	-0.3	0.950	0.138	0.950	0.140	0.939	0.135	0.944	0.135	0.936	0.135
	0	0.950	0.139	0.950	0.141	0.940	0.135	0.946	0.135	0.936	0.136
	0.1	0.956	0.139	0.956	0.141	0.945	0.135	0.950	0.135	0.940	0.136
	0.3	0.964	0.139	0.963	0.141	0.956	0.135	0.960	0.135	0.954	0.136
	0.5	0.980	0.138	0.980	0.141	0.974	0.135	0.976	0.135	0.970	0.135
	0.9	1.000	0.138	1.000	0.141	1.000	0.135	1.000	0.135	1.000	0.135

Table A.2. Results of Naive Independence Assumption with: Normal, 3-class, Good:Good, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.936	0.805	0.957	0.846	0.876	0.713	0.909	0.707	0.880	0.707
	0	0.942	0.804	0.963	0.845	0.884	0.712	0.913	0.706	0.889	0.706
	0.1	0.943	0.806	0.964	0.846	0.885	0.711	0.919	0.706	0.890	0.705
	0.3	0.957	0.806	0.973	0.847	0.906	0.714	0.929	0.709	0.911	0.710
	0.5	0.972	0.805	0.984	0.846	0.932	0.711	0.944	0.706	0.936	0.707
	0.9	0.994	0.803	0.999	0.846	0.988	0.709	0.980	0.704	0.991	0.709
20	-0.3	0.943	0.576	0.953	0.589	0.912	0.534	0.927	0.533	0.913	0.532
	0	0.947	0.575	0.957	0.589	0.913	0.533	0.933	0.532	0.917	0.531
	0.1	0.948	0.575	0.957	0.589	0.916	0.532	0.932	0.531	0.917	0.530
	0.3	0.962	0.575	0.969	0.588	0.933	0.531	0.946	0.531	0.935	0.530
	0.5	0.978	0.575	0.983	0.589	0.956	0.532	0.966	0.531	0.958	0.530
	0.9	0.998	0.574	1.000	0.587	0.997	0.529	0.995	0.528	0.997	0.527
50	-0.3	0.947	0.366	0.951	0.369	0.932	0.354	0.939	0.354	0.934	0.354
	0	0.943	0.366	0.948	0.369	0.928	0.353	0.936	0.353	0.931	0.353
	0.1	0.953	0.366	0.956	0.369	0.938	0.353	0.943	0.354	0.939	0.354
	0.3	0.965	0.366	0.966	0.369	0.952	0.354	0.958	0.354	0.952	0.354
	0.5	0.977	0.366	0.979	0.370	0.970	0.354	0.973	0.354	0.970	0.354
	0.9	1.000	0.366	1.000	0.369	1.000	0.353	0.999	0.353	1.000	0.353
100	-0.3	0.949	0.259	0.950	0.260	0.943	0.255	0.947	0.255	0.944	0.255
	0	0.945	0.259	0.947	0.260	0.939	0.255	0.943	0.255	0.939	0.255
	0.1	0.957	0.259	0.958	0.260	0.949	0.255	0.954	0.255	0.951	0.255
	0.3	0.966	0.259	0.967	0.260	0.960	0.255	0.963	0.255	0.960	0.255
	0.5	0.977	0.259	0.978	0.260	0.974	0.255	0.976	0.255	0.974	0.255
	0.9	1.000	0.259	1.000	0.260	1.000	0.254	1.000	0.255	1.000	0.255

Table A.3. Results of Naive Independence Assumption with: Normal, 3-class Good:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.941	0.552	0.962	0.626	0.887	0.493	0.908	0.490	0.874	0.501
	0	0.950	0.555	0.968	0.628	0.896	0.495	0.916	0.491	0.884	0.502
	0.1	0.950	0.553	0.969	0.627	0.898	0.493	0.915	0.490	0.885	0.501
	0.3	0.958	0.553	0.976	0.626	0.910	0.493	0.922	0.489	0.903	0.500
	0.5	0.974	0.554	0.987	0.628	0.938	0.494	0.945	0.490	0.933	0.501
	0.9	0.994	0.552	0.999	0.626	0.985	0.490	0.979	0.486	0.989	0.496
20	-0.3	0.945	0.391	0.952	0.413	0.914	0.364	0.925	0.364	0.905	0.368
	0	0.952	0.391	0.958	0.413	0.923	0.364	0.935	0.364	0.918	0.368
	0.1	0.955	0.391	0.964	0.413	0.929	0.364	0.936	0.364	0.921	0.367
	0.3	0.964	0.391	0.970	0.413	0.940	0.364	0.947	0.364	0.935	0.367
	0.5	0.979	0.390	0.984	0.412	0.961	0.363	0.963	0.363	0.959	0.366
	0.9	0.998	0.390	0.999	0.412	0.996	0.361	0.992	0.361	0.997	0.364
50	-0.3	0.946	0.247	0.948	0.251	0.932	0.239	0.938	0.239	0.931	0.240
	0	0.950	0.247	0.953	0.252	0.935	0.239	0.941	0.239	0.933	0.240
	0.1	0.953	0.247	0.954	0.252	0.944	0.239	0.947	0.239	0.941	0.240
	0.3	0.964	0.247	0.966	0.252	0.955	0.239	0.959	0.239	0.953	0.240
	0.5	0.980	0.247	0.982	0.251	0.974	0.238	0.976	0.239	0.972	0.240
	0.9	1.000	0.246	1.000	0.251	0.999	0.238	0.999	0.239	1.000	0.239
100	-0.3	0.947	0.174	0.948	0.176	0.941	0.171	0.942	0.172	0.939	0.172
	0	0.952	0.174	0.953	0.176	0.945	0.171	0.949	0.172	0.944	0.172
	0.1	0.958	0.174	0.959	0.176	0.952	0.171	0.953	0.172	0.950	0.172
	0.3	0.964	0.174	0.965	0.176	0.960	0.171	0.961	0.172	0.958	0.172
	0.5	0.978	0.174	0.979	0.176	0.976	0.171	0.976	0.172	0.975	0.172
	0.9	1.000	0.174	1.000	0.176	1.000	0.171	1.000	0.172	1.000	0.172

Table A.4. Results of Naive Independence Assumption with: Normal, 3-class Good:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.938	1.106	0.932	0.868	0.902	0.805	0.899	0.804	0.894	0.786
	0	0.943	1.026	0.939	0.868	0.911	0.806	0.910	0.806	0.904	0.787
	0.1	0.942	1.133	0.937	0.867	0.908	0.804	0.909	0.804	0.904	0.785
	0.3	0.958	1.014	0.957	0.868	0.932	0.803	0.925	0.803	0.925	0.785
	0.5	0.968	1.057	0.964	0.866	0.947	0.804	0.933	0.803	0.943	0.785
	0.9	0.991	1.125	0.993	0.867	0.988	0.805	0.966	0.804	0.987	0.787
20	-0.3	0.936	0.613	0.934	0.609	0.915	0.582	0.917	0.583	0.912	0.576
	0	0.948	0.622	0.945	0.609	0.929	0.582	0.929	0.583	0.929	0.576
	0.1	0.949	0.617	0.948	0.610	0.934	0.584	0.931	0.585	0.930	0.578
	0.3	0.955	0.614	0.955	0.609	0.942	0.582	0.938	0.583	0.941	0.576
	0.5	0.974	0.617	0.974	0.609	0.964	0.582	0.959	0.583	0.963	0.577
	0.9	0.997	0.620	0.996	0.610	0.995	0.582	0.988	0.582	0.994	0.576
50	-0.3	0.942	0.385	0.942	0.385	0.931	0.378	0.937	0.378	0.932	0.377
	0	0.953	0.386	0.951	0.386	0.945	0.378	0.946	0.379	0.944	0.377
	0.1	0.950	0.385	0.948	0.386	0.942	0.378	0.941	0.379	0.941	0.377
	0.3	0.965	0.386	0.963	0.386	0.959	0.378	0.958	0.379	0.958	0.378
	0.5	0.980	0.385	0.979	0.386	0.974	0.378	0.974	0.379	0.974	0.377
	0.9	0.999	0.386	0.999	0.386	0.999	0.378	0.997	0.379	0.999	0.377
100	-0.3	0.941	0.273	0.942	0.273	0.935	0.270	0.938	0.270	0.936	0.270
	0	0.954	0.273	0.950	0.273	0.949	0.270	0.949	0.271	0.948	0.270
	0.1	0.955	0.273	0.955	0.273	0.950	0.270	0.952	0.271	0.951	0.270
	0.3	0.963	0.273	0.963	0.273	0.959	0.271	0.960	0.271	0.959	0.270
	0.5	0.978	0.273	0.977	0.273	0.975	0.270	0.976	0.271	0.975	0.270
	0.9	0.999	0.273	0.999	0.273	0.998	0.271	0.998	0.271	0.998	0.271

Table A.5. Results of Naive Independence Assumption with: Normal, 3-class Fair:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.912	0.707	0.955	0.719	0.885	0.659	0.899	0.656	0.900	0.659
	0	0.921	0.706	0.962	0.719	0.897	0.659	0.909	0.656	0.912	0.659
	0.1	0.931	0.708	0.966	0.719	0.904	0.660	0.913	0.656	0.920	0.659
	0.3	0.939	0.707	0.971	0.719	0.914	0.659	0.923	0.655	0.928	0.659
	0.5	0.953	0.708	0.981	0.719	0.935	0.661	0.932	0.658	0.949	0.661
	0.9	0.979	0.708	0.995	0.720	0.973	0.659	0.961	0.656	0.983	0.660
20	-0.3	0.928	0.510	0.950	0.505	0.905	0.483	0.924	0.482	0.920	0.480
	0	0.939	0.510	0.959	0.506	0.912	0.483	0.936	0.481	0.929	0.480
	0.1	0.933	0.509	0.953	0.505	0.911	0.481	0.931	0.480	0.926	0.478
	0.3	0.953	0.510	0.971	0.505	0.932	0.483	0.947	0.481	0.945	0.480
	0.5	0.961	0.510	0.977	0.505	0.946	0.482	0.954	0.480	0.958	0.479
	0.9	0.989	0.510	0.995	0.505	0.983	0.481	0.980	0.479	0.988	0.479
50	-0.3	0.935	0.327	0.944	0.325	0.922	0.318	0.934	0.318	0.931	0.318
	0	0.940	0.326	0.948	0.325	0.927	0.318	0.940	0.318	0.938	0.318
	0.1	0.946	0.327	0.954	0.325	0.934	0.318	0.942	0.318	0.943	0.318
	0.3	0.960	0.326	0.966	0.325	0.951	0.318	0.958	0.318	0.957	0.318
	0.5	0.968	0.326	0.974	0.325	0.963	0.318	0.966	0.318	0.969	0.318
	0.9	0.994	0.326	0.996	0.325	0.993	0.318	0.992	0.318	0.995	0.318
100	-0.3	0.940	0.232	0.945	0.231	0.935	0.229	0.939	0.229	0.940	0.229
	0	0.949	0.232	0.952	0.231	0.945	0.229	0.947	0.229	0.949	0.229
	0.1	0.950	0.231	0.953	0.231	0.945	0.229	0.948	0.229	0.948	0.229
	0.3	0.962	0.232	0.964	0.231	0.957	0.229	0.960	0.229	0.962	0.229
	0.5	0.974	0.232	0.976	0.231	0.972	0.229	0.973	0.229	0.974	0.229
	0.9	0.995	0.232	0.997	0.231	0.994	0.229	0.995	0.229	0.995	0.229

Table A.6. Results of Naive Independence Assumption with: Normal, 3-class Fair:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.927	0.921	0.954	0.930	0.878	0.842	0.925	0.841	0.892	0.840
	0	0.938	0.921	0.959	0.930	0.893	0.843	0.935	0.841	0.906	0.840
	0.1	0.942	0.920	0.963	0.930	0.894	0.841	0.935	0.839	0.905	0.838
	0.3	0.957	0.920	0.974	0.930	0.919	0.840	0.952	0.837	0.927	0.837
	0.5	0.972	0.920	0.985	0.930	0.945	0.839	0.968	0.838	0.949	0.838
	0.9	1.000	0.921	1.000	0.930	0.999	0.837	1.000	0.836	0.999	0.837
20	-0.3	0.935	0.661	0.946	0.662	0.909	0.624	0.927	0.624	0.913	0.624
	0	0.940	0.661	0.951	0.662	0.914	0.623	0.933	0.623	0.920	0.623
	0.1	0.947	0.661	0.956	0.661	0.921	0.623	0.940	0.623	0.927	0.623
	0.3	0.967	0.661	0.974	0.661	0.946	0.623	0.961	0.623	0.949	0.624
	0.5	0.983	0.661	0.986	0.662	0.969	0.623	0.978	0.623	0.972	0.624
	0.9	1.000	0.661	1.000	0.661	1.000	0.622	1.000	0.622	1.000	0.623
50	-0.3	0.938	0.422	0.942	0.421	0.929	0.410	0.935	0.411	0.931	0.411
	0	0.945	0.422	0.951	0.422	0.934	0.411	0.941	0.412	0.935	0.412
	0.1	0.955	0.422	0.960	0.422	0.944	0.410	0.951	0.411	0.946	0.411
	0.3	0.968	0.422	0.970	0.421	0.960	0.411	0.966	0.411	0.961	0.412
	0.5	0.985	0.422	0.986	0.422	0.979	0.410	0.982	0.410	0.980	0.411
	0.9	1.000	0.422	1.000	0.421	1.000	0.411	1.000	0.411	1.000	0.411
100	-0.3	0.937	0.299	0.938	0.299	0.930	0.295	0.934	0.295	0.931	0.295
	0	0.946	0.299	0.947	0.299	0.941	0.295	0.944	0.295	0.942	0.295
	0.1	0.954	0.299	0.956	0.299	0.950	0.295	0.953	0.295	0.951	0.295
	0.3	0.966	0.299	0.966	0.299	0.960	0.295	0.963	0.295	0.960	0.295
	0.5	0.983	0.299	0.984	0.299	0.980	0.295	0.981	0.295	0.980	0.295
	0.9	1.000	0.299	1.000	0.299	1.000	0.295	1.000	0.295	1.000	0.295

Table A.7. Results of Naive Independence Assumption with: Normal, 3-class Good:Poor, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.940	0.649	0.958	0.707	0.889	0.590	0.921	0.587	0.881	0.591
	0	0.947	0.649	0.963	0.708	0.899	0.591	0.931	0.588	0.894	0.592
	0.1	0.957	0.649	0.973	0.709	0.914	0.590	0.940	0.588	0.905	0.592
	0.3	0.963	0.647	0.975	0.706	0.925	0.588	0.950	0.586	0.919	0.589
	0.5	0.981	0.647	0.989	0.707	0.951	0.587	0.972	0.585	0.948	0.588
	0.9	1.000	0.645	1.000	0.706	0.999	0.582	1.000	0.581	0.999	0.583
20	-0.3	0.941	0.458	0.949	0.476	0.916	0.433	0.928	0.434	0.910	0.435
	0	0.950	0.457	0.956	0.476	0.928	0.434	0.938	0.434	0.923	0.436
	0.1	0.957	0.457	0.963	0.476	0.935	0.433	0.945	0.434	0.929	0.435
	0.3	0.970	0.458	0.974	0.477	0.953	0.434	0.961	0.434	0.948	0.435
	0.5	0.985	0.457	0.988	0.476	0.976	0.434	0.980	0.434	0.973	0.435
	0.9	1.000	0.456	1.000	0.475	1.000	0.430	1.000	0.431	1.000	0.431
50	-0.3	0.938	0.289	0.940	0.293	0.928	0.281	0.931	0.282	0.924	0.282
	0	0.953	0.289	0.954	0.293	0.942	0.282	0.946	0.282	0.941	0.283
	0.1	0.954	0.289	0.957	0.293	0.944	0.282	0.948	0.282	0.942	0.282
	0.3	0.970	0.289	0.972	0.293	0.962	0.282	0.967	0.282	0.961	0.282
	0.5	0.985	0.288	0.986	0.293	0.981	0.282	0.982	0.282	0.979	0.282
	0.9	1.000	0.289	1.000	0.293	1.000	0.281	1.000	0.281	1.000	0.281
100	-0.3	0.941	0.204	0.942	0.205	0.937	0.201	0.937	0.202	0.936	0.202
	0	0.947	0.204	0.948	0.205	0.942	0.201	0.943	0.201	0.941	0.201
	0.1	0.952	0.204	0.952	0.205	0.948	0.201	0.948	0.201	0.945	0.202
	0.3	0.971	0.204	0.971	0.205	0.967	0.201	0.968	0.201	0.965	0.201
	0.5	0.986	0.204	0.985	0.205	0.982	0.201	0.983	0.202	0.982	0.202
	0.9	1.000	0.204	1.000	0.205	1.000	0.201	1.000	0.201	1.000	0.201

Table A.8. Results of Naive Independence Assumption with: Normal, 3-class Good:Poor, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.975	0.460	0.956	0.532	0.877	0.385	0.919	0.378	0.787	0.409
	0	0.979	0.460	0.963	0.533	0.887	0.383	0.932	0.377	0.805	0.405
	0.1	0.983	0.460	0.967	0.535	0.898	0.385	0.933	0.379	0.818	0.406
	0.3	0.989	0.459	0.976	0.533	0.914	0.383	0.949	0.377	0.839	0.402
	0.5	0.995	0.456	0.986	0.532	0.940	0.380	0.965	0.375	0.876	0.396
	0.9	1.000	0.448	1.000	0.529	0.998	0.371	1.000	0.369	0.988	0.378
20	-0.3	0.962	0.329	0.952	0.356	0.908	0.296	0.931	0.296	0.862	0.302
	0	0.966	0.328	0.955	0.355	0.914	0.295	0.935	0.295	0.867	0.301
	0.1	0.970	0.328	0.961	0.355	0.925	0.295	0.944	0.295	0.881	0.301
	0.3	0.979	0.328	0.973	0.355	0.940	0.294	0.956	0.295	0.905	0.300
	0.5	0.991	0.326	0.985	0.353	0.965	0.293	0.976	0.294	0.939	0.298
	0.9	1.000	0.322	1.000	0.351	1.000	0.288	1.000	0.289	0.999	0.291
50	-0.3	0.945	0.208	0.941	0.215	0.924	0.199	0.933	0.200	0.904	0.201
	0	0.952	0.208	0.948	0.215	0.932	0.199	0.940	0.200	0.915	0.201
	0.1	0.960	0.209	0.956	0.215	0.940	0.199	0.947	0.200	0.923	0.201
	0.3	0.972	0.208	0.969	0.215	0.956	0.199	0.963	0.199	0.943	0.200
	0.5	0.987	0.208	0.987	0.215	0.977	0.199	0.981	0.200	0.970	0.200
	0.9	1.000	0.207	1.000	0.214	1.000	0.197	1.000	0.198	1.000	0.198
100	-0.3	0.943	0.148	0.941	0.150	0.930	0.144	0.934	0.145	0.919	0.145
	0	0.951	0.148	0.949	0.150	0.940	0.144	0.944	0.145	0.931	0.145
	0.1	0.959	0.148	0.959	0.150	0.948	0.144	0.952	0.145	0.940	0.145
	0.3	0.971	0.148	0.971	0.150	0.964	0.144	0.965	0.145	0.957	0.145
	0.5	0.986	0.148	0.985	0.150	0.983	0.144	0.984	0.145	0.978	0.145
	0.9	1.000	0.147	1.000	0.150	1.000	0.144	1.000	0.144	1.000	0.144

Table A.9. Results of Naive Independence Assumption with: Normal, 2-class, Good:Good, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.855	0.396	0.966	0.677	0.855	0.565	0.926	0.567	0.751	0.564
	0	0.866	0.395	0.972	0.675	0.861	0.560	0.937	0.563	0.771	0.563
	0.1	0.871	0.396	0.974	0.679	0.865	0.561	0.942	0.563	0.780	0.560
	0.3	0.882	0.395	0.984	0.677	0.875	0.557	0.956	0.562	0.797	0.558
	0.5	0.895	0.393	0.993	0.679	0.886	0.553	0.976	0.562	0.833	0.559
	0.9	0.935	0.388	1.000	0.691	0.927	0.539	0.999	0.567	0.930	0.558
20	-0.3	0.896	0.293	0.958	0.561	0.923	0.474	0.933	0.495	0.841	0.488
	0	0.913	0.294	0.966	0.561	0.931	0.470	0.943	0.493	0.852	0.487
	0.1	0.902	0.293	0.962	0.562	0.922	0.472	0.941	0.496	0.849	0.486
	0.3	0.919	0.293	0.979	0.562	0.930	0.469	0.964	0.494	0.870	0.487
	0.5	0.934	0.293	0.988	0.564	0.937	0.467	0.979	0.495	0.896	0.488
	0.9	0.960	0.288	1.000	0.564	0.960	0.454	1.000	0.493	0.962	0.483
50	-0.3	0.932	0.192	0.939	0.335	0.977	0.302	0.926	0.303	0.899	0.315
	0	0.950	0.192	0.957	0.332	0.979	0.297	0.946	0.300	0.919	0.311
	0.1	0.956	0.192	0.962	0.331	0.979	0.297	0.953	0.301	0.930	0.312
	0.3	0.967	0.192	0.972	0.332	0.983	0.297	0.964	0.301	0.942	0.312
	0.5	0.981	0.191	0.985	0.329	0.989	0.292	0.982	0.298	0.965	0.307
	0.9	0.996	0.191	1.000	0.331	0.996	0.285	1.000	0.299	0.996	0.302
100	-0.3	0.946	0.136	0.944	0.165	0.970	0.164	0.940	0.157	0.927	0.161
	0	0.954	0.136	0.952	0.165	0.974	0.164	0.948	0.157	0.938	0.161
	0.1	0.961	0.136	0.960	0.164	0.975	0.162	0.955	0.155	0.943	0.159
	0.3	0.970	0.136	0.969	0.165	0.980	0.163	0.965	0.156	0.956	0.160
	0.5	0.989	0.136	0.988	0.164	0.992	0.162	0.986	0.156	0.980	0.159
	0.9	1.000	0.136	1.000	0.163	1.000	0.159	1.000	0.154	1.000	0.155

Table A.10. Results of Naive Independence Assumption with: Normal, 2-class, Good:Good, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.922	0.594	0.951	0.638	0.863	0.525	0.885	0.518	0.852	0.542
	0	0.939	0.595	0.962	0.639	0.880	0.520	0.902	0.515	0.871	0.531
	0.1	0.939	0.594	0.965	0.638	0.881	0.518	0.905	0.513	0.877	0.529
	0.3	0.950	0.593	0.974	0.638	0.900	0.524	0.911	0.518	0.893	0.538
	0.5	0.964	0.592	0.983	0.636	0.922	0.515	0.929	0.510	0.926	0.526
	0.9	0.979	0.588	0.995	0.635	0.963	0.511	0.958	0.507	0.980	0.522
20	-0.3	0.932	0.429	0.943	0.443	0.899	0.399	0.916	0.399	0.894	0.404
	0	0.945	0.428	0.955	0.442	0.918	0.398	0.929	0.397	0.909	0.402
	0.1	0.950	0.428	0.959	0.442	0.923	0.398	0.934	0.398	0.917	0.403
	0.3	0.963	0.428	0.973	0.442	0.939	0.398	0.948	0.397	0.933	0.403
	0.5	0.976	0.427	0.984	0.442	0.958	0.397	0.963	0.397	0.957	0.402
	0.9	0.993	0.426	0.998	0.440	0.989	0.395	0.988	0.395	0.994	0.400
50	-0.3	0.941	0.274	0.943	0.277	0.927	0.266	0.932	0.266	0.923	0.267
	0	0.950	0.274	0.954	0.277	0.941	0.266	0.944	0.266	0.937	0.268
	0.1	0.953	0.273	0.956	0.276	0.943	0.265	0.947	0.266	0.941	0.267
	0.3	0.972	0.273	0.974	0.276	0.962	0.265	0.964	0.265	0.959	0.267
	0.5	0.982	0.273	0.985	0.276	0.978	0.265	0.978	0.265	0.977	0.267
	0.9	0.999	0.272	0.999	0.276	0.998	0.264	0.998	0.265	0.999	0.266
100	-0.3	0.935	0.194	0.937	0.195	0.927	0.191	0.932	0.192	0.926	0.192
	0	0.950	0.194	0.951	0.195	0.944	0.191	0.945	0.191	0.944	0.192
	0.1	0.959	0.194	0.959	0.195	0.953	0.191	0.956	0.191	0.952	0.192
	0.3	0.967	0.194	0.969	0.195	0.963	0.192	0.964	0.192	0.963	0.192
	0.5	0.983	0.194	0.983	0.195	0.980	0.191	0.980	0.191	0.980	0.192
	0.9	1.000	0.194	1.000	0.195	1.000	0.191	1.000	0.191	1.000	0.191

Table A.11. Results of Naive Independence Assumption with: Normal, 2-class Good:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.830	0.488	0.963	0.622	0.836	0.577	0.900	0.559	0.826	0.555
	0	0.840	0.487	0.969	0.621	0.845	0.575	0.915	0.557	0.837	0.551
	0.1	0.846	0.486	0.971	0.622	0.850	0.574	0.920	0.558	0.843	0.551
	0.3	0.858	0.486	0.977	0.622	0.865	0.572	0.922	0.554	0.854	0.552
	0.5	0.876	0.488	0.987	0.625	0.883	0.572	0.942	0.557	0.875	0.553
	0.9	0.913	0.481	0.997	0.628	0.928	0.558	0.966	0.552	0.939	0.548
20	-0.3	0.890	0.367	0.956	0.507	0.909	0.460	0.926	0.468	0.885	0.453
	0	0.896	0.366	0.962	0.506	0.909	0.459	0.939	0.468	0.890	0.451
	0.1	0.903	0.365	0.966	0.506	0.915	0.459	0.938	0.467	0.896	0.452
	0.3	0.911	0.365	0.973	0.508	0.924	0.458	0.950	0.468	0.909	0.451
	0.5	0.927	0.366	0.987	0.508	0.933	0.457	0.968	0.468	0.927	0.450
	0.9	0.957	0.362	0.998	0.504	0.967	0.446	0.991	0.460	0.973	0.445
50	-0.3	0.932	0.238	0.944	0.315	0.956	0.292	0.932	0.295	0.922	0.294
	0	0.944	0.238	0.951	0.313	0.962	0.291	0.941	0.294	0.934	0.293
	0.1	0.948	0.238	0.959	0.314	0.965	0.291	0.947	0.295	0.936	0.294
	0.3	0.962	0.238	0.970	0.313	0.972	0.290	0.960	0.294	0.954	0.293
	0.5	0.980	0.238	0.986	0.312	0.983	0.288	0.979	0.293	0.974	0.291
	0.9	0.996	0.237	1.000	0.309	0.995	0.284	0.999	0.289	0.996	0.286
100	-0.3	0.934	0.169	0.937	0.184	0.945	0.181	0.931	0.179	0.928	0.179
	0	0.946	0.169	0.948	0.184	0.954	0.181	0.943	0.179	0.939	0.179
	0.1	0.953	0.169	0.956	0.183	0.960	0.179	0.952	0.177	0.945	0.177
	0.3	0.968	0.169	0.970	0.184	0.972	0.181	0.963	0.178	0.961	0.178
	0.5	0.983	0.169	0.983	0.184	0.984	0.181	0.982	0.178	0.979	0.178
	0.9	1.000	0.169	1.000	0.183	1.000	0.180	1.000	0.178	1.000	0.178

Table A.12. Results of Naive Independence Assumption with: Normal, 2-class Good:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.927	0.704	0.950	0.731	0.870	0.639	0.907	0.634	0.852	0.642
	0	0.946	0.704	0.962	0.732	0.892	0.633	0.929	0.629	0.879	0.635
	0.1	0.951	0.703	0.967	0.731	0.897	0.631	0.934	0.627	0.883	0.632
	0.3	0.971	0.703	0.981	0.730	0.932	0.638	0.956	0.633	0.914	0.640
	0.5	0.987	0.700	0.990	0.729	0.952	0.628	0.974	0.624	0.941	0.630
	0.9	1.000	0.696	1.000	0.726	0.999	0.622	1.000	0.620	0.997	0.622
20	-0.3	0.923	0.509	0.936	0.516	0.896	0.481	0.913	0.480	0.891	0.482
	0	0.949	0.509	0.957	0.516	0.925	0.481	0.939	0.480	0.918	0.482
	0.1	0.954	0.508	0.962	0.515	0.934	0.479	0.947	0.479	0.928	0.481
	0.3	0.969	0.508	0.976	0.515	0.950	0.479	0.962	0.479	0.944	0.480
	0.5	0.988	0.508	0.991	0.515	0.974	0.479	0.982	0.479	0.969	0.480
	0.9	1.000	0.506	1.000	0.514	1.000	0.477	1.000	0.477	1.000	0.477
50	-0.3	0.928	0.326	0.933	0.327	0.918	0.318	0.925	0.319	0.914	0.319
	0	0.949	0.326	0.952	0.327	0.937	0.318	0.945	0.318	0.935	0.319
	0.1	0.953	0.326	0.957	0.327	0.945	0.317	0.951	0.318	0.943	0.318
	0.3	0.971	0.326	0.974	0.327	0.966	0.318	0.969	0.318	0.964	0.319
	0.5	0.989	0.325	0.990	0.326	0.984	0.317	0.987	0.318	0.982	0.318
	0.9	1.000	0.325	1.000	0.326	1.000	0.317	1.000	0.317	1.000	0.317
100	-0.3	0.929	0.231	0.932	0.232	0.924	0.229	0.927	0.229	0.923	0.229
	0	0.950	0.232	0.952	0.232	0.946	0.229	0.949	0.229	0.945	0.229
	0.1	0.956	0.231	0.957	0.232	0.951	0.229	0.954	0.229	0.951	0.229
	0.3	0.971	0.231	0.972	0.232	0.969	0.228	0.970	0.229	0.968	0.229
	0.5	0.989	0.231	0.989	0.231	0.987	0.228	0.988	0.229	0.987	0.229
	0.9	1.000	0.231	1.000	0.232	1.000	0.229	1.000	0.229	1.000	0.229

Table A.13. Results of Naive Independence Assumption with: Normal, 2-class Fair:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.863	0.564	0.963	0.585	0.841	0.586	0.919	0.558	0.867	0.571
	0	0.880	0.563	0.969	0.586	0.859	0.585	0.933	0.560	0.876	0.568
	0.1	0.888	0.564	0.976	0.587	0.864	0.585	0.942	0.559	0.881	0.570
	0.3	0.918	0.564	0.985	0.588	0.899	0.586	0.959	0.562	0.910	0.571
	0.5	0.945	0.562	0.991	0.589	0.930	0.581	0.975	0.558	0.934	0.571
	0.9	0.987	0.554	1.000	0.594	0.984	0.577	1.000	0.561	0.984	0.576
20	-0.3	0.900	0.426	0.952	0.458	0.888	0.442	0.927	0.441	0.901	0.439
	0	0.916	0.425	0.958	0.457	0.901	0.440	0.939	0.440	0.911	0.439
	0.1	0.921	0.424	0.968	0.457	0.914	0.440	0.949	0.440	0.919	0.438
	0.3	0.940	0.425	0.978	0.457	0.928	0.440	0.961	0.441	0.935	0.439
	0.5	0.960	0.424	0.990	0.458	0.951	0.439	0.981	0.440	0.957	0.438
	0.9	0.991	0.423	1.000	0.459	0.989	0.437	1.000	0.440	0.992	0.439
50	-0.3	0.934	0.277	0.943	0.293	0.937	0.282	0.933	0.284	0.929	0.285
	0	0.947	0.277	0.955	0.294	0.946	0.283	0.945	0.285	0.941	0.286
	0.1	0.950	0.277	0.958	0.293	0.951	0.282	0.947	0.284	0.944	0.285
	0.3	0.970	0.276	0.977	0.293	0.971	0.282	0.970	0.284	0.965	0.285
	0.5	0.982	0.276	0.987	0.293	0.982	0.283	0.982	0.285	0.978	0.286
	0.9	1.000	0.275	1.000	0.292	1.000	0.281	1.000	0.283	1.000	0.284
100	-0.3	0.932	0.196	0.936	0.198	0.930	0.195	0.930	0.196	0.926	0.196
	0	0.952	0.196	0.953	0.198	0.948	0.195	0.950	0.195	0.947	0.195
	0.1	0.955	0.196	0.957	0.198	0.952	0.195	0.952	0.195	0.950	0.195
	0.3	0.969	0.196	0.970	0.198	0.965	0.195	0.966	0.195	0.964	0.196
	0.5	0.987	0.196	0.988	0.198	0.986	0.195	0.985	0.195	0.983	0.195
	0.9	1.000	0.196	1.000	0.198	1.000	0.194	1.000	0.195	1.000	0.195

Table A.14. Results of Naive Independence Assumption with: Normal, 2-class Fair:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.915	0.697	0.944	0.700	0.865	0.648	0.892	0.643	0.895	0.639
	0	0.931	0.698	0.954	0.702	0.886	0.645	0.905	0.640	0.911	0.636
	0.1	0.934	0.698	0.956	0.701	0.892	0.645	0.910	0.640	0.915	0.636
	0.3	0.945	0.699	0.966	0.702	0.911	0.651	0.923	0.646	0.935	0.646
	0.5	0.954	0.697	0.974	0.701	0.922	0.645	0.930	0.640	0.946	0.639
	0.9	0.978	0.697	0.990	0.702	0.965	0.644	0.957	0.640	0.983	0.643
20	-0.3	0.932	0.507	0.940	0.507	0.909	0.486	0.920	0.486	0.920	0.486
	0	0.941	0.507	0.949	0.507	0.920	0.486	0.931	0.486	0.932	0.486
	0.1	0.948	0.508	0.955	0.507	0.928	0.486	0.938	0.486	0.940	0.485
	0.3	0.958	0.507	0.966	0.507	0.942	0.487	0.946	0.487	0.952	0.487
	0.5	0.972	0.506	0.978	0.506	0.958	0.485	0.961	0.485	0.967	0.485
	0.9	0.989	0.507	0.993	0.507	0.985	0.486	0.982	0.486	0.992	0.486
50	-0.3	0.935	0.325	0.936	0.325	0.927	0.320	0.930	0.321	0.927	0.321
	0	0.951	0.325	0.952	0.325	0.945	0.320	0.947	0.320	0.947	0.321
	0.1	0.951	0.325	0.953	0.325	0.944	0.320	0.945	0.320	0.947	0.320
	0.3	0.968	0.325	0.970	0.325	0.963	0.320	0.963	0.321	0.965	0.321
	0.5	0.975	0.325	0.976	0.325	0.972	0.319	0.972	0.320	0.975	0.320
	0.9	0.995	0.325	0.996	0.325	0.995	0.320	0.994	0.320	0.996	0.320
100	-0.3	0.933	0.231	0.934	0.231	0.931	0.229	0.932	0.230	0.934	0.230
	0	0.947	0.231	0.948	0.231	0.945	0.229	0.945	0.230	0.946	0.230
	0.1	0.952	0.231	0.953	0.231	0.947	0.229	0.949	0.230	0.949	0.230
	0.3	0.970	0.231	0.970	0.231	0.967	0.229	0.966	0.230	0.967	0.230
	0.5	0.978	0.231	0.979	0.231	0.975	0.229	0.976	0.230	0.976	0.230
	0.9	0.996	0.231	0.996	0.231	0.996	0.229	0.996	0.229	0.997	0.229

Table A.15. Results of Naive Independence Assumption with: Normal, 2-class Good:Poor, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.827	0.415	0.943	0.532	0.891	0.549	0.915	0.505	0.883	0.498
	0	0.845	0.417	0.952	0.533	0.894	0.546	0.926	0.505	0.890	0.500
	0.1	0.848	0.416	0.956	0.536	0.897	0.549	0.935	0.509	0.895	0.505
	0.3	0.866	0.421	0.962	0.535	0.905	0.551	0.944	0.509	0.908	0.506
	0.5	0.880	0.418	0.974	0.532	0.908	0.548	0.955	0.505	0.912	0.503
	0.9	0.914	0.422	0.992	0.532	0.922	0.550	0.978	0.505	0.937	0.502
20	-0.3	0.877	0.300	0.940	0.416	0.902	0.395	0.928	0.390	0.909	0.384
	0	0.889	0.301	0.952	0.414	0.908	0.397	0.943	0.390	0.919	0.382
	0.1	0.888	0.302	0.954	0.416	0.908	0.397	0.945	0.391	0.917	0.382
	0.3	0.907	0.302	0.964	0.416	0.923	0.398	0.961	0.390	0.932	0.382
	0.5	0.924	0.303	0.975	0.416	0.935	0.400	0.971	0.392	0.941	0.383
	0.9	0.960	0.305	0.994	0.415	0.962	0.402	0.992	0.391	0.966	0.380
50	-0.3	0.917	0.196	0.938	0.257	0.929	0.248	0.933	0.245	0.925	0.243
	0	0.929	0.196	0.954	0.259	0.934	0.250	0.949	0.247	0.938	0.245
	0.1	0.930	0.197	0.954	0.259	0.937	0.250	0.951	0.246	0.940	0.245
	0.3	0.955	0.197	0.969	0.257	0.955	0.249	0.968	0.245	0.959	0.243
	0.5	0.965	0.198	0.978	0.260	0.966	0.252	0.978	0.247	0.970	0.245
	0.9	0.993	0.198	0.996	0.259	0.993	0.252	0.997	0.248	0.994	0.244
100	-0.3	0.924	0.141	0.938	0.151	0.921	0.153	0.933	0.149	0.924	0.148
	0	0.936	0.141	0.949	0.150	0.933	0.152	0.948	0.148	0.940	0.147
	0.1	0.943	0.141	0.954	0.150	0.937	0.152	0.953	0.148	0.943	0.147
	0.3	0.960	0.141	0.966	0.151	0.955	0.153	0.967	0.149	0.960	0.148
	0.5	0.975	0.142	0.980	0.152	0.968	0.154	0.982	0.150	0.975	0.149
	0.9	0.998	0.142	0.999	0.151	0.996	0.153	0.999	0.149	0.998	0.147

Table A.16. Results of Naive Independence Assumption with: Normal, 2-class Good:Poor, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.860	0.641	0.996	1.349	0.978	0.937	0.978	0.955	0.924	1.000
	0	0.865	0.637	0.999	1.344	0.980	0.931	0.980	0.948	0.924	NA
	0.1	0.866	0.636	0.997	1.347	0.979	0.934	0.983	0.952	0.928	NA
	0.3	0.866	0.640	0.998	1.346	0.979	0.932	0.985	0.945	0.934	NA
	0.5	0.881	0.637	1.000	1.343	0.986	0.930	0.992	0.946	0.955	NA
	0.9	0.897	0.635	1.000	1.351	0.992	0.919	1.000	0.953	0.995	0.983
20	-0.3	0.850	0.454	0.993	1.224	0.939	0.440	0.943	0.402	0.905	NA
	0	0.854	0.453	0.995	1.220	0.940	0.436	0.945	0.397	0.909	NA
	0.1	0.862	0.453	0.996	1.233	0.945	0.440	0.952	0.401	0.912	NA
	0.3	0.863	0.452	0.994	1.228	0.948	0.437	0.959	0.398	0.919	NA
	0.5	0.872	0.452	0.998	1.230	0.965	0.436	0.974	0.396	0.945	NA
	0.9	0.906	0.452	1.000	1.230	0.999	0.434	1.000	0.398	0.998	0.400
50	-0.3	0.910	0.294	0.992	1.060	0.940	0.271	0.935	0.247	0.911	NA
	0	0.918	0.295	0.994	1.051	0.945	0.271	0.946	0.247	0.921	NA
	0.1	0.914	0.294	0.995	1.057	0.954	0.271	0.949	0.247	0.931	NA
	0.3	0.922	0.295	0.996	1.058	0.957	0.271	0.956	0.248	0.934	NA
	0.5	0.934	0.294	0.999	1.049	0.977	0.271	0.978	0.247	0.964	0.248
	0.9	0.953	0.294	1.000	1.060	0.999	0.270	1.000	0.246	0.999	NA
100	-0.3	0.936	0.210	0.988	0.740	0.959	0.208	0.941	0.181	0.926	NA
	0	0.940	0.211	0.989	0.726	0.957	0.209	0.943	0.182	0.928	0.182
	0.1	0.941	0.210	0.990	0.724	0.962	0.208	0.944	0.181	0.929	0.182
	0.3	0.950	0.210	0.993	0.728	0.969	0.208	0.956	0.181	0.945	NA
	0.5	0.960	0.210	0.998	0.733	0.983	0.208	0.977	0.181	0.967	NA
	0.9	0.972	0.210	1.000	0.725	0.999	0.206	1.000	0.181	0.999	NA

Table A.17. Results of Naive Independence Assumption with: Gamma, 3-class, Good:Good, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.828	0.366	0.997	0.817	0.977	0.539	0.966	0.492	0.876	0.554
	0	0.834	0.367	0.998	0.812	0.979	0.538	0.967	0.493	0.883	0.559
	0.1	0.833	0.366	0.998	0.817	0.979	0.540	0.971	0.494	0.884	0.553
	0.3	0.829	0.367	0.998	0.816	0.979	0.542	0.976	0.494	0.890	0.560
	0.5	0.842	0.363	0.999	0.817	0.987	0.539	0.988	0.493	0.918	0.553
	0.9	0.885	0.359	1.000	0.812	0.997	0.524	1.000	0.491	0.996	0.519
20	-0.3	0.911	0.264	0.991	0.504	0.909	0.220	0.942	0.207	0.877	0.212
	0	0.911	0.263	0.991	0.506	0.914	0.219	0.944	0.207	0.882	0.212
	0.1	0.911	0.263	0.991	0.502	0.920	0.220	0.948	0.207	0.890	0.211
	0.3	0.920	0.263	0.992	0.501	0.921	0.219	0.951	0.206	0.891	0.210
	0.5	0.923	0.261	0.996	0.504	0.949	0.219	0.973	0.207	0.929	0.210
	0.9	0.949	0.259	1.000	0.497	0.999	0.216	1.000	0.205	0.998	0.206
50	-0.3	0.972	0.168	0.977	0.215	0.914	0.137	0.938	0.137	0.903	0.138
	0	0.972	0.168	0.979	0.215	0.924	0.137	0.943	0.137	0.913	0.138
	0.1	0.973	0.168	0.978	0.216	0.922	0.137	0.944	0.137	0.913	0.138
	0.3	0.977	0.168	0.982	0.214	0.926	0.137	0.951	0.137	0.918	0.138
	0.5	0.986	0.168	0.991	0.214	0.957	0.137	0.972	0.137	0.947	0.137
	0.9	0.996	0.168	1.000	0.212	1.000	0.136	1.000	0.136	1.000	0.137
100	-0.3	0.973	0.119	0.971	0.124	0.931	0.102	0.942	0.102	0.921	0.102
	0	0.974	0.119	0.972	0.124	0.933	0.102	0.945	0.102	0.923	0.102
	0.1	0.977	0.119	0.975	0.124	0.935	0.101	0.947	0.102	0.926	0.102
	0.3	0.977	0.119	0.976	0.124	0.940	0.101	0.951	0.101	0.932	0.102
	0.5	0.990	0.119	0.990	0.124	0.966	0.101	0.974	0.101	0.960	0.102
	0.9	1.000	0.119	1.000	0.124	1.000	0.101	1.000	0.101	1.000	0.101

Table A.18. Results of Naive Independence Assumption with: Gamma, 3-class, Good:Good, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.520	0.789	0.920	1.256	0.745	1.061	0.720	1.062	0.741	1.101
	0	0.515	0.781	0.930	1.257	0.744	1.057	0.735	1.060	0.742	1.090
	0.1	0.523	0.887	0.930	1.254	0.743	1.057	0.730	1.056	0.736	1.090
	0.3	0.525	0.769	0.936	1.257	0.756	1.058	0.745	1.060	0.745	NA
	0.5	0.545	0.835	0.952	1.262	0.771	1.062	0.766	1.064	0.748	1.102
	0.9	0.577	0.787	0.975	1.258	0.808	1.047	0.803	1.055	0.754	1.090
20	-0.3	0.278	0.545	0.818	1.074	0.425	0.635	0.275	0.632	0.277	NA
	0	0.276	0.544	0.831	1.076	0.412	0.634	0.268	0.631	0.271	0.637
	0.1	0.276	0.543	0.839	1.076	0.416	0.634	0.266	0.631	0.272	NA
	0.3	0.261	0.560	0.846	1.072	0.406	0.634	0.258	0.632	0.264	NA
	0.5	0.256	0.560	0.840	1.075	0.400	0.634	0.236	0.632	0.249	NA
	0.9	0.181	0.568	0.848	1.081	0.344	0.633	0.131	0.632	0.152	0.649
50	-0.3	0.029	0.346	0.639	0.771	0.050	0.393	0.020	0.369	0.023	NA
	0	0.023	0.346	0.637	0.770	0.045	0.393	0.016	0.369	0.019	0.373
	0.1	0.022	0.346	0.649	0.775	0.044	0.392	0.015	0.369	0.018	0.373
	0.3	0.015	0.346	0.632	0.764	0.037	0.392	0.011	0.369	0.013	NA
	0.5	0.009	0.346	0.627	0.767	0.026	0.393	0.006	0.370	0.008	0.374
	0.9	0.000	0.346	0.617	0.769	0.004	0.392	0.000	0.369	0.000	NA
100	-0.3	0.001	0.246	0.401	0.512	0.002	0.307	0.001	0.268	0.001	0.271
	0	0.000	0.246	0.385	0.504	0.001	0.307	0.000	0.268	0.000	0.271
	0.1	0.000	0.246	0.396	0.508	0.001	0.307	0.000	0.269	0.000	NA
	0.3	0.000	0.246	0.400	0.511	0.001	0.307	0.000	0.268	0.000	NA
	0.5	0.000	0.246	0.395	0.509	0.000	0.307	0.000	0.268	0.000	0.270
	0.9	0.000	0.246	0.401	0.512	0.000	0.306	0.000	0.267	0.000	NA

Table A.19. Results of Naive Independence Assumption with: Gamma, 3-class Good:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.270	0.390	0.754	0.867	0.607	0.598	0.407	0.585	0.446	0.630
	0	0.268	0.389	0.753	0.864	0.602	0.597	0.404	0.585	0.439	0.625
	0.1	0.267	0.389	0.751	0.865	0.602	0.599	0.411	0.584	0.445	0.627
	0.3	0.258	0.389	0.745	0.868	0.600	0.598	0.399	0.585	0.437	0.629
	0.5	0.253	0.389	0.745	0.865	0.611	0.596	0.390	0.584	0.428	0.623
	0.9	0.228	0.382	0.718	0.863	0.572	0.586	0.323	0.580	0.361	0.615
20	-0.3	0.083	0.281	0.342	0.583	0.122	0.289	0.051	0.278	0.074	0.280
	0	0.080	0.281	0.338	0.582	0.121	0.290	0.051	0.278	0.072	0.280
	0.1	0.075	0.281	0.332	0.581	0.119	0.289	0.048	0.278	0.071	0.279
	0.3	0.077	0.281	0.330	0.578	0.121	0.291	0.046	0.280	0.070	0.281
	0.5	0.062	0.280	0.307	0.577	0.099	0.288	0.033	0.277	0.054	0.282
	0.9	0.032	0.279	0.251	0.574	0.060	0.287	0.011	0.277	0.024	0.279
50	-0.3	0.001	0.182	0.035	0.276	0.001	0.174	0.001	0.173	0.002	0.174
	0	0.002	0.182	0.036	0.276	0.002	0.174	0.001	0.173	0.002	0.174
	0.1	0.001	0.182	0.037	0.276	0.001	0.174	0.001	0.173	0.001	0.174
	0.3	0.001	0.182	0.033	0.277	0.001	0.174	0.000	0.173	0.001	0.174
	0.5	0.001	0.181	0.034	0.277	0.001	0.173	0.000	0.172	0.001	0.173
	0.9	0.000	0.181	0.028	0.271	0.000	0.174	0.000	0.173	0.000	0.173
100	-0.3	0.000	0.129	0.002	0.146	0.000	0.127	0.000	0.127	0.000	0.128
	0	0.000	0.129	0.001	0.145	0.000	0.127	0.000	0.127	0.000	0.128
	0.1	0.000	0.129	0.001	0.146	0.000	0.128	0.000	0.127	0.000	0.128
	0.3	0.000	0.129	0.001	0.146	0.000	0.127	0.000	0.127	0.000	0.128
	0.5	0.000	0.129	0.001	0.146	0.000	0.127	0.000	0.127	0.000	0.128
	0.9	0.000	0.129	0.001	0.145	0.000	0.127	0.000	0.127	0.000	0.128

Table A.20. Results of Naive Independence Assumption with: Gamma, 3-class Good:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.809	0.899	0.996	1.219	0.877	1.163	0.965	1.156	0.941	1.169
	0	0.837	0.915	0.998	1.223	0.900	1.164	0.978	1.159	0.951	1.162
	0.1	0.848	0.936	0.999	1.223	0.910	1.163	0.985	1.159	0.957	1.160
	0.3	0.863	1.053	0.999	1.224	0.926	1.162	0.992	1.163	0.964	1.162
	0.5	0.889	0.963	1.000	1.222	0.941	1.157	0.997	1.161	0.971	1.158
	0.9	0.935	0.914	1.000	1.235	0.968	1.159	1.000	1.179	0.984	1.173
20	-0.3	0.871	0.622	0.992	0.932	0.914	0.785	0.941	0.801	0.924	NA
	0	0.900	0.627	0.998	0.935	0.936	0.783	0.962	0.799	0.944	0.794
	0.1	0.906	0.619	0.997	0.934	0.942	0.782	0.970	0.799	0.949	0.792
	0.3	0.930	0.625	0.999	0.937	0.953	0.784	0.981	0.800	0.959	NA
	0.5	0.950	0.626	1.000	0.938	0.966	0.780	0.993	0.797	0.970	NA
	0.9	0.979	0.617	1.000	0.942	0.983	0.778	1.000	0.801	0.985	0.795
50	-0.3	0.889	0.391	0.950	0.482	0.931	0.484	0.932	0.469	0.909	NA
	0	0.911	0.391	0.964	0.483	0.944	0.485	0.954	0.470	0.927	NA
	0.1	0.916	0.391	0.968	0.484	0.950	0.484	0.958	0.469	0.934	NA
	0.3	0.944	0.391	0.983	0.484	0.960	0.483	0.975	0.467	0.948	NA
	0.5	0.972	0.391	0.993	0.484	0.974	0.484	0.990	0.468	0.968	NA
	0.9	0.999	0.391	1.000	0.484	0.986	0.484	1.000	0.468	0.986	0.463
100	-0.3	0.883	0.277	0.899	0.282	0.943	0.381	0.928	0.341	0.908	0.337
	0	0.912	0.277	0.924	0.282	0.959	0.381	0.955	0.342	0.935	NA
	0.1	0.917	0.277	0.929	0.282	0.961	0.380	0.960	0.342	0.935	NA
	0.3	0.942	0.277	0.948	0.282	0.969	0.381	0.974	0.342	0.950	0.338
	0.5	0.969	0.277	0.974	0.282	0.977	0.381	0.990	0.342	0.968	0.338
	0.9	1.000	0.277	1.000	0.282	0.980	0.381	1.000	0.342	0.980	0.337

Table A.21. Results of Naive Independence Assumption with:Gamma, 3-class Fair:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.695	0.411	0.998	0.916	0.952	0.651	0.953	0.659	0.926	0.685
	0	0.716	0.412	0.998	0.918	0.960	0.649	0.972	0.659	0.938	0.677
	0.1	0.720	0.411	0.998	0.915	0.962	0.649	0.975	0.657	0.940	0.680
	0.3	0.722	0.410	0.999	0.921	0.965	0.649	0.984	0.660	0.948	0.684
	0.5	0.748	0.409	0.999	0.918	0.976	0.644	0.992	0.656	0.966	0.676
	0.9	0.842	0.405	1.000	0.915	0.991	0.635	1.000	0.656	0.997	0.671
20	-0.3	0.777	0.299	0.989	0.659	0.917	0.345	0.925	0.338	0.895	0.337
	0	0.788	0.299	0.993	0.659	0.928	0.346	0.945	0.339	0.909	0.338
	0.1	0.799	0.299	0.994	0.660	0.933	0.347	0.953	0.340	0.915	0.339
	0.3	0.814	0.299	0.996	0.656	0.944	0.345	0.964	0.338	0.927	0.338
	0.5	0.829	0.297	0.997	0.659	0.969	0.345	0.981	0.339	0.960	0.338
	0.9	0.900	0.296	1.000	0.658	1.000	0.343	1.000	0.338	1.000	0.339
50	-0.3	0.885	0.194	0.963	0.337	0.910	0.204	0.925	0.203	0.902	0.204
	0	0.896	0.194	0.966	0.332	0.923	0.204	0.939	0.203	0.918	0.204
	0.1	0.901	0.194	0.971	0.336	0.927	0.204	0.944	0.203	0.923	0.204
	0.3	0.918	0.194	0.978	0.335	0.943	0.204	0.957	0.203	0.938	NA
	0.5	0.942	0.194	0.989	0.334	0.967	0.204	0.977	0.203	0.965	0.204
	0.9	0.980	0.194	1.000	0.331	1.000	0.203	1.000	0.202	1.000	0.202
100	-0.3	0.904	0.138	0.927	0.167	0.923	0.149	0.929	0.149	0.919	0.149
	0	0.914	0.138	0.934	0.165	0.931	0.149	0.940	0.149	0.925	0.150
	0.1	0.925	0.138	0.943	0.168	0.941	0.149	0.946	0.149	0.933	0.149
	0.3	0.936	0.138	0.954	0.168	0.954	0.149	0.960	0.149	0.947	0.150
	0.5	0.964	0.138	0.973	0.167	0.975	0.149	0.980	0.149	0.970	0.149
	0.9	0.999	0.138	1.000	0.167	1.000	0.149	1.000	0.149	1.000	0.149

Table A.22. Results of Naive Independence Assumption with:Gamma, 3-class Fair:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.197	15.647	0.608	1.352	0.351	1.169	0.425	1.213	0.393	1.228
	0	0.193	3.328	0.620	1.352	0.358	1.171	0.426	1.215	0.398	NA
	0.1	0.188	27.779	0.617	1.355	0.351	1.174	0.421	1.220	0.397	1.229
	0.3	0.194	5.221	0.615	1.354	0.358	1.170	0.431	1.219	0.396	1.228
	0.5	0.171	7.245	0.617	1.357	0.342	1.173	0.410	1.219	0.379	NA
	0.9	0.124	13.852	0.602	1.357	0.317	1.164	0.391	1.220	0.344	1.238
20	-0.3	0.070	0.834	0.525	1.092	0.051	0.695	0.033	0.698	0.029	0.711
	0	0.068	0.636	0.516	1.090	0.049	0.697	0.032	0.700	0.028	NA
	0.1	0.067	2.108	0.512	1.085	0.050	0.694	0.028	0.696	0.027	0.709
	0.3	0.061	1.577	0.511	1.091	0.051	0.694	0.029	0.697	0.031	NA
	0.5	0.062	1.381	0.506	1.089	0.044	0.696	0.023	0.698	0.027	0.715
	0.9	0.050	0.835	0.510	1.090	0.028	0.694	0.016	0.698	0.030	NA
50	-0.3	0.024	0.366	0.491	0.784	0.001	0.404	0.000	0.389	0.000	0.391
	0	0.023	0.365	0.494	0.786	0.000	0.404	0.000	0.389	0.000	0.390
	0.1	0.025	0.365	0.493	0.783	0.000	0.404	0.000	0.389	0.000	0.390
	0.3	0.020	0.367	0.483	0.781	0.000	0.404	0.000	0.390	0.000	NA
	0.5	0.018	0.366	0.494	0.782	0.000	0.404	0.000	0.390	0.000	0.390
	0.9	0.013	0.373	0.499	0.781	0.000	0.402	0.000	0.388	0.000	NA
100	-0.3	0.012	0.259	0.308	0.517	0.000	0.290	0.000	0.279	0.000	NA
	0	0.008	0.259	0.312	0.516	0.000	0.290	0.000	0.279	0.000	0.279
	0.1	0.008	0.259	0.316	0.516	0.000	0.290	0.000	0.279	0.000	NA
	0.3	0.008	0.259	0.326	0.522	0.000	0.290	0.000	0.279	0.000	NA
	0.5	0.007	0.259	0.329	0.519	0.000	0.290	0.000	0.280	0.000	0.280
	0.9	0.006	0.259	0.329	0.515	0.000	0.290	0.000	0.279	0.000	0.279

Table A.23. Results of Naive Independence Assumption with: Gamma, 3-class Good:Poor, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.157	0.632	0.403	0.971	0.261	0.856	0.247	0.837	0.189	0.852
	0	0.153	0.629	0.401	0.971	0.262	0.856	0.237	0.834	0.190	0.850
	0.1	0.156	0.630	0.402	0.968	0.251	0.853	0.242	0.836	0.184	0.851
	0.3	0.150	0.629	0.406	0.973	0.264	0.859	0.234	0.840	0.188	0.856
	0.5	0.138	0.631	0.394	0.971	0.254	0.854	0.224	0.836	0.177	0.857
	0.9	0.087	0.629	0.361	0.970	0.225	0.850	0.194	0.836	0.164	0.867
20	-0.3	0.039	0.470	0.154	0.696	0.015	0.513	0.007	0.512	0.003	0.525
	0	0.035	0.471	0.150	0.691	0.015	0.514	0.006	0.515	0.004	0.527
	0.1	0.035	0.470	0.146	0.691	0.015	0.515	0.007	0.515	0.005	0.527
	0.3	0.035	0.472	0.154	0.691	0.015	0.516	0.007	0.516	0.005	0.529
	0.5	0.030	0.470	0.155	0.690	0.010	0.513	0.004	0.513	0.003	0.526
	0.9	0.019	0.471	0.161	0.690	0.009	0.512	0.002	0.513	0.004	0.529
50	-0.3	0.002	0.310	0.021	0.385	0.000	0.331	0.000	0.327	0.000	0.334
	0	0.002	0.310	0.021	0.383	0.000	0.331	0.000	0.327	0.000	0.333
	0.1	0.003	0.310	0.024	0.383	0.000	0.332	0.000	0.328	0.000	0.335
	0.3	0.003	0.310	0.024	0.384	0.000	0.331	0.000	0.327	0.000	0.334
	0.5	0.002	0.310	0.025	0.383	0.000	0.331	0.000	0.327	0.000	0.334
	0.9	0.001	0.310	0.023	0.382	0.000	0.331	0.000	0.327	0.000	0.334
100	-0.3	0.000	0.221	0.001	0.235	0.000	0.246	0.000	0.240	0.000	0.245
	0	0.000	0.221	0.001	0.234	0.000	0.246	0.000	0.240	0.000	0.245
	0.1	0.000	0.221	0.001	0.234	0.000	0.246	0.000	0.240	0.000	0.246
	0.3	0.000	0.221	0.001	0.234	0.000	0.247	0.000	0.241	0.000	0.246
	0.5	0.000	0.221	0.001	0.234	0.000	0.247	0.000	0.241	0.000	0.246
	0.9	0.000	0.221	0.001	0.235	0.000	0.246	0.000	0.240	0.000	0.246

Table A.24. Results of Naive Independence Assumption with: Gamma, 3-class Good:Poor, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.993	0.280	0.950	0.387	0.947	0.334	0.931	0.263	0.735	0.403
	0	0.994	0.279	0.957	0.388	0.947	0.335	0.936	0.262	0.738	0.398
	0.1	0.996	0.279	0.958	0.390	0.950	0.336	0.941	0.268	0.758	0.393
	0.3	0.997	0.278	0.970	0.390	0.958	0.335	0.949	0.266	0.771	0.389
	0.5	0.998	0.274	0.983	0.387	0.974	0.330	0.970	0.261	0.820	0.369
	0.9	1.000	0.264	1.000	0.382	0.999	0.317	0.999	0.258	0.974	0.309
20	-0.3	0.980	0.200	0.942	0.232	0.905	0.174	0.929	0.172	0.805	0.185
	0	0.982	0.200	0.945	0.232	0.906	0.174	0.932	0.172	0.820	0.184
	0.1	0.985	0.200	0.953	0.232	0.915	0.173	0.939	0.171	0.826	0.183
	0.3	0.989	0.199	0.959	0.231	0.925	0.172	0.950	0.171	0.839	0.182
	0.5	0.995	0.198	0.976	0.231	0.951	0.172	0.968	0.170	0.877	0.180
	0.9	1.000	0.193	1.000	0.228	1.000	0.167	1.000	0.167	0.993	0.171
50	-0.3	0.950	0.127	0.931	0.136	0.922	0.122	0.933	0.122	0.871	0.126
	0	0.957	0.127	0.939	0.136	0.931	0.122	0.940	0.122	0.878	0.126
	0.1	0.959	0.127	0.941	0.136	0.932	0.121	0.942	0.122	0.883	0.125
	0.3	0.968	0.127	0.954	0.136	0.946	0.122	0.954	0.122	0.901	0.125
	0.5	0.983	0.127	0.973	0.136	0.966	0.121	0.973	0.122	0.933	0.125
	0.9	1.000	0.125	1.000	0.134	1.000	0.119	1.000	0.119	0.999	0.120
100	-0.3	0.941	0.090	0.930	0.093	0.936	0.091	0.940	0.091	0.902	0.092
	0	0.946	0.090	0.935	0.094	0.937	0.091	0.943	0.091	0.909	0.092
	0.1	0.947	0.090	0.939	0.093	0.939	0.091	0.944	0.091	0.909	0.092
	0.3	0.958	0.090	0.951	0.093	0.953	0.091	0.957	0.091	0.930	0.092
	0.5	0.973	0.090	0.968	0.093	0.971	0.091	0.974	0.091	0.949	0.092
	0.9	1.000	0.090	1.000	0.093	1.000	0.090	1.000	0.090	1.000	0.091

Table A.25. Results of Naive Independence Assumption with: Gamma, 2-class, Good:Good, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.988	0.175	0.951	0.259	0.978	0.408	0.928	0.211	0.751	0.394
	0	0.989	0.174	0.953	0.257	0.977	0.403	0.929	0.204	0.747	0.375
	0.1	0.990	0.174	0.959	0.255	0.977	0.408	0.943	0.212	0.756	0.368
	0.3	0.994	0.171	0.969	0.254	0.984	0.403	0.951	0.212	0.786	NA
	0.5	0.997	0.170	0.982	0.253	0.988	0.403	0.970	0.212	0.825	0.339
	0.9	1.000	0.165	1.000	0.251	0.999	0.385	0.999	0.200	0.975	0.257
20	-0.3	0.967	0.122	0.932	0.143	0.921	0.119	0.927	0.115	0.811	NA
	0	0.971	0.123	0.936	0.144	0.923	0.119	0.931	0.115	0.811	NA
	0.1	0.975	0.122	0.943	0.143	0.929	0.119	0.941	0.114	0.825	0.127
	0.3	0.983	0.122	0.955	0.142	0.939	0.118	0.948	0.113	0.841	0.125
	0.5	0.989	0.121	0.970	0.142	0.957	0.117	0.965	0.113	0.874	0.124
	0.9	1.000	0.119	1.000	0.141	1.000	0.114	1.000	0.111	0.993	0.115
50	-0.3	0.935	0.078	0.918	0.083	0.929	0.081	0.932	0.081	0.866	0.085
	0	0.936	0.078	0.919	0.082	0.935	0.080	0.937	0.080	0.872	0.085
	0.1	0.941	0.078	0.925	0.082	0.937	0.081	0.939	0.081	0.877	0.085
	0.3	0.954	0.078	0.938	0.082	0.949	0.080	0.953	0.080	0.893	0.084
	0.5	0.976	0.077	0.965	0.082	0.971	0.080	0.973	0.080	0.929	0.083
	0.9	1.000	0.077	1.000	0.082	1.000	0.078	1.000	0.079	0.999	0.080
100	-0.3	0.921	0.055	0.912	0.057	0.938	0.060	0.937	0.060	0.896	0.061
	0	0.928	0.055	0.920	0.057	0.944	0.059	0.944	0.060	0.903	0.061
	0.1	0.930	0.055	0.922	0.057	0.947	0.060	0.945	0.060	0.908	0.061
	0.3	0.947	0.055	0.939	0.057	0.959	0.060	0.958	0.060	0.928	0.061
	0.5	0.965	0.055	0.958	0.056	0.974	0.059	0.975	0.059	0.949	0.060
	0.9	1.000	0.055	1.000	0.056	1.000	0.059	1.000	0.059	1.000	0.059

Table A.26. Results of Naive Independence Assumption with: Gamma, 2-class, Good:Good, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.684	0.356	0.838	0.467	0.724	0.408	0.579	0.345	0.651	0.482
	0	0.686	0.354	0.831	0.464	0.721	0.404	0.579	0.343	0.650	0.471
	0.1	0.687	0.353	0.846	0.464	0.723	0.404	0.581	0.344	0.652	0.467
	0.3	0.698	0.354	0.858	0.467	0.730	0.405	0.587	0.346	0.668	0.463
	0.5	0.690	0.351	0.861	0.463	0.729	0.402	0.581	0.343	0.668	0.460
	0.9	0.709	0.342	0.908	0.460	0.749	0.390	0.598	0.332	0.746	0.433
20	-0.3	0.599	0.256	0.700	0.281	0.527	0.226	0.501	0.222	0.579	0.236
	0	0.603	0.256	0.705	0.281	0.530	0.226	0.503	0.222	0.585	0.236
	0.1	0.595	0.255	0.703	0.281	0.518	0.225	0.488	0.222	0.571	0.236
	0.3	0.599	0.255	0.707	0.280	0.527	0.224	0.496	0.221	0.575	0.233
	0.5	0.618	0.255	0.727	0.281	0.536	0.225	0.509	0.222	0.598	0.234
	0.9	0.634	0.251	0.777	0.278	0.533	0.220	0.495	0.217	0.616	0.226
50	-0.3	0.356	0.165	0.411	0.171	0.349	0.158	0.323	0.158	0.403	0.162
	0	0.357	0.164	0.412	0.171	0.350	0.158	0.326	0.158	0.405	0.162
	0.1	0.355	0.164	0.413	0.170	0.345	0.158	0.321	0.157	0.408	0.162
	0.3	0.352	0.164	0.409	0.170	0.339	0.157	0.313	0.157	0.394	0.161
	0.5	0.341	0.164	0.401	0.170	0.329	0.157	0.304	0.157	0.389	0.161
	0.9	0.308	0.163	0.371	0.170	0.293	0.156	0.266	0.156	0.362	0.159
100	-0.3	0.134	0.117	0.154	0.119	0.159	0.117	0.146	0.118	0.202	0.119
	0	0.131	0.117	0.153	0.119	0.152	0.118	0.140	0.118	0.200	0.120
	0.1	0.128	0.117	0.150	0.119	0.153	0.118	0.139	0.118	0.196	0.119
	0.3	0.127	0.117	0.150	0.119	0.149	0.118	0.136	0.118	0.198	0.120
	0.5	0.106	0.117	0.125	0.119	0.132	0.117	0.117	0.117	0.177	0.119
	0.9	0.045	0.116	0.056	0.118	0.079	0.116	0.069	0.116	0.117	0.118

Table A.27. Results of Naive Independence Assumption with: Gamma, 2-class Good:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.542	0.219	0.740	0.303	0.811	0.465	0.452	0.271	0.554	0.446
	0	0.544	0.217	0.736	0.301	0.809	0.465	0.451	0.266	0.557	0.434
	0.1	0.540	0.216	0.735	0.301	0.813	0.465	0.446	0.272	0.553	0.434
	0.3	0.545	0.217	0.753	0.302	0.806	0.460	0.444	0.269	0.557	0.407
	0.5	0.543	0.215	0.760	0.300	0.810	0.461	0.446	0.270	0.566	0.415
	0.9	0.572	0.210	0.816	0.298	0.806	0.448	0.452	0.265	0.612	0.380
20	-0.3	0.405	0.156	0.507	0.174	0.399	0.156	0.361	0.150	0.455	0.165
	0	0.409	0.156	0.508	0.174	0.405	0.155	0.367	0.150	0.453	0.165
	0.1	0.402	0.155	0.500	0.173	0.398	0.155	0.360	0.149	0.448	0.165
	0.3	0.395	0.155	0.495	0.173	0.392	0.155	0.355	0.149	0.443	0.165
	0.5	0.394	0.155	0.497	0.173	0.384	0.154	0.348	0.148	0.441	0.162
	0.9	0.381	0.152	0.513	0.171	0.366	0.151	0.330	0.146	0.432	0.155
50	-0.3	0.158	0.100	0.188	0.103	0.203	0.106	0.194	0.105	0.278	0.111
	0	0.151	0.100	0.180	0.103	0.196	0.105	0.189	0.105	0.267	0.110
	0.1	0.154	0.100	0.180	0.103	0.199	0.105	0.190	0.105	0.268	0.111
	0.3	0.135	0.099	0.163	0.103	0.183	0.105	0.174	0.104	0.252	0.110
	0.5	0.129	0.099	0.152	0.103	0.177	0.105	0.171	0.105	0.245	0.109
	0.9	0.073	0.099	0.093	0.102	0.138	0.103	0.133	0.103	0.203	0.106
100	-0.3	0.026	0.071	0.030	0.072	0.059	0.078	0.056	0.078	0.104	0.080
	0	0.026	0.071	0.030	0.072	0.059	0.078	0.058	0.078	0.105	0.080
	0.1	0.025	0.071	0.030	0.072	0.056	0.078	0.057	0.078	0.102	0.080
	0.3	0.023	0.071	0.027	0.072	0.056	0.078	0.057	0.078	0.098	0.080
	0.5	0.018	0.071	0.021	0.072	0.048	0.078	0.047	0.078	0.091	0.080
	0.9	0.003	0.070	0.003	0.072	0.026	0.077	0.028	0.077	0.056	0.079

Table A.28. Results of Naive Independence Assumption with: Gamma, 2-class Good:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.975	0.418	0.952	0.534	0.927	0.468	0.925	0.414	0.780	0.505
	0	0.976	0.417	0.957	0.534	0.930	0.465	0.935	0.413	0.797	0.498
	0.1	0.982	0.417	0.964	0.535	0.937	0.464	0.940	0.413	0.805	0.492
	0.3	0.988	0.415	0.973	0.534	0.947	0.462	0.958	0.413	0.835	0.484
	0.5	0.994	0.413	0.985	0.534	0.964	0.462	0.972	0.413	0.866	0.478
	0.9	1.000	0.402	1.000	0.529	0.999	0.451	1.000	0.407	0.984	0.440
20	-0.3	0.953	0.301	0.941	0.324	0.889	0.269	0.926	0.265	0.840	0.274
	0	0.962	0.302	0.951	0.325	0.901	0.269	0.934	0.265	0.853	0.274
	0.1	0.961	0.303	0.947	0.325	0.899	0.271	0.935	0.266	0.852	0.275
	0.3	0.974	0.302	0.965	0.325	0.920	0.269	0.954	0.265	0.879	0.273
	0.5	0.986	0.300	0.980	0.324	0.950	0.268	0.975	0.264	0.916	0.272
	0.9	1.000	0.297	1.000	0.322	0.999	0.262	1.000	0.260	0.997	0.263
50	-0.3	0.932	0.195	0.926	0.200	0.905	0.188	0.924	0.188	0.876	0.191
	0	0.943	0.195	0.939	0.200	0.923	0.188	0.939	0.188	0.898	0.190
	0.1	0.947	0.195	0.944	0.200	0.927	0.188	0.943	0.188	0.904	0.190
	0.3	0.963	0.195	0.960	0.200	0.945	0.187	0.960	0.187	0.923	0.190
	0.5	0.976	0.194	0.974	0.200	0.961	0.187	0.973	0.187	0.945	0.189
	0.9	1.000	0.194	1.000	0.200	1.000	0.186	1.000	0.186	1.000	0.187
100	-0.3	0.929	0.139	0.928	0.140	0.925	0.139	0.934	0.139	0.908	0.141
	0	0.935	0.139	0.934	0.141	0.933	0.140	0.941	0.140	0.915	0.141
	0.1	0.942	0.139	0.939	0.140	0.935	0.139	0.945	0.140	0.920	0.141
	0.3	0.952	0.139	0.952	0.140	0.950	0.140	0.958	0.140	0.933	0.141
	0.5	0.976	0.139	0.975	0.141	0.973	0.140	0.979	0.140	0.963	0.141
	0.9	1.000	0.138	1.000	0.140	1.000	0.139	1.000	0.139	1.000	0.139

Table A.29. Results of Naive Independence Assumption with: Gamma, 2-class Fair:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.959	0.256	0.947	0.341	0.975	0.516	0.925	0.323	0.785	0.440
	0	0.966	0.255	0.952	0.345	0.977	0.520	0.936	0.328	0.806	0.421
	0.1	0.970	0.255	0.958	0.341	0.979	0.508	0.941	0.316	0.818	0.412
	0.3	0.977	0.255	0.966	0.342	0.982	0.514	0.953	0.327	0.837	0.408
	0.5	0.987	0.253	0.982	0.338	0.988	0.507	0.972	0.319	0.876	0.392
	0.9	0.999	0.246	1.000	0.337	0.999	0.488	1.000	0.311	0.989	0.344
20	-0.3	0.932	0.184	0.925	0.201	0.913	0.187	0.926	0.181	0.839	0.194
	0	0.937	0.183	0.931	0.200	0.915	0.185	0.929	0.179	0.848	0.192
	0.1	0.948	0.183	0.939	0.201	0.927	0.186	0.941	0.180	0.863	0.192
	0.3	0.958	0.183	0.953	0.200	0.942	0.186	0.953	0.179	0.887	0.191
	0.5	0.976	0.182	0.969	0.200	0.961	0.184	0.971	0.178	0.918	0.189
	0.9	1.000	0.181	1.000	0.199	0.999	0.181	1.000	0.176	0.997	0.180
50	-0.3	0.906	0.117	0.902	0.120	0.920	0.126	0.926	0.126	0.879	0.130
	0	0.916	0.117	0.912	0.120	0.930	0.126	0.935	0.125	0.888	0.130
	0.1	0.927	0.117	0.923	0.120	0.938	0.125	0.945	0.125	0.900	0.129
	0.3	0.943	0.117	0.939	0.120	0.950	0.125	0.956	0.125	0.918	0.129
	0.5	0.963	0.117	0.960	0.120	0.969	0.125	0.974	0.125	0.942	0.129
	0.9	1.000	0.116	1.000	0.119	1.000	0.123	1.000	0.123	1.000	0.124
100	-0.3	0.901	0.083	0.899	0.084	0.931	0.093	0.932	0.093	0.903	0.095
	0	0.907	0.083	0.905	0.084	0.934	0.093	0.935	0.093	0.905	0.095
	0.1	0.917	0.083	0.914	0.084	0.943	0.093	0.948	0.093	0.914	0.095
	0.3	0.934	0.083	0.932	0.084	0.955	0.093	0.959	0.093	0.933	0.094
	0.5	0.958	0.083	0.957	0.084	0.974	0.092	0.976	0.092	0.958	0.094
	0.9	1.000	0.083	1.000	0.084	1.000	0.091	1.000	0.092	1.000	0.092

Table A.30. Results of Naive Independence Assumption with: Gamma, 2-class Fair:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.876	0.595	0.938	0.716	0.876	0.692	0.898	0.672	0.902	0.721
	0	0.888	0.594	0.949	0.717	0.891	0.694	0.908	0.673	0.922	0.724
	0.1	0.892	0.595	0.952	0.714	0.892	0.691	0.910	0.670	0.924	0.726
	0.3	0.900	0.594	0.958	0.713	0.899	0.689	0.914	0.669	0.932	0.728
	0.5	0.914	0.595	0.968	0.715	0.912	0.689	0.920	0.671	0.944	0.733
	0.9	0.938	0.593	0.981	0.715	0.936	0.687	0.934	0.670	0.973	0.752
20	-0.3	0.881	0.432	0.904	0.457	0.878	0.466	0.917	0.459	0.894	0.474
	0	0.889	0.433	0.908	0.458	0.883	0.465	0.921	0.459	0.899	0.473
	0.1	0.886	0.433	0.912	0.458	0.882	0.466	0.920	0.459	0.902	0.474
	0.3	0.899	0.433	0.923	0.458	0.892	0.467	0.930	0.460	0.912	0.475
	0.5	0.914	0.432	0.938	0.457	0.904	0.465	0.937	0.459	0.924	0.474
	0.9	0.947	0.432	0.971	0.456	0.937	0.464	0.953	0.457	0.953	0.473
50	-0.3	0.823	0.279	0.829	0.281	0.857	0.316	0.906	0.315	0.854	0.318
	0	0.835	0.279	0.845	0.280	0.876	0.316	0.921	0.314	0.870	0.318
	0.1	0.840	0.279	0.848	0.280	0.871	0.316	0.921	0.314	0.870	0.318
	0.3	0.847	0.279	0.856	0.280	0.883	0.316	0.930	0.315	0.881	0.318
	0.5	0.867	0.278	0.876	0.280	0.898	0.315	0.940	0.314	0.897	0.318
	0.9	0.911	0.278	0.921	0.280	0.935	0.315	0.967	0.313	0.940	0.317
100	-0.3	0.741	0.198	0.744	0.199	0.814	0.234	0.861	0.234	0.793	0.236
	0	0.748	0.198	0.751	0.199	0.823	0.234	0.870	0.233	0.801	0.235
	0.1	0.751	0.198	0.755	0.199	0.823	0.233	0.871	0.233	0.802	0.235
	0.3	0.763	0.198	0.768	0.198	0.836	0.233	0.883	0.233	0.814	0.235
	0.5	0.779	0.198	0.783	0.198	0.850	0.233	0.900	0.233	0.833	0.235
	0.9	0.812	0.198	0.817	0.199	0.885	0.233	0.929	0.233	0.871	0.235

Table A.31. Results of Naive Independence Assumption with: Gamma, 2-class Good:Poor, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.656	0.327	0.862	0.419	0.906	0.683	0.817	0.522	0.883	0.633
	0	0.651	0.325	0.861	0.420	0.903	0.684	0.815	0.519	0.886	0.643
	0.1	0.659	0.328	0.868	0.422	0.906	0.684	0.815	0.514	0.888	0.636
	0.3	0.665	0.328	0.877	0.421	0.907	0.684	0.819	0.521	0.902	0.642
	0.5	0.681	0.329	0.884	0.421	0.911	0.683	0.826	0.521	0.911	0.660
	0.9	0.719	0.332	0.906	0.418	0.920	0.666	0.830	0.506	0.940	0.683
20	-0.3	0.668	0.246	0.763	0.258	0.752	0.331	0.805	0.310	0.829	0.307
	0	0.667	0.247	0.767	0.258	0.753	0.331	0.801	0.309	0.831	0.307
	0.1	0.673	0.247	0.775	0.258	0.762	0.329	0.808	0.309	0.834	0.306
	0.3	0.677	0.246	0.775	0.258	0.769	0.331	0.810	0.310	0.843	0.308
	0.5	0.696	0.247	0.795	0.259	0.780	0.331	0.816	0.310	0.855	0.309
	0.9	0.722	0.248	0.824	0.259	0.800	0.328	0.824	0.309	0.869	0.306
50	-0.3	0.649	0.164	0.694	0.174	0.775	0.241	0.819	0.237	0.838	0.236
	0	0.657	0.165	0.701	0.174	0.780	0.241	0.819	0.237	0.841	0.237
	0.1	0.656	0.164	0.700	0.174	0.783	0.241	0.826	0.236	0.839	0.236
	0.3	0.663	0.164	0.714	0.174	0.789	0.242	0.827	0.237	0.844	0.236
	0.5	0.677	0.165	0.722	0.175	0.802	0.242	0.836	0.237	0.859	0.237
	0.9	0.694	0.164	0.744	0.174	0.814	0.240	0.845	0.237	0.868	0.237
100	-0.3	0.611	0.118	0.658	0.138	0.789	0.191	0.812	0.193	0.837	0.198
	0	0.604	0.118	0.653	0.138	0.794	0.191	0.816	0.193	0.842	0.198
	0.1	0.601	0.118	0.650	0.137	0.784	0.191	0.806	0.192	0.834	0.197
	0.3	0.613	0.118	0.663	0.137	0.796	0.190	0.813	0.192	0.847	0.197
	0.5	0.622	0.118	0.674	0.138	0.804	0.192	0.824	0.193	0.849	0.199
	0.9	0.640	0.118	0.687	0.137	0.814	0.191	0.832	0.192	0.861	0.198

Table A.32. Results of Naive Independence Assumption with: Gamma, 2-class Good:Poor, Cost 2

Appendix B. Derivation of partial derivatives for Delta Method for test comparisons

$$
\begin{aligned}
\left(\frac{\partial BC}{\partial \mu_1}\right) &= \frac{\partial}{\partial \mu_1}\left[\begin{array}{c}
c_{2|1}p_1\left(\Phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)-\Phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right)+c_{3|1}p_1\left(\Phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right)+\\
c_{1|2}p_2\left(\Phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right)+c_{3|2}p_2\left(\Phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right)+\\
c_{1|3}p_3\left(\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right)+c_{2|3}p_3\left(\Phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)-\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right)
\end{array}\right]\\
&= \left[\begin{array}{c}
c_{2|1}p_1\frac{\partial}{\partial\mu_1}\left[\Phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\right]-c_{2|1}p_1\frac{\partial}{\partial\mu_1}\left[\Phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right]+c_{3|1}p_1\frac{\partial}{\partial\mu_1}\left[\Phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right]+\\
c_{1|2}p_2\frac{\partial}{\partial\mu_1}\left[\Phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right]+c_{3|2}p_2\frac{\partial}{\partial\mu_1}\left[\Phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right]+\\
c_{1|3}p_3\frac{\partial}{\partial\mu_1}\left[\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]+c_{2|3}p_3\frac{\partial}{\partial\mu_1}\left[\Phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\right]-c_{2|3}p_3\frac{\partial}{\partial\mu_1}\left[\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]
\end{array}\right]\\
&= \left[\begin{array}{c}
c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\frac{\partial}{\partial\mu_1}\left[\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\right]-c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\frac{\partial}{\partial\mu_1}\left[\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right]+\\
c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\frac{\partial}{\partial\mu_1}\left[\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right]+c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\frac{\partial}{\partial\mu_1}\left[\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right]+\\
c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\frac{\partial}{\partial\mu_1}\left[\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right]+c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial\mu_1}\left[\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]+\\
c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial\mu_1}\left[\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\right]-c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial\mu_1}\left[\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]
\end{array}\right]\\
&= \left[\begin{array}{c}
c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_2}{\partial\mu_1}\sigma_1^{-1}-\sigma_1^{-1}\right]-c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_1}{\partial\mu_1}\sigma_1^{-1}-\sigma_1^{-1}\right]+\\
c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[\sigma_1^{-1}-\frac{\partial\theta_2}{\partial\mu_1}\sigma_1^{-1}\right]+c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial\theta_1}{\partial\mu_1}\sigma_2^{-1}\right]+\\
c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\frac{\partial\theta_2}{\partial\mu_1}\sigma_2^{-1}\right]+c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\mu_1}\sigma_3^{-1}\right]+\\
c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_2}{\partial\mu_1}\sigma_3^{-1}\right]-c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\mu_1}\sigma_3^{-1}\right]
\end{array}\right]
\end{aligned}
$$

Pulling out the standard deviations and using $\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)=\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)$ we get

$$
\begin{aligned}
\left(\frac{\partial BC}{\partial \mu_1}\right) &= \left[\begin{array}{c}
\sigma_1^{-1}\left[\begin{array}{c}
c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_2}{\partial\mu_1}-1\right]-c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_1}{\partial\mu_1}-1\right]+\\
c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[1-\frac{\partial\theta_2}{\partial\mu_1}\right]
\end{array}\right]+\\
\sigma_2^{-1}\left[c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial\theta_1}{\partial\mu_1}\right]+c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\frac{\partial\theta_2}{\partial\mu_1}\right]\right]+\\
\sigma_3^{-1}\left[c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\mu_1}\right]+c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_2}{\partial\mu_1}\right]-c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\mu_1}\right]\right]
\end{array}\right]\\
&= \left[\begin{array}{c}
\sigma_1^{-1}\left[\left[\frac{\partial\theta_2}{\partial\mu_1}-1\right]\left(c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)-c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right)-c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_1}{\partial\mu_1}-1\right]\right]+\\
\sigma_2^{-1}\left[c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial\theta_1}{\partial\mu_1}\right]+c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\frac{\partial\theta_2}{\partial\mu_1}\right]\right]+\\
\sigma_3^{-1}\left[\left[\frac{\partial\theta_1}{\partial\mu_1}\right]\left(c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)-c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right)+c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_2}{\partial\mu_1}\right]\right]
\end{array}\right]\\
&= \left[\begin{array}{c}
\sigma_1^{-1}\left[\left[\frac{\partial\theta_2}{\partial\mu_1}-1\right]\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left(c_{2|1}p_1-c_{3|1}p_1\right)-c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_1}{\partial\mu_1}-1\right]\right]+\\
\sigma_2^{-1}\left[c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial\theta_1}{\partial\mu_1}\right]+c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\frac{\partial\theta_2}{\partial\mu_1}\right]\right]+\\
\sigma_3^{-1}\left[\left[\frac{\partial\theta_1}{\partial\mu_1}\right]\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left(c_{1|3}p_3-c_{2|3}p_3\right)+c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_2}{\partial\mu_1}\right]\right]
\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\partial BC}{\partial \mu_2}\right) &= \frac{\partial}{\partial \mu_2}\left[\begin{array}{c}
c_{2|1}p_1\left(\Phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)-\Phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right)+c_{3|1}p_1\left(\Phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right)+ \\
c_{1|2}p_2\left(\Phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right)+c_{3|2}p_2\left(\Phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right)+ \\
c_{1|3}p_3\left(\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right)+c_{2|3}p_3\left(\Phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)-\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right)
\end{array}\right] \\
&= \left[\begin{array}{c}
c_{2|1}p_1\frac{\partial}{\partial\mu_2}\left[\Phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\right]-c_{2|1}p_1\frac{\partial}{\partial\mu_2}\left[\Phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right]+c_{3|1}p_1\frac{\partial}{\partial\mu_2}\left[\Phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right]+ \\
c_{1|2}p_2\frac{\partial}{\partial\mu_2}\left[\Phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right]+c_{3|2}p_2\frac{\partial}{\partial\mu_2}\left[\Phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right]+ \\
c_{1|3}p_3\frac{\partial}{\partial\mu_2}\left[\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]+c_{2|3}p_3\frac{\partial}{\partial\mu_2}\left[\Phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\right]-c_{2|3}p_3\frac{\partial}{\partial\mu_2}\left[\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\frac{\partial}{\partial\mu_2}\left[\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\right]-c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\frac{\partial}{\partial\mu_2}\left[\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right]+ \\
c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\frac{\partial}{\partial\mu_2}\left[\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right]+c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\frac{\partial}{\partial\mu_2}\left[\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right]+ \\
c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\frac{\partial}{\partial\mu_2}\left[\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right]+c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial\mu_2}\left[\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]+ \\
c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial\mu_2}\left[\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\right]-c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial\mu_2}\left[\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_2}{\partial\mu_2}\sigma_1^{-1}\right]-c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_1}{\partial\mu_2}\sigma_1^{-1}\right]+ \\
c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[-\frac{\partial\theta_2}{\partial\mu_2}\sigma_1^{-1}\right]+c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial\theta_1}{\partial\mu_2}\sigma_2^{-1}-\sigma_2^{-1}\right]+ \\
c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[\sigma_2^{-1}-\frac{\partial\theta_2}{\partial\mu_2}\sigma_2^{-1}\right]+c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\mu_2}\sigma_3^{-1}\right]+ \\
c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_2}{\partial\mu_2}\sigma_3^{-1}\right]-c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\mu_2}\sigma_3^{-1}\right]
\end{array}\right]
\end{aligned}
$$

Pulling out the standard deviations and using $\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)=\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)$ we get

$$
\begin{aligned}
\left(\frac{\partial BC}{\partial \mu_2}\right) &= \left[\begin{array}{c}
\sigma_1^{-1}\left[c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_2}{\partial\mu_2}\right]-c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_1}{\partial\mu_2}\right]+c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[-\frac{\partial\theta_2}{\partial\mu_2}\right]\right]+ \\
\sigma_2^{-1}\left[c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial\theta_1}{\partial\mu_2}-1\right]+c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[1-\frac{\partial\theta_2}{\partial\mu_2}\right]\right]+ \\
\sigma_3^{-1}\left[c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\mu_2}\right]+c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_2}{\partial\mu_2}\right]-c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\mu_2}\right]\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
\sigma_1^{-1}\left[\left[\frac{\partial\theta_2}{\partial\mu_2}\right]\left(c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)-c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right)-c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_1}{\partial\mu_2}\right]\right]+ \\
\sigma_2^{-1}\left[c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial\theta_1}{\partial\mu_2}-1\right]+c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[1-\frac{\partial\theta_2}{\partial\mu_2}\right]\right]+ \\
\sigma_3^{-1}\left[\left[\frac{\partial\theta_1}{\partial\mu_2}\right]\left(c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)-c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right)+c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_2}{\partial\mu_2}\right]\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
\sigma_1^{-1}\left[\left[\frac{\partial\theta_2}{\partial\mu_2}\right]\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left(c_{2|1}p_1-c_{3|1}p_1\right)-c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_1}{\partial\mu_2}\right]\right]+ \\
\sigma_2^{-1}\left[c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial\theta_1}{\partial\mu_2}-1\right]+c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[1-\frac{\partial\theta_2}{\partial\mu_2}\right]\right]+ \\
\sigma_3^{-1}\left[\left[\frac{\partial\theta_1}{\partial\mu_2}\right]\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left(c_{1|3}p_3-c_{2|3}p_3\right)+c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_2}{\partial\mu_2}\right]\right]
\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\partial BC}{\partial \mu_3}\right) &= \frac{\partial}{\partial \mu_3}\left[\begin{array}{c}
c_{2|1}p_1\left(\Phi\left(\tfrac{\theta_2-\mu_1}{\sigma_1}\right)-\Phi\left(\tfrac{\theta_1-\mu_1}{\sigma_1}\right)\right)+c_{3|1}p_1\left(\Phi\left(\tfrac{\mu_1-\theta_2}{\sigma_1}\right)\right)+ \\
c_{1|2}p_2\left(\Phi\left(\tfrac{\theta_1-\mu_2}{\sigma_2}\right)\right)+c_{3|2}p_2\left(\Phi\left(\tfrac{\mu_2-\theta_2}{\sigma_2}\right)\right)+ \\
c_{1|3}p_3\left(\Phi\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)\right)+c_{2|3}p_3\left(\Phi\left(\tfrac{\theta_2-\mu_3}{\sigma_3}\right)-\Phi\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)\right)
\end{array}\right] \\
&= \left[\begin{array}{c}
c_{2|1}p_1\tfrac{\partial}{\partial\mu_3}\left[\Phi\left(\tfrac{\theta_2-\mu_1}{\sigma_1}\right)\right]-c_{2|1}p_1\tfrac{\partial}{\partial\mu_3}\left[\Phi\left(\tfrac{\theta_1-\mu_1}{\sigma_1}\right)\right]+c_{3|1}p_1\tfrac{\partial}{\partial\mu_3}\left[\Phi\left(\tfrac{\mu_1-\theta_2}{\sigma_1}\right)\right]+ \\
c_{1|2}p_2\tfrac{\partial}{\partial\mu_3}\left[\Phi\left(\tfrac{\theta_1-\mu_2}{\sigma_2}\right)\right]+c_{3|2}p_2\tfrac{\partial}{\partial\mu_3}\left[\Phi\left(\tfrac{\mu_2-\theta_2}{\sigma_2}\right)\right]+ \\
c_{1|3}p_3\tfrac{\partial}{\partial\mu_3}\left[\Phi\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)\right]+c_{2|3}p_3\tfrac{\partial}{\partial\mu_3}\left[\Phi\left(\tfrac{\theta_2-\mu_3}{\sigma_3}\right)\right]-c_{2|3}p_3\tfrac{\partial}{\partial\mu_3}\left[\Phi\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
c_{2|1}p_1\phi\left(\tfrac{\theta_2-\mu_1}{\sigma_1}\right)\tfrac{\partial}{\partial\mu_3}\left[\left(\tfrac{\theta_2-\mu_1}{\sigma_1}\right)\right]-c_{2|1}p_1\phi\left(\tfrac{\theta_1-\mu_1}{\sigma_1}\right)\tfrac{\partial}{\partial\mu_3}\left[\left(\tfrac{\theta_1-\mu_1}{\sigma_1}\right)\right]+ \\
c_{3|1}p_1\phi\left(\tfrac{\mu_1-\theta_2}{\sigma_1}\right)\tfrac{\partial}{\partial\mu_3}\left[\left(\tfrac{\mu_1-\theta_2}{\sigma_1}\right)\right]+c_{1|2}p_2\phi\left(\tfrac{\theta_1-\mu_2}{\sigma_2}\right)\tfrac{\partial}{\partial\mu_3}\left[\left(\tfrac{\theta_1-\mu_2}{\sigma_2}\right)\right]+ \\
c_{3|2}p_2\phi\left(\tfrac{\mu_2-\theta_2}{\sigma_2}\right)\tfrac{\partial}{\partial\mu_3}\left[\left(\tfrac{\mu_2-\theta_2}{\sigma_2}\right)\right]+c_{1|3}p_3\phi\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)\tfrac{\partial}{\partial\mu_3}\left[\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)\right]+ \\
c_{2|3}p_3\phi\left(\tfrac{\theta_2-\mu_3}{\sigma_3}\right)\tfrac{\partial}{\partial\mu_3}\left[\left(\tfrac{\theta_2-\mu_3}{\sigma_3}\right)\right]-c_{2|3}p_3\phi\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)\tfrac{\partial}{\partial\mu_3}\left[\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
c_{2|1}p_1\phi\left(\tfrac{\theta_2-\mu_1}{\sigma_1}\right)\left[\tfrac{\partial\theta_2}{\partial\mu_3}\sigma_1^{-1}\right]-c_{2|1}p_1\phi\left(\tfrac{\theta_1-\mu_1}{\sigma_1}\right)\left[\tfrac{\partial\theta_1}{\partial\mu_3}\sigma_1^{-1}\right]+ \\
c_{3|1}p_1\phi\left(\tfrac{\mu_1-\theta_2}{\sigma_1}\right)\left[-\tfrac{\partial\theta_2}{\partial\mu_3}\sigma_1^{-1}\right]+c_{1|2}p_2\phi\left(\tfrac{\theta_1-\mu_2}{\sigma_2}\right)\left[\tfrac{\partial\theta_1}{\partial\mu_3}\sigma_2^{-1}\right]+ \\
c_{3|2}p_2\phi\left(\tfrac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\tfrac{\partial\theta_2}{\partial\mu_3}\sigma_2^{-1}\right]+c_{1|3}p_3\phi\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)\left[\tfrac{\partial\theta_1}{\partial\mu_3}\sigma_3^{-1}-\sigma_3^{-1}\right]+ \\
c_{2|3}p_3\phi\left(\tfrac{\theta_2-\mu_3}{\sigma_3}\right)\left[\tfrac{\partial\theta_2}{\partial\mu_3}\sigma_3^{-1}-\sigma_3^{-1}\right]-c_{2|3}p_3\phi\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)\left[\tfrac{\partial\theta_1}{\partial\mu_3}\sigma_3^{-1}-\sigma_3^{-1}\right]
\end{array}\right]
\end{aligned}
$$

Pulling out the standard deviations and using $\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)=\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)$ we get

$$
\begin{aligned}
\left(\frac{\partial BC}{\partial \mu_3}\right) &= \left[\begin{array}{c}
\sigma_1^{-1}\left[c_{2|1}p_1\phi\left(\tfrac{\theta_2-\mu_1}{\sigma_1}\right)\left[\tfrac{\partial\theta_2}{\partial\mu_3}\right]-c_{2|1}p_1\phi\left(\tfrac{\theta_1-\mu_1}{\sigma_1}\right)\left[\tfrac{\partial\theta_1}{\partial\mu_3}\right]+c_{3|1}p_1\phi\left(\tfrac{\mu_1-\theta_2}{\sigma_1}\right)\left[-\tfrac{\partial\theta_2}{\partial\mu_3}\right]\right]+ \\
\sigma_2^{-1}\left[c_{1|2}p_2\phi\left(\tfrac{\theta_1-\mu_2}{\sigma_2}\right)\left[\tfrac{\partial\theta_1}{\partial\mu_3}\right]+c_{3|2}p_2\phi\left(\tfrac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\tfrac{\partial\theta_2}{\partial\mu_3}\right]\right]+ \\
\sigma_3^{-1}\left[\begin{array}{c}
c_{1|3}p_3\phi\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)\left[\tfrac{\partial\theta_1}{\partial\mu_3}-1\right]+c_{2|3}p_3\phi\left(\tfrac{\theta_2-\mu_3}{\sigma_3}\right)\left[\tfrac{\partial\theta_2}{\partial\mu_3}-1\right] \\
-c_{2|3}p_3\phi\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)\left[\tfrac{\partial\theta_1}{\partial\mu_3}-1\right]
\end{array}\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
\sigma_1^{-1}\left[\left[\tfrac{\partial\theta_2}{\partial\mu_3}\right]\left(c_{2|1}p_1\phi\left(\tfrac{\theta_2-\mu_1}{\sigma_1}\right)-c_{3|1}p_1\phi\left(\tfrac{\mu_1-\theta_2}{\sigma_1}\right)\right)-c_{2|1}p_1\phi\left(\tfrac{\theta_1-\mu_1}{\sigma_1}\right)\left[\tfrac{\partial\theta_1}{\partial\mu_3}\right]\right]+ \\
\sigma_2^{-1}\left[c_{1|2}p_2\phi\left(\tfrac{\theta_1-\mu_2}{\sigma_2}\right)\left[\tfrac{\partial\theta_1}{\partial\mu_3}\right]+c_{3|2}p_2\phi\left(\tfrac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\tfrac{\partial\theta_2}{\partial\mu_3}\right]\right]+ \\
\sigma_3^{-1}\left[\left[\tfrac{\partial\theta_1}{\partial\mu_3}-1\right]\left(c_{1|3}p_3\phi\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)-c_{2|3}p_3\phi\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)\right)+c_{2|3}p_3\phi\left(\tfrac{\theta_2-\mu_3}{\sigma_3}\right)\left[\tfrac{\partial\theta_2}{\partial\mu_3}-1\right]\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
\sigma_1^{-1}\left[\left[\tfrac{\partial\theta_2}{\partial\mu_3}\right]\phi\left(\tfrac{\theta_2-\mu_1}{\sigma_1}\right)\left(c_{2|1}p_1-c_{3|1}p_1\right)-c_{2|1}p_1\phi\left(\tfrac{\theta_1-\mu_1}{\sigma_1}\right)\left[\tfrac{\partial\theta_1}{\partial\mu_3}\right]\right]+ \\
\sigma_2^{-1}\left[c_{1|2}p_2\phi\left(\tfrac{\theta_1-\mu_2}{\sigma_2}\right)\left[\tfrac{\partial\theta_1}{\partial\mu_3}\right]+c_{3|2}p_2\phi\left(\tfrac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\tfrac{\partial\theta_2}{\partial\mu_3}\right]\right]+ \\
\sigma_3^{-1}\left[\left[\tfrac{\partial\theta_1}{\partial\mu_3}-1\right]\phi\left(\tfrac{\theta_1-\mu_3}{\sigma_3}\right)\left(c_{1|3}p_3-c_{2|3}p_3\right)+c_{2|3}p_3\phi\left(\tfrac{\theta_2-\mu_3}{\sigma_3}\right)\left[\tfrac{\partial\theta_2}{\partial\mu_3}-1\right]\right]
\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\partial BC}{\partial \sigma_1}\right) &= \frac{\partial}{\partial \sigma_1}\left[\begin{array}{c}
c_{2|1}p_1\left(\Phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)-\Phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right)+c_{3|1}p_1\left(\Phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right)+\\
c_{1|2}p_2\left(\Phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right)+c_{3|2}p_2\left(\Phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right)+\\
c_{1|3}p_3\left(\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right)+c_{2|3}p_3\left(\Phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)-\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right)
\end{array}\right]\\
&= \left[\begin{array}{c}
c_{2|1}p_1\frac{\partial}{\partial\sigma_1}\left[\Phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\right]-c_{2|1}p_1\frac{\partial}{\partial\sigma_1}\left[\Phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right]+c_{3|1}p_1\frac{\partial}{\partial\sigma_1}\left[\Phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right]+\\
c_{1|2}p_2\frac{\partial}{\partial\sigma_1}\left[\Phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right]+c_{3|2}p_2\frac{\partial}{\partial\sigma_1}\left[\Phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right]+\\
c_{1|3}p_3\frac{\partial}{\partial\sigma_1}\left[\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]+c_{2|3}p_3\frac{\partial}{\partial\sigma_1}\left[\Phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\right]-c_{2|3}p_3\frac{\partial}{\partial\sigma_1}\left[\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]
\end{array}\right]\\
&= \left[\begin{array}{c}
c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\frac{\partial}{\partial\sigma_1}\left[\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\right]-c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\frac{\partial}{\partial\sigma_1}\left[\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right]+\\
c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\frac{\partial}{\partial\sigma_1}\left[\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right]+c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\frac{\partial}{\partial\sigma_1}\left[\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right]+\\
c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\frac{\partial}{\partial\sigma_1}\left[\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right]+c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial\sigma_1}\left[\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]+\\
c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial\sigma_1}\left[\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\right]-c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial\sigma_1}\left[\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]
\end{array}\right]\\
&= \left[\begin{array}{c}
c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{1}{\sigma_1}\frac{\partial}{\partial\sigma_1}(\theta_2-\mu_1)+(\theta_2-\mu_1)\frac{\partial}{\partial\sigma_1}(\frac{1}{\sigma_1})\right]-\\
c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{1}{\sigma_1}\frac{\partial}{\partial\sigma_1}(\theta_1-\mu_1)+(\theta_1-\mu_1)\frac{\partial}{\partial\sigma_1}(\frac{1}{\sigma_1})\right]+\\
c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[\frac{1}{\sigma_1}\frac{\partial}{\partial\sigma_1}(\mu_1-\theta_2)+(\mu_1-\theta_2)\frac{\partial}{\partial\sigma_1}\frac{1}{\sigma_1}\right]+\\
c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial\theta_1}{\partial\sigma_1}\sigma_2^{-1}\right]+\\
c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\frac{\partial\theta_2}{\partial\sigma_1}\sigma_2^{-1}\right]+\\
c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\sigma_1}\sigma_3^{-1}\right]+\\
c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_2}{\partial\sigma_1}\sigma_3^{-1}\right]-\\
c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\sigma_1}\sigma_3^{-1}\right]
\end{array}\right]\\
&= \left[\begin{array}{c}
c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\sigma_1^{-1}\frac{\partial\theta_2}{\partial\sigma_1}-(\theta_2-\mu_1)\sigma_1^{-2}\right]-c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{1}{\sigma_1}\frac{\partial\theta_1}{\partial\sigma_1}-(\theta_1-\mu_1)\sigma_1^{-2}\right]+\\
c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[\frac{1}{\sigma_1}\frac{-\partial\theta_2}{\partial\sigma_1}-(\mu_1-\theta_2)\sigma_1^{-2}\right]+c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial\theta_1}{\partial\sigma_1}\sigma_2^{-1}\right]+\\
c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\frac{\partial\theta_2}{\partial\sigma_1}\sigma_2^{-1}\right]+c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\sigma_1}\sigma_3^{-1}\right]+\\
c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_2}{\partial\sigma_1}\sigma_3^{-1}\right]-c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\sigma_1}\sigma_3^{-1}\right]
\end{array}\right]
\end{aligned}
$$

and continuing to simplify:

$$
\begin{aligned}
\left(\frac{\partial BC}{\partial \sigma_1}\right) &= \begin{bmatrix}
\sigma_1^{-1} c_{2|1} p_1 \phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_2}{\partial \sigma_1} - (\frac{\theta_2-\mu_1}{\sigma_1})\right] - \\
\sigma_1^{-1} c_{2|1} p_1 \phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_1}{\partial \sigma_1} - (\frac{\theta_1-\mu_1}{\sigma_1})\right] + \\
\sigma_1^{-1} c_{3|1} p_1 \phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[\frac{-\partial \theta_2}{\partial \sigma_1} - (\frac{\mu_1-\theta_2}{\sigma_1})\right] + \\
\sigma_2^{-1} c_{1|2} p_2 \phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial \theta_1}{\partial \sigma_1}\right] + \\
\sigma_2^{-1} c_{3|2} p_2 \phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\frac{\partial \theta_2}{\partial \sigma_1}\right] + \\
\sigma_3^{-1} c_{1|3} p_3 \phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_1}{\partial \sigma_1}\right] + \\
\sigma_3^{-1} c_{2|3} p_3 \phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_2}{\partial \sigma_1}\right] - \\
\sigma_3^{-1} c_{2|3} p_3 \phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_1}{\partial \sigma_1}\right]
\end{bmatrix} \\
&= \begin{bmatrix}
\sigma_1^{-1} \begin{bmatrix} c_{2|1} p_1 \phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_2}{\partial \sigma_1} - (\frac{\theta_2-\mu_1}{\sigma_1})\right] - c_{2|1} p_1 \phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_1}{\partial \sigma_1} - (\frac{\theta_1-\mu_1}{\sigma_1})\right] + \\ c_{3|1} p_1 \phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[\frac{-\partial \theta_2}{\partial \sigma_1} - (\frac{\mu_1-\theta_2}{\sigma_1})\right] \end{bmatrix} + \\
\sigma_2^{-1}\left[c_{1|2} p_2 \phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial \theta_1}{\partial \sigma_1}\right] + c_{3|2} p_2 \phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\frac{\partial \theta_2}{\partial \sigma_1}\right]\right] + \\
\sigma_3^{-1}\left[c_{1|3} p_3 \phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_1}{\partial \sigma_1}\right] + c_{2|3} p_3 \phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_2}{\partial \sigma_1}\right] - c_{2|3} p_3 \phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_1}{\partial \sigma_1}\right]\right]
\end{bmatrix} \\
&= \begin{bmatrix}
\sigma_1^{-1}\left[\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left(c_{2|1}p_1 - c_{3|1}p_1\right)\left[\frac{\partial \theta_2}{\partial \sigma_1} - (\frac{\theta_2-\mu_1}{\sigma_1})\right] - c_{2|1} p_1 \phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_1}{\partial \sigma_1} - (\frac{\theta_1-\mu_1}{\sigma_1})\right]\right] + \\
\sigma_2^{-1}\left[c_{1|2} p_2 \phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial \theta_1}{\partial \sigma_1}\right] - c_{3|2} p_2 \phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[\frac{\partial \theta_2}{\partial \sigma_1}\right]\right] + \\
\sigma_3^{-1}\left[\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_1}{\partial \sigma_1}\right]\left(c_{1|3}p_3 - c_{2|3}p_3\right) + c_{2|3} p_3 \phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_2}{\partial \sigma_1}\right]\right]
\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\partial BC}{\partial \sigma 2}\right) &= \frac{\partial}{\partial \sigma_2}\begin{bmatrix}
c_{2|1}p_1\left(\Phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)-\Phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right)+c_{3|1}p_1\left(\Phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right)+ \\
c_{1|2}p_2\left(\Phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right)+c_{3|2}p_2\left(\Phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right)+ \\
c_{1|3}p_3\left(\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right)+c_{2|3}p_3\left(\Phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)-\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right)
\end{bmatrix} \\
&= \begin{bmatrix}
c_{2|1}p_1\frac{\partial}{\partial\sigma_2}\left[\Phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\right]-c_{2|1}p_1\frac{\partial}{\partial\sigma_2}\left[\Phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right]+c_{3|1}p_1\frac{\partial}{\partial\sigma_2}\left[\Phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right]+ \\
c_{1|2}p_2\frac{\partial}{\partial\sigma_2}\left[\Phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right]+c_{3|2}p_2\frac{\partial}{\partial\sigma_2}\left[\Phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right]+ \\
c_{1|3}p_3\frac{\partial}{\partial\sigma_2}\left[\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]+c_{2|3}p_3\frac{\partial}{\partial\sigma_2}\left[\Phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\right]-c_{2|3}p_3\frac{\partial}{\partial\sigma_2}\left[\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]
\end{bmatrix} \\
&= \begin{bmatrix}
c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\frac{\partial}{\partial\sigma_2}\left[\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\right]-c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\frac{\partial}{\partial\sigma_2}\left[\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right]+ \\
c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\frac{\partial}{\partial\sigma_2}\left[\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right]+c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\frac{\partial}{\partial\sigma_2}\left[\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right]+ \\
c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\frac{\partial}{\partial\sigma_2}\left[\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right]+c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial\sigma_2}\left[\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]+ \\
c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial\sigma_2}\left[\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\right]-c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial\sigma_2}\left[\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]
\end{bmatrix} \\
&= \begin{bmatrix}
c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_2}{\partial\sigma_2}\sigma_1^{-1}\right]- \\
c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_1}{\partial\sigma_2}\sigma_1^{-1}\right]+ \\
c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[-\frac{\partial\theta_2}{\partial\sigma_2}\sigma_1^{-1}\right]+ \\
c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{1}{\sigma_2}\frac{\partial}{\partial\sigma_2}(\theta_1-\mu_2)+(\theta_1-\mu_2)\frac{\partial}{\partial\sigma_2}\left(\frac{1}{\sigma_2}\right)\right]+ \\
c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[\frac{1}{\sigma_2}\frac{\partial}{\partial\sigma_2}(\mu_2-\theta_2)+(\mu_2-\theta_2)\frac{\partial}{\partial\sigma_2}\left(\frac{1}{\sigma_2}\right)\right]+ \\
c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\sigma_2}\sigma_3^{-1}\right]+ \\
c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_2}{\partial\sigma_2}\sigma_3^{-1}\right]- \\
c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\sigma_2}\sigma_3^{-1}\right]
\end{bmatrix} \\
&= \begin{bmatrix}
c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_2}{\partial\sigma_2}\sigma_1^{-1}\right]- \\
c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial\theta_1}{\partial\sigma_2}\sigma_1^{-1}\right]+ \\
c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[-\frac{\partial\theta_2}{\partial\sigma_2}\sigma_1^{-1}\right]+ \\
c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\sigma_2^{-1}\frac{\partial\theta_1}{\partial\sigma_2}-(\theta_1-\mu_2)\sigma_2^{-2}\right]+ \\
c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[\sigma_2^{-1}\frac{-\partial\theta_2}{\partial\sigma_2}-(\mu_2-\theta_2)\sigma_2^{-2}\right]+ \\
c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\sigma_2}\sigma_3^{-1}\right]+ \\
c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_2}{\partial\sigma_2}\sigma_3^{-1}\right]- \\
c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial\theta_1}{\partial\sigma_2}\sigma_3^{-1}\right]
\end{bmatrix}
\end{aligned}
$$

and continuing to simplify:

$$
\begin{aligned}
\left(\frac{\partial BC}{\partial \sigma_2}\right) &= \left[\begin{array}{c}
\sigma_1^{-1} c_{2|1} p_1 \phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_2}{\partial \sigma_2}\right] - \\
\sigma_1^{-1} c_{2|1} p_1 \phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_1}{\partial \sigma_2}\right] + \\
\sigma_1^{-1} c_{3|1} p_1 \phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[-\frac{\partial \theta_2}{\partial \sigma_2}\right] + \\
\sigma_2^{-1} c_{1|2} p_2 \phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial \theta_1}{\partial \sigma_2} - \left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right] + \\
\sigma_2^{-1} c_{3|2} p_2 \phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[\frac{-\partial \theta_2}{\partial \sigma_2} - \left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right] + \\
\sigma_3^{-1} c_{1|3} p_3 \phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_1}{\partial \sigma_2}\right] + \\
\sigma_3^{-1} c_{2|3} p_3 \phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_2}{\partial \sigma_2}\right] - \\
\sigma_3^{-1} c_{2|3} p_3 \phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_1}{\partial \sigma_2}\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
\sigma_1^{-1}\left[c_{2|1} p_1 \phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_2}{\partial \sigma_2}\right] - c_{2|1} p_1 \phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_1}{\partial \sigma_2}\right] + c_{3|1} p_1 \phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[-\frac{\partial \theta_2}{\partial \sigma_2}\right]\right] + \\
\sigma_2^{-1}\left[c_{1|2} p_2 \phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial \theta_1}{\partial \sigma_2} - \left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right] + c_{3|2} p_2 \phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[\frac{-\partial \theta_2}{\partial \sigma_2} - \left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right]\right] + \\
\sigma_3^{-1}\left[c_{1|3} p_3 \phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_1}{\partial \sigma_2}\right] + c_{2|3} p_3 \phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_2}{\partial \sigma_2}\right] - c_{2|3} p_3 \phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_1}{\partial \sigma_2}\right]\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
\sigma_1^{-1}\left[\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_2}{\partial \sigma_2}\right]\left(c_{2|1} p_1 - c_{3|1} p_1\right) - c_{2|1} p_1 \phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_1}{\partial \sigma_2}\right]\right] + \\
\sigma_2^{-1}\left[c_{1|2} p_2 \phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial \theta_1}{\partial \sigma_2} - \left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right] + c_{3|2} p_2 \phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[\frac{-\partial \theta_2}{\partial \sigma_2} - \left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right]\right] + \\
\sigma_3^{-1}\left[\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_1}{\partial \sigma_2}\right]\left(c_{1|3} p_3 - c_{2|3} p_3\right) + c_{2|3} p_3 \phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_2}{\partial \sigma_2}\right]\right]
\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\partial BC}{\partial \sigma_3}\right) &= \frac{\partial}{\partial \sigma_3}\left[\begin{array}{c}
c_{2|1}p_1\left(\Phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)-\Phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right)+c_{3|1}p_1\left(\Phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right)+ \\
c_{1|2}p_2\left(\Phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right)+c_{3|2}p_2\left(\Phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right)+ \\
c_{1|3}p_3\left(\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right)+c_{2|3}p_3\left(\Phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)-\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right)
\end{array}\right] \\
&= \left[\begin{array}{c}
c_{2|1}p_1\frac{\partial}{\partial \sigma_3}\left[\Phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\right]-c_{2|1}p_1\frac{\partial}{\partial \sigma_3}\left[\Phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right]+c_{3|1}p_1\frac{\partial}{\partial \sigma_3}\left[\Phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right]+ \\
c_{1|2}p_2\frac{\partial}{\partial \sigma_3}\left[\Phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right]+c_{3|2}p_2\frac{\partial}{\partial \sigma_3}\left[\Phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right]+ \\
c_{1|3}p_3\frac{\partial}{\partial \sigma_3}\left[\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]+c_{2|3}p_3\frac{\partial}{\partial \sigma_3}\left[\Phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\right]-c_{2|3}p_3\frac{\partial}{\partial \sigma_3}\left[\Phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\frac{\partial}{\partial \sigma_3}\left[\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\right]-c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\frac{\partial}{\partial \sigma_3}\left[\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\right]+ \\
c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\frac{\partial}{\partial \sigma_3}\left[\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\right]+c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\frac{\partial}{\partial \sigma_3}\left[\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\right]+ \\
c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\frac{\partial}{\partial \sigma_3}\left[\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\right]+c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial \sigma_3}\left[\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]+ \\
c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial \sigma_3}\left[\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\right]-c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\frac{\partial}{\partial \sigma_3}\left[\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_2}{\partial \sigma_3}\sigma_1^{-1}\right]- \\
c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_1}{\partial \sigma_3}\sigma_1^{-1}\right]+ \\
c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[-\frac{\partial \theta_2}{\partial \sigma_3}\sigma_1^{-1}\right]+ \\
c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial \theta_1}{\partial \sigma_3}\sigma_2^{-1}\right]+ \\
c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\frac{\partial \theta_2}{\partial \sigma_3}\sigma_2^{-1}\right]+ \\
c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{1}{\sigma_3}\frac{\partial}{\partial \sigma_3}(\theta_1-\mu_3)+(\theta_1-\mu_3)\frac{\partial}{\partial \sigma_3}\left(\frac{1}{\sigma_3}\right)\right]+ \\
c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{1}{\sigma_3}\frac{\partial}{\partial \sigma_3}(\theta_2-\mu_3)+(\theta_2-\mu_3)\frac{\partial}{\partial \sigma_3}\left(\frac{1}{\sigma_3}\right)\right]- \\
c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{1}{\sigma_3}\frac{\partial}{\partial \sigma_3}(\theta_1-\mu_3)+(\theta_1-\mu_3)\frac{\partial}{\partial \sigma_3}\left(\frac{1}{\sigma_3}\right)\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
c_{2|1}p_1\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_2}{\partial \sigma_3}\sigma_1^{-1}\right]- \\
c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_1}{\partial \sigma_3}\sigma_1^{-1}\right]+ \\
c_{3|1}p_1\phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[-\frac{\partial \theta_2}{\partial \sigma_3}\sigma_1^{-1}\right]+ \\
c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial \theta_1}{\partial \sigma_3}\sigma_2^{-1}\right]+ \\
c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\frac{\partial \theta_2}{\partial \sigma_3}\sigma_2^{-1}\right]+ \\
c_{1|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\sigma_3^{-1}\frac{\partial \theta_1}{\partial \sigma_3}-(\theta_1-\mu_3)\sigma_3^{-2}\right]+ \\
c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\sigma_3^{-1}\frac{\partial \theta_2}{\partial \sigma_3}-(\theta_2-\mu_3)\sigma_3^{-2}\right]- \\
c_{2|3}p_3\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\sigma_3^{-1}\frac{\partial \theta_1}{\partial \sigma_3}-(\theta_1-\mu_3)\sigma_3^{-2}\right]
\end{array}\right]
\end{aligned}
$$

and continuing to simplify:

$$
\begin{aligned}
\left(\frac{\partial BC}{\partial \sigma_3}\right) &= \left[\begin{array}{c}
\sigma_1^{-1} c_{2|1} p_1 \phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_2}{\partial \sigma_3}\right] - \\
\sigma_1^{-1} c_{2|1} p_1 \phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_1}{\partial \sigma_3}\right] + \\
\sigma_1^{-1} c_{3|1} p_1 \phi\left(\frac{\mu_1-\theta_2}{\sigma_1}\right)\left[-\frac{\partial \theta_2}{\partial \sigma_3}\right] + \\
\sigma_2^{-1} c_{1|2} p_2 \phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial \theta_1}{\partial \sigma_3}\right] + \\
\sigma_2^{-1} c_{3|2} p_2 \phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\frac{\partial \theta_2}{\partial \sigma_3}\right] + \\
\sigma_3^{-1} c_{1|3} p_3 \phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_1}{\partial \sigma_3} - \left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right] + \\
\sigma_3^{-1} c_{2|3} p_3 \phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_2}{\partial \sigma_3} - \left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\right] - \\
\sigma_3^{-1} c_{2|3} p_3 \phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_1}{\partial \sigma_3} - \left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
\sigma_1^{-1}\left[\phi\left(\frac{\theta_2-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_2}{\partial \sigma_3}\right]\left(c_{2|1}p_1 - c_{3|1}p_1\right) - c_{2|1}p_1\phi\left(\frac{\theta_1-\mu_1}{\sigma_1}\right)\left[\frac{\partial \theta_1}{\partial \sigma_3}\right]\right] + \\
\sigma_2^{-1}\left[c_{1|2}p_2\phi\left(\frac{\theta_1-\mu_2}{\sigma_2}\right)\left[\frac{\partial \theta_1}{\partial \sigma_3}\right] + c_{3|2}p_2\phi\left(\frac{\mu_2-\theta_2}{\sigma_2}\right)\left[-\frac{\partial \theta_2}{\partial \sigma_3}\right]\right] + \\
\sigma_3^{-1}\left[\phi\left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_1}{\partial \sigma_3} - \left(\frac{\theta_1-\mu_3}{\sigma_3}\right)\right]\left(c_{1|3}p_3 - c_{2|3}p_3\right) + c_{2|3}p_3\phi\left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\left[\frac{\partial \theta_2}{\partial \sigma_3} - \left(\frac{\theta_2-\mu_3}{\sigma_3}\right)\right]\right]
\end{array}\right]
\end{aligned}
$$

Appendix C. Results from Comparisons of tests accounting for correlation

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.955	0.644	0.962	0.611	0.862	0.552	0.928	0.543	0.835	0.555
	0	0.948	0.644	0.961	0.611	0.858	0.553	0.922	0.544	0.836	0.556
	0.1	0.949	0.633	0.966	0.600	0.853	0.543	0.917	0.534	0.830	0.546
	0.3	0.952	0.600	0.964	0.566	0.857	0.512	0.924	0.504	0.828	0.516
	0.5	0.954	0.544	0.963	0.511	0.858	0.464	0.920	0.456	0.822	0.469
	0.9	0.966	0.283	0.968	0.261	0.885	0.248	0.925	0.235	0.825	0.250
20	-0.3	0.950	0.465	0.956	0.448	0.900	0.422	0.934	0.421	0.888	0.424
	0	0.948	0.464	0.954	0.447	0.897	0.422	0.935	0.420	0.886	0.423
	0.1	0.950	0.458	0.957	0.440	0.901	0.414	0.933	0.413	0.891	0.416
	0.3	0.952	0.433	0.961	0.416	0.901	0.391	0.935	0.390	0.888	0.393
	0.5	0.949	0.390	0.961	0.375	0.900	0.352	0.931	0.352	0.883	0.355
	0.9	0.958	0.198	0.956	0.188	0.907	0.177	0.936	0.177	0.884	0.179
50	-0.3	0.947	0.298	0.953	0.293	0.928	0.285	0.940	0.286	0.924	0.286
	0	0.949	0.298	0.953	0.293	0.930	0.285	0.942	0.285	0.926	0.286
	0.1	0.948	0.294	0.953	0.288	0.925	0.281	0.938	0.281	0.920	0.281
	0.3	0.952	0.277	0.952	0.272	0.929	0.264	0.942	0.264	0.922	0.265
	0.5	0.950	0.248	0.954	0.244	0.931	0.237	0.943	0.237	0.925	0.238
	0.9	0.953	0.124	0.951	0.121	0.931	0.118	0.943	0.118	0.921	0.119
100	-0.3	0.947	0.212	0.953	0.209	0.938	0.207	0.945	0.207	0.935	0.207
	0	0.952	0.211	0.952	0.209	0.940	0.206	0.948	0.207	0.938	0.207
	0.1	0.950	0.208	0.951	0.206	0.938	0.203	0.946	0.203	0.936	0.204
	0.3	0.949	0.196	0.949	0.194	0.938	0.192	0.944	0.192	0.936	0.192
	0.5	0.947	0.176	0.948	0.174	0.937	0.172	0.943	0.172	0.933	0.173
	0.9	0.951	0.088	0.950	0.087	0.938	0.085	0.945	0.086	0.934	0.086

Table C.1. Results adjusting for correlation with: Normal, 3-class, Good:Good, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.956	0.428	0.965	0.427	0.878	0.375	0.927	0.369	0.836	0.389
	0	0.957	0.422	0.965	0.418	0.868	0.372	0.925	0.366	0.826	0.384
	0.1	0.957	0.416	0.970	0.412	0.869	0.366	0.925	0.360	0.829	0.378
	0.3	0.957	0.393	0.968	0.389	0.873	0.345	0.922	0.338	0.825	0.358
	0.5	0.965	0.358	0.971	0.352	0.882	0.314	0.926	0.307	0.832	0.328
	0.9	0.973	0.185	0.969	0.182	0.906	0.171	0.932	0.159	0.825	0.178
20	-0.3	0.952	0.307	0.957	0.299	0.906	0.280	0.935	0.279	0.889	0.282
	0	0.953	0.304	0.956	0.295	0.902	0.277	0.932	0.276	0.884	0.279
	0.1	0.954	0.300	0.956	0.291	0.910	0.273	0.937	0.272	0.891	0.275
	0.3	0.954	0.282	0.954	0.273	0.901	0.256	0.929	0.255	0.879	0.258
	0.5	0.954	0.254	0.956	0.245	0.902	0.229	0.930	0.229	0.878	0.232
	0.9	0.963	0.129	0.956	0.123	0.912	0.116	0.935	0.115	0.878	0.117
50	-0.3	0.951	0.196	0.952	0.193	0.929	0.188	0.942	0.188	0.923	0.189
	0	0.950	0.195	0.951	0.191	0.930	0.186	0.939	0.186	0.923	0.187
	0.1	0.949	0.191	0.952	0.188	0.929	0.183	0.940	0.183	0.921	0.184
	0.3	0.953	0.180	0.952	0.177	0.934	0.172	0.944	0.172	0.925	0.173
	0.5	0.951	0.162	0.951	0.158	0.931	0.154	0.942	0.155	0.922	0.155
	0.9	0.956	0.080	0.949	0.079	0.933	0.076	0.942	0.076	0.922	0.077
100	-0.3	0.952	0.139	0.949	0.138	0.940	0.136	0.947	0.136	0.937	0.137
	0	0.953	0.138	0.950	0.137	0.943	0.135	0.949	0.135	0.938	0.135
	0.1	0.949	0.136	0.949	0.134	0.939	0.133	0.945	0.133	0.937	0.133
	0.3	0.949	0.128	0.948	0.126	0.939	0.125	0.945	0.125	0.933	0.125
	0.5	0.949	0.115	0.948	0.113	0.937	0.112	0.943	0.112	0.934	0.112
	0.9	0.952	0.057	0.954	0.056	0.941	0.055	0.945	0.055	0.935	0.056

Table C.2. Results adjusting for correlation with:Normal, 3-class, Good:Good, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.933	0.791	0.961	0.769	0.878	0.720	0.907	0.714	0.881	0.716
	0	0.931	0.778	0.961	0.756	0.876	0.709	0.907	0.703	0.880	0.705
	0.1	0.925	0.765	0.960	0.742	0.871	0.698	0.904	0.692	0.874	0.694
	0.3	0.930	0.725	0.960	0.703	0.877	0.663	0.905	0.658	0.881	0.662
	0.5	0.932	0.661	0.958	0.640	0.885	0.607	0.908	0.602	0.890	0.610
	0.9	0.918	0.394	0.937	0.387	0.905	0.383	0.872	0.377	0.918	0.413
20	-0.3	0.941	0.574	0.952	0.561	0.913	0.539	0.929	0.539	0.916	0.538
	0	0.944	0.565	0.951	0.551	0.911	0.531	0.930	0.530	0.913	0.529
	0.1	0.942	0.556	0.951	0.542	0.910	0.522	0.928	0.521	0.912	0.521
	0.3	0.939	0.525	0.951	0.512	0.907	0.493	0.921	0.493	0.909	0.493
	0.5	0.939	0.477	0.953	0.465	0.912	0.448	0.924	0.448	0.916	0.449
	0.9	0.933	0.277	0.943	0.274	0.917	0.267	0.906	0.267	0.925	0.270
50	-0.3	0.942	0.369	0.947	0.365	0.928	0.358	0.938	0.359	0.928	0.359
	0	0.950	0.364	0.946	0.359	0.938	0.353	0.946	0.354	0.939	0.354
	0.1	0.948	0.357	0.950	0.353	0.932	0.347	0.940	0.348	0.933	0.348
	0.3	0.951	0.337	0.951	0.333	0.935	0.327	0.941	0.328	0.937	0.328
	0.5	0.947	0.305	0.947	0.301	0.936	0.296	0.943	0.297	0.935	0.297
	0.9	0.945	0.174	0.947	0.173	0.939	0.171	0.931	0.172	0.942	0.172
100	-0.3	0.946	0.262	0.948	0.260	0.939	0.259	0.942	0.259	0.939	0.259
	0	0.946	0.259	0.949	0.257	0.939	0.254	0.944	0.255	0.938	0.255
	0.1	0.952	0.254	0.951	0.252	0.945	0.250	0.949	0.250	0.947	0.250
	0.3	0.948	0.239	0.948	0.238	0.940	0.236	0.942	0.236	0.940	0.236
	0.5	0.949	0.216	0.949	0.215	0.940	0.213	0.944	0.213	0.942	0.213
	0.9	0.950	0.123	0.946	0.123	0.947	0.122	0.945	0.122	0.947	0.123

Table C.3. Results adjusting for correlation with: 3-class Good:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.939	0.552	0.956	0.548	0.889	0.504	0.912	0.500	0.879	0.513
	0	0.940	0.536	0.955	0.531	0.889	0.491	0.911	0.487	0.874	0.500
	0.1	0.942	0.528	0.958	0.522	0.892	0.483	0.910	0.479	0.879	0.492
	0.3	0.939	0.498	0.964	0.492	0.893	0.455	0.905	0.451	0.879	0.465
	0.5	0.939	0.458	0.959	0.452	0.895	0.421	0.902	0.417	0.883	0.433
	0.9	0.926	0.281	0.947	0.283	0.917	0.274	0.886	0.267	0.926	0.301
20	-0.3	0.943	0.394	0.951	0.388	0.917	0.372	0.929	0.372	0.910	0.375
	0	0.949	0.386	0.949	0.379	0.919	0.364	0.930	0.364	0.912	0.368
	0.1	0.949	0.378	0.947	0.371	0.920	0.356	0.932	0.356	0.911	0.360
	0.3	0.948	0.356	0.949	0.350	0.918	0.335	0.927	0.335	0.911	0.339
	0.5	0.948	0.324	0.955	0.318	0.924	0.305	0.927	0.305	0.916	0.310
	0.9	0.944	0.194	0.947	0.193	0.936	0.187	0.916	0.187	0.941	0.195
50	-0.3	0.949	0.251	0.951	0.249	0.938	0.244	0.942	0.245	0.934	0.245
	0	0.949	0.245	0.951	0.243	0.938	0.238	0.942	0.239	0.934	0.239
	0.1	0.946	0.241	0.949	0.238	0.936	0.234	0.938	0.234	0.933	0.235
	0.3	0.946	0.226	0.952	0.224	0.934	0.220	0.938	0.221	0.931	0.222
	0.5	0.948	0.205	0.951	0.203	0.936	0.199	0.939	0.200	0.932	0.201
	0.9	0.945	0.120	0.948	0.120	0.942	0.119	0.936	0.119	0.945	0.120
100	-0.3	0.950	0.178	0.945	0.177	0.944	0.175	0.944	0.175	0.942	0.176
	0	0.949	0.174	0.946	0.173	0.944	0.171	0.945	0.172	0.943	0.172
	0.1	0.947	0.170	0.949	0.170	0.940	0.168	0.943	0.168	0.938	0.169
	0.3	0.952	0.160	0.949	0.160	0.945	0.158	0.948	0.158	0.945	0.159
	0.5	0.948	0.145	0.947	0.144	0.945	0.143	0.947	0.143	0.944	0.144
	0.9	0.949	0.085	0.947	0.085	0.946	0.084	0.943	0.084	0.946	0.085

Table C.4. Results adjusting for correlation with: Normal, 3-class Good:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.932	1.256	0.943	0.856	0.907	0.828	0.905	0.827	0.901	0.811
	0	0.934	1.739	0.943	0.827	0.906	0.799	0.905	0.798	0.902	0.782
	0.1	0.934	1.031	0.946	0.814	0.908	0.785	0.900	0.785	0.900	0.769
	0.3	0.938	0.913	0.944	0.773	0.912	0.751	0.907	0.751	0.908	0.736
	0.5	0.932	1.595	0.936	0.721	0.919	0.704	0.897	0.704	0.916	0.691
	0.9	0.925	0.801	0.915	0.545	0.943	0.545	0.881	0.548	0.949	0.548
20	-0.3	0.943	0.629	0.945	0.617	0.923	0.602	0.927	0.602	0.920	0.597
	0	0.943	0.606	0.948	0.596	0.926	0.581	0.928	0.582	0.924	0.576
	0.1	0.943	0.602	0.949	0.586	0.927	0.571	0.927	0.571	0.922	0.566
	0.3	0.943	0.569	0.946	0.555	0.929	0.542	0.927	0.543	0.925	0.537
	0.5	0.942	0.519	0.943	0.513	0.931	0.501	0.924	0.502	0.932	0.498
	0.9	0.938	0.377	0.932	0.371	0.948	0.367	0.912	0.369	0.947	0.369
50	-0.3	0.945	0.398	0.943	0.397	0.938	0.392	0.940	0.392	0.937	0.391
	0	0.944	0.384	0.947	0.383	0.937	0.377	0.940	0.378	0.935	0.377
	0.1	0.953	0.376	0.947	0.375	0.946	0.370	0.947	0.371	0.945	0.369
	0.3	0.949	0.355	0.951	0.354	0.944	0.350	0.943	0.351	0.943	0.350
	0.5	0.948	0.326	0.948	0.325	0.943	0.321	0.940	0.322	0.942	0.321
	0.9	0.946	0.224	0.944	0.226	0.951	0.225	0.935	0.226	0.952	0.227
100	-0.3	0.949	0.283	0.951	0.282	0.947	0.280	0.946	0.281	0.945	0.280
	0	0.953	0.272	0.950	0.272	0.947	0.270	0.949	0.271	0.948	0.270
	0.1	0.951	0.267	0.948	0.267	0.946	0.264	0.948	0.265	0.947	0.264
	0.3	0.948	0.252	0.948	0.251	0.943	0.250	0.946	0.250	0.943	0.250
	0.5	0.947	0.230	0.949	0.230	0.943	0.228	0.944	0.229	0.942	0.229
	0.9	0.947	0.157	0.948	0.158	0.951	0.158	0.943	0.158	0.952	0.158

Table C.5. Results adjusting for correlation with: Normal, 3-class Fair:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.921	0.719	0.946	0.695	0.901	0.678	0.912	0.674	0.912	0.678
	0	0.919	0.696	0.950	0.673	0.893	0.657	0.904	0.654	0.910	0.658
	0.1	0.921	0.685	0.950	0.659	0.899	0.647	0.907	0.644	0.914	0.650
	0.3	0.918	0.654	0.947	0.631	0.897	0.620	0.903	0.616	0.913	0.625
	0.5	0.926	0.616	0.948	0.594	0.904	0.588	0.907	0.584	0.924	0.595
	0.9	0.918	0.488	0.933	0.478	0.919	0.483	0.896	0.482	0.942	0.505
20	-0.3	0.932	0.523	0.947	0.507	0.906	0.498	0.930	0.496	0.924	0.495
	0	0.937	0.505	0.951	0.489	0.911	0.480	0.931	0.479	0.924	0.478
	0.1	0.935	0.497	0.947	0.481	0.913	0.473	0.931	0.472	0.928	0.471
	0.3	0.931	0.473	0.949	0.457	0.907	0.450	0.926	0.449	0.924	0.449
	0.5	0.936	0.443	0.951	0.429	0.913	0.423	0.928	0.422	0.933	0.423
	0.9	0.936	0.345	0.946	0.338	0.922	0.337	0.926	0.337	0.940	0.341
50	-0.3	0.945	0.337	0.949	0.332	0.932	0.329	0.943	0.329	0.941	0.329
	0	0.944	0.325	0.949	0.321	0.931	0.318	0.941	0.318	0.942	0.318
	0.1	0.946	0.320	0.950	0.315	0.934	0.313	0.943	0.313	0.944	0.313
	0.3	0.942	0.304	0.948	0.300	0.932	0.298	0.940	0.298	0.942	0.298
	0.5	0.943	0.283	0.950	0.280	0.933	0.278	0.941	0.278	0.943	0.278
	0.9	0.941	0.218	0.946	0.217	0.935	0.216	0.940	0.217	0.947	0.217
100	-0.3	0.945	0.239	0.950	0.238	0.939	0.237	0.943	0.238	0.943	0.238
	0	0.945	0.231	0.948	0.230	0.941	0.229	0.945	0.229	0.947	0.229
	0.1	0.944	0.227	0.947	0.226	0.940	0.225	0.943	0.225	0.944	0.225
	0.3	0.953	0.216	0.952	0.215	0.949	0.214	0.950	0.214	0.954	0.215
	0.5	0.948	0.201	0.949	0.200	0.944	0.199	0.945	0.200	0.946	0.200
	0.9	0.942	0.154	0.948	0.154	0.942	0.154	0.943	0.154	0.947	0.155

Table C.6. Results adjusting for correlation with: Normal, 3-class Fair:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.929	0.922	0.962	0.907	0.886	0.862	0.928	0.859	0.898	0.860
	0	0.930	0.891	0.969	0.878	0.886	0.836	0.926	0.834	0.896	0.834
	0.1	0.926	0.868	0.968	0.858	0.881	0.815	0.921	0.812	0.889	0.813
	0.3	0.937	0.817	0.968	0.805	0.894	0.768	0.928	0.766	0.899	0.767
	0.5	0.933	0.740	0.970	0.724	0.890	0.691	0.925	0.690	0.892	0.690
	0.9	0.939	0.375	0.973	0.374	0.909	0.367	0.931	0.361	0.891	0.365
20	-0.3	0.938	0.672	0.951	0.662	0.913	0.642	0.931	0.641	0.917	0.642
	0	0.938	0.650	0.952	0.639	0.918	0.622	0.933	0.621	0.921	0.622
	0.1	0.939	0.635	0.953	0.625	0.913	0.606	0.931	0.606	0.917	0.607
	0.3	0.942	0.594	0.956	0.584	0.919	0.568	0.937	0.568	0.922	0.569
	0.5	0.943	0.530	0.955	0.523	0.917	0.506	0.935	0.506	0.918	0.507
	0.9	0.946	0.263	0.959	0.262	0.925	0.253	0.940	0.254	0.922	0.254
50	-0.3	0.948	0.433	0.950	0.430	0.937	0.424	0.943	0.425	0.939	0.425
	0	0.944	0.419	0.952	0.416	0.933	0.410	0.940	0.411	0.935	0.411
	0.1	0.940	0.409	0.951	0.406	0.933	0.401	0.939	0.401	0.934	0.401
	0.3	0.951	0.382	0.953	0.379	0.942	0.374	0.948	0.374	0.943	0.374
	0.5	0.947	0.340	0.950	0.337	0.935	0.332	0.942	0.333	0.937	0.333
	0.9	0.947	0.167	0.955	0.166	0.938	0.163	0.944	0.164	0.937	0.164

Table C.7. Results adjusting for correlation with: Normal, 3-class Good:Poor, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.940	0.660	0.957	0.655	0.897	0.614	0.925	0.611	0.889	0.616
	0	0.940	0.630	0.961	0.623	0.893	0.588	0.924	0.585	0.885	0.590
	0.1	0.938	0.616	0.957	0.609	0.893	0.572	0.921	0.570	0.881	0.574
	0.3	0.941	0.573	0.959	0.566	0.895	0.532	0.924	0.530	0.884	0.535
	0.5	0.945	0.513	0.969	0.505	0.898	0.477	0.925	0.475	0.882	0.481
	0.9	0.954	0.263	0.969	0.258	0.920	0.251	0.933	0.246	0.889	0.251
20	-0.3	0.947	0.471	0.951	0.467	0.926	0.451	0.936	0.451	0.918	0.453
	0	0.949	0.450	0.951	0.446	0.927	0.431	0.936	0.431	0.921	0.433
	0.1	0.946	0.440	0.952	0.436	0.928	0.421	0.937	0.421	0.922	0.423
	0.3	0.947	0.409	0.953	0.404	0.922	0.391	0.933	0.391	0.914	0.393
	0.5	0.950	0.363	0.950	0.358	0.927	0.347	0.935	0.347	0.916	0.349
	0.9	0.949	0.181	0.954	0.178	0.925	0.173	0.933	0.172	0.910	0.173
50	-0.3	0.947	0.299	0.952	0.298	0.939	0.294	0.942	0.294	0.937	0.295
	0	0.948	0.287	0.951	0.285	0.938	0.281	0.944	0.281	0.935	0.281
	0.1	0.945	0.279	0.950	0.278	0.937	0.274	0.940	0.274	0.933	0.274
	0.3	0.950	0.259	0.951	0.258	0.940	0.254	0.944	0.254	0.937	0.255
	0.5	0.951	0.230	0.949	0.228	0.940	0.225	0.943	0.225	0.935	0.226
	0.9	0.950	0.112	0.950	0.112	0.939	0.110	0.942	0.110	0.935	0.110
100	-0.3	0.948	0.212	0.953	0.211	0.944	0.210	0.946	0.210	0.943	0.210
	0	0.950	0.203	0.948	0.203	0.946	0.201	0.947	0.202	0.944	0.202
	0.1	0.951	0.198	0.948	0.197	0.944	0.196	0.946	0.196	0.942	0.196
	0.3	0.947	0.184	0.952	0.183	0.943	0.182	0.943	0.182	0.941	0.182
	0.5	0.952	0.162	0.949	0.162	0.948	0.161	0.949	0.161	0.945	0.161
	0.9	0.948	0.079	0.949	0.079	0.941	0.078	0.945	0.078	0.940	0.078

Table C.8. Results adjusting for correlation with: Normal, 3-class Good:Poor, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.976	0.468	0.944	0.447	0.881	0.400	0.919	0.394	0.790	0.425
	0	0.978	0.447	0.944	0.423	0.878	0.381	0.915	0.375	0.780	0.406
	0.1	0.978	0.436	0.944	0.412	0.880	0.372	0.918	0.366	0.784	0.398
	0.3	0.982	0.406	0.941	0.381	0.890	0.345	0.918	0.339	0.790	0.370
	0.5	0.985	0.365	0.949	0.339	0.883	0.309	0.913	0.303	0.771	0.333
	0.9	0.986	0.185	0.948	0.170	0.909	0.161	0.918	0.153	0.766	0.176
20	-0.3	0.962	0.338	0.948	0.328	0.909	0.307	0.930	0.308	0.865	0.314
	0	0.958	0.323	0.947	0.312	0.904	0.294	0.924	0.295	0.858	0.301
	0.1	0.965	0.315	0.946	0.304	0.913	0.286	0.932	0.286	0.865	0.293
	0.3	0.965	0.294	0.948	0.283	0.910	0.266	0.928	0.267	0.857	0.273
	0.5	0.971	0.261	0.949	0.250	0.916	0.235	0.932	0.236	0.862	0.242
	0.9	0.975	0.129	0.951	0.123	0.923	0.115	0.931	0.116	0.853	0.120
50	-0.3	0.956	0.217	0.948	0.214	0.936	0.209	0.945	0.209	0.920	0.210
	0	0.954	0.208	0.949	0.204	0.934	0.199	0.941	0.199	0.915	0.200
	0.1	0.955	0.202	0.950	0.198	0.934	0.193	0.942	0.194	0.916	0.195
	0.3	0.949	0.188	0.947	0.184	0.930	0.180	0.936	0.180	0.910	0.181
	0.5	0.958	0.166	0.948	0.162	0.934	0.158	0.942	0.159	0.914	0.160
	0.9	0.961	0.081	0.951	0.079	0.940	0.077	0.942	0.077	0.909	0.078
100	-0.3	0.951	0.154	0.950	0.153	0.940	0.151	0.945	0.151	0.933	0.152
	0	0.950	0.147	0.948	0.146	0.940	0.144	0.944	0.144	0.932	0.145
	0.1	0.951	0.143	0.948	0.142	0.943	0.140	0.946	0.140	0.933	0.141
	0.3	0.953	0.133	0.952	0.132	0.943	0.130	0.946	0.130	0.930	0.130
	0.5	0.951	0.118	0.948	0.116	0.940	0.115	0.945	0.115	0.928	0.115
	0.9	0.958	0.057	0.950	0.056	0.946	0.056	0.948	0.056	0.934	0.056

Table C.9. Results adjusting for correlation with: Normal, 2-class, Good:Good, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.859	0.401	0.958	0.618	0.860	0.577	0.928	0.576	0.759	0.570
	0	0.870	0.387	0.956	0.604	0.865	0.557	0.928	0.559	0.760	0.554
	0.1	0.861	0.378	0.956	0.596	0.858	0.555	0.927	0.557	0.753	0.556
	0.3	0.878	0.358	0.962	0.575	0.871	0.530	0.933	0.539	0.754	0.535
	0.5	0.884	0.324	0.962	0.543	0.870	0.497	0.935	0.510	0.741	0.515
	0.9	0.923	0.181	0.976	0.412	0.910	0.364	0.957	0.399	0.759	0.415
20	-0.3	0.902	0.302	0.957	0.532	0.927	0.487	0.936	0.508	0.848	0.500
	0	0.904	0.289	0.959	0.511	0.923	0.471	0.942	0.492	0.847	0.485
	0.1	0.908	0.284	0.958	0.508	0.929	0.465	0.940	0.486	0.845	0.480
	0.3	0.906	0.265	0.961	0.487	0.927	0.444	0.943	0.467	0.838	0.463
	0.5	0.913	0.238	0.964	0.462	0.929	0.419	0.946	0.447	0.841	0.443
	0.9	0.937	0.123	0.977	0.342	0.950	0.303	0.960	0.336	0.857	0.337
50	-0.3	0.945	0.198	0.953	0.314	0.978	0.308	0.944	0.309	0.918	0.320
	0	0.947	0.190	0.952	0.302	0.977	0.296	0.942	0.298	0.915	0.310
	0.1	0.945	0.186	0.953	0.300	0.977	0.293	0.939	0.296	0.912	0.307
	0.3	0.945	0.173	0.954	0.285	0.977	0.280	0.942	0.283	0.912	0.295
	0.5	0.951	0.154	0.957	0.264	0.978	0.259	0.946	0.261	0.916	0.274
	0.9	0.953	0.076	0.961	0.163	0.978	0.172	0.951	0.163	0.915	0.173
100	-0.3	0.953	0.141	0.947	0.162	0.973	0.170	0.947	0.163	0.936	0.167
	0	0.954	0.136	0.947	0.154	0.973	0.163	0.948	0.155	0.935	0.160
	0.1	0.953	0.132	0.950	0.151	0.971	0.160	0.947	0.152	0.934	0.157
	0.3	0.952	0.123	0.951	0.142	0.972	0.150	0.946	0.143	0.930	0.148
	0.5	0.951	0.109	0.946	0.128	0.966	0.136	0.941	0.127	0.928	0.132
	0.9	0.956	0.053	0.955	0.065	0.968	0.078	0.947	0.065	0.931	0.068

Table C.10. Results adjusting for correlation with: Normal, 2-class, Good:Good, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.927	0.612	0.944	0.593	0.876	0.552	0.896	0.545	0.862	0.569
	0	0.930	0.582	0.941	0.560	0.878	0.524	0.892	0.518	0.863	0.543
	0.1	0.927	0.566	0.942	0.544	0.878	0.511	0.893	0.504	0.867	0.531
	0.3	0.929	0.531	0.942	0.507	0.876	0.477	0.889	0.471	0.870	0.499
	0.5	0.922	0.479	0.940	0.455	0.873	0.432	0.879	0.425	0.871	0.456
	0.9	0.897	0.296	0.933	0.283	0.876	0.281	0.859	0.273	0.918	0.317
20	-0.3	0.941	0.448	0.942	0.437	0.915	0.420	0.929	0.419	0.909	0.425
	0	0.944	0.424	0.943	0.412	0.916	0.397	0.927	0.397	0.909	0.403
	0.1	0.938	0.411	0.946	0.399	0.910	0.385	0.919	0.385	0.903	0.391
	0.3	0.941	0.385	0.946	0.373	0.912	0.361	0.921	0.360	0.908	0.367
	0.5	0.939	0.345	0.947	0.334	0.909	0.323	0.916	0.323	0.911	0.331
	0.9	0.925	0.210	0.941	0.206	0.914	0.203	0.907	0.202	0.938	0.213
50	-0.3	0.947	0.288	0.950	0.285	0.935	0.280	0.941	0.280	0.931	0.281
	0	0.947	0.273	0.947	0.269	0.935	0.265	0.940	0.265	0.931	0.266
	0.1	0.945	0.265	0.944	0.262	0.933	0.257	0.939	0.258	0.932	0.259
	0.3	0.945	0.247	0.948	0.243	0.932	0.240	0.936	0.240	0.929	0.242
	0.5	0.945	0.221	0.950	0.218	0.933	0.214	0.936	0.215	0.931	0.217
	0.9	0.939	0.133	0.946	0.132	0.934	0.131	0.930	0.131	0.945	0.134
100	-0.3	0.951	0.205	0.951	0.204	0.946	0.202	0.949	0.202	0.943	0.203
	0	0.949	0.194	0.950	0.192	0.943	0.191	0.944	0.191	0.942	0.192
	0.1	0.947	0.188	0.948	0.187	0.940	0.186	0.941	0.186	0.938	0.187
	0.3	0.948	0.175	0.946	0.174	0.941	0.172	0.942	0.172	0.938	0.173
	0.5	0.951	0.157	0.954	0.155	0.942	0.154	0.945	0.154	0.941	0.155
	0.9	0.946	0.094	0.949	0.094	0.944	0.094	0.942	0.094	0.949	0.095

Table C.11. Results adjusting for correlation with: Normal, 2-class Good:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.830	0.494	0.952	0.600	0.844	0.592	0.906	0.570	0.832	0.568
	0	0.837	0.477	0.954	0.586	0.844	0.574	0.911	0.554	0.832	0.554
	0.1	0.833	0.467	0.954	0.575	0.846	0.565	0.906	0.547	0.831	0.545
	0.3	0.837	0.441	0.954	0.552	0.847	0.536	0.906	0.524	0.828	0.520
	0.5	0.845	0.407	0.955	0.517	0.854	0.500	0.902	0.491	0.835	0.492
	0.9	0.835	0.264	0.943	0.380	0.856	0.373	0.888	0.374	0.869	0.390
20	-0.3	0.895	0.377	0.954	0.495	0.912	0.472	0.931	0.480	0.889	0.465
	0	0.894	0.362	0.957	0.483	0.912	0.459	0.932	0.468	0.886	0.451
	0.1	0.897	0.353	0.956	0.474	0.912	0.448	0.930	0.457	0.885	0.441
	0.3	0.898	0.330	0.959	0.449	0.911	0.425	0.934	0.437	0.888	0.419
	0.5	0.895	0.301	0.959	0.421	0.916	0.396	0.929	0.410	0.888	0.392
	0.9	0.882	0.184	0.957	0.308	0.916	0.287	0.920	0.304	0.916	0.282
50	-0.3	0.945	0.249	0.952	0.312	0.964	0.303	0.941	0.307	0.933	0.305
	0	0.945	0.237	0.953	0.299	0.965	0.293	0.944	0.296	0.936	0.295
	0.1	0.939	0.231	0.954	0.293	0.958	0.286	0.937	0.290	0.928	0.289
	0.3	0.942	0.215	0.949	0.275	0.960	0.268	0.939	0.271	0.930	0.269
	0.5	0.946	0.193	0.957	0.251	0.964	0.245	0.945	0.248	0.938	0.245
	0.9	0.932	0.114	0.959	0.171	0.956	0.171	0.940	0.169	0.948	0.158
100	-0.3	0.947	0.177	0.954	0.187	0.955	0.189	0.945	0.186	0.942	0.186
	0	0.946	0.168	0.950	0.178	0.951	0.180	0.944	0.177	0.937	0.178
	0.1	0.949	0.164	0.951	0.175	0.955	0.177	0.944	0.174	0.939	0.174
	0.3	0.946	0.153	0.950	0.162	0.953	0.165	0.943	0.162	0.938	0.162
	0.5	0.944	0.137	0.951	0.146	0.950	0.149	0.940	0.145	0.936	0.145
	0.9	0.943	0.080	0.953	0.089	0.953	0.095	0.944	0.089	0.950	0.087

Table C.12. Results adjusting for correlation with: Normal, 2-class Good:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.931	0.742	0.944	0.728	0.888	0.687	0.915	0.682	0.868	0.693
	0	0.940	0.689	0.946	0.672	0.888	0.634	0.919	0.630	0.864	0.641
	0.1	0.943	0.666	0.949	0.648	0.894	0.611	0.921	0.607	0.867	0.617
	0.3	0.943	0.611	0.949	0.593	0.883	0.560	0.910	0.556	0.855	0.568
	0.5	0.947	0.540	0.951	0.522	0.885	0.493	0.911	0.489	0.845	0.501
	0.9	0.961	0.268	0.954	0.258	0.909	0.250	0.921	0.245	0.849	0.255
20	-0.3	0.945	0.543	0.951	0.535	0.918	0.518	0.935	0.517	0.913	0.519
	0	0.944	0.503	0.947	0.493	0.919	0.478	0.934	0.478	0.912	0.480
	0.1	0.946	0.486	0.948	0.476	0.921	0.462	0.934	0.461	0.911	0.464
	0.3	0.949	0.444	0.951	0.434	0.920	0.420	0.935	0.420	0.906	0.422
	0.5	0.950	0.390	0.947	0.380	0.917	0.368	0.931	0.368	0.902	0.370
	0.9	0.954	0.189	0.948	0.184	0.929	0.178	0.936	0.178	0.902	0.180
50	-0.3	0.947	0.350	0.950	0.348	0.937	0.343	0.944	0.344	0.937	0.344
	0	0.942	0.325	0.953	0.322	0.932	0.318	0.936	0.318	0.929	0.318
	0.1	0.943	0.313	0.946	0.310	0.932	0.306	0.938	0.306	0.929	0.306
	0.3	0.951	0.286	0.951	0.283	0.939	0.278	0.945	0.279	0.935	0.279
	0.5	0.951	0.250	0.948	0.247	0.940	0.243	0.945	0.244	0.936	0.244
	0.9	0.954	0.119	0.953	0.117	0.942	0.115	0.945	0.116	0.934	0.116
100	-0.3	0.947	0.250	0.948	0.249	0.942	0.247	0.944	0.247	0.941	0.247
	0	0.950	0.231	0.946	0.230	0.945	0.228	0.948	0.229	0.946	0.229
	0.1	0.947	0.223	0.947	0.222	0.941	0.220	0.945	0.221	0.941	0.221
	0.3	0.947	0.203	0.951	0.202	0.940	0.200	0.942	0.201	0.937	0.201
	0.5	0.948	0.177	0.950	0.176	0.945	0.174	0.946	0.175	0.942	0.175
	0.9	0.954	0.084	0.952	0.083	0.948	0.083	0.949	0.083	0.943	0.083

Table C.13. Results adjusting for correlation with: Normal, 2-class Fair:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.865	0.579	0.956	0.602	0.849	0.614	0.925	0.582	0.874	0.596
	0	0.872	0.549	0.959	0.577	0.849	0.580	0.927	0.553	0.865	0.567
	0.1	0.877	0.537	0.960	0.565	0.851	0.569	0.929	0.543	0.863	0.558
	0.3	0.893	0.505	0.961	0.536	0.858	0.534	0.930	0.513	0.850	0.529
	0.5	0.910	0.453	0.969	0.487	0.877	0.486	0.936	0.469	0.848	0.491
	0.9	0.952	0.247	0.979	0.297	0.916	0.304	0.958	0.295	0.835	0.343
20	-0.3	0.911	0.446	0.955	0.476	0.903	0.467	0.938	0.464	0.916	0.463
	0	0.911	0.421	0.958	0.451	0.899	0.439	0.936	0.438	0.905	0.437
	0.1	0.918	0.409	0.956	0.440	0.905	0.426	0.938	0.427	0.908	0.425
	0.3	0.918	0.379	0.957	0.411	0.903	0.397	0.940	0.399	0.902	0.398
	0.5	0.929	0.337	0.962	0.370	0.913	0.355	0.942	0.360	0.900	0.360
	0.9	0.946	0.170	0.972	0.216	0.940	0.203	0.956	0.213	0.903	0.219
50	-0.3	0.944	0.293	0.952	0.307	0.946	0.301	0.942	0.303	0.941	0.304
	0	0.946	0.275	0.950	0.289	0.948	0.282	0.942	0.284	0.938	0.285
	0.1	0.945	0.267	0.949	0.280	0.948	0.274	0.942	0.276	0.937	0.277
	0.3	0.945	0.245	0.950	0.258	0.946	0.252	0.942	0.254	0.936	0.255
	0.5	0.949	0.216	0.950	0.228	0.953	0.223	0.946	0.225	0.939	0.226
	0.9	0.948	0.105	0.958	0.117	0.954	0.115	0.943	0.115	0.931	0.116
100	-0.3	0.949	0.208	0.949	0.209	0.946	0.208	0.947	0.208	0.945	0.208
	0	0.946	0.196	0.949	0.196	0.945	0.195	0.944	0.195	0.941	0.195
	0.1	0.945	0.190	0.949	0.190	0.942	0.189	0.942	0.189	0.940	0.189
	0.3	0.951	0.174	0.950	0.175	0.948	0.173	0.948	0.174	0.943	0.174
	0.5	0.952	0.153	0.947	0.153	0.948	0.152	0.947	0.152	0.943	0.153
	0.9	0.954	0.074	0.951	0.074	0.949	0.074	0.948	0.073	0.941	0.073

Table C.14. Results adjusting for correlation with: Normal, 2-class Fair:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.926	0.730	0.942	0.705	0.887	0.685	0.908	0.678	0.912	0.676
	0	0.916	0.689	0.938	0.663	0.876	0.647	0.897	0.641	0.905	0.641
	0.1	0.922	0.672	0.942	0.648	0.885	0.633	0.904	0.628	0.913	0.628
	0.3	0.923	0.634	0.938	0.611	0.887	0.600	0.899	0.596	0.918	0.600
	0.5	0.922	0.586	0.939	0.567	0.890	0.557	0.895	0.554	0.920	0.562
	0.9	0.927	0.447	0.934	0.446	0.915	0.449	0.900	0.452	0.940	0.464
20	-0.3	0.936	0.536	0.947	0.527	0.918	0.517	0.929	0.517	0.927	0.515
	0	0.938	0.504	0.947	0.496	0.921	0.486	0.927	0.485	0.928	0.485
	0.1	0.944	0.491	0.945	0.483	0.923	0.472	0.933	0.472	0.934	0.473
	0.3	0.943	0.461	0.944	0.452	0.920	0.444	0.929	0.444	0.934	0.446
	0.5	0.936	0.423	0.946	0.416	0.920	0.411	0.924	0.411	0.935	0.414
	0.9	0.936	0.317	0.940	0.318	0.932	0.317	0.922	0.319	0.947	0.322
50	-0.3	0.944	0.346	0.948	0.344	0.938	0.340	0.938	0.341	0.940	0.341
	0	0.948	0.324	0.948	0.323	0.940	0.320	0.943	0.320	0.944	0.320
	0.1	0.946	0.316	0.948	0.314	0.940	0.311	0.941	0.312	0.943	0.312
	0.3	0.943	0.296	0.950	0.294	0.938	0.292	0.939	0.292	0.941	0.293
	0.5	0.942	0.271	0.950	0.269	0.937	0.268	0.939	0.268	0.941	0.269
	0.9	0.945	0.200	0.947	0.201	0.946	0.201	0.940	0.202	0.950	0.203
100	-0.3	0.950	0.246	0.952	0.245	0.947	0.244	0.948	0.244	0.947	0.244
	0	0.945	0.231	0.947	0.229	0.942	0.229	0.945	0.229	0.942	0.229
	0.1	0.950	0.224	0.944	0.224	0.946	0.223	0.946	0.223	0.946	0.223
	0.3	0.950	0.210	0.952	0.209	0.946	0.209	0.948	0.209	0.950	0.209
	0.5	0.945	0.192	0.947	0.191	0.943	0.191	0.945	0.191	0.948	0.192
	0.9	0.948	0.142	0.949	0.142	0.950	0.142	0.945	0.142	0.951	0.143

Table C.15. Results adjusting for correlation with: Normal, 2-class Good:Poor, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.833	0.426	0.951	0.534	0.892	0.565	0.927	0.519	0.893	0.521
	0	0.836	0.410	0.949	0.523	0.889	0.547	0.923	0.503	0.883	0.502
	0.1	0.838	0.403	0.947	0.512	0.889	0.539	0.921	0.496	0.880	0.494
	0.3	0.847	0.390	0.950	0.498	0.893	0.518	0.923	0.480	0.887	0.473
	0.5	0.851	0.370	0.948	0.475	0.892	0.497	0.919	0.461	0.884	0.453
	0.9	0.869	0.309	0.943	0.404	0.901	0.436	0.921	0.401	0.891	NA
20	-0.3	0.876	0.311	0.953	0.415	0.902	0.408	0.941	0.400	0.917	0.397
	0	0.890	0.300	0.955	0.402	0.908	0.395	0.940	0.389	0.913	0.381
	0.1	0.895	0.294	0.952	0.394	0.916	0.389	0.941	0.383	0.918	NA
	0.3	0.892	0.279	0.952	0.386	0.909	0.379	0.939	0.373	0.910	0.358
	0.5	0.904	0.264	0.951	0.368	0.922	0.361	0.939	0.356	0.913	0.335
	0.9	0.917	0.211	0.951	0.322	0.934	0.318	0.942	0.311	0.918	NA
50	-0.3	0.919	0.205	0.949	0.257	0.930	0.257	0.943	0.253	0.935	0.254
	0	0.928	0.196	0.952	0.249	0.937	0.249	0.948	0.246	0.935	0.245
	0.1	0.931	0.192	0.950	0.244	0.939	0.246	0.947	0.242	0.939	0.240
	0.3	0.937	0.182	0.952	0.235	0.942	0.236	0.948	0.232	0.938	0.227
	0.5	0.938	0.169	0.956	0.223	0.943	0.226	0.943	0.221	0.930	0.213
	0.9	0.948	0.127	0.955	0.189	0.960	0.193	0.951	0.184	0.942	0.163
100	-0.3	0.931	0.147	0.953	0.155	0.929	0.159	0.946	0.155	0.936	0.156
	0	0.939	0.141	0.950	0.148	0.932	0.153	0.949	0.149	0.937	0.149
	0.1	0.936	0.137	0.951	0.145	0.935	0.150	0.948	0.145	0.939	0.144
	0.3	0.940	0.129	0.948	0.137	0.934	0.142	0.950	0.137	0.939	0.136
	0.5	0.943	0.119	0.953	0.128	0.938	0.132	0.947	0.127	0.936	0.124
	0.9	0.951	0.087	0.956	0.098	0.949	0.103	0.955	0.097	0.945	0.093

Table C.16. Results adjusting for correlation with: Normal, 2-class Good:Poor, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.856	0.650	0.976	1.505	0.978	0.948	0.973	0.961	0.914	1.005
	0	0.850	0.617	0.977	1.485	0.977	0.935	0.973	0.952	0.913	NA
	0.1	0.855	0.610	0.976	1.471	0.978	0.919	0.976	0.928	0.915	NA
	0.3	0.852	0.586	0.980	1.462	0.980	0.899	0.978	0.907	0.913	NA
	0.5	0.851	0.518	0.985	1.414	0.981	0.862	0.976	0.858	0.901	NA
	0.9	0.867	0.288	0.990	1.247	0.991	0.685	0.971	0.609	0.873	NA
20	-0.3	0.856	0.466	0.916	1.228	0.941	0.447	0.944	0.408	0.905	NA
	0	0.856	0.448	0.930	1.216	0.940	0.436	0.945	0.398	0.904	NA
	0.1	0.853	0.437	0.929	1.210	0.941	0.431	0.944	0.392	0.904	NA
	0.3	0.862	0.418	0.932	1.197	0.941	0.415	0.947	0.374	0.901	NA
	0.5	0.854	0.366	0.939	1.176	0.944	0.384	0.943	0.338	0.896	NA
	0.9	0.861	0.195	0.962	1.090	0.975	0.264	0.942	0.183	0.885	NA
50	-0.3	0.919	0.305	0.872	1.057	0.946	0.277	0.944	0.254	0.922	NA
	0	0.919	0.293	0.884	1.039	0.948	0.271	0.947	0.247	0.924	NA
	0.1	0.912	0.285	0.884	1.039	0.948	0.267	0.947	0.243	0.923	NA
	0.3	0.909	0.272	0.897	1.032	0.947	0.258	0.943	0.232	0.916	0.233
	0.5	0.905	0.237	0.899	1.004	0.945	0.235	0.934	0.207	0.908	NA
	0.9	0.906	0.121	0.927	0.917	0.970	0.152	0.941	0.107	0.911	0.107
100	-0.3	0.941	0.219	0.864	0.849	0.961	0.212	0.943	0.185	0.926	NA
	0	0.941	0.210	0.866	0.820	0.959	0.208	0.944	0.182	0.928	NA
	0.1	0.938	0.205	0.868	0.812	0.959	0.205	0.943	0.178	0.928	NA
	0.3	0.940	0.195	0.871	0.791	0.964	0.199	0.944	0.171	0.926	0.172
	0.5	0.930	0.169	0.880	0.761	0.964	0.184	0.943	0.153	0.925	NA
	0.9	0.922	0.085	0.890	0.603	0.978	0.124	0.940	0.079	0.922	NA

Table C.17. Results adjusting for correlation with: Gamma, 3-class, Good:Good, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.826	0.362	0.966	0.950	0.976	0.538	0.965	0.491	0.881	0.553
	0	0.825	0.356	0.969	0.940	0.977	0.540	0.963	0.493	0.872	0.555
	0.1	0.826	0.348	0.969	0.937	0.978	0.534	0.963	0.484	0.865	0.555
	0.3	0.830	0.348	0.971	0.930	0.977	0.526	0.964	0.478	0.863	0.549
	0.5	0.840	0.293	0.977	0.886	0.980	0.503	0.966	0.443	0.855	0.523
	0.9	0.875	0.153	0.978	0.753	0.991	0.409	0.964	0.306	0.846	0.413
20	-0.3	0.911	0.262	0.904	0.547	0.908	0.221	0.936	0.208	0.874	0.212
	0	0.909	0.259	0.914	0.546	0.910	0.219	0.940	0.207	0.877	0.211
	0.1	0.912	0.253	0.912	0.536	0.909	0.216	0.939	0.203	0.875	0.208
	0.3	0.908	0.253	0.912	0.543	0.909	0.215	0.939	0.203	0.874	0.207
	0.5	0.914	0.212	0.914	0.506	0.914	0.191	0.943	0.177	0.870	0.182
	0.9	0.934	0.107	0.915	0.382	0.928	0.114	0.944	0.093	0.866	0.101
50	-0.3	0.973	0.169	0.839	0.199	0.920	0.138	0.943	0.138	0.908	0.138
	0	0.971	0.168	0.844	0.200	0.918	0.137	0.943	0.137	0.910	0.138
	0.1	0.969	0.164	0.840	0.195	0.916	0.135	0.940	0.135	0.904	0.136
	0.3	0.972	0.163	0.847	0.199	0.915	0.134	0.941	0.134	0.903	0.135
	0.5	0.967	0.136	0.851	0.176	0.914	0.117	0.941	0.117	0.898	0.118
	0.9	0.959	0.068	0.848	0.113	0.913	0.060	0.938	0.060	0.895	0.060
100	-0.3	0.973	0.121	0.818	0.120	0.929	0.102	0.942	0.102	0.921	0.102
	0	0.975	0.119	0.829	0.120	0.931	0.101	0.942	0.102	0.923	0.102
	0.1	0.970	0.116	0.832	0.116	0.929	0.100	0.941	0.100	0.922	0.101
	0.3	0.972	0.116	0.832	0.116	0.935	0.099	0.945	0.099	0.925	0.100
	0.5	0.965	0.097	0.836	0.097	0.930	0.087	0.943	0.087	0.921	0.087
	0.9	0.958	0.048	0.830	0.050	0.929	0.045	0.943	0.045	0.916	0.045

Table C.18. Results adjusting for correlation with: Gamma, 3-class, Good:Good, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.505	0.828	0.578	1.372	0.734	1.091	0.694	1.090	0.722	1.128
	0	0.468	0.758	0.560	1.353	0.719	1.055	0.678	1.054	0.692	1.087
	0.1	0.463	0.736	0.544	1.340	0.709	1.037	0.668	1.036	0.677	NA
	0.3	0.425	0.722	0.546	1.339	0.701	1.012	0.640	1.013	0.629	1.049
	0.5	0.350	0.702	0.521	1.304	0.677	0.976	0.575	0.976	0.561	NA
	0.9	0.085	0.490	0.436	1.256	0.563	0.845	0.228	0.837	0.216	NA
20	-0.3	0.276	0.564	0.166	0.815	0.407	0.656	0.268	0.654	0.268	NA
	0	0.230	0.537	0.150	0.786	0.364	0.634	0.227	0.631	0.225	NA
	0.1	0.211	0.523	0.148	0.787	0.345	0.621	0.202	0.617	0.209	0.626
	0.3	0.180	0.500	0.129	0.758	0.317	0.602	0.167	0.600	0.181	NA
	0.5	0.108	0.452	0.116	0.726	0.258	0.568	0.105	0.562	0.119	0.577
	0.9	0.008	0.303	0.069	0.629	0.126	0.476	0.008	0.465	0.019	0.502
50	-0.3	0.019	0.365	0.002	0.305	0.035	0.407	0.014	0.386	0.016	0.390
	0	0.015	0.345	0.003	0.290	0.031	0.392	0.010	0.369	0.011	0.375
	0.1	0.010	0.336	0.002	0.283	0.024	0.386	0.006	0.362	0.008	NA
	0.3	0.005	0.319	0.002	0.272	0.015	0.372	0.004	0.346	0.005	NA
	0.5	0.001	0.284	0.001	0.249	0.009	0.352	0.001	0.322	0.001	NA
	0.9	0.000	0.186	0.001	0.182	0.000	0.291	0.000	0.246	0.000	NA
100	-0.3	0.000	0.259	0.000	0.148	0.001	0.316	0.000	0.279	0.000	0.283
	0	0.000	0.245	0.000	0.141	0.000	0.307	0.000	0.268	0.000	NA
	0.1	0.000	0.239	0.000	0.138	0.000	0.303	0.000	0.263	0.000	0.268
	0.3	0.000	0.226	0.000	0.129	0.000	0.294	0.000	0.253	0.000	0.259
	0.5	0.000	0.201	0.000	0.119	0.000	0.279	0.000	0.234	0.000	NA
	0.9	0.000	0.130	0.000	0.095	0.000	0.239	0.000	0.180	0.000	NA

Table C.19. Results adjusting for correlation with: Gamma, 3-class Good:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.001	0.389	0.082	1.031	0.001	0.603	0.001	0.588	0.003	0.630
	0	0.000	0.382	0.075	1.015	0.001	0.596	0.001	0.584	0.004	0.626
	0.1	0.000	0.374	0.073	1.012	0.002	0.592	0.001	0.577	0.003	0.617
	0.3	0.000	0.372	0.065	0.998	0.001	0.581	0.001	0.564	0.003	0.606
	0.5	0.000	0.320	0.049	0.957	0.002	0.563	0.001	0.541	0.004	0.592
	0.9	0.000	0.204	0.010	0.833	0.001	0.473	0.001	0.433	0.003	0.490
20	-0.3	0.000	0.284	0.004	0.526	0.000	0.295	0.000	0.285	0.000	0.287
	0	0.000	0.278	0.004	0.526	0.000	0.289	0.000	0.279	0.000	0.280
	0.1	0.000	0.272	0.003	0.522	0.000	0.286	0.000	0.275	0.000	0.278
	0.3	0.000	0.271	0.002	0.516	0.000	0.281	0.000	0.269	0.000	0.272
	0.5	0.000	0.232	0.001	0.513	0.000	0.258	0.000	0.245	0.000	0.250
	0.9	0.000	0.143	0.000	0.482	0.000	0.201	0.000	0.184	0.000	0.194
50	-0.3	0.000	0.185	0.000	0.200	0.000	0.177	0.000	0.177	0.000	0.178
	0	0.000	0.181	0.000	0.201	0.000	0.173	0.000	0.173	0.000	0.174
	0.1	0.000	0.177	0.000	0.202	0.000	0.171	0.000	0.170	0.000	0.171
	0.3	0.000	0.175	0.000	0.203	0.000	0.168	0.000	0.167	0.000	0.168
	0.5	0.000	0.150	0.000	0.202	0.000	0.153	0.000	0.152	0.000	0.153
	0.9	0.000	0.090	0.000	0.200	0.000	0.111	0.000	0.110	0.000	0.111
100	-0.3	0.000	0.131	0.000	0.113	0.000	0.130	0.000	0.130	0.000	0.130
	0	0.000	0.129	0.000	0.112	0.000	0.128	0.000	0.127	0.000	0.128
	0.1	0.000	0.126	0.000	0.112	0.000	0.126	0.000	0.126	0.000	0.127
	0.3	0.000	0.125	0.000	0.111	0.000	0.124	0.000	0.123	0.000	0.124
	0.5	0.000	0.106	0.000	0.109	0.000	0.113	0.000	0.112	0.000	0.113
	0.9	0.000	0.063	0.000	0.106	0.000	0.082	0.000	0.081	0.000	0.082

Table C.20. Results adjusting for correlation with: Gamma, 3-class Good:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.822	1.021	0.953	1.275	0.886	1.230	0.969	1.215	0.953	1.232
	0	0.830	0.880	0.962	1.240	0.892	1.159	0.977	1.155	0.941	1.159
	0.1	0.825	0.836	0.966	1.221	0.903	1.134	0.979	1.134	0.944	1.138
	0.3	0.839	0.856	0.970	1.190	0.905	1.074	0.982	1.084	0.933	NA
	0.5	0.838	0.780	0.978	1.126	0.909	0.998	0.987	1.019	0.930	1.024
	0.9	0.871	0.422	0.993	0.918	0.935	0.726	0.990	0.768	0.904	NA
20	-0.3	0.896	0.666	0.964	1.012	0.933	0.839	0.956	0.855	0.944	0.853
	0	0.893	0.618	0.966	0.976	0.929	0.780	0.961	0.795	0.937	0.788
	0.1	0.891	0.589	0.972	0.961	0.939	0.758	0.959	0.775	0.941	0.769
	0.3	0.896	0.550	0.976	0.931	0.938	0.713	0.964	0.730	0.935	0.724
	0.5	0.898	0.482	0.983	0.883	0.943	0.651	0.967	0.665	0.929	NA
	0.9	0.899	0.247	0.994	0.709	0.967	0.445	0.973	0.429	0.924	NA
50	-0.3	0.916	0.422	0.987	0.679	0.942	0.517	0.949	0.506	0.927	0.499
	0	0.915	0.390	0.988	0.641	0.943	0.484	0.954	0.469	0.924	NA
	0.1	0.912	0.376	0.990	0.626	0.948	0.472	0.951	0.454	0.928	0.451
	0.3	0.908	0.348	0.990	0.588	0.946	0.442	0.953	0.419	0.923	NA
	0.5	0.902	0.299	0.989	0.537	0.949	0.401	0.953	0.368	0.918	NA
	0.9	0.902	0.144	0.971	0.297	0.979	0.272	0.958	0.186	0.920	NA
100	-0.3	0.912	0.299	0.981	0.388	0.951	0.402	0.947	0.369	0.928	NA
	0	0.910	0.276	0.973	0.354	0.958	0.382	0.953	0.342	0.931	NA
	0.1	0.905	0.266	0.971	0.340	0.956	0.372	0.947	0.330	0.925	NA
	0.3	0.910	0.246	0.963	0.310	0.961	0.352	0.953	0.305	0.929	NA
	0.5	0.904	0.212	0.957	0.260	0.965	0.325	0.955	0.268	0.928	0.269
	0.9	0.891	0.101	0.921	0.112	0.976	0.241	0.956	0.133	0.921	0.142

Table C.21. Results adjusting for correlation with: Gamma, 3-class Fair:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.701	0.413	0.966	0.914	0.959	0.673	0.962	0.680	0.939	0.704
	0	0.707	0.402	0.969	0.907	0.957	0.647	0.967	0.654	0.927	0.673
	0.1	0.708	0.392	0.973	0.903	0.959	0.639	0.966	0.644	0.931	0.667
	0.3	0.713	0.389	0.980	0.900	0.956	0.615	0.967	0.621	0.916	0.641
	0.5	0.720	0.335	0.987	0.871	0.960	0.583	0.972	0.587	0.911	0.617
	0.9	0.792	0.186	0.994	0.771	0.975	0.439	0.976	0.413	0.887	0.465
20	-0.3	0.780	0.304	0.955	0.703	0.929	0.361	0.941	0.354	0.911	0.353
	0	0.786	0.296	0.964	0.698	0.929	0.347	0.945	0.339	0.907	0.338
	0.1	0.791	0.288	0.965	0.695	0.925	0.339	0.946	0.331	0.906	0.331
	0.3	0.792	0.284	0.965	0.691	0.928	0.326	0.947	0.318	0.902	0.318
	0.5	0.798	0.240	0.973	0.659	0.930	0.293	0.950	0.283	0.897	0.285
	0.9	0.846	0.126	0.978	0.542	0.947	0.183	0.950	0.158	0.884	0.169
50	-0.3	0.892	0.200	0.929	0.383	0.921	0.213	0.936	0.212	0.917	0.212
	0	0.894	0.193	0.926	0.372	0.923	0.204	0.939	0.202	0.918	0.203
	0.1	0.897	0.188	0.926	0.369	0.925	0.199	0.941	0.198	0.919	0.199
	0.3	0.902	0.184	0.922	0.366	0.921	0.189	0.937	0.188	0.913	0.189
	0.5	0.899	0.155	0.930	0.334	0.924	0.166	0.941	0.165	0.916	0.166
	0.9	0.892	0.077	0.927	0.234	0.926	0.085	0.938	0.083	0.905	0.083
100	-0.3	0.914	0.142	0.900	0.187	0.932	0.156	0.939	0.155	0.927	0.156
	0	0.915	0.138	0.895	0.182	0.935	0.149	0.943	0.149	0.927	0.149
	0.1	0.912	0.134	0.900	0.177	0.931	0.145	0.938	0.145	0.923	0.146
	0.3	0.922	0.131	0.892	0.174	0.933	0.139	0.942	0.138	0.928	0.139
	0.5	0.909	0.110	0.898	0.152	0.934	0.122	0.942	0.122	0.926	0.122
	0.9	0.905	0.054	0.883	0.088	0.936	0.061	0.941	0.061	0.923	0.061

Table C.22. Results adjusting for correlation with: Gamma, 3-class Fair:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.059	4.974	0.102	1.425	0.074	1.183	0.097	1.228	0.097	NA
	0	0.058	5.914	0.094	1.412	0.069	1.170	0.090	1.212	0.088	NA
	0.1	0.059	5.098	0.086	1.405	0.068	1.157	0.085	1.203	0.083	1.214
	0.3	0.055	4.925	0.072	1.389	0.064	1.143	0.077	1.187	0.074	1.202
	0.5	0.054	9.341	0.052	1.376	0.053	1.102	0.061	1.146	0.061	1.164
	0.9	0.056	4.995	0.015	1.367	0.032	0.984	0.044	1.021	0.040	1.077
20	-0.3	0.030	0.713	0.004	1.148	0.002	0.704	0.002	0.706	0.001	NA
	0	0.031	0.819	0.003	1.134	0.002	0.694	0.002	0.696	0.001	0.710
	0.1	0.034	0.678	0.002	1.129	0.003	0.689	0.002	0.691	0.001	NA
	0.3	0.035	0.784	0.002	1.127	0.002	0.679	0.001	0.681	0.001	0.699
	0.5	0.031	0.629	0.001	1.121	0.002	0.643	0.001	0.641	0.001	0.669
	0.9	0.030	0.616	0.000	1.141	0.001	0.561	0.000	0.551	0.001	NA
50	-0.3	0.003	0.428	0.000	0.792	0.000	0.407	0.000	0.393	0.000	NA
	0	0.003	0.364	0.000	0.780	0.000	0.403	0.000	0.389	0.000	0.390
	0.1	0.003	0.358	0.000	0.775	0.000	0.400	0.000	0.385	0.000	NA
	0.3	0.005	0.356	0.000	0.763	0.000	0.393	0.000	0.377	0.000	NA
	0.5	0.003	0.316	0.000	0.752	0.000	0.370	0.000	0.353	0.000	0.356
	0.9	0.004	0.242	0.000	0.717	0.000	0.308	0.000	0.286	0.000	0.291
100	-0.3	0.000	0.262	0.000	0.473	0.000	0.292	0.000	0.282	0.000	NA
	0	0.000	0.259	0.000	0.468	0.000	0.290	0.000	0.279	0.000	0.279
	0.1	0.000	0.253	0.000	0.466	0.000	0.287	0.000	0.276	0.000	0.276
	0.3	0.000	0.250	0.000	0.463	0.000	0.281	0.000	0.270	0.000	NA
	0.5	0.000	0.224	0.000	0.449	0.000	0.265	0.000	0.253	0.000	NA
	0.9	0.000	0.167	0.000	0.407	0.000	0.218	0.000	0.203	0.000	0.204

Table C.23. Results adjusting for correlation with: Gamma, 3-class Good:Poor, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.000	0.619	0.089	0.888	0.000	0.855	0.000	0.834	0.001	0.850
	0	0.000	0.626	0.081	0.869	0.001	0.854	0.001	0.834	0.001	0.846
	0.1	0.000	0.614	0.073	0.857	0.000	0.843	0.001	0.824	0.001	0.842
	0.3	0.000	0.620	0.062	0.833	0.000	0.843	0.001	0.824	0.001	0.844
	0.5	0.000	0.562	0.046	0.800	0.000	0.820	0.001	0.800	0.002	0.825
	0.9	0.000	0.483	0.010	0.698	0.000	0.743	0.000	0.721	0.001	0.768
20	-0.3	0.000	0.467	0.004	0.460	0.000	0.514	0.000	0.515	0.000	0.528
	0	0.000	0.468	0.005	0.462	0.000	0.515	0.000	0.515	0.000	0.529
	0.1	0.000	0.461	0.003	0.456	0.000	0.510	0.000	0.509	0.000	0.524
	0.3	0.000	0.467	0.002	0.455	0.000	0.508	0.000	0.508	0.000	0.522
	0.5	0.000	0.421	0.001	0.442	0.000	0.485	0.000	0.485	0.000	0.503
	0.9	0.000	0.357	0.000	0.430	0.000	0.444	0.000	0.443	0.000	0.469
50	-0.3	0.000	0.308	0.000	0.202	0.000	0.331	0.000	0.327	0.000	0.334
	0	0.000	0.309	0.000	0.197	0.000	0.330	0.000	0.326	0.000	0.334
	0.1	0.000	0.304	0.000	0.199	0.000	0.329	0.000	0.325	0.000	0.332
	0.3	0.000	0.308	0.000	0.194	0.000	0.329	0.000	0.324	0.000	0.332
	0.5	0.000	0.278	0.000	0.190	0.000	0.313	0.000	0.308	0.000	0.317
	0.9	0.000	0.235	0.000	0.171	0.000	0.285	0.000	0.279	0.000	0.289
100	-0.3	0.000	0.220	0.000	0.094	0.000	0.246	0.000	0.240	0.000	0.245
	0	0.000	0.221	0.000	0.089	0.000	0.246	0.000	0.241	0.000	0.247
	0.1	0.000	0.217	0.000	0.089	0.000	0.244	0.000	0.238	0.000	0.245
	0.3	0.000	0.220	0.000	0.084	0.000	0.244	0.000	0.238	0.000	0.244
	0.5	0.000	0.198	0.000	0.077	0.000	0.234	0.000	0.227	0.000	0.234
	0.9	0.000	0.167	0.000	0.062	0.000	0.213	0.000	0.205	0.000	0.213

Table C.24. Results adjusting for correlation with: Gamma, 3-class Good:Poor, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.994	0.280	0.912	0.670	0.952	0.341	0.925	0.271	0.730	0.415
	0	0.994	0.272	0.920	0.639	0.944	0.332	0.923	0.260	0.723	0.405
	0.1	0.993	0.267	0.913	0.629	0.949	0.331	0.920	0.262	0.726	0.401
	0.3	0.994	0.253	0.916	0.602	0.942	0.318	0.924	0.247	0.713	0.389
	0.5	0.993	0.229	0.923	0.533	0.951	0.305	0.919	0.229	0.709	0.381
	0.9	0.994	0.120	0.911	0.244	0.966	0.228	0.924	0.139	0.694	0.282
20	-0.3	0.986	0.202	0.881	0.281	0.911	0.175	0.933	0.174	0.815	0.186
	0	0.982	0.196	0.884	0.263	0.904	0.171	0.929	0.170	0.812	0.182
	0.1	0.980	0.193	0.877	0.250	0.906	0.170	0.929	0.168	0.806	0.180
	0.3	0.983	0.182	0.879	0.231	0.901	0.161	0.925	0.159	0.802	0.171
	0.5	0.984	0.164	0.876	0.192	0.912	0.144	0.931	0.142	0.803	0.154
	0.9	0.983	0.085	0.877	0.081	0.923	0.076	0.926	0.073	0.778	0.081
50	-0.3	0.956	0.130	0.858	0.129	0.928	0.124	0.939	0.124	0.877	0.127
	0	0.954	0.127	0.861	0.125	0.924	0.122	0.934	0.122	0.868	0.126
	0.1	0.954	0.124	0.862	0.122	0.929	0.120	0.941	0.120	0.877	0.124
	0.3	0.954	0.116	0.864	0.114	0.928	0.112	0.937	0.113	0.874	0.116
	0.5	0.951	0.105	0.861	0.102	0.924	0.102	0.931	0.102	0.866	0.105
	0.9	0.958	0.054	0.856	0.051	0.938	0.052	0.940	0.052	0.866	0.054
100	-0.3	0.947	0.093	0.855	0.092	0.936	0.092	0.943	0.093	0.905	0.094
	0	0.947	0.090	0.859	0.089	0.938	0.090	0.941	0.091	0.905	0.092
	0.1	0.940	0.088	0.858	0.087	0.934	0.089	0.939	0.089	0.902	0.091
	0.3	0.942	0.083	0.869	0.082	0.938	0.085	0.942	0.085	0.906	0.086
	0.5	0.939	0.075	0.857	0.073	0.937	0.076	0.940	0.077	0.901	0.078
	0.9	0.939	0.038	0.857	0.036	0.936	0.039	0.936	0.039	0.893	0.040

Table C.25. Results adjusting for correlation with: Gamma, 2-class, Good:Good, Cost 1

n_j	Corr	Delta Cov	Delta Len	Generalized Cov	Generalized Len	AN Cov	AN Len	BP Cov	BP Len	BCa Cov	BCa Len
10	-0.3	0.991	0.174	0.931	0.878	0.980	0.406	0.925	0.210	0.744	0.410
	0	0.989	0.168	0.935	0.839	0.976	0.402	0.919	0.208	0.730	0.392
	0.1	0.988	0.166	0.933	0.803	0.978	0.407	0.922	0.217	0.733	0.398
	0.3	0.991	0.155	0.934	0.739	0.975	0.392	0.922	0.195	0.736	0.374
	0.5	0.991	0.141	0.933	0.625	0.978	0.377	0.918	0.179	0.720	0.373
	0.9	0.992	0.075	0.920	0.201	0.981	0.325	0.927	0.120	0.713	0.320
20	-0.3	0.971	0.124	0.876	0.223	0.926	0.121	0.933	0.117	0.812	0.131
	0	0.969	0.121	0.882	0.208	0.918	0.120	0.924	0.115	0.803	0.129
	0.1	0.969	0.119	0.885	0.188	0.925	0.117	0.928	0.112	0.806	0.125
	0.3	0.971	0.111	0.883	0.162	0.918	0.110	0.923	0.105	0.798	0.119
	0.5	0.970	0.101	0.881	0.125	0.921	0.100	0.923	0.095	0.792	0.109
	0.9	0.975	0.052	0.869	0.050	0.938	0.055	0.928	0.049	0.786	0.059
50	-0.3	0.941	0.080	0.861	0.079	0.934	0.082	0.934	0.082	0.870	0.086
	0	0.935	0.077	0.850	0.076	0.932	0.081	0.933	0.081	0.868	0.085
	0.1	0.935	0.076	0.861	0.074	0.933	0.079	0.935	0.079	0.870	0.084
	0.3	0.938	0.071	0.861	0.069	0.935	0.074	0.936	0.074	0.872	0.078
	0.5	0.939	0.064	0.863	0.062	0.937	0.067	0.936	0.067	0.867	0.071
	0.9	0.941	0.033	0.856	0.031	0.942	0.034	0.935	0.034	0.858	0.037
100	-0.3	0.929	0.057	0.849	0.056	0.942	0.061	0.942	0.061	0.902	0.063
	0	0.922	0.055	0.858	0.054	0.943	0.060	0.941	0.060	0.901	0.061
	0.1	0.923	0.054	0.862	0.053	0.942	0.059	0.940	0.059	0.905	0.060
	0.3	0.928	0.051	0.860	0.050	0.946	0.056	0.944	0.056	0.906	0.057
	0.5	0.920	0.046	0.855	0.044	0.945	0.050	0.941	0.050	0.898	0.052
	0.9	0.921	0.023	0.849	0.022	0.947	0.025	0.940	0.026	0.894	0.026

Table C.26. Results adjusting for correlation with: Gamma, 2-class, Good:Good, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.684	0.360	0.838	0.763	0.728	0.415	0.591	0.353	0.655	0.493
	0	0.667	0.344	0.828	0.726	0.718	0.405	0.579	0.344	0.653	0.479
	0.1	0.659	0.337	0.830	0.712	0.715	0.398	0.570	0.338	0.652	0.474
	0.3	0.636	0.320	0.826	0.680	0.704	0.387	0.554	0.324	0.647	0.465
	0.5	0.603	0.293	0.801	0.627	0.685	0.367	0.520	0.301	0.646	0.453
	0.9	0.377	0.175	0.631	0.335	0.570	0.278	0.324	0.195	0.627	0.411
20	-0.3	0.610	0.262	0.731	0.365	0.539	0.231	0.513	0.228	0.585	0.243
	0	0.585	0.252	0.713	0.339	0.526	0.225	0.498	0.222	0.576	0.236
	0.1	0.583	0.248	0.718	0.324	0.520	0.221	0.492	0.218	0.576	0.232
	0.3	0.555	0.234	0.697	0.304	0.499	0.210	0.467	0.207	0.560	0.222
	0.5	0.494	0.213	0.653	0.260	0.446	0.192	0.416	0.188	0.528	0.203
	0.9	0.215	0.127	0.379	0.131	0.222	0.117	0.179	0.111	0.335	0.129
50	-0.3	0.380	0.170	0.549	0.168	0.365	0.162	0.342	0.162	0.416	0.166
	0	0.356	0.164	0.519	0.161	0.352	0.158	0.326	0.158	0.403	0.162
	0.1	0.349	0.160	0.503	0.157	0.348	0.155	0.322	0.155	0.405	0.159
	0.3	0.308	0.151	0.470	0.148	0.312	0.147	0.288	0.147	0.373	0.151
	0.5	0.239	0.138	0.421	0.134	0.248	0.134	0.227	0.134	0.322	0.138
	0.9	0.030	0.082	0.125	0.084	0.053	0.080	0.040	0.079	0.107	0.085
100	-0.3	0.150	0.121	0.343	0.119	0.166	0.120	0.151	0.120	0.212	0.122
	0	0.140	0.117	0.316	0.115	0.164	0.118	0.149	0.118	0.213	0.120
	0.1	0.121	0.115	0.310	0.113	0.150	0.116	0.134	0.116	0.197	0.118
	0.3	0.102	0.108	0.270	0.106	0.125	0.110	0.114	0.110	0.174	0.112
	0.5	0.067	0.098	0.204	0.096	0.095	0.100	0.084	0.100	0.139	0.102
	0.9	0.001	0.058	0.017	0.061	0.005	0.060	0.004	0.059	0.016	0.062

Table C.27. Results adjusting for correlation with: Gamma, 2-class Good:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.552	0.221	0.779	0.978	0.814	0.465	0.470	0.272	0.573	0.463
	0	0.526	0.212	0.768	0.935	0.802	0.461	0.451	0.272	0.563	0.448
	0.1	0.522	0.209	0.756	0.889	0.803	0.454	0.441	0.260	0.565	0.450
	0.3	0.491	0.197	0.738	0.828	0.786	0.444	0.420	0.249	0.551	0.453
	0.5	0.447	0.179	0.686	0.721	0.775	0.433	0.384	0.230	0.541	0.437
	0.9	0.225	0.109	0.372	0.277	0.691	0.369	0.235	0.162	0.529	0.459
20	-0.3	0.413	0.160	0.549	0.287	0.408	0.160	0.369	0.154	0.452	0.170
	0	0.402	0.153	0.511	0.251	0.404	0.155	0.365	0.149	0.457	0.165
	0.1	0.377	0.150	0.499	0.230	0.386	0.151	0.349	0.145	0.440	0.162
	0.3	0.347	0.142	0.462	0.202	0.359	0.144	0.325	0.137	0.428	0.154
	0.5	0.300	0.129	0.397	0.167	0.323	0.132	0.290	0.126	0.397	0.143
	0.9	0.088	0.077	0.132	0.077	0.146	0.084	0.122	0.075	0.250	0.096
50	-0.3	0.156	0.103	0.245	0.100	0.200	0.108	0.192	0.108	0.263	0.113
	0	0.154	0.099	0.227	0.096	0.202	0.105	0.193	0.105	0.265	0.110
	0.1	0.145	0.097	0.208	0.093	0.188	0.102	0.180	0.102	0.260	0.108
	0.3	0.116	0.091	0.186	0.088	0.165	0.098	0.160	0.097	0.240	0.103
	0.5	0.085	0.083	0.136	0.079	0.137	0.089	0.132	0.089	0.211	0.094
	0.9	0.006	0.049	0.013	0.049	0.027	0.053	0.025	0.053	0.073	0.059
100	-0.3	0.031	0.073	0.077	0.072	0.064	0.080	0.062	0.080	0.108	0.082
	0	0.027	0.071	0.058	0.069	0.061	0.078	0.061	0.078	0.106	0.080
	0.1	0.021	0.069	0.056	0.067	0.054	0.076	0.054	0.077	0.096	0.079
	0.3	0.018	0.065	0.042	0.063	0.046	0.072	0.045	0.072	0.090	0.075
	0.5	0.010	0.059	0.023	0.057	0.031	0.066	0.033	0.066	0.071	0.069
	0.9	0.000	0.035	0.001	0.035	0.002	0.040	0.002	0.040	0.012	0.042

Table C.28. Results adjusting for correlation with: Gamma, 2-class Good:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.973	0.425	0.911	0.838	0.934	0.479	0.930	0.427	0.788	0.521
	0	0.974	0.405	0.914	0.811	0.931	0.462	0.927	0.411	0.779	0.500
	0.1	0.977	0.396	0.919	0.795	0.932	0.455	0.930	0.401	0.779	0.492
	0.3	0.974	0.370	0.920	0.751	0.933	0.438	0.929	0.384	0.775	0.477
	0.5	0.975	0.332	0.928	0.689	0.938	0.409	0.928	0.349	0.764	0.450
	0.9	0.977	0.173	0.925	0.359	0.957	0.293	0.931	0.211	0.748	0.326
20	-0.3	0.960	0.313	0.877	0.436	0.900	0.280	0.931	0.276	0.851	0.285
	0	0.955	0.298	0.876	0.404	0.889	0.269	0.927	0.265	0.837	0.275
	0.1	0.954	0.292	0.874	0.386	0.894	0.265	0.929	0.261	0.843	0.270
	0.3	0.954	0.271	0.878	0.350	0.896	0.246	0.929	0.242	0.843	0.252
	0.5	0.954	0.244	0.875	0.301	0.893	0.221	0.927	0.217	0.835	0.226
	0.9	0.955	0.123	0.865	0.119	0.896	0.115	0.923	0.110	0.811	0.117
50	-0.3	0.947	0.203	0.854	0.201	0.924	0.195	0.942	0.195	0.903	0.197
	0	0.938	0.194	0.852	0.190	0.914	0.187	0.933	0.187	0.892	0.190
	0.1	0.939	0.189	0.858	0.185	0.919	0.184	0.937	0.184	0.893	0.186
	0.3	0.939	0.177	0.855	0.171	0.920	0.173	0.938	0.173	0.891	0.176
	0.5	0.939	0.158	0.854	0.151	0.921	0.155	0.938	0.155	0.891	0.157
	0.9	0.936	0.079	0.851	0.075	0.925	0.078	0.939	0.078	0.887	0.080
100	-0.3	0.940	0.145	0.858	0.143	0.934	0.144	0.943	0.144	0.919	0.145
	0	0.935	0.139	0.850	0.136	0.929	0.139	0.937	0.140	0.909	0.141
	0.1	0.933	0.135	0.857	0.133	0.928	0.137	0.936	0.137	0.912	0.138
	0.3	0.931	0.126	0.847	0.123	0.932	0.129	0.940	0.129	0.911	0.130
	0.5	0.926	0.113	0.851	0.109	0.927	0.116	0.934	0.116	0.904	0.117
	0.9	0.929	0.056	0.838	0.054	0.933	0.058	0.942	0.058	0.905	0.059

Table C.29. Results adjusting for correlation with: Gamma, 2-class Fair:Fair, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.961	0.261	0.936	1.061	0.980	0.524	0.930	0.332	0.799	0.459
	0	0.957	0.250	0.937	1.010	0.973	0.508	0.926	0.317	0.796	0.429
	0.1	0.960	0.243	0.939	0.975	0.977	0.504	0.924	0.313	0.784	0.420
	0.3	0.962	0.227	0.944	0.922	0.975	0.492	0.927	0.296	0.786	0.406
	0.5	0.965	0.205	0.941	0.807	0.979	0.476	0.927	0.276	0.781	NA
	0.9	0.970	0.108	0.926	0.306	0.985	0.390	0.927	0.174	0.767	0.313
20	-0.3	0.936	0.190	0.889	0.332	0.918	0.194	0.931	0.188	0.845	0.201
	0	0.938	0.181	0.888	0.297	0.916	0.185	0.928	0.179	0.843	0.192
	0.1	0.937	0.177	0.887	0.276	0.918	0.182	0.928	0.175	0.843	0.189
	0.3	0.931	0.165	0.883	0.234	0.911	0.171	0.922	0.164	0.832	0.178
	0.5	0.934	0.148	0.873	0.196	0.922	0.155	0.928	0.148	0.834	0.163
	0.9	0.943	0.076	0.879	0.073	0.931	0.084	0.925	0.075	0.820	0.086
50	-0.3	0.919	0.122	0.856	0.121	0.930	0.130	0.935	0.130	0.890	0.134
	0	0.913	0.117	0.852	0.113	0.930	0.126	0.936	0.126	0.886	0.130
	0.1	0.917	0.114	0.860	0.110	0.931	0.123	0.936	0.123	0.894	0.127
	0.3	0.911	0.106	0.846	0.102	0.927	0.115	0.934	0.115	0.880	0.119
	0.5	0.908	0.095	0.851	0.091	0.929	0.104	0.931	0.103	0.878	0.108
	0.9	0.904	0.048	0.848	0.045	0.928	0.052	0.930	0.052	0.872	0.055
100	-0.3	0.914	0.087	0.855	0.086	0.938	0.096	0.940	0.096	0.911	0.098
	0	0.908	0.083	0.850	0.082	0.937	0.092	0.938	0.093	0.909	0.094
	0.1	0.909	0.081	0.851	0.080	0.937	0.090	0.939	0.090	0.908	0.092
	0.3	0.905	0.076	0.844	0.074	0.935	0.085	0.937	0.085	0.905	0.087
	0.5	0.902	0.067	0.841	0.065	0.936	0.076	0.939	0.076	0.904	0.078
	0.9	0.898	0.034	0.840	0.032	0.941	0.039	0.940	0.039	0.901	0.040

Table C.30. Results adjusting for correlation with: Gamma, 2-class Fair:Fair, Cost 2

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.889	0.608	0.716	0.671	0.889	0.709	0.907	0.688	0.915	0.733
	0	0.883	0.589	0.705	0.649	0.888	0.691	0.906	0.670	0.918	0.722
	0.1	0.882	0.580	0.702	0.646	0.889	0.682	0.902	0.661	0.923	0.723
	0.3	0.873	0.559	0.695	0.628	0.882	0.664	0.902	0.642	0.922	0.709
	0.5	0.873	0.533	0.680	0.608	0.883	0.646	0.902	0.623	0.927	0.701
	0.9	0.869	0.456	0.637	0.555	0.883	0.585	0.902	0.568	0.928	0.666
20	-0.3	0.888	0.447	0.363	0.387	0.884	0.481	0.918	0.474	0.897	0.487
	0	0.881	0.432	0.348	0.380	0.875	0.467	0.916	0.460	0.893	0.475
	0.1	0.878	0.424	0.341	0.376	0.875	0.460	0.917	0.454	0.897	0.470
	0.3	0.877	0.410	0.333	0.367	0.874	0.447	0.918	0.441	0.893	0.457
	0.5	0.881	0.388	0.332	0.355	0.879	0.428	0.917	0.422	0.903	0.441
	0.9	0.855	0.331	0.303	0.315	0.865	0.385	0.918	0.379	0.898	0.397
50	-0.3	0.846	0.288	0.033	0.178	0.880	0.325	0.924	0.323	0.879	0.326
	0	0.836	0.278	0.037	0.172	0.868	0.316	0.921	0.314	0.869	0.318
	0.1	0.826	0.274	0.033	0.172	0.866	0.312	0.916	0.310	0.863	0.314
	0.3	0.829	0.264	0.029	0.165	0.870	0.302	0.917	0.300	0.864	0.305
	0.5	0.819	0.251	0.028	0.156	0.862	0.290	0.916	0.289	0.860	0.293
	0.9	0.774	0.212	0.023	0.139	0.843	0.257	0.910	0.255	0.848	0.258
100	-0.3	0.760	0.204	0.000	0.104	0.827	0.239	0.880	0.239	0.809	0.241
	0	0.757	0.198	0.001	0.103	0.824	0.232	0.869	0.232	0.803	0.234
	0.1	0.751	0.195	0.000	0.102	0.821	0.230	0.866	0.229	0.800	0.232
	0.3	0.733	0.188	0.000	0.101	0.815	0.224	0.864	0.223	0.789	0.226
	0.5	0.711	0.178	0.000	0.099	0.803	0.215	0.860	0.215	0.777	0.218
	0.9	0.649	0.151	0.001	0.097	0.767	0.189	0.833	0.188	0.746	0.190

Table C.31. Results adjusting for correlation with: Gamma, 2-class Good:Poor, Cost 1

		Delta		Generalized		AN		BP		BCa	
n_j	Corr	Cov	Len	Cov	Len	Cov	Len	Cov	Len	Cov	Len
10	-0.3	0.654	0.332	0.930	1.064	0.911	0.689	0.820	0.520	0.888	0.632
	0	0.655	0.325	0.930	1.047	0.904	0.679	0.820	0.514	0.889	0.636
	0.1	0.654	0.320	0.923	1.013	0.903	0.679	0.817	0.511	0.890	0.639
	0.3	0.633	0.309	0.911	0.998	0.899	0.673	0.807	0.507	0.893	0.668
	0.5	0.647	0.302	0.901	0.919	0.900	0.654	0.812	0.491	0.908	0.678
	0.9	0.640	0.276	0.830	0.649	0.901	0.618	0.818	0.464	0.911	0.717
20	-0.3	0.678	0.254	0.783	0.376	0.764	0.338	0.809	0.316	0.837	0.312
	0	0.671	0.245	0.777	0.354	0.758	0.330	0.815	0.308	0.837	0.305
	0.1	0.665	0.242	0.769	0.340	0.759	0.328	0.806	0.306	0.835	0.306
	0.3	0.669	0.236	0.741	0.308	0.761	0.319	0.809	0.299	0.839	0.298
	0.5	0.643	0.226	0.716	0.280	0.742	0.310	0.791	0.289	0.819	0.291
	0.9	0.637	0.204	0.647	0.150	0.755	0.289	0.807	0.272	0.825	0.275
50	-0.3	0.666	0.169	0.582	0.090	0.784	0.247	0.826	0.242	0.843	0.240
	0	0.657	0.164	0.573	0.088	0.780	0.241	0.819	0.237	0.839	0.236
	0.1	0.655	0.162	0.570	0.088	0.783	0.239	0.823	0.234	0.846	0.235
	0.3	0.648	0.157	0.560	0.085	0.779	0.234	0.820	0.230	0.845	0.231
	0.5	0.628	0.151	0.560	0.081	0.768	0.227	0.813	0.223	0.838	0.224
	0.9	0.595	0.134	0.528	0.078	0.762	0.209	0.808	0.206	0.828	0.210
100	-0.3	0.615	0.122	0.439	0.059	0.789	0.195	0.813	0.196	0.836	0.200
	0	0.613	0.118	0.434	0.058	0.798	0.191	0.819	0.193	0.844	0.199
	0.1	0.607	0.117	0.446	0.058	0.790	0.190	0.812	0.191	0.836	0.197
	0.3	0.588	0.113	0.435	0.057	0.781	0.185	0.804	0.187	0.833	0.193
	0.5	0.568	0.108	0.433	0.056	0.774	0.180	0.794	0.181	0.826	0.189
	0.9	0.533	0.096	0.411	0.055	0.763	0.166	0.785	0.169	0.821	0.178

Table C.32. Results adjusting for correlation with: Gamma, 2-class Good:Poor, Cost 2

Appendix D. Delta Method Variance Calculations for BP and BN Strategies in Sequential Testing

D.1 Derivation of BP

$$
\begin{aligned}
P_{TP} = P(S=1|T=1) &= P_1\left[(X_1 > \theta_1) \cup \left[(X_1 < \theta_1 \cap (X_2 > \theta_2))\right]\right] \\
&= P_1(X_1 > \theta_1) + P_1\left[(X_1 < \theta_1 \cap (X_2 > \theta_2))\right] \\
&= 1 - F_{X_{a},1}(\theta_1) + \left[F_{X_{a},1}(\theta_1) - F_{X_{a,1}X_{a,2}}(\theta_1, \theta_2)\right] \\
&= 1 - F_{X_{a,1}X_{a,2}}(\theta_1, \theta_2) \\
P_{FP} = P(S=1|T=0) &= 1 - F_{X_{b},1}(\theta_1) + F_{X_{b},1}(\theta_1) - F_{X_{b,1},X_{b,2}}(\theta_1, \theta_2) \\
&= 1 - F_{X_{b,1}X_{b,2}}(\theta_1, \theta_2)
\end{aligned}
$$

Therefore the BP function would be given by:

$$
\begin{aligned}
BC =& c_{b|a} p_a F_{X_a,1,2}(\theta_1, \theta_2) + c_2 p_{1|2}\left(1 - F_{X_b,1,2}(\theta_1, \theta_2)\right) \\
=& c_{b|a} p_a \Phi\left[\frac{\theta_1 - \mu_{a,1}}{\sigma_{a,1}}, \frac{\theta_2 - \mu_{a,2}}{\sigma_{a,2}}, \rho_a\right] + c_{a|b} p_b \left(1 - \Phi\left[\frac{\theta_1 - \mu_{b,1}}{\sigma_{b,1}}, \frac{\theta_2 - \mu_{b,2}}{\sigma_{b,2}}, \rho_b\right]\right)
\end{aligned}
$$

D.1.1 Partial Derivatives - Multivariate Chain Rule.

The chain rule for higher order functions is as follows: Define $f = \Phi(z_1, z_2)$ where $z_1 = \frac{\theta_1 - \mu_{a,1}}{\sigma_{a,1}}$ and $z_2 = \frac{\theta_2 - \mu_{a,2}}{\sigma_{a,2}}$

$$
\frac{\partial f}{\partial \mu_{a,1}} = \frac{\partial f}{\partial z_1}\frac{\partial z_1}{\partial \mu_{a,1}} + \frac{\partial f}{\partial z_2}\frac{\partial z_2}{\partial \mu_{a,1}}
$$

Taking the derivative of $\Phi(z_1, z_2)$ w.r.t. each of it's parameters can be accomplished as follows:

$$\frac{\partial\Phi(z_1,z_2)}{\partial z_1} = \frac{\partial}{\partial z_1}\int_{-\infty}^{z_1}\int_{-\infty}^{z_2}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp^{-\frac{u^2-\rho uv+v^2}{2\pi(1-\rho^2)}}\partial v\partial u$$

By the fundemental theorem of calculus, this can simplify to:

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\int_{-\infty}^{z_2}\exp^{-\frac{z_1^2-2\rho z_1 v+v^2}{2(1-\rho^2)}}\partial v$$

completing the square:

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\int_{-\infty}^{z_2}\exp^{-\frac{(v-\rho z_1)^2+z_1^2(1-\rho^2)}{2(1-\rho^2)}}\partial v$$

rearranging terms:

$$\begin{aligned}&=\frac{1}{\sqrt{2\pi}}\exp^{-\frac{z_1^2}{2}}\int_{-\infty}^{z_2}\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\exp^{-\frac{(v-\rho z_1)^2}{2(1-\rho^2)}}\partial v\\ \frac{\partial\Phi(z_1,z_2)}{\partial z_1}&=\phi(z_1)\Phi\left(z_2;\rho z_1,\sqrt{1-\rho^2}\right)\end{aligned}$$

Similarly,

$$\frac{\partial\Phi(z_1,z_2)}{\partial z_2} = \phi(z_2)\Phi\left(z_1;\rho z_2,\sqrt{1-\rho^2}\right)$$

Using these results, we can evaluate the partial derivative of $\mu_{a,1}$ as:

$$
\begin{aligned}
\frac{\partial \Phi\left[z_{a,1}, z_{a,2}\right]}{\partial \mu_{a,1}} =& \phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2}\right) \frac{\partial z_{a,1}}{\partial \mu_{a,1}} \\
&+ \phi(z_{a,2}) \Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right) \frac{\partial z_{a,2}}{\partial \mu_{a,1}} \\
=& \phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2}\right) \frac{\partial}{\partial \mu_{a,1}} \left(\frac{\theta_1 - \mu_{a,1}}{\sigma_{a,1}}\right) \sigma_{a,1}^{-1} \\
&+ \phi(z_{a,2}) \Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_2}{\partial \mu_{a,1}} \sigma_{a,2}^{-1} \\
=& \phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2}\right) \left(\frac{\partial \theta_1}{\partial \mu_{a,1}} - 1\right) \sigma_{a,1}^{-1} \\
&+ \phi(z_{a,2}) \Phi\left(z_{a,1} \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_2}{\partial \mu_{a,1}} \sigma_{a,2}^{-1} \\
\frac{\partial \Phi\left[z_{b,1}, z_{b,2}\right]}{\partial \mu_{a,1}} =& \phi(z_{b,1}) \Phi\left(z_{b,2}; \rho_b z_{b,1}, \sqrt{1-\rho_b^2}\right) \frac{\partial \theta_1}{\partial \mu_{a,1}} \sigma_{b,1}^{-1} \\
&+ \phi(z_{b,1}) \Phi\left(z_{b,2}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2}\right) \frac{\partial \theta_2}{\partial \mu_{a,1}} \sigma_{b,2}^{-1}
\end{aligned}
$$

Thus we can arrive at:

$$
\begin{aligned}
\frac{\partial BC}{\partial \mu_{a,1}} =& c_a p_{b|a} \left(\phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2}\right) \left(\frac{\partial \theta_1}{\partial \mu_{a,1}} - 1\right) \sigma_{a,1}^{-1}\right. \\
&\left. + \phi(z_{a,2}) \Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_2}{\partial \mu_{a,1}} \sigma_{a,2}^{-1}\right) \\
&- c_b p_{a|b} \left(\phi(z_{b,1}) \Phi\left(z_{b,2}; \rho_b z_1, \sqrt{1-\rho_b^2}\right) \frac{\partial \theta_1}{\partial \mu_{a,1}} \sigma_{b,1}^{-1}\right. \\
&\left. + \phi(z_{b,2}) \Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2}\right) \frac{\partial \theta_2}{\partial \mu_{a,1}} \sigma_{b,2}^{-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial BC}{\partial \mu_{a,2}} =& c_a p_{b|a} \left(\phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_1, \sqrt{1-\rho_a^2} \right) \frac{\partial \theta_1}{\partial \mu_{a,2}} \sigma_{a,1}^{-1} \right. \\
&\left. + \phi(z_{a,2}) \Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2} \right) \left(\frac{\partial \theta_2}{\partial \mu_{a,2}} - 1 \right) \sigma_{a,2}^{-1} \right) \\
&- c_b p_{a|b} \left(\phi(z_{b,1}) \Phi\left(z_{b,2}; \rho_b z_1, \sqrt{1-\rho_b^2} \right) \frac{\partial \theta_1}{\partial \mu_{a,1}} \sigma_{b,1}^{-1} \right. \\
&\left. + \phi(z_{b,2}) \Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2} \right) \frac{\partial \theta_2}{\partial \mu_{a,1}} \sigma_{b,2}^{-1} \right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial BC}{\partial \mu_{b,1}} =& c_a p_{b|a} \left(\phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_1, \sqrt{1-\rho_a^2} \right) \frac{\partial \theta_1}{\partial \mu_{b,1}} \sigma_{a,1}^{-1} \right. \\
&\left. + \phi(z_{a,2}) \Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2} \right) \frac{\partial \theta_2}{\partial \mu_{b,1}} \sigma_{a,2}^{-1} \right) \\
&- c_b p_{a|b} \left(\phi(z_{b,1}) \Phi\left(z_{b,2}; \rho_b z_1, \sqrt{1-\rho_b^2} \right) \left(\frac{\partial \theta_1}{\partial \mu_{b,1}} - 1 \right) \sigma_{b,1}^{-1} \right. \\
&\left. + \phi(z_{b,2}) \Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2} \right) \frac{\partial \theta_2}{\partial \mu_{b,1}} \sigma_{b,2}^{-1} \right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial BC}{\partial \mu_{b,2}} =& c_a p_{b|a} \left(\phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_1, \sqrt{1-\rho_a^2} \right) \frac{\partial \theta_1}{\partial \mu_{b,2}} \sigma_{a,1}^{-1} \right. \\
&\left. + \phi(z_{a,2}) \Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2} \right) \frac{\partial \theta_2}{\partial \mu_{b,2}} \sigma_{a,2}^{-1} \right) \\
&- c_b p_{a|b} \left(\phi(z_{b,1}) \Phi\left(z_{b,2}; \rho_b z_1, \sqrt{1-\rho_b^2} \right) \frac{\partial \theta_1}{\partial \mu_{b,2}} \sigma_{b,1}^{-1} \right. \\
&\left. + \phi(z_{b,2}) \Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2} \right) \left(\frac{\partial \theta_2}{\partial \mu_{b,2}} - 1 \right) \sigma_{b,2}^{-1} \right)
\end{aligned}
$$

Similarly, we can evaluate the partial derivative of $\sigma_{a,1}$ as:

$$
\begin{aligned}
\frac{\partial \Phi\left[z_{a,1}, z_{a,2}\right]}{\partial \sigma_{a,1}} =& \phi(z_{a,1})\Phi\left(z_{a,2}; \rho_a z_1, \sqrt{1-\rho_a^2}\right)\frac{\partial z_1}{\partial \sigma_{a,1}} \\
&+ \phi(z_{a,2})\Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right)\frac{\partial z_2}{\partial \sigma_{a,1}} \\
=& \phi(z_{a,1})\Phi\left(z_{a,2}; \rho_a z_1, \sqrt{1-\rho_a^2}\right)\left(\sigma_{a,1}^{-1}\frac{\partial \theta_1}{\partial \sigma_{a,1}} - \sigma_{a,1}^{-2}\left(\theta_1 - \mu_{a,1}\right)\right) \\
&+ \phi(z_{a,2})\Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right)\frac{\partial \theta_2}{\partial \sigma_{a,1}}\sigma_{a,2}^{-1} \\
\frac{\partial \Phi\left[z_{b,1}, z_{b,2}\right]}{\partial \sigma_{a,1}} =& \phi(z_{b,1})\Phi\left(z_{b,2}; \rho_b z_{b,1}, \sqrt{1-\rho_b^2}\right)\frac{\partial \theta_1}{\partial \sigma_{a,1}}\sigma_{b,1}^{-1} \\
&+ \phi(z_{b,1})\Phi\left(z_{b,2}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2}\right)\frac{\partial \theta_2}{\partial \sigma_{a,1}}\sigma_{b,2}^{-1}
\end{aligned}
$$

Thus we can arrive at:

$$
\begin{aligned}
\frac{\partial BC}{\partial \sigma_{a,1}} =& c_{b|a}p_a\left(\phi(z_{a,1})\Phi\left(z_{a,2}; \rho_a z_1, \sqrt{1-\rho_a^2}\right)\left(\sigma_{a,1}^{-1}\frac{\partial \theta_1}{\partial \sigma_{a,1}} - \sigma_{a,1}^{-2}\left(\theta_1 - \mu_{a,1}\right)\right)\right. \\
&\left.+\phi(z_{a,2})\Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right)\frac{\partial \theta_2}{\partial \sigma_{a,1}}\sigma_{a,2}^{-1}\right) \\
&- c_{a|b}p_b\left(\phi(z_{b,1})\Phi\left(z_{b,2}; \rho_b z_1, \sqrt{1-\rho_b^2}\right)\frac{\partial \theta_1}{\partial \sigma_{a,1}}\sigma_{b,1}^{-1}\right. \\
&\left.+\phi(z_{b,2})\Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2}\right)\frac{\partial \theta_2}{\partial \sigma_{a,1}}\sigma_{b,2}^{-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial BC}{\partial \sigma_{a,2}} =& c_{b|a}p_a \left(\phi(z_{a,1})\Phi\left(z_{a,2}; \rho_a z_1, \sqrt{1-\rho_a^2}\right)\frac{\partial \theta_1}{\partial \sigma_{a,2}}\sigma_{a,1}^{-1} \right. \\
&\left. + \phi(z_{a,2})\Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right)\left(\sigma_{a,2}^{-1}\frac{\partial \theta_2}{\partial \sigma_{a,2}} - \sigma_{a,2}^{-2}\left(\theta_2 - \mu_{a,2}\right)\right)\right) \\
&- c_{a|b}p_b \left(\phi(z_{b,1})\Phi\left(z_{b,2}; \rho_b z_1, \sqrt{1-\rho_b^2}\right)\frac{\partial \theta_1}{\partial \sigma_{a,2}}\sigma_{b,1}^{-1} \right. \\
&\left. + \phi(z_{b,2})\Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2}\right)\frac{\partial \theta_2}{\partial \sigma_{a,2}}\sigma_{b,2}^{-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial BC}{\partial \sigma_{b,1}} =& c_{b|a}p_a \left(\phi(z_{a,1})\Phi\left(z_{a,2}; \rho_a z_1, \sqrt{1-\rho_a^2}\right)\frac{\partial \theta_1}{\partial \sigma_{b,1}}\sigma_{a,1}^{-1} \right. \\
&\left. + \phi(z_{a,2})\Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right)\frac{\partial \theta_2}{\partial \sigma_{b,1}}\sigma_{a,2}^{-1}\right) \\
&- c_{a|b}p_b \left(\phi(z_{b,1})\Phi\left(z_{b,2}; \rho_b z_1, \sqrt{1-\rho_b^2}\right)\left(\sigma_{b,1}^{-1}\frac{\partial \theta_1}{\partial \sigma_{b,1}} - \sigma_{b,1}^{-2}\left(\theta_1 - \mu_{b,1}\right)\right) \right. \\
&\left. + \phi(z_{b,2})\Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2}\right)\frac{\partial \theta_2}{\partial \sigma_{b,1}}\sigma_{b,2}^{-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial BC}{\partial \sigma_{b,2}} =& c_{b|a}p_a \left(\phi(z_{a,1})\Phi\left(z_{a,2}; \rho_a z_1, \sqrt{1-\rho_a^2}\right)\frac{\partial \theta_1}{\partial \sigma_{b,2}}\sigma_{a,1}^{-1} \right. \\
&\left. + \phi(z_{a,2})\Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right)\frac{\partial \theta_2}{\partial \sigma_{b,2}}\sigma_{a,2}^{-1}\right) \\
&- c_{a|b}p_b \left(\phi(z_{b,1})\Phi\left(z_{b,2}; \rho_b z_1, \sqrt{1-\rho_b^2}\right)\frac{\partial \theta_1}{\partial \sigma_{b,2}}\sigma_{b,1}^{-1} \right. \\
&\left. + \phi(z_{b,2})\Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2}\right)\left(\sigma_{b,2}^{-1}\frac{\partial \theta_2}{\partial \sigma_{b,2}} - \sigma_{b,2}^{-2}\left(\theta_2 - \mu_{b,2}\right)\right)\right)
\end{aligned}
$$

Finally, using the identity:

$$
\frac{\partial}{\partial \rho}\Phi(z_1, z_2) = \phi(z_1, z_2)
$$

We can arrive at the following result:

$$
\begin{aligned}
\frac{\partial BC}{\partial \rho_a} =& c_{b|a} p_a \left(\phi(z_{a,1}) \Phi \left(z_{a,2}; \rho_a z_1, \sqrt{1-\rho_a^2} \right) \frac{\partial \theta_1}{\partial \rho_a} \sigma_{a,1}^{-1} \right. \\
& \left. + \phi(z_{a,2}) \Phi \left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2} \right) \frac{\partial \theta_2}{\partial \rho_a} \sigma_{a,2}^{-1} \right) \\
& - c_{a|b} p_b \left(\phi(z_{b,1}) \Phi \left(z_{b,2}; \rho_b z_1, \sqrt{1-\rho_b^2} \right) \frac{\partial \theta_1}{\partial \rho_a} \sigma_{b,1}^{-1} \right. \\
& \left. + \phi(z_{b,2}) \Phi \left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2} \right) \frac{\partial \theta_2}{\partial \rho_a} \sigma_{b,2}^{-1} \right) \\
\frac{\partial BC}{\partial \rho_b} =& c_{b|a} p_a \left(\phi(z_{a,1}) \Phi \left(z_{a,2}; \rho_a z_1, \sqrt{1-\rho_a^2} \right) \frac{\partial \theta_1}{\partial \rho_b} \sigma_{a,1}^{-1} \right. \\
& \left. + \phi(z_{a,2}) \Phi \left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2} \right) \frac{\partial \theta_2}{\partial \rho_b} \sigma_{a,2}^{-1} \right) \\
& - c_{a|b} p_b \left(\phi(z_{b,1}) \Phi \left(z_{b,2}; \rho_b z_1, \sqrt{1-\rho_b^2} \right) \frac{\partial \theta_1}{\partial \rho_b} \sigma_{b,1}^{-1} \right. \\
& \left. + \phi(z_{b,2}) \Phi \left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2} \right) \frac{\partial \theta_2}{\partial \rho_b} \sigma_{b,2}^{-1} \right)
\end{aligned}
$$

D.1.2 Variance and Covariance and the Delta Application.

Variance of the means of the required terms for the delta method are well known and given by:

$$Var(\mu_{i,j}) = \frac{\sigma_{i,j}^2}{n_j} \forall i \in \{a, b\}, j \in \{1, 2\}.$$

Similarly, variance for the variance terms is a well known result:

$$Var(\sigma_{i,j}^2) = \frac{\sigma_{i,j}^2}{2(n_j - 1)} \forall i \in \{a, b\}, j \in \{1, 2\}.$$

Variances for the correlation is found in Lehman and Casella (1998) to be asymptotically equal to:

$$Var(\rho_i) = \left(1 - \rho_i^2\right)^2 \forall i \in \{a, b\}.$$

Covariances must be computed for terms that are not mutually independent.

Those are given by:

$$Cov(\mu_{i,1}, \mu_{i,2}) = \hat{\rho}\sqrt{Var(\mu_{i,1})Var(\mu_{i,1})} \forall i \in \{a, b\},$$

$$Cov(\sigma_{i,1}, \sigma_{i,2}) = \hat{\rho}^2\sqrt{Var(\sigma_{i,1})Var(\sigma_{i,1})} \forall i \in \{a, b\}.$$

The second result is defined using the law of total variance from Fisher's ANOVA work. Thus, we can find the variance of BC as an application of the Delta Method approximation provided by Klien (1950),

$$\begin{aligned} Var(BC) \approx & \sum_{i=a}^{b}\sum_{j=1}^{2}\left(\frac{\partial BC}{\partial \mu_{i,j}}\right)^2 Var(\mu_{i,j}) + \sum_{i=a}^{b}\sum_{j=1}^{2}\left(\frac{\partial BC}{\partial \sigma_{i,j}}\right)^2 Var(\sigma_{i,j}) + \sum_{i=a}^{b}\left(\frac{\partial BC}{\partial \rho_i}\right)^2 Var(\rho_i) \\ & + \sum_{i=a}^{b}\left(\frac{\partial BC}{\partial \mu_{i,1}}\right)\left(\frac{\partial BC}{\partial \mu_{i,2}}\right) Cov(\mu_{i,1}, \mu_{i,2}) + \sum_{i=a}^{b}\left(\frac{\partial BC}{\partial \sigma_{i,1}}\right)\left(\frac{\partial BC}{\partial \sigma_{i,2}}\right) Cov(\sigma_{i,1}, \sigma_{i,2}) \end{aligned}$$

D.2 Derivation of BN

$$\begin{aligned} P_{TP} =& 1 - F_{X_{a,1}}(\theta_1) - F_{X_{a,2}}(\theta_2) + F_{X_{a,1},X_{a,2}}(\theta_1, \theta_2) \\ P_{FP} =& 1 - F_{X_{b,1}}(\theta_1) - F_{X_{b,2}}(\theta_2) + F_{X_{b,1},X_{b,2}}(\theta_1, \theta_2) \end{aligned}$$

Therefore the BN function would be given by:

$$\begin{aligned} BC =& c_{b|1}p_a\left(F_{X_{a,1}}(\theta_1) + F_{X_{a,2}}(\theta_2) - F_{X_{a,1},X_{a,2}}(\theta_1, \theta_2)\right) \\ & + c_{a|b}p_b\left(1 - F_{X_{b,1}}(\theta_1) - F_{X_{b,2}}(\theta_2) + F_{X_{b,1},X_{b,2}}(\theta_1, \theta_2)\right) \\ =& c_a p_{b|a}\left[\Phi(z_{a,1}) + \Phi(z_{a,2}) - \Phi(z_{a,1}, z_{a,1})\right] \\ & + c_b p_{a|b}\left[1 - \Phi(z_{b,1}) - \Phi(z_{b,2}) + \Phi(z_{b,1}, z_{b,2})\right] \end{aligned}$$

D.2.1 Partial Derivatives.

Partial derivatives for BN follow identical methodologies as those for BP.

$$
\begin{aligned}
\frac{\partial BC}{\partial \mu_{a,1}} =& c_{b|a} p_a \left[\phi\left(z_{a,1}\right) \left(\frac{\partial \theta_1}{\partial \mu_{a,1}} - 1 \right) \sigma_{a,1}^{-1} + \phi\left(z_{a,2}\right) \frac{\partial \theta_2}{\partial \mu_{a,1}} \sigma_{a,2}^{-1} \right. \\
& -\phi(z_{a,2}) \Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2} \right) \frac{\partial \theta_2}{\partial \mu_{a,1}} \sigma_{a,2}^{-1} \\
& \left. - \phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2} \right) \left(\frac{\partial \theta_1}{\partial \mu_{a,1}} - 1 \right) \sigma_{a,1}^{-1} \right] \\
& - c_{a|b} p_b \left[\phi\left(z_{b,1}\right) \frac{\partial \theta_1}{\partial \mu_{a,1}} \sigma_{b,1}^{-1} + \phi\left(z_{b,2}\right) \frac{\partial \theta_2}{\partial \mu_{a,1}} \sigma_{b,2}^{-1} \right. \\
& -\phi(z_{b,2}) \Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2} \right) \frac{\partial \theta_2}{\partial \mu_{a,1}} \sigma_{b,2}^{-1} \\
& \left. - \phi(z_{b,1}) \Phi\left(z_{b,2}; \rho_b z_{b,1}, \sqrt{1-\rho_b^2} \right) \frac{\partial \theta_1}{\partial \mu_{a,1}} \sigma_{b,1}^{-1} \right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial BC}{\partial \mu_{a,2}} =& c_{b|a} p_a \left[\phi\left(z_{a,1}\right) \frac{\partial \theta_1}{\partial \mu_{a,2}} \sigma_{a,1}^{-1} + \phi\left(z_{a,2}\right) \left(\frac{\partial \theta_2}{\partial \mu_{a,2}} - 1 \right) \sigma_{a,2}^{-1} \right. \\
& -\phi(z_{a,2}) \Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2} \right) \left(\frac{\partial \theta_2}{\partial \mu_{a,2}} - 1 \right) \sigma_{a,2}^{-1} \\
& \left. - \phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2} \right) \frac{\partial \theta_1}{\partial \mu_{a,2}} \sigma_{a,1}^{-1} \right] \\
& - c_{a|b} p_b \left[\phi\left(z_{b,1}\right) \frac{\partial \theta_1}{\partial \mu_{a,2}} \sigma_{b,1}^{-1} + \phi\left(z_{b,2}\right) \frac{\partial \theta_2}{\partial \mu_{a,2}} \sigma_{b,2}^{-1} \right. \\
& -\phi(z_{b,2}) \Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2} \right) \frac{\partial \theta_2}{\partial \mu_{a,2}} \sigma_{b,2}^{-1} \\
& \left. - \phi(z_{b,1}) \Phi\left(z_{b,2}; \rho_b z_{b,1}, \sqrt{1-\rho_b^2} \right) \frac{\partial \theta_1}{\partial \mu_{a,2}} \sigma_{b,1}^{-1} \right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial BC}{\partial \mu_{b,1}} =& c_{b|a} p_a \left[\phi\left(z_{a,1}\right) \frac{\partial \theta_1}{\partial \mu_{b,1}} \sigma_{a,1}^{-1} + \phi\left(z_{a,2}\right) \frac{\partial \theta_2}{\partial \mu_{b,1}} \sigma_{a,2}^{-1} \right. \\
& -\phi(z_{a,2}) \Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_2}{\partial \mu_{b,1}} \sigma_{a,2}^{-1} \\
& \left. -\ \phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_1}{\partial \mu_{b,1}} \sigma_{a,1}^{-1} \right] \\
& - c_{a|b} p_b \left[\phi\left(z_{b,1}\right) \left(\frac{\partial \theta_1}{\partial \mu_{b,1}} - 1 \right) \sigma_{b,1}^{-1} \right. \\
& + \phi\left(z_{b,2}\right) \frac{\partial \theta_2}{\partial \mu_{b,1}} \sigma_{b,2}^{-1} - \phi(z_{b,2}) \Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2}\right) \frac{\partial \theta_2}{\partial \mu_{b,1}} \sigma_{b,2}^{-1} \\
& \left. -\ \phi(z_{b,1}) \Phi\left(z_{b,2}; \rho_b z_{b,1}, \sqrt{1-\rho_b^2}\right) \left(\frac{\partial \theta_1}{\partial \mu_{b,1}} - 1 \right) \sigma_{b,1}^{-1} \right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial BC}{\partial \mu_{b,2}} =& c_{b|a} p_a \left[\phi\left(z_{a,1}\right) \frac{\partial \theta_1}{\partial \mu_{b,2}} \sigma_{a,1}^{-1} + \phi\left(z_{a,2}\right) \frac{\partial \theta_2}{\partial \mu_{b,2}} \sigma_{a,2}^{-1} \right. \\
& -\phi(z_{a,2}) \Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_2}{\partial \mu_{b,2}} \sigma_{a,2}^{-1} \\
& \left. -\ \phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_1}{\partial \mu_{b,2}} \sigma_{a,1}^{-1} \right] \\
& - c_{a|b} p_b \left[\phi\left(z_{b,1}\right) \frac{\partial \theta_1}{\partial \mu_{b,2}} \sigma_{b,1}^{-1} + \phi\left(z_{b,2}\right) \left(\frac{\partial \theta_2}{\partial \mu_{b,2}} - 1 \right) \sigma_{b,2}^{-1} \right. \\
& -\phi(z_{b,2}) \Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2}\right) \left(\frac{\partial \theta_2}{\partial \mu_{b,2}} - 1 \right) \sigma_{b,2}^{-1} \\
& \left. -\ \phi(z_{b,1}) \Phi\left(z_{b,2}; \rho_b z_{b,1}, \sqrt{1-\rho_b^2}\right) \frac{\partial \theta_1}{\partial \mu_{b,2}} \sigma_{b,1}^{-1} \right]
\end{aligned}
$$

Partial derivatives with respect to $\sigma_{i,j}$

$$\begin{aligned}\frac{\partial BC}{\partial \sigma_{a,1}} =& c_{b|a}p_a\Bigg[\phi\left(z_{a,1}\right)\left(\sigma_{a,1}^{-1}\frac{\partial \theta_1}{\partial \sigma_{a,1}} - \sigma_{a,1}^{-2}(\theta_1-\mu_{a,1})\right)\\ &+\phi\left(z_{a,2}\right)\frac{\partial \theta_2}{\partial \sigma_{a,1}}\sigma_{a,2}^{-1} - \phi(z_{a,2})\Phi\left(z_{a,1};\rho_a z_{a,2},\sqrt{1-\rho_a^2}\right)\frac{\partial \theta_2}{\partial \sigma_{a,1}}\sigma_{a,2}^{-1}\\ &-\phi(z_{a,1})\Phi\left(z_{a,2};\rho_a z_{a,1},\sqrt{1-\rho_a^2}\right)\left(\sigma_{a,1}^{-1}\frac{\partial \theta_1}{\partial \sigma_{a,1}} - \sigma_{a,1}^{-2}(\theta_1-\mu_{a,1})\right)\Bigg]\\ &- c_{a|b}p_b\Bigg[\phi\left(z_{b,1}\right)\frac{\partial \theta_1}{\partial \mu_{a,1}}\sigma_{b,1}^{-1} + \phi\left(z_{b,2}\right)\frac{\partial \theta_2}{\partial \sigma_{a,1}}\sigma_{b,2}^{-1}\\ &-\phi(z_{b,2})\Phi\left(z_{b,1};\rho_b z_{b,2},\sqrt{1-\rho_b^2}\right)\frac{\partial \theta_2}{\partial \sigma_{a,1}}\sigma_{b,2}^{-1}\\ &-\phi(z_{b,1})\Phi\left(z_{b,2};\rho_b z_{b,1},\sqrt{1-\rho_b^2}\right)\frac{\partial \theta_1}{\partial \sigma_{a,1}}\sigma_{b,1}^{-1}\Bigg]\end{aligned}$$

$$\begin{aligned}\frac{\partial BC}{\partial \sigma_{a,2}} =& c_{b|a}p_a\Bigg[\phi\left(z_{a,1}\right)\frac{\partial \theta_1}{\partial \sigma_{a,2}}\sigma_{a,1}^{-1} + \phi\left(z_{a,2}\right)\left(\sigma_{a,2}^{-1}\frac{\partial \theta_2}{\partial \sigma_{a,2}} - \sigma_{a,2}^{-2}(\theta_2-\mu_{a,2})\right)\\ &-\phi(z_{a,2})\Phi\left(z_{a,1};\rho_a z_{a,2},\sqrt{1-\rho_a^2}\right)\left(\sigma_{a,2}^{-1}\frac{\partial \theta_2}{\partial \sigma_{a,2}} - \sigma_{a,2}^{-2}(\theta_2-\mu_{a,2})\right)\\ &-\phi(z_{a,1})\Phi\left(z_{a,2};\rho_a z_{a,1},\sqrt{1-\rho_a^2}\right)\frac{\partial \theta_1}{\partial \sigma_{a,2}}\sigma_{a,1}^{-1}\Bigg]\\ &- c_{a|b}p_b\Bigg[\phi\left(z_{b,1}\right)\frac{\partial \theta_1}{\partial \sigma_{a,2}}\sigma_{b,1}^{-1} + \phi\left(z_{b,2}\right)\frac{\partial \theta_2}{\partial \sigma_{a,2}}\sigma_{b,2}^{-1}\\ &-\phi(z_{b,2})\Phi\left(z_{b,1};\rho_b z_{b,2},\sqrt{1-\rho_b^2}\right)\frac{\partial \theta_2}{\partial \sigma_{a,2}}\sigma_{b,2}^{-1}\\ &-\phi(z_{b,1})\Phi\left(z_{b,2};\rho_b z_{b,1},\sqrt{1-\rho_b^2}\right)\frac{\partial \theta_1}{\partial \sigma_{a,2}}\sigma_{b,1}^{-1}\Bigg]\end{aligned}$$

$$
\begin{aligned}
\frac{\partial BC}{\partial \sigma_{b,1}} =& c_{b|a} p_a \left[\phi\left(z_{a,1}\right) \frac{\partial \theta_1}{\partial \sigma_{b,1}} \sigma_{a,1}^{-1} + \phi\left(z_{a,2}\right) \frac{\partial \theta_2}{\partial \sigma_{b,1}} \sigma_{a,2}^{-1} \right. \\
& -\phi(z_{a,2}) \Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_2}{\partial \sigma_{b,1}} \sigma_{a,2}^{-1} \\
& \left. - \phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_1}{\partial \sigma_{b,1}} \sigma_{a,1}^{-1} \right] \\
& - c_{a|b} p_b \left[\phi\left(z_{b,1}\right) \left(\sigma_{b,1}^{-1} \frac{\partial \theta_1}{\partial \sigma_{b,1}} - \sigma_{b,1}^{-2} (\theta_1 - \mu_{b,1}) \right) + \phi\left(z_{b,2}\right) \frac{\partial \theta_2}{\partial \sigma_{b,1}} \sigma_{b,2}^{-1} \right. \\
& -\phi(z_{b,2}) \Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2}\right) \frac{\partial \theta_2}{\partial \sigma_{b,1}} \sigma_{b,2}^{-1} \\
& \left. - \phi(z_{b,1}) \Phi\left(z_{b,2}; \rho_b z_{b,1}, \sqrt{1-\rho_b^2}\right) \left(\sigma_{b,1}^{-1} \frac{\partial \theta_1}{\partial \sigma_{b,1}} - \sigma_{b,1}^{-2} (\theta_1 - \mu_{b,1}) \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial BC}{\partial \sigma_{b,2}} =& c_{b|a} p_a \left[\phi\left(z_{a,1}\right) \frac{\partial \theta_1}{\partial \sigma_{b,2}} \sigma_{a,1}^{-1} + \phi\left(z_{a,2}\right) \frac{\partial \theta_2}{\partial \sigma_{b,2}} \sigma_{a,2}^{-1} \right. \\
& -\phi(z_{a,2}) \Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_2}{\partial \sigma_{b,2}} \sigma_{a,2}^{-1} \\
& \left. - \phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_1}{\partial \sigma_{b,2}} \sigma_{a,1}^{-1} \right] \\
& - c_{a|b} p_b \left[\phi\left(z_{b,1}\right) \frac{\partial \theta_1}{\partial \mu_{b,2}} \sigma_{b,1}^{-1} + \phi\left(z_{b,2}\right) \left(\sigma_{b,2}^{-1} \frac{\partial \theta_2}{\partial \sigma_{b,2}} - \sigma_{b,2}^{-2} (\theta_2 - \mu_{b,2}) \right) \right. \\
& -\phi(z_{b,2}) \Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2}\right) \left(\sigma_{b,2}^{-1} \frac{\partial \theta_2}{\partial \sigma_{b,2}} - \sigma_{b,2}^{-2} (\theta_2 - \mu_{b,2}) \right) \\
& \left. -\phi(z_{b,1}) \Phi\left(z_{b,2}; \rho_b z_{b,1}, \sqrt{1-\rho_b^2}\right) \frac{\partial \theta_1}{\partial \mu_{b,2}} \sigma_{b,1}^{-1} \right]
\end{aligned}
$$

Partial derivatives with respect to ρ_i

$$\begin{aligned}\frac{\partial BC}{\partial \rho_a} =& c_{b|a} p_a \Bigg[\phi\left(z_{a,1}\right) \frac{\partial \theta_1}{\partial \rho_a} \sigma_{a,1}^{-1} + \phi\left(z_{a,2}\right) \frac{\partial \theta_2}{\partial \rho_a} \sigma_{a,2}^{-1} \\ & -\phi(z_{a,2}) \Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_2}{\partial \rho_a} \sigma_{a,2}^{-1} \\ & - \phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_1}{\partial \rho_a} \sigma_{a,1}^{-1} \Bigg] \\ & - c_{a|b} p_b \Bigg[\phi\left(z_{b,1}\right) \frac{\partial \theta_1}{\partial \rho_a} \sigma_{b,1}^{-1} + \phi\left(z_{b,2}\right) \frac{\partial \theta_2}{\partial \rho_a} \sigma_{b,2}^{-1} \\ & -\phi(z_{b,2}) \Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2}\right) \frac{\partial \theta_2}{\partial \rho_a} \sigma_{b,2}^{-1} \\ & - \phi(z_{b,1}) \Phi\left(z_{b,2}; \rho_b z_{b,1}, \sqrt{1-\rho_b^2}\right) \frac{\partial \theta_1}{\partial \rho_a} \sigma_{b,1}^{-1} \Bigg]\end{aligned}$$

$$\begin{aligned}\frac{\partial BC}{\partial \rho_b} =& c_{b|a} p_a \Bigg[\phi\left(z_{a,1}\right) \frac{\partial \theta_1}{\partial \rho_b} \sigma_{a,1}^{-1} + \phi\left(z_{a,2}\right) \frac{\partial \theta_2}{\partial \rho_b} \sigma_{a,2}^{-1} \\ & -\phi(z_{a,2}) \Phi\left(z_{a,1}; \rho_a z_{a,2}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_2}{\partial \rho_b} \sigma_{a,2}^{-1} \\ & - \phi(z_{a,1}) \Phi\left(z_{a,2}; \rho_a z_{a,1}, \sqrt{1-\rho_a^2}\right) \frac{\partial \theta_1}{\partial \rho_b} \sigma_{a,1}^{-1} \Bigg] \\ & - c_{a|b} p_b \Bigg[\phi\left(z_{b,1}\right) \frac{\partial \theta_1}{\partial \rho_b} \sigma_{b,1}^{-1} + \phi\left(z_{b,2}\right) \frac{\partial \theta_2}{\partial \rho_b} \sigma_{b,2}^{-1} \\ & -\phi(z_{b,2}) \Phi\left(z_{b,1}; \rho_b z_{b,2}, \sqrt{1-\rho_b^2}\right) \frac{\partial \theta_2}{\partial \rho_b} \sigma_{b,2}^{-1} \\ & - \phi(z_{b,1}) \Phi\left(z_{b,2}; \rho_b z_{b,1}, \sqrt{1-\rho_b^2}\right) \frac{\partial \theta_1}{\partial \rho_b} \sigma_{b,1}^{-1} \Bigg]\end{aligned}$$

Appendix E. Results from Sequential Testing

E.1 BP Strategy Result Tables

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.935	0.056	0.5	1000			0.922	0.566
	500	0.946	0.084	0.933	0.079		500	0.95	0.089	0.912	0.08
	250	0.943	0.119	0.928	0.112		250	0.942	0.126	0.925	0.114
	100	0.939	0.187	0.926	0.178		100	0.945	0.198	0.932	0.184
	50	0.93	0.263	0.933	0.253		50	0.923	0.281	0.916	0.263
0.2	1000			0.920	0.056	0.8	1000			0.914	0.057
	500	0.954	0.087	0.936	0.08		500	0.946	0.091	0.912	0.082
	250	0.948	0.122	0.929	0.113		250	0.946	0.129	0.923	0.118
	100	0.935	0.192	0.927	0.18		100	0.943	0.204	0.934	0.19
	50	0.922	0.271	0.916	0.256		50	0.945	0.291	0.931	0.271

Table E.1. Results for the BP strategy for a Poor, Poor test comparison at equal cost

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.944	0.049	0.5	1000			0.944	0.052
	500	0.952	0.07	0.947	0.069		500	0.951	0.073	0.95	0.073
	250	0.951	0.098	0.948	0.097		250	0.939	0.103	0.939	0.103
	100	0.951	0.155	0.943	0.153		100	0.951	0.164	0.946	0.163
	50	0.937	0.217	0.931	0.214		50	0.945	0.231	0.935	0.228
0.2	1000			0.945	0.051	0.8	1000			0.953	0.052
	500	0.953	0.072	0.952	0.072		500	0.955	0.073	0.956	0.073
	250	0.957	0.102	0.955	0.101		250	0.95	0.104	0.947	0.104
	100	0.946	0.16	0.94	0.159		100	0.949	0.165	0.945	0.163
	50	0.95	0.225	0.938	0.221		50	0.958	0.234	0.95	0.23

Table E.2. Results for the BP strategy for a Poor, Good test comparison at equal cost

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.927	0.037	0.5	1000			0.919	0.041
	500	0.955	0.056	0.929	0.052		500	0.944	0.066	0.906	0.058
	250	0.947	0.079	0.928	0.073		250	0.934	0.093	0.905	0.082
	100	0.941	0.125	0.921	0.115		100	0.942	0.147	0.917	0.131
	50	0.929	0.175	0.906	0.163		50	0.934	0.205	0.902	0.187
0.2	1000			0.917	0.038	0.8	1000			0.912	0.044
	500	0.936	0.06	0.909	0.054		500	0.948	0.071	0.915	0.062
	250	0.953	0.085	0.921	0.077		250	0.946	0.100	0.910	0.088
	100	0.941	0.134	0.912	0.121		100	0.939	0.157	0.910	0.142
	50	0.931	0.187	0.9	0.172		50	0.936	0.221	0.907	0.204

Table E.3. Results for the BP strategy for a Good, Good test comparison at equal cost

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.915	0.064	0.5	1000			0.903	0.064
	500	0.953	0.103	0.917	0.091		500	0.940	0.110	0.887	0.092
	250	0.941	0.145	0.902	0.129		250	0.939	0.154	0.889	0.132
	100	0.940	0.228	0.920	0.207		100	0.934	0.243	0.905	0.215
	50	0.924	0.322	0.918	0.297		50	0.918	0.342	0.897	0.310
0.2	1000			0.910	0.064	0.8	1000			0.895	0.067
	500	0.947	0.106	0.905	0.091		500	0.948	0.112	0.905	0.096
	250	0.939	0.150	0.901	0.129		250	0.943	0.158	0.908	0.140
	100	0.947	0.235	0.917	0.209		100	0.947	0.249	0.922	0.228
	50	0.927	0.331	0.909	0.301		50	0.933	0.345	0.911	0.324

Table E.4. Results for the BP strategy for a Poor, Poor test comparison at unequal cost

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.946	0.064	0.5	1000			0.952	0.071
	500	0.951	0.093	0.947	0.091		500	0.950	0.100	0.948	0.100
	250	0.943	0.131	0.936	0.128		250	0.945	0.141	0.941	0.141
	100	0.949	0.206	0.939	0.201		100	0.943	0.222	0.941	0.221
	50	0.941	0.288	0.931	0.282		50	0.945	0.313	0.933	0.308
0.2	1000			0.944	0.068	0.8	1000			0.955	0.071
	500	0.948	0.097	0.945	0.096		500	0.955	0.100	0.957	0.100
	250	0.942	0.137	0.936	0.135		250	0.954	0.141	0.952	0.141
	100	0.945	0.215	0.942	0.212		100	0.950	0.223	0.945	0.222
	50	0.932	0.301	0.921	0.294		50	0.945	0.317	0.935	0.313

Table E.5. Results for the BP strategy for a Poor, Good test comparison at unequal cost

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.916	0.047	0.5	1000			0.910	0.054
	500	0.936	0.075	0.899	0.067		500	0.958	0.090	0.908	0.076
	250	0.946	0.106	0.915	0.095		250	0.952	0.126	0.916	0.108
	100	0.936	0.166	0.911	0.149		100	0.937	0.198	0.899	0.172
	50	0.932	0.232	0.897	0.211		50	0.924	0.277	0.883	0.245
0.2	1000			0.914	0.050	0.8	1000			0.896	0.058
	500	0.950	0.082	0.912	0.071		500	0.950	0.096	0.906	0.082
	250	0.947	0.115	0.911	0.100		250	0.938	0.136	0.903	0.118
	100	0.946	0.181	0.906	0.159		100	0.933	0.213	0.906	0.190
	50	0.938	0.252	0.906	0.225		50	0.939	0.300	0.915	0.273

Table E.6. Results for the BP strategy for a Good, Good test comparison at unequal cost

E.2 BN Strategy Result Tables

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.935	0.056	0.5	1000			0.922	0.057
	500	0.952	0.084	0.935	0.079		500	0.950	0.089	0.927	0.081
	250	0.939	0.119	0.926	0.113		250	0.951	0.126	0.937	0.116
	100	0.937	0.187	0.932	0.181		100	0.933	0.198	0.927	0.190
	50	0.924	0.263	0.929	0.259		50	0.931	0.281	0.929	0.276
0.2	1000			0.921	0.056	0.8	1000			0.914	0.057
	500	0.947	0.087	0.922	0.080		500	0.940	0.091	0.921	0.084
	250	0.949	0.122	0.930	0.114		250	0.938	0.129	0.922	0.133
	100	0.941	0.192	0.932	0.183		100	0.938	0.204	0.934	0.202
	50	0.926	0.270	0.923	0.264		50	0.910	0.290	0.909	0.289

Table E.7. Results for the BN strategy for a Poor, Poor test comparison at equal cost

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.943	0.049	0.5	1000			0.944	0.052
	500	0.948	0.070	0.943	0.068		500	0.946	0.073	0.944	0.073
	250	0.942	0.098	0.937	0.095		250	0.948	0.104	0.943	0.104
	100	0.946	0.155	0.934	0.151		100	0.950	0.164	0.947	0.163
	50	0.942	0.217	0.928	0.211		50	0.944	0.232	0.932	0.228
0.2	1000			0.945	0.051	0.8	1000			0.953	0.052
	500	0.948	0.072	0.938	0.071		500	0.958	0.073	0.958	0.073
	250	0.948	0.102	0.945	0.100		250	0.942	0.104	0.940	0.104
	100	0.939	0.160	0.927	0.157		100	0.948	0.165	0.943	0.163
	50	0.937	0.224	0.922	0.218		50	0.953	0.233	0.937	0.230

Table E.8. Results for the BN strategy for a Poor, Good test comparison at equal cost

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.928	0.037	0.5	1000			0.919	0.041
	500	0.947	0.056	0.928	0.052		500	0.943	0.066	0.901	0.058
	250	0.948	0.079	0.928	0.073		250	0.950	0.093	0.905	0.082
	100	0.942	0.125	0.924	0.116		100	0.940	0.146	0.909	0.131
	50	0.935	0.175	0.905	0.163		50	0.931	0.206	0.905	0.188
0.2	1000			0.917	0.038	0.8	1000			0.912	0.0436
	500	0.949	0.060	0.916	0.054		500	0.940	0.071	0.908	0.062
	250	0.946	0.085	0.919	0.077		250	0.946	0.100	0.909	0.088
	100	0.945	0.134	0.914	0.122		100	0.937	0.157	0.907	0.142
	50	0.925	0.188	0.891	0.171		50	0.935	0.220	0.911	0.204

Table E.9. Results for the BN strategy for a Good, Good test comparison at equal cost

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.940	0.077	0.5	1000			0.934	0.075
	500	0.958	0.110	0.954	0.109		500	0.942	0.111	0.931	0.107
	250	0.952	0.156	0.951	0.155		250	0.949	0.157	0.945	0.170
	100	0.948	0.243	0.952	0.248		100	0.944	0.246	0.944	0.259
	50	0.931	0.341	0.937	0.356		50	0.924	0.345	0.931	0.354
0.2	1000			0.948	0.076	0.8	1000			0.939	0.075
	500	0.949	0.111	0.942	0.108		500	0.949	0.112	0.936	0.122
	250	0.940	0.156	0.930	0.153		250	0.947	0.158	0.944	0.155
	100	0.947	0.244	0.949	0.247		100	0.940	0.249	0.943	0.256
	50	0.937	0.343	0.941	0.354		50	0.933	0.347	0.936	0.709

Table E.10. Results for the BN strategy for a Poor, Poor test comparison at unequal cost

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.946	0.064	0.5	1000			0.952	0.071
	500	0.942	0.096	0.947	0.098		500	0.951	0.100	0.958	0.103
	250	0.948	0.136	0.950	0.138		250	0.955	0.141	0.959	0.146
	100	0.944	0.214	0.945	0.218		100	0.952	0.223	0.951	0.229
	50	0.932	0.300	0.924	0.304		50	0.937	0.315	0.933	0.321
0.2	1000			0.944	0.068	0.8	1000			0.955	0.071
	500	0.954	0.098	0.958	0.101		500	0.954	0.100	0.963	0.103
	250	0.945	0.139	0.949	0.143		250	0.943	0.141	0.949	0.146
	100	0.945	0.219	0.942	0.224		100	0.951	0.224	0.952	0.230
	50	0.942	0.308	0.937	0.312		50	0.946	0.317	0.941	0.323

Table E.11. Results for the BN strategy for a Poor, Good test comparison at unequal cost

		Generlized		Delta				Generlized		Delta	
ρ	n_j	Cov	Len	Cov	Len	ρ	n_j	Cov	Len	Cov	Len
0	1000			0.925	0.053	0.5	1000			0.925	0.058
	500	0.947	0.079	0.935	0.075		500	0.952	0.090	0.928	0.082
	250	0.950	0.112	0.936	0.106		250	0.948	0.128	0.924	0.116
	100	0.943	0.176	0.929	0.167		100	0.941	0.201	0.915	0.185
	50	0.937	0.247	0.916	0.236		50	0.934	0.282	0.908	0.264
0.2	1000			0.933	0.055	0.8	1000			0.914	0.061
	500	0.956	0.084	0.934	0.078		500	0.947	0.096	0.926	0.086
	250	0.945	0.118	0.926	0.110		250	0.951	0.136	0.927	0.123
	100	0.949	0.186	0.928	0.174		100	0.947	0.214	0.933	0.260
	50	0.941	0.262	0.920	0.247		50	0.928	0.301	0.908	0.284

Table E.12. Results for the BN strategy for a Good, Good test comparison at unequal cost

Appendix F. R Code

F.1 Correlated Delta Method for 2 classes

```
p1<#Set Prevalence Class 1
p2<#Set Prevalence Class 2
w21<# Set cost 2|1
w12<# Set cost 1|2
YA=#Class 1 for Test 1
XA=#Class 2 for Test 1
Y=#Class 1 for Test 2
X=#Class 2 for Test 2
n1=#Size of Class 1 (both tests)
n2=#Size of Class 2 (both tests)

start=c(.1)
L=c(-1000)
U=c(1000)
gmu1=mean(YA)
gmu2=mean(XA)
gsig1=sd(YA)
gsig2=sd(XA)
f=function(par){abs(1-pnorm(par,gmu1,gsig1))*(p1*w21)+
    abs(pnorm(par,gmu2,gsig2))*(p2*w12)}
x=optim(start,f,lower=-1000,upper=1000,method="L-BFGS-B")
c1=x$par
EBCA=x$value
vm1=(gsig1^2)/n1
vm2=(gsig2^2)/n2
vs1=(gsig1^2)/(2*(n1-1))
vs2=(gsig2^2)/(2*(n2-1))
#calculate Partial Derivatives
g=function(par){abs(1-pnorm(par,mux,sigx))*(p1*w21)+
    abs(pnorm(par,muy,sigy))*(p2*w12)}
mux=gmu1+.0001
muy=gmu2
sigx=gsig1
sigy=gsig2
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par
mux=gmu1-.0001
muy=gmu2
sigx=gsig1
sigy=gsig2
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par
dc1m1=(o1p-o1m)/.0002
mux=gmu1
muy=gmu2+.0001
sigx=gsig1
```

```
sigy=gsig2
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par
mux=gmu1
muy=gmu2-.0001
sigx=gsig1
sigy=gsig2
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par
dc1m2=(o1p-o1m)/.0002
mux=gmu1
muy=gmu2
sigx=gsig1+.0001
sigy=gsig2
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par
mux=gmu1
muy=gmu2
sigx=gsig1-.0001
sigy=gsig2
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par
dc1s1=(o1p-o1m)/.0002
mux=gmu1
muy=gmu2
sigx=gsig1
sigy=gsig2+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par
mux=gmu1
muy=gmu2
sigx=gsig1
sigy=gsig2-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par
dc1s2=(o1p-o1m)/.0002
##calc dbcmu1 using equation to compare to estimate
dp1<-(1/gsig1)*(-w21*p1*dnorm((c1-gmu1)/gsig1)*(dc1m1-1))
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*dc1m1)
dbcm1<-dp1+dp2
##calc dbcmu2 using equation to compare to estimate
dp1<-(1/gsig1)*(-w21*p1*dnorm((c1-gmu1)/gsig1)*dc1m2)
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*(dc1m2-1))
dbcm2<-dp1+dp2
##calc dbcs1 using equation to compare to estimate
dp1<-(1/gsig1)*(-w21*p1*dnorm((c1-gmu1)/gsig1)*
            (dc1s1-((c1-gmu1)/gsig1)))
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*dc1s1)
dbcs1<-dp1+dp2
##calc dbcs2 using equation to compare to estimate
dp1<-(1/gsig1)*(-w21*p1*dnorm((c1-gmu1)/gsig1)*dc1s2)
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*
```

```
            (dc1s2-((c1-gmu2)/gsig2)))
dbcs2<-dp1+dp2
VBCA=(dbcm1^2)*vm1+(dbcs1^2)*vs1+(dbcm2^2)*vm2+
            (dbcs2^2)*vs2
dbcm1a=dbcm1
dbcm2a=dbcm2
dbcs1a=dbcs1
dbcs2a=dbcs2
vm1a=vm1
vs1a=vs1
vm2a=vm2
vs2a=vs2
#Repeat for Test 2
gmu1=mean(Y)
gmu2=mean(X)
gsig1=sd(Y)
gsig2=sd(X)
f=function(par){abs(1-pnorm(par,gmu1,gsig1))*(p1*w21)+
    abs(pnorm(par,gmu2,gsig2))*(p2*w12)}
x=optim(start,f,lower=-1000,upper=1000,method="L-BFGS-B")
c1=x$par
EBC=x$value
vm1=(gsig1^2)/n1
vm2=(gsig2^2)/n2
vs1=(gsig1^2)/(2*(n1-1))
vs2=(gsig2^2)/(2*(n2-1))
g=function(par){abs(1-pnorm(par,mux,sigx))*(p1*w21)+
    abs(pnorm(par,muy,sigy))*(p2*w12)}
mux=gmu1+.0001
muy=gmu2
sigx=gsig1
sigy=gsig2
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par
mux=gmu1-.0001
muy=gmu2
sigx=gsig1
sigy=gsig2
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par
dc1m1=(o1p-o1m)/.0002
mux=gmu1
muy=gmu2+.0001
sigx=gsig1
sigy=gsig2
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par
mux=gmu1
muy=gmu2-.0001
sigx=gsig1
sigy=gsig2
x=nlminb(start,g,lower=L,upper=U)
```

```
o1m=x$par
dc1m2=(o1p-o1m)/.0002
mux=gmu1
muy=gmu2
sigx=gsig1+.0001
sigy=gsig2
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par
mux=gmu1
muy=gmu2
sigx=gsig1-.0001
sigy=gsig2
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par
dc1s1=(o1p-o1m)/.0002
mux=gmu1
muy=gmu2
sigx=gsig1
sigy=gsig2+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par
mux=gmu1
muy=gmu2
sigx=gsig1
sigy=gsig2-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par
dc1s2=(o1p-o1m)/.0002
##calc dbcmu1 using equation to compare to estimate
dp1<-(1/gsig1)*(-w21*p1*dnorm((c1-gmu1)/gsig1)*(dc1m1-1))
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*dc1m1)
dbcm1<-dp1+dp2
##calc dbcmu2 using equation to compare to estimate
dp1<-(1/gsig1)*(-w21*p1*dnorm((c1-gmu1)/gsig1)*dc1m2)
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*(dc1m2-1))
dbcm2<-dp1+dp2
##calc dbcs1 using equation to compare to estimate
dp1<-(1/gsig1)*(-w21*p1*dnorm((c1-gmu1)/gsig1)*
       (dc1s1-((c1-gmu1)/gsig1)))
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*dc1s1)
dbcs1<-dp1+dp2
##calc dbcs2 using equation to compare to estimate
dp1<-(1/gsig1)*(-w21*p1*dnorm((c1-gmu1)/gsig1)*dc1s2)
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*
       (dc1s2-((c1-gmu2)/gsig2)))
dbcs2<-dp1+dp2
VBC=(dbcm1^2)*vm1+(dbcs1^2)*vs1+(dbcm2^2)*vm2+
       (dbcs2^2)*vs2
#Calculate the Correlation
corest1=cor(X,XA)
corest2=cor(Y,YA)
corrs=dbcm1*dbcm1a*corest1*sqrt(vm1a)*sqrt(vm1)+
```

```
      dbcm2*dbcm2a*corest2*sqrt(vm2a)*sqrt(vm2)+
      dbcs1a*dbcs1*corest1^2*sqrt(vs1a)*sqrt(vs1)+
      dbcs2a*dbcs2*corest2^2*sqrt(vs2a)*sqrt(vs2)
#Produce Results
VETA=VBCA+VBC-2*corrs
EETA=EBCA-EBC
W=(EETA-TV)/sqrt(VETA)
lowconf=EETA-qnorm(.025,lower.tail=F)*sqrt(VETA)
highconf=EETA+qnorm(.025,lower.tail=F)*sqrt(VETA)
```

F.2 Correlated Generalized Method for 2 Classes

```
library(mvtnorm)

p1<#Set Prevalence Class 1
p2<#Set Prevalence Class 2
w21<# Set cost 2|1
w12<# Set cost 1|2
YA=#Class 1 for Test 1
XA=#Class 2 for Test 1
Y=#Class 1 for Test 2
X=#Class 2 for Test 2
n1=#Size of Class 1 (both tests)
n2=#Size of Class 2 (both tests)
start=c(.1)
L=c(-1000)
U=c(1000)
K=2500
ybar1a=mean(XA)
ybar2a=mean(YA)
var1a=var(XA)
var2a=var(YA)
ybar1=mean(X)
ybar2=mean(Y)
var1=var(X)
var2=var(Y)
cov1=cov(XA,X)
cov2=cov(YA,Y)
co1=cor(XA,X)
co2=cor(YA,Y)
draw1=rWishart(K,n1-1,matrix(c(1,co1,co1,1),2,2))
draw2=rWishart(K,n2-1,matrix(c(1,co2,co2,1),2,2))
t1=rmvt(K,matrix(c(1,co1,co1,1),ncol=2)*(n1-3)/(n1-1),df=n1-1)
t2=rmvt(K,matrix(c(1,co1,co1,1),ncol=2)*(n2-3)/(n2-1),df=n2-1)
Rs1a=c(rep((n1-1)*var1a,K))/draw1[1,1,]
Rs2a=c(rep((n2-1)*var2a,K))/draw2[1,1,]
Rm1a=c(rep(ybar1a,K))-(t1[,1]*(sqrt(var1a/n1)))
Rm2a=c(rep(ybar2a,K))-(t2[,1]*(sqrt(var2a/n2)))
f=function(x){
  hun2=function(par){
    abs(1-pnorm(par,x[1],x[2]))*(p1*w21)+
      abs(pnorm(par,x[3],x[4]))*(p2*w12)}
```

```
  y=optim(start,hun2,lower=L,upper=U,method="L-BFGS-B")
  BC=y$value
  return(BC)
}
ap1=cbind(Rm1a,sqrt(Rs1a),Rm2a,sqrt(Rs2a))
RbcA=apply(ap1,1,FUN=f)
Rs1=c(rep((n1-1)*var1,K))/draw1[2,2,]
Rs2=c(rep((n2-1)*var2,K))/draw2[2,2,]
Rm1=c(rep(ybar1,K))-(t1[,2]*(sqrt(var1/n1)))
Rm2=c(rep(ybar2,K))-(t2[,2]*(sqrt(var2/n2)))
ap1=cbind(Rm1,sqrt(Rs1),Rm2,sqrt(Rs2))
Rbc=apply(ap1,1,FUN=f)
Reta=RbcA-Rbc
lowconf=quantile(Reta,.025)
hiconf=quantile(Reta,.975)
```

F.3 Correlated Delta Method for 3 classes

```
p1<#Set Prevalence Class 1
p2<#Set Prevalence Class 2
p3<#Set Prevalence Class 3
w21<# Set cost 2|1
w31<# Set cost 3|1
w12<# Set cost 1|2
w32<# Set cost 3|2
w13<# Set cost 1|3
w23<# Set cost 2|3
YA=#Class 1 for Test 1
XA=#Class 2 for Test 1
ZA=#Class 3 for Test 1
Y=#Class 1 for Test 2
X=#Class 2 for Test 2
Z=#Class 3 for Test 2
n1=#Size of Class 1 (both tests)
n2=#Size of Class 2 (both tests)
n3=#Size of Class 3 (both tests)
start=c(.1,0)
L=c(-1000,-1000)
U=c(1000,1000)
gmu1=mean(YA)
gmu2=mean(XA)
gmu3=mean(ZA)
gsig1=sd(YA)
gsig2=sd(XA)
gsig3=sd(ZA)
f=function(par){abs(pnorm(par[2],gmu1,gsig1)-
    pnorm(par[1],gmu1,gsig1))*(p1*w21)+
    abs(1-pnorm(par[2],gmu1,gsig1))*(p1*w31)+
    abs(pnorm(par[1],gmu2,gsig2))*(p2*w12)+
    abs(1-pnorm(par[2],gmu2,gsig2))*(p2*w32)+
    abs(pnorm(par[1],gmu3,gsig3))*(p3*w13)+
    abs(pnorm(par[2],gmu3,gsig3)-
```

```
    pnorm(par[1],gmu3,gsig3))*(p3*w23)}
x=optim(start,f,lower=-1000,upper=1000,method="L-BFGS-B")
c1=x$par[1]
c2=x$par[2]
EBCA=x$value
vm1=(gsig1^2)/n1
vm2=(gsig2^2)/n2
vm3=(gsig3^2)/n3
vs1=(gsig1^2)/(2*(n1-1))
vs2=(gsig2^2)/(2*(n2-1))
vs3=(gsig3^2)/(2*(n3-1))
g=function(par){abs(pnorm(par[2],mux,sigx)-
    pnorm(par[1],mux,sigx))*(p1*w21)+
    abs(1-pnorm(par[2],mux,sigx))*(p1*w31)+
    abs(pnorm(par[1],muy,sigy))*(p2*w12)+
    abs(1-pnorm(par[2],muy,sigy))*(p2*w32)+
    abs(pnorm(par[1],muz,sigz))*(p3*w13)+
    abs(pnorm(par[2],muz,sigz)-
    pnorm(par[1],gmu3,gsig3))*(p3*w23)}
mux=gmu1+.0001
muy=gmu2
muz=gmu3
sigx=gsig1
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mux=gmu1-.0001
muy=gmu2
muz=gmu3
sigx=gsig1
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1m1=(o1p-o1m)/.0002
dc2m1=(o2p-o2m)/.0002
mux=gmu1
muy=gmu2+.0001
muz=gmu3
sigx=gsig1
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mux=gmu1
muy=gmu2-.0001
muz=gmu3
sigx=gsig1
```

```
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1m2=(o1p-o1m)/.0002
dc2m2=(o2p-o2m)/.0002
mux=gmu1
muy=gmu2
muz=gmu3+.0001
sigx=gsig1
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mux=gmu1
muy=gmu2
muz=gmu3-.0001
sigx=gsig1
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1m3=(o1p-o1m)/.0002
dc2m3=(o2p-o2m)/.0002
mux=gmu1
muy=gmu2
muz=gmu3
sigx=gsig1+.0001
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mux=gmu1
muy=gmu2
muz=gmu3
sigx=gsig1-.0001
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1s1=(o1p-o1m)/.0002
dc2s1=(o2p-o2m)/.0002
mux=gmu1
muy=gmu2
muz=gmu3
sigx=gsig1
sigy=gsig2+.0001
```

```
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mux=gmu1
muy=gmu2
muz=gmu3
sigx=gsig1
sigy=gsig2-.0001
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1s2=(o1p-o1m)/.0002
dc2s2=(o2p-o2m)/.0002
mux=gmu1
muy=gmu2
muz=gmu3
sigx=gsig1
sigy=gsig2
sigz=gsig3+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mux=gmu1
muy=gmu2
muz=gmu3
sigx=gsig1
sigy=gsig2
sigz=gsig3-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1s3=(o1p-o1m)/.0002
dc2s3=(o2p-o2m)/.0002
##calc dbcmu1 using equation to compare to estimate
dp1<-(1/gsig1)*((dc2m1-1)*dnorm((c2-gmu1)/gsig1)*
        (w21*p1-w31*p1)-
        w21*p1*dnorm((c1-gmu1)/gsig1)*(dc1m1-1))
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*dc1m1+
        w32*p2*dnorm((gmu2-c2)/gsig2)*(-dc2m1))
dp3<-(1/gsig3)*((dc1m1)*dnorm((c1-gmu3)/gsig3)*(w13*p3-w23*p3)...
    +
        w23*p3*dnorm((c2-gmu3)/gsig3)*dc2m1)
dbcm1<-dp1+dp2+dp3
##calc dbcmu2 using equation to compare to estimate
dp1<-(1/gsig1)*((dc2m2)*dnorm((c2-gmu1)/gsig1)*(w21*p1-w31*p1)...
    -
        w21*p1*dnorm((c1-gmu1)/gsig1)*dc1m2)
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*(dc1m2-1)+
        w32*p2*dnorm((gmu2-c2)/gsig2)*(1-dc2m2))
dp3<-(1/gsig3)*(dc1m2*dnorm((c1-gmu3)/gsig3)*(w13*p3-w23*p3)+
```

```
        w23*p3*dnorm((c2-gmu3)/gsig3)*dc2m2)
dbcm2<-dp1+dp2+dp3
##calc dbcmu3 using equation to compare to estimate
dp1<-(1/gsig1)*(dc2m3*dnorm((c2-gmu1)/gsig1)*(w21*p1-w31*p1)-
        w21*p1*dnorm((c1-gmu1)/gsig1)*dc1m3)
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*dc1m3+
        w32*p2*dnorm((gmu2-c2)/gsig2)*(-dc2m3))
dp3<-(1/gsig3)*((dc1m3-1)*dnorm((c1-gmu3)/gsig3)*
        (w13*p3-w23*p3)+
        w23*p3*dnorm((c2-gmu3)/gsig3)*(dc2m3-1))
dbcm3<-dp1+dp2+dp3
##calc dbcs1 using equation to compare to estimate
dp1<-(1/gsig1)*(dnorm((c2-gmu1)/gsig1)*
        (w21*p1*(dc2s1-((c2-gmu1)/gsig1))+
        w31*p1*(-dc2s1-((gmu1-c2)/gsig1)))
        -w21*p1*dnorm((c1-gmu1)/gsig1)*
        (dc1s1-((c1-gmu1)/gsig1)))
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*
        dc1s1-w32*p2*dnorm((gmu2-c2)/gsig2)*dc2s1)
dp3<-(1/gsig3)*(dnorm((c1-gmu3)/gsig3)*dc1s1*(w13*p3-w23*p3)+
        w23*p3*dnorm((c2-gmu3)/gsig3)*dc2s1)
dbcs1<-dp1+dp2+dp3
##calc dbcs2 using equation to compare to estimate
dp1<-(1/gsig1)*(dnorm((c2-gmu1)/gsig1)*dc2s2*(w21*p1-w31*p1)-
        w21*p1*dnorm((c1-gmu1)/gsig1)*dc1s2)
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*
        (dc1s2-((c1-gmu2)/gsig2))+
        w32*p2*dnorm((gmu2-c2)/gsig2)*
        (-dc2s2-((gmu2-c2)/gsig2)))
dp3<-(1/gsig3)*(dnorm((c1-gmu3)/gsig3)*dc1s2*(w13*p3-w23*p3)+
        w23*p3*dnorm((c2-gmu3)/gsig3)*dc2s2)
dbcs2<-dp1+dp2+dp3
##calc dbcs3 using equation to compare to estimate
dp1<-(1/gsig1)*(dnorm((c2-gmu1)/gsig1)*dc2s3*(w21*p1-w31*p1)-
        w21*p1*dnorm((c1-gmu1)/gsig1)*dc1s3)
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*
        dc1s3+w32*p2*dnorm((gmu2-c2)/gsig2)*(-dc2s3))
dp3<-(1/gsig3)*(dnorm((c1-gmu3)/gsig3)*(dc1s3-
        ((c1-gmu3)/gsig3))*(w13*p3-w23*p3)+
        w23*p3*dnorm((c2-gmu3)/gsig3)*
        (dc2s3-((c2-gmu3)/gsig3)))
dbcs3<-dp1+dp2+dp3
VBCA=(dbcm1^2)*vm1+(dbcs1^2)*vs1+(dbcm2^2)*vm2+
  (dbcs2^2)*vs2+(dbcm3^2)*vm3+(dbcs3^2)*vs3
dbcm1a=dbcm1
dbcm2a=dbcm2
dbcm3a=dbcm3
dbcs1a=dbcs1
dbcs2a=dbcs2
dbcs3a=dbcs3
vm1a=vm1
vs1a=vs1
```

```
vm2a=vm2
vs2a=vs2
vm3a=vm3
vs3a=vs3
#Repeat for Test 2
gmu1=mean(Y)
gmu2=mean(X)
gmu3=mean(Z)
gsig1=sd(Y)
gsig2=sd(X)
gsig3=sd(Z)
f=function(par){abs(pnorm(par[2],gmu1,gsig1)-
    pnorm(par[1],gmu1,gsig1))*(p1*w21)+
    abs(1-pnorm(par[2],gmu1,gsig1))*(p1*w31)+
    abs(pnorm(par[1],gmu2,gsig2))*(p2*w12)+
    abs(1-pnorm(par[2],gmu2,gsig2))*(p2*w32)+
    abs(pnorm(par[1],gmu3,gsig3))*(p3*w13)+
    abs(pnorm(par[2],gmu3,gsig3)-
    pnorm(par[1],gmu3,gsig3))*(p3*w23)}
x=optim(start,f,lower=-1000,upper=1000,method="L-BFGS-B")
c1=x$par[1]
c2=x$par[2]
EBC=x$value
vm1=(gsig1^2)/n1
vm2=(gsig2^2)/n2
vm3=(gsig3^2)/n3
vs1=(gsig1^2)/(2*(n1-1))
vs2=(gsig2^2)/(2*(n2-1))
vs3=(gsig3^2)/(2*(n3-1))
mux=gmu1+.0001
muy=gmu2
muz=gmu3
sigx=gsig1
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mux=gmu1-.0001
muy=gmu2
muz=gmu3
sigx=gsig1
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1m1=(o1p-o1m)/.0002
dc2m1=(o2p-o2m)/.0002
mux=gmu1
muy=gmu2+.0001
muz=gmu3
```

```
sigx=gsig1
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mux=gmu1
muy=gmu2-.0001
muz=gmu3
sigx=gsig1
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1m2=(o1p-o1m)/.0002
dc2m2=(o2p-o2m)/.0002
mux=gmu1
muy=gmu2
muz=gmu3+.0001
sigx=gsig1
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mux=gmu1
muy=gmu2
muz=gmu3-.0001
sigx=gsig1
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1m3=(o1p-o1m)/.0002
dc2m3=(o2p-o2m)/.0002
mux=gmu1
muy=gmu2
muz=gmu3
sigx=gsig1+.0001
sigy=gsig2
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mux=gmu1
muy=gmu2
muz=gmu3
sigx=gsig1-.0001
sigy=gsig2
sigz=gsig3
```

```
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1s1=(o1p-o1m)/.0002
dc2s1=(o2p-o2m)/.0002
mux=gmu1
muy=gmu2
muz=gmu3
sigx=gsig1
sigy=gsig2+.0001
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mux=gmu1
muy=gmu2
muz=gmu3
sigx=gsig1
sigy=gsig2-.0001
sigz=gsig3
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1s2=(o1p-o1m)/.0002
dc2s2=(o2p-o2m)/.0002
mux=gmu1
muy=gmu2
muz=gmu3
sigx=gsig1
sigy=gsig2
sigz=gsig3+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mux=gmu1
muy=gmu2
muz=gmu3
sigx=gsig1
sigy=gsig2
sigz=gsig3-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1s3=(o1p-o1m)/.0002
dc2s3=(o2p-o2m)/.0002
##calc dbcmu1 using equation to compare to estimate
dp1<-(1/gsig1)*((dc2m1-1)*dnorm((c2-gmu1)/gsig1)*
        (w21*p1-w31*p1)-
        w21*p1*dnorm((c1-gmu1)/gsig1)*(dc1m1-1))
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*dc1m1+
        w32*p2*dnorm((gmu2-c2)/gsig2)*(-dc2m1))
dp3<-(1/gsig3)*((dc1m1)*dnorm((c1-gmu3)/gsig3)*(w13*p3-w23*p3)...
```

```
    +
         w23*p3*dnorm((c2-gmu3)/gsig3)*dc2m1)
dbcm1<-dp1+dp2+dp3
##calc dbcmu2 using equation to compare to estimate
dp1<-(1/gsig1)*((dc2m2)*dnorm((c2-gmu1)/gsig1)*(w21*p1-w31*p1)...
    -
         w21*p1*dnorm((c1-gmu1)/gsig1)*dc1m2)
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*(dc1m2-1)+
         w32*p2*dnorm((gmu2-c2)/gsig2)*(1-dc2m2))
dp3<-(1/gsig3)*(dc1m2*dnorm((c1-gmu3)/gsig3)*(w13*p3-w23*p3)+
         w23*p3*dnorm((c2-gmu3)/gsig3)*dc2m2)
dbcm2<-dp1+dp2+dp3
##calc dbcmu3 using equation to compare to estimate
dp1<-(1/gsig1)*(dc2m3*dnorm((c2-gmu1)/gsig1)*(w21*p1-w31*p1)-
         w21*p1*dnorm((c1-gmu1)/gsig1)*dc1m3)
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*dc1m3+
         w32*p2*dnorm((gmu2-c2)/gsig2)*(-dc2m3))
dp3<-(1/gsig3)*((dc1m3-1)*dnorm((c1-gmu3)/gsig3)*
         (w13*p3-w23*p3)+
         w23*p3*dnorm((c2-gmu3)/gsig3)*(dc2m3-1))
dbcm3<-dp1+dp2+dp3
##calc dbcs1 using equation to compare to estimate
dp1<-(1/gsig1)*(dnorm((c2-gmu1)/gsig1)*
         (w21*p1*(dc2s1-((c2-gmu1)/gsig1))+
         w31*p1*(-dc2s1-((gmu1-c2)/gsig1)))
         -w21*p1*dnorm((c1-gmu1)/gsig1)*
         (dc1s1-((c1-gmu1)/gsig1)))
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*dc1s1-
         w32*p2*dnorm((gmu2-c2)/gsig2)*dc2s1)
dp3<-(1/gsig3)*(dnorm((c1-gmu3)/gsig3)*dc1s1*(w13*p3-w23*p3)+
         w23*p3*dnorm((c2-gmu3)/gsig3)*dc2s1)
dbcs1<-dp1+dp2+dp3
##calc dbcs2 using equation to compare to estimate
dp1<-(1/gsig1)*(dnorm((c2-gmu1)/gsig1)*dc2s2*(w21*p1-w31*p1)-
         w21*p1*dnorm((c1-gmu1)/gsig1)*dc1s2)
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*
         (dc1s2-((c1-gmu2)/gsig2))+
         w32*p2*dnorm((gmu2-c2)/gsig2)*
         (-dc2s2-((gmu2-c2)/gsig2)))
dp3<-(1/gsig3)*(dnorm((c1-gmu3)/gsig3)*dc1s2*(w13*p3-w23*p3)+
         w23*p3*dnorm((c2-gmu3)/gsig3)*dc2s2)
dbcs2<-dp1+dp2+dp3
##calc dbcs3 using equation to compare to estimate
dp1<-(1/gsig1)*(dnorm((c2-gmu1)/gsig1)*dc2s3*(w21*p1-w31*p1)-
         w21*p1*dnorm((c1-gmu1)/gsig1)*dc1s3)
dp2<-(1/gsig2)*(w12*p2*dnorm((c1-gmu2)/gsig2)*dc1s3+
         w32*p2*dnorm((gmu2-c2)/gsig2)*(-dc2s3))
dp3<-(1/gsig3)*(dnorm((c1-gmu3)/gsig3)*
         (dc1s3-((c1-gmu3)/gsig3))*(w13*p3-w23*p3)+
         w23*p3*dnorm((c2-gmu3)/gsig3)*
         (dc2s3-((c2-gmu3)/gsig3)))
dbcs3<-dp1+dp2+dp3
```

```
VBC=(dbcm1^2)*vm1+(dbcs1^2)*vs1+(dbcm2^2)*vm2+
    (dbcs2^2)*vs2+(dbcm3^2)*vm3+(dbcs3^2)*vs3
#Calculate Correlation
corest1=cor(X,XA)
corest2=cor(Y,YA)
corest3=cor(Z,ZA)
corrs=dbcm1*dbcm1a*corest1*sqrt(vm1a)*sqrt(vm1)+
  dbcm2*dbcm2a*corest2*sqrt(vm2a)*sqrt(vm2)+
  dbcm3*dbcm3a*corest3*sqrt(vm3a)*sqrt(vm3)+
  dbcs1a*dbcs1*corest1^2*sqrt(vs1a)*sqrt(vs1)+
  dbcs2a*dbcs2*corest2^2*sqrt(vs2a)*sqrt(vs2)+
  dbcs3a*dbcs3*corest3^2*sqrt(vs3a)*sqrt(vs3)
VETA=VBCA+VBC-2*corrs
EETA=EBCA-EBC
W=(EETA-TV)/sqrt(VETA)
deltap=pnorm(W,lower.tail=F)
lowconf=EETA-qnorm(.025,lower.tail=F)*sqrt(VETA)
highconf=EETA+qnorm(.025,lower.tail=F)*sqrt(VETA)
```

F.4 Correlated Generalized Method for 3 Classes

```
p1<#Set Prevalence Class 1
p2<#Set Prevalence Class 2
p3<#Set Prevalence Class 3
w21<# Set cost 2|1
w31<# Set cost 3|1
w12<# Set cost 1|2
w32<# Set cost 3|2
w13<# Set cost 1|3
w23<# Set cost 2|3
YA=#Class 1 for Test 1
XA=#Class 2 for Test 1
ZA=#Class 3 for Test 1
Y=#Class 1 for Test 2
X=#Class 2 for Test 2
Z=#Class 3 for Test 2
n1=#Size of Class 1 (both tests)
n2=#Size of Class 2 (both tests)
n3=#Size of Class 3 (both tests)
start=c(.1,0)
L=c(-1000,-1000)
U=c(1000,1000)
cov1=cov(XA,X)
cov2=cov(YA,Y)
cov3=cov(ZA,Z)
K=2500
ybar1a=mean(XA)
ybar2a=mean(YA)
ybar3a=mean(ZA)
var1a=var(XA)
var2a=var(YA)
var3a=var(ZA)
```

```
ybar1=mean(X)
ybar2=mean(Y)
ybar3=mean(Z)
var1=var(X)
var2=var(Y)
var3=var(Z)
draw1=rWishart(K,n1-1,matrix(c(var1a,cov1,cov1,var1)/
                              (n1-1),2,2))
draw2=rWishart(K,n2-1,matrix(c(var2a,cov2,cov2,var2)/
                              (n2-2),2,2))
draw3=rWishart(K,n3-1,matrix(c(var3a,cov3,cov3,var3)/
                              (n3-3),2,2))
co1=cor(XA,X)
co2=cor(YA,Y)
co3=cor(ZA,Z)
t1=rt(K,n1-1)
t2=rt(K,n2-1)
t3=rt(K,n3-1)
Rs1a=draw1[1,1,]
Rs2a=draw2[1,1,]
Rs3a=draw3[1,1,]
Rm1a=c(rep(ybar1a,K))-(t1*(sqrt(var1a/n1)))
Rm2a=c(rep(ybar2a,K))-(t2*(sqrt(var2a/n2)))
Rm3a=c(rep(ybar3a,K))-(t3*(sqrt(var3a/n3)))
f=function(x){
  hun2=function(par){
    abs(pnorm(par[2],x[1],x[2])-
      pnorm(par[1],x[1],x[2]))*(p1*w21)+
      abs(1-pnorm(par[2],x[1],x[2]))*(p1*w31)+
      abs(pnorm(par[1],x[3],x[4]))*(p2*w12)+
      abs(1-pnorm(par[2],x[3],x[4]))*(p2*w32)+
      abs(pnorm(par[1],x[5],x[6]))*(p3*w13)+
      abs(pnorm(par[2],x[5],x[6])-
      pnorm(par[1],x[5],x[6]))*(p3*w23)}
  y=optim(start,hun2,lower=L,upper=U,method="L-BFGS-B")
  BC=y$value
  return(BC)
}
ap1=cbind(Rm1a,sqrt(Rs1a),Rm2a,sqrt(Rs2a),Rm3a,sqrt(Rs3a))
RbcA=apply(ap1,1,FUN=f)
t1=rt(K,n1-1)
t2=rt(K,n2-1)
t3=rt(K,n3-1)
Rs1=draw1[2,2,]
Rs2=draw2[2,2,]
Rs3=draw3[2,2,]
Rm1=c(rep(ybar1,K))+co1*(var1/var1a)*(Rm1a-ybar1a)-
  (t1*(sqrt(var1a*(1-co1)^2/n1)))
Rm2=c(rep(ybar2,K))+co2*(var2/var2a)*(Rm2a-ybar2a)-
  (t1*(sqrt(var2a*(1-co2)^2/n2)))
Rm3=c(rep(ybar3,K))+co3*(var3/var3a)*(Rm3a-ybar3a)-
  (t1*(sqrt(var2a*(1-co3)^2/n3)))
```

```
ap1=cbind(Rm1,sqrt(Rs1),Rm2,sqrt(Rs2),Rm3,sqrt(Rs3))
Rbc=apply(ap1,1,FUN=f)
Reta=RbcA-Rbc
genp=length(which(Reta<TV))/K
lowconf=quantile(Reta,.025)
hiconf=quantile(Reta,.975)
```

F.5 Sequential Delta Method, BP Strategy

```
#Calculate and Set Parameters
p1<#Set Prevalence Class 1
p2<#Set Prevalence Class 2
w21<# Set cost 2|1
w12<# Set cost 1|2
YA=#Class 1 for Test 1
XA=#Class 2 for Test 1
Y=#Class 1 for Test 2
X=#Class 2 for Test 2
n1=#Size of Class 1 (both tests)
n2=#Size of Class 2 (both tests)
start=c(.1,0)
L=c(-1000,-1000)
U=c(1000,1000)
gmua1=mean(XA1)
gmua2=mean(XA2)
gmub1=mean(XB1)
gmub2=mean(XB2)
gsiga1=sd(XA1)
gsiga2=sd(XA2)
gsigb1=sd(XB1)
gsigb2=sd(XB2)
gRhoa=cor(XA1,XA2)
gRhob=cor(XB1,XB2)
vma1=(gsiga1^2)/n1
vma2=(gsiga2^2)/n1
vmb1=(gsigb1^2)/n2
vmb2=(gsigb2^2)/n2
vsa1=(gsiga1^2)/(2*(n1-1))
vsa2=(gsiga2^2)/(2*(n1-1))
vsb1=(gsigb1^2)/(2*(n2-1))
vsb2=(gsigb2^2)/(2*(n2-1))
vrhoa=(1-gRhoa^2)^2/sqrt(n1)
vrhob=(1-gRhob^2)^2/sqrt(n2)
#Define BC Function
g=function(par){
  (1-pmvnorm(lower=-Inf,upper=par,mean=c(mua1,mua2),
      sigma=matrix(c(siga1^2,Rhoa*siga1*siga2,Rhoa*siga1*siga2...
         ,
      siga2^2),nrow=2)))*(p1*w21)+
    (pmvnorm(lower=-Inf,upper=par,mean=c(mub1,mub2),
      sigma=matrix(c(sigb1^2,Rhob*sigb1*sigb2,Rhob*sigb1*sigb2...
         ,
```

```
      sigb2^2),nrow=2)))*(p2*w12)}
mua1=gmua1
mua2=gmua2
mub1=gmub1
mub2=gmub2
siga1=gsiga1
siga2=gsiga2
sigb1=gsigb1
sigb2=gsigb2
Rhoa=gRhoa
Rhob=gRhob
x=nlminb(start,g,lower=L,upper=U)
c1=x$par[1]
c2=x$par[2]
EBC=x$objective
#Partial Derivatives
#Partial with respect to mua1
mua1=gmua1+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mua1=gmua1-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1ma1=(o1p-o1m)/.0002
dc2ma1=(o2p-o2m)/.0002
mua1=gmua1
#partial with respect to mua2
mua2=gmua2+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mua2=gmua2-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1ma2=(o1p-o1m)/.0002
dc2ma2=(o2p-o2m)/.0002
mua2=gmua2
#Partial with respect to mub1
mub1=gmub1+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mub1=gmub1-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1mb1=(o1p-o1m)/.0002
dc2mb1=(o2p-o2m)/.0002
mub1=gmub1
```

```
#partial with respect to mub2
mub2=gmub2+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mub2=gmub2-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1mb2=(o1p-o1m)/.0002
dc2mb2=(o2p-o2m)/.0002
mub2=gmub2
#Partial with respect to siga1
siga1=gsiga1+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
siga1=gsiga1-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1sa1=(o1p-o1m)/.0002
dc2sa1=(o2p-o2m)/.0002
siga1=gsiga1
#partial with respect to siga2
siga2=gsiga2+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
siga2=gsiga2-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1sa2=(o1p-o1m)/.0002
dc2sa2=(o2p-o2m)/.0002
siga2=gsiga2
#Partial with respect to sigb1
sigb1=gsigb1+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
sigb1=gsigb1-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1sb1=(o1p-o1m)/.0002
dc2sb1=(o2p-o2m)/.0002
sigb1=gsigb1
#partial with respect to sigb2
sigb2=gsigb2+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
```

```
o2p=x$par[2]
sigb2=gsigb2-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1sb2=(o1p-o1m)/.0002
dc2sb2=(o2p-o2m)/.0002
sigb2=gsigb2
#Partial with respect to Rhoa
Rhoa=gRhoa+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
Rhoa=gRhoa-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1ra=(o1p-o1m)/.0002
dc2ra=(o2p-o2m)/.0002
Rhoa=gRhoa
#partial with respect to sigb2
Rhob=gRhob+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
Rhob=gRhob-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1rb=(o1p-o1m)/.0002
dc2rb=(o2p-o2m)/.0002
Rhob=gRhob
za1=(c1-gmua1)/gsiga1
za2=(c2-gmua2)/gsiga2
zb1=(c1-gmub1)/gsigb1
zb2=(c2-gmub2)/gsigb2
##Full Partial Derivative Calculations
##calc dbcmua1 using equation to compare to estimate
dp1<-w21*p1*dnorm(za1)*pnorm(za2,mean=gRhoa*za1,
        sd=sqrt(1-gRhoa^2))*(dc1ma1-1)/gsiga1
dp2<-w21*p1*dnorm(za2)*pnorm(za1,mean=gRhoa*za2,
        sd=sqrt(1-gRhoa^2))*(dc2ma1)/gsiga2
dp3<-w12*p2*dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,
        sd=sqrt(1-gRhob^2))*(dc1ma1)/gsigb1
dp4<-w12*p2*dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,
        sd=sqrt(1-gRhob^2))*(dc2ma1)/gsigb2
dbcma1<-dp3+dp4-dp1-dp2
##calc dbcmua2 using equation to compare to estimate
dp1<-w21*p1*dnorm(za1)*pnorm(za2,mean=gRhoa*za1,
        sd=sqrt(1-gRhoa^2))*(dc1ma2)/gsiga1
dp2<-w21*p1*dnorm(za2)*pnorm(za1,mean=gRhoa*za2,
        sd=sqrt(1-gRhoa^2))*(dc2ma2-1)/gsiga2
```

```
dp3<-w12*p2*dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,
        sd=sqrt(1-gRhob^2))*(dc1ma2)/gsigb1
dp4<-w12*p2*dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,
        sd=sqrt(1-gRhob^2))*(dc2ma2)/gsigb2
dbcma2<-dp3+dp4-dp1-dp2
##calc dbcmub1 using equation to compare to estimate
dp1<-w21*p1*dnorm(za1)*pnorm(za2,mean=gRhoa*za1,
        sd=sqrt(1-gRhoa^2))*(dc1mb1)/gsiga1
dp2<-w21*p1*dnorm(za2)*pnorm(za1,mean=gRhoa*za2,
        sd=sqrt(1-gRhoa^2))*(dc2mb1)/gsiga2
dp3<-w12*p2*dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,
        sd=sqrt(1-gRhob^2))*(dc1mb1-1)/gsigb1
dp4<-w12*p2*dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,
        sd=sqrt(1-gRhob^2))*(dc2mb1)/gsigb2
dbcmb1<-dp3+dp4-dp1-dp2
##calc dbcmub2 using equation to compare to estimate
dp1<-w21*p1*dnorm(za1)*pnorm(za2,mean=gRhoa*za1,
        sd=sqrt(1-gRhoa^2))*(dc1mb2)/gsiga1
dp2<-w21*p1*dnorm(za2)*pnorm(za1,mean=gRhoa*za2,
        sd=sqrt(1-gRhoa^2))*(dc2mb2)/gsiga2
dp3<-w12*p2*dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,
        sd=sqrt(1-gRhob^2))*(dc1mb2)/gsigb1
dp4<-w12*p2*dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,
        sd=sqrt(1-gRhob^2))*(dc2mb2-1)/gsigb2
dbcmb2<-dp3+dp4-dp1-dp2
##calc dbcsa1 using equation to compare to estimate
dp1<-w21*p1*dnorm(za1)*pnorm(za2,mean=gRhoa*za1,
        sd=sqrt(1-gRhoa^2))*(dc1sa1-(c1-gmua1)/gsiga1)/gsiga1
dp2<-w21*p1*dnorm(za2)*pnorm(za1,mean=gRhoa*za2,
        sd=sqrt(1-gRhoa^2))*(dc2sa1)/gsiga2
dp3<-w12*p2*dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,
        sd=sqrt(1-gRhob^2))*(dc1sa1)/gsigb1
dp4<-w12*p2*dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,
        sd=sqrt(1-gRhob^2))*(dc2sa1)/gsigb2
dbcsa1<-dp3+dp4-dp1-dp2

##calc dbcsa2 using equation to compare to estimate
dp1<-w21*p1*dnorm(za1)*pnorm(za2,mean=gRhoa*za1,
        sd=sqrt(1-gRhoa^2))*(dc1sa2)/gsiga1
dp2<-w21*p1*dnorm(za2)*pnorm(za1,mean=gRhoa*za2,
        sd=sqrt(1-gRhoa^2))*(dc2sa2-(c2-gmua2)/gsiga2)/gsiga2
dp3<-w12*p2*dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,
        sd=sqrt(1-gRhob^2))*(dc1sa2)/gsigb1
dp4<-w12*p2*dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,
        sd=sqrt(1-gRhob^2))*(dc2sa2)/gsigb2
dbcsa2<-dp3+dp4-dp1-dp2
##calc dbcsb1 using equation to compare to estimate
dp1<-w21*p1*dnorm(za1)*pnorm(za2,mean=gRhoa*za1,
        sd=sqrt(1-gRhoa^2))*(dc1sb1)/gsiga1
dp2<-w21*p1*dnorm(za2)*pnorm(za1,mean=gRhoa*za2,
        sd=sqrt(1-gRhoa^2))*(dc2sb1)/gsiga2
dp3<-w12*p2*dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,
```

```
          sd=sqrt(1-gRhob^2))*(dc1sb1-(c1-gmub1)/gsigb1)/gsigb1
dp4<-w12*p2*dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,
          sd=sqrt(1-gRhob^2))*(dc2sb1)/gsigb2
dbcsb1<-dp3+dp4-dp1-dp2
##calc dbcsb2 using equation to compare to estimate
dp1<-w21*p1*dnorm(za1)*pnorm(za2,mean=gRhoa*za1,
          sd=sqrt(1-gRhoa^2))*(dc1sb2)/gsiga1
dp2<-w21*p1*dnorm(za2)*pnorm(za1,mean=gRhoa*za2,
          sd=sqrt(1-gRhoa^2))*(dc2sb2)/gsiga2
dp3<-w12*p2*dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,
          sd=sqrt(1-gRhob^2))*(dc1sb2)/gsigb1
dp4<-w12*p2*dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,
          sd=sqrt(1-gRhob^2))*(dc2sb2-(c2-gmub2)/gsigb2)/gsigb2
dbcsb2<-dp3+dp4-dp1-dp2
##calc dbcra using equation to compare to estimate
dp1<-w21*p1*dnorm(za1)*pnorm(za2,mean=gRhoa*za1,
          sd=sqrt(1-gRhoa^2))*(dc1ra)/gsiga1
dp2<-w21*p1*dnorm(za2)*pnorm(za1,mean=gRhoa*za2,
          sd=sqrt(1-gRhoa^2))*(dc2ra)/gsiga2
dp3<-w12*p2*dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,
          sd=sqrt(1-gRhob^2))*(dc1ra)/gsigb1
dp4<-w12*p2*dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,
          sd=sqrt(1-gRhob^2))*(dc2ra)/gsigb2
dbcra<-dp3+dp4-dp1-dp2
##calc dbcrb using equation to compare to estimate
dp1<-w21*p1*dnorm(za1)*pnorm(za2,mean=gRhoa*za1,
          sd=sqrt(1-gRhoa^2))*(dc1rb)/gsiga1
dp2<-w21*p1*dnorm(za2)*pnorm(za1,mean=gRhoa*za2,
          sd=sqrt(1-gRhoa^2))*(dc2rb)/gsiga2
dp3<-w12*p2*dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,
          sd=sqrt(1-gRhob^2))*(dc1rb)/gsigb1
dp4<-w12*p2*dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,
          sd=sqrt(1-gRhob^2))*(dc2rb)/gsigb2
dbcrb<-dp3+dp4-dp1-dp2
#Calculate Variance
VBC=(dbcma1)^2*vma1+(dbcma2)^2*vma2+(dbcmb1)^2*vmb1+
  (dbcmb2)^2*vmb2+(dbcsa1)^2*vsa1+(dbcsa2)^2*vsa2+
  (dbcsb1)^2*vsb1+(dbcsb2)^2*vsb2+(dbcra)^2*vrhoa+
  (dbcrb)^2*vrhob+dbcma1*dbcma2*gRhoa*sqrt(vma1*vma2)+
  dbcmb1*dbcmb2*gRhob*sqrt(vmb1*vmb2)+
  dbcsa1*dbcsa2*gRhoa^2*sqrt(vsa1*vsa2)+
  dbcsb1*dbcsb2*gRhob^2*sqrt(vsb1*vsb2)
VETA=VBC
EETA=EBC
#Confidence Intervals
LowConf=EETA-qnorm(.025,lower.tail=F)*sqrt(VETA)
HighConf=EETA+qnorm(.025,lower.tail=F)*sqrt(VETA)
```

F.6 Sequential Delta Method, BN Strategy

```
#Calculate and Set Parameters
p1<#Set Prevalence Class 1
```

```
p2<#Set Prevalence Class 2
w21<# Set cost 2|1
w12<# Set cost 1|2
YA=#Class 1 for Test 1
XA=#Class 2 for Test 1
Y=#Class 1 for Test 2
X=#Class 2 for Test 2
n1=#Size of Class 1 (both tests)
n2=#Size of Class 2 (both tests)
start=c(.1,0)
L=c(-10,-10)
U=c(10,10)
gmua1=mean(XA1)
gmua2=mean(XA2)
gmub1=mean(XB1)
gmub2=mean(XB2)
gsiga1=sd(XA1)
gsiga2=sd(XA2)
gsigb1=sd(XB1)
gsigb2=sd(XB2)
gRhoa=cor(XA1,XA2)
gRhob=cor(XB1,XB2)
vma1=(gsiga1^2)/n1
vma2=(gsiga2^2)/n1
vmb1=(gsigb1^2)/n2
vmb2=(gsigb2^2)/n2
vsa1=(gsiga1^2)/(2*(n1-1))
vsa2=(gsiga2^2)/(2*(n1-1))
vsb1=(gsigb1^2)/(2*(n2-1))
vsb2=(gsigb2^2)/(2*(n2-1))
vrhoa=(1-gRhoa^2)^2/sqrt(n1)
vrhob=(1-gRhob^2)^2/sqrt(n2)
#Define BC Function
fp=function(par){1-pnorm(par[1],mean=mua1,sd=siga1)-
    pnorm(par[2],mean=mua2,sd=siga2)+
    (pmvnorm(lower=-Inf,upper=par, mean=c(mua1,mua2),
    sigma=matrix(c(siga1^2,Rhoa*siga1*siga2,
    Rhoa*siga1*siga2,siga2^2),nrow=2)))}
tp=function(par){1-pnorm(par[1],mean=mub1,sd=sigb1)-
    pnorm(par[2],mean=mub2,sd=sigb2)+
    (pmvnorm(lower=-Inf,upper=par, mean=c(mub1,mub2),
    sigma=matrix(c(sigb1^2,Rhob*sigb1*sigb2,
    Rhob*sigb1*sigb2,sigb2^2),nrow=2)))}
g=function(par){p2*w12*(1-tp(par))+p1*w21*(fp(par))}
mua1=gmua1
mua2=gmua2
mub1=gmub1
mub2=gmub2
siga1=gsiga1
siga2=gsiga2
sigb1=gsigb1
sigb2=gsigb2
```

```
Rhoa=gRhoa
Rhob=gRhob
x=nlminb(start,g,lower=L,upper=U)
c1=x$par[1]
c2=x$par[2]
EBC=x$objective
##Caclulate Partials
#Partial with respect to mua1
mua1=gmua1+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mua1=gmua1-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1ma1=(o1p-o1m)/.0002
dc2ma1=(o2p-o2m)/.0002
mua1=gmua1
#partial with respect to mua2
mua2=gmua2+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mua2=gmua2-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1ma2=(o1p-o1m)/.0002
dc2ma2=(o2p-o2m)/.0002
mua2=gmua2
#Partial with respect to mub1
mub1=gmub1+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mub1=gmub1-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1mb1=(o1p-o1m)/.0002
dc2mb1=(o2p-o2m)/.0002
mub1=gmub1
#partial with respect to mub2
mub2=gmub2+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
mub2=gmub2-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
```

```
dc1mb2=(o1p-o1m)/.0002
dc2mb2=(o2p-o2m)/.0002
mub2=gmub2
#Partial with respect to siga1
siga1=gsiga1+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
siga1=gsiga1-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1sa1=(o1p-o1m)/.0002
dc2sa1=(o2p-o2m)/.0002
siga1=gsiga1
#partial with respect to siga2
siga2=gsiga2+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
siga2=gsiga2-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1sa2=(o1p-o1m)/.0002
dc2sa2=(o2p-o2m)/.0002
siga2=gsiga2
#Partial with respect to sigb1
sigb1=gsigb1+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
sigb1=gsigb1-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1sb1=(o1p-o1m)/.0002
dc2sb1=(o2p-o2m)/.0002
sigb1=gsigb1
#partial with respect to sigb2
sigb2=gsigb2+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
sigb2=gsigb2-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1sb2=(o1p-o1m)/.0002
dc2sb2=(o2p-o2m)/.0002
sigb2=gsigb2
#Partial with respect to Rhoa
```

```
Rhoa=gRhoa+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
Rhoa=gRhoa-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1ra=(o1p-o1m)/.0002
dc2ra=(o2p-o2m)/.0002
Rhoa=gRhoa
#partial with respect to sigb2
Rhob=gRhob+.0001
x=nlminb(start,g,lower=L,upper=U)
o1p=x$par[1]
o2p=x$par[2]
Rhob=gRhob-.0001
x=nlminb(start,g,lower=L,upper=U)
o1m=x$par[1]
o2m=x$par[2]
dc1rb=(o1p-o1m)/.0002
dc2rb=(o2p-o2m)/.0002
Rhob=gRhob
za1=(c1-gmua1)/gsiga1
za2=(c2-gmua2)/gsiga2
zb1=(c1-gmub1)/gsigb1
zb2=(c2-gmub2)/gsigb2
##Caculate Partial Derivatives
##calc dbcmua1 using equation to compare to estimate
dp1=dnorm(za1)*(dc1ma1-1)/gsiga1
dp2=dnorm(za2)*(dc2ma1)/gsiga2
dp3=dnorm(za2)*pnorm(za1,mean=gRhoa*za2,sd=sqrt(1-gRhoa^2))*
  (dc2ma1)/gsiga2
dp4=dnorm(za1)*pnorm(za2,mean=gRhoa*za1,sd=sqrt(1-gRhoa^2))*
  (dc1ma1-1)/gsiga1
dp5=dnorm(zb1)*(dc1ma1)/gsigb1
dp6=dnorm(zb2)*(dc2ma1)/gsigb2
dp7=dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,sd=sqrt(1-gRhob^2))*
  (dc1ma1)/gsigb1
dp8=dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,sd=sqrt(1-gRhob^2))*
  (dc2ma1)/gsigb2
dbcma1<-p1*w21*(dp1+dp2-dp3-dp4)-p2*w12*(dp5+dp6-dp7-dp8)
##calc dbcmua2 using equation to compare to estimate
dp1=dnorm(za1)*(dc1ma2)/gsiga1
dp2=dnorm(za2)*(dc2ma2-1)/gsiga2
dp3=dnorm(za2)*pnorm(za1,mean=gRhoa*za2,sd=sqrt(1-gRhoa^2))*
  (dc2ma2-1)/gsiga2
dp4=dnorm(za1)*pnorm(za2,mean=gRhoa*za1,sd=sqrt(1-gRhoa^2))*
  (dc1ma2)/gsiga1
dp5=dnorm(zb1)*(dc1ma2)/gsigb1
dp6=dnorm(zb2)*(dc2ma2)/gsigb2
dp7=dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,sd=sqrt(1-gRhob^2))*
```

```
  (dc1ma2)/gsigb1
dp8=dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,sd=sqrt(1-gRhob^2))*
  (dc2ma2)/gsigb2
dbcma2<-p1*w21*(dp1+dp2-dp3-dp4)-p2*w12*(dp5+dp6-dp7-dp8)
##calc dbcmub1 using equation to compare to estimate
dp1=dnorm(za1)*(dc1mb1)/gsiga1
dp2=dnorm(za2)*(dc2mb1)/gsiga2
dp3=dnorm(za2)*pnorm(za1,mean=gRhoa*za2,sd=sqrt(1-gRhoa^2))*
  (dc2mb1)/gsiga2
dp4=dnorm(za1)*pnorm(za2,mean=gRhoa*za1,sd=sqrt(1-gRhoa^2))*
  (dc1mb1)/gsiga1
dp5=dnorm(zb1)*(dc1mb1-1)/gsigb1
dp6=dnorm(zb2)*(dc2mb1)/gsigb2
dp7=dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,sd=sqrt(1-gRhob^2))*
  (dc1mb1-1)/gsigb1
dp8=dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,sd=sqrt(1-gRhob^2))*
  (dc2mb1)/gsigb2
dbcmb1<-p1*w21*(dp1+dp2-dp3-dp4)-p2*w12*(dp5+dp6-dp7-dp8)
##calc dbcmub2 using equation to compare to estimate
dp1=dnorm(za1)*(dc1mb2)/gsiga1
dp2=dnorm(za2)*(dc2mb2)/gsiga2
dp3=dnorm(za2)*pnorm(za1,mean=gRhoa*za2,sd=sqrt(1-gRhoa^2))*
  (dc2mb2)/gsiga2
dp4=dnorm(za1)*pnorm(za2,mean=gRhoa*za1,sd=sqrt(1-gRhoa^2))*
  (dc1mb2)/gsiga1
dp5=dnorm(zb1)*(dc1mb2)/gsigb1
dp6=dnorm(zb2)*(dc2mb2-1)/gsigb2
dp7=dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,sd=sqrt(1-gRhob^2))*
  (dc1mb2)/gsigb1
dp8=dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,sd=sqrt(1-gRhob^2))*
  (dc2mb2-1)/gsigb2
dbcmb2<-p1*w21*(dp1+dp2-dp3-dp4)-p2*w12*(dp5+dp6-dp7-dp8)
##calc dbcsa1 using equation to compare to estimate
dp1=dnorm(za1)*(dc1sa1-(c1-gmua1)/gsiga1)/gsiga1
dp2=dnorm(za2)*(dc2sa1)/gsiga2
dp3=dnorm(za2)*pnorm(za1,mean=gRhoa*za2,sd=sqrt(1-gRhoa^2))*
  (dc2sa1)/gsiga2
dp4=dnorm(za1)*pnorm(za2,mean=gRhoa*za1,sd=sqrt(1-gRhoa^2))*
  (dc1sa1-(c1-gmua1)/gsiga1)/gsiga1
dp5=dnorm(zb1)*(dc1sa1)/gsigb1
dp6=dnorm(zb2)*(dc2sa1)/gsigb2
dp7=dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,sd=sqrt(1-gRhob^2))*
  (dc1sa1)/gsigb1
dp8=dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,sd=sqrt(1-gRhob^2))*
  (dc2sa1)/gsigb2
dbcsa1<-p1*w21*(dp1+dp2-dp3-dp4)-p2*w12*(dp5+dp6-dp7-dp8)
##calc dbcsa2 using equation to compare to estimate
dp1=dnorm(za1)*(dc1sa2)/gsiga1
dp2=dnorm(za2)*(dc2sa2-(c2-gmua2)/gsiga2)/gsiga2
dp3=dnorm(za2)*pnorm(za1,mean=gRhoa*za2,sd=sqrt(1-gRhoa^2))*
  (dc2sa2-(c2-gmua2)/gsiga2)/gsiga2
dp4=dnorm(za1)*pnorm(za2,mean=gRhoa*za1,sd=sqrt(1-gRhoa^2))*
```

```
  (dc1sa2)/gsiga1
dp5=dnorm(zb1)*(dc1sa1)/gsigb1
dp6=dnorm(zb2)*(dc2sa1)/gsigb2
dp7=dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,sd=sqrt(1-gRhob^2))*
  (dc1sa1)/gsigb1
dp8=dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,sd=sqrt(1-gRhob^2))*
  (dc2sa1)/gsigb2
dbcsa2<-p1*w21*(dp1+dp2-dp3-dp4)-p2*w12*(dp5+dp6-dp7-dp8)
##calc dbcsb1 using equation to compare to estimate
dp1=dnorm(za1)*(dc1sb1)/gsiga1
dp2=dnorm(za2)*(dc2sb1)/gsiga2
dp3=dnorm(za2)*pnorm(za1,mean=gRhoa*za2,sd=sqrt(1-gRhoa^2))*
  (dc2sb1)/gsiga2
dp4=dnorm(za1)*pnorm(za2,mean=gRhoa*za1,sd=sqrt(1-gRhoa^2))*
  (dc1sb1)/gsiga1
dp5=dnorm(zb1)*(dc1sb1-(c1-gmub1)/gsigb1)/gsigb1
dp6=dnorm(zb2)*(dc2sb1)/gsigb2
dp7=dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,sd=sqrt(1-gRhob^2))*
  (dc1sb1-(c1-gmub1)/gsigb1)/gsigb1
dp8=dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,sd=sqrt(1-gRhob^2))*
  (dc2sb1)/gsigb2
dbcsb1<-p1*w21*(dp1+dp2-dp3-dp4)-p2*w12*(dp5+dp6-dp7-dp8)
##calc dbcsb2 using equation to compare to estimate
dp1=dnorm(za1)*(dc1sb2)/gsiga1
dp2=dnorm(za2)*(dc2sb2)/gsiga2
dp3=dnorm(za2)*pnorm(za1,mean=gRhoa*za2,sd=sqrt(1-gRhoa^2))*
  (dc2sb2)/gsiga2
dp4=dnorm(za1)*pnorm(za2,mean=gRhoa*za1,sd=sqrt(1-gRhoa^2))*
  (dc1sb2)/gsiga1
dp5=dnorm(zb1)*(dc1sb2)/gsigb1
dp6=dnorm(zb2)*(dc2sb2-(c2-gmub2)/gsigb2)/gsigb2
dp7=dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,sd=sqrt(1-gRhob^2))*
  (dc1sb2)/gsigb1
dp8=dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,sd=sqrt(1-gRhob^2))*
  (dc2sb2-(c2-gmub2)/gsigb2)/gsigb2
dbcsb2<-p1*w21*(dp1+dp2-dp3-dp4)-p2*w12*(dp5+dp6-dp7-dp8)
##calc dbcra using equation to compare to estimate
dp1=dnorm(za1)*(dc1ra)/gsiga1
dp2=dnorm(za2)*(dc2ra)/gsiga2
dp3=dnorm(za2)*pnorm(za1,mean=gRhoa*za2,sd=sqrt(1-gRhoa^2))*
  (dc2ra)/gsiga2
dp4=dnorm(za1)*pnorm(za2,mean=gRhoa*za1,sd=sqrt(1-gRhoa^2))*
  (dc1ra)/gsiga1
dp5=dnorm(zb1)*(dc1ra)/gsigb1
dp6=dnorm(zb2)*(dc2ra)/gsigb2
dp7=dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,sd=sqrt(1-gRhob^2))*
  (dc1ra)/gsigb1
dp8=dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,sd=sqrt(1-gRhob^2))*
  (dc2ra)/gsigb2
dbcra<-p1*w21*(dp1+dp2-dp3-dp4)-p2*w12*(dp5+dp6-dp7-dp8)
##calc dbcrb using equation to compare to estimate
dp1=dnorm(za1)*(dc1rb)/gsiga1
```

```
dp2=dnorm(za2)*(dc2rb)/gsiga2
dp3=dnorm(za2)*pnorm(za1,mean=gRhoa*za2,sd=sqrt(1-gRhoa^2))*
  (dc2rb)/gsiga2
dp4=dnorm(za1)*pnorm(za2,mean=gRhoa*za1,sd=sqrt(1-gRhoa^2))*
  (dc1rb)/gsiga1
dp5=dnorm(zb1)*(dc1rb)/gsigb1
dp6=dnorm(zb2)*(dc2rb)/gsigb2
dp7=dnorm(zb1)*pnorm(zb2,mean=gRhob*zb1,sd=sqrt(1-gRhob^2))*
  (dc1rb)/gsigb1
dp8=dnorm(zb2)*pnorm(zb1,mean=gRhob*zb2,sd=sqrt(1-gRhob^2))*
  (dc2rb)/gsigb2
dbcrb<-p1*w21*(dp1+dp2-dp3-dp4)-p2*w12*(dp5+dp6-dp7-dp8)
#Calculate Variance
VBC=(dbcma1)^2*vma1+(dbcma2)^2*vma2+(dbcmb1)^2*vmb1+
  (dbcmb2)^2*vmb2+(dbcsa1)^2*vsa1+(dbcsa2)^2*vsa2+
  (dbcsb1)^2*vsb1+(dbcsb2)^2*vsb2+(dbcra)^2*vrhoa+
  (dbcrb)^2*vrhob+dbcma1*dbcma2*gRhoa*sqrt(vma1*vma2)+
  dbcmb1*dbcmb2*gRhob*sqrt(vmb1*vmb2)+
  dbcsa1*dbcsa2*gRhoa^2*sqrt(vsa1*vsa2)+
  dbcsb1*dbcsb2*gRhob^2*sqrt(vsb1*vsb2)
VETA=VBC
EETA=EBC
#Calculate CI
LowConf=EETA-qnorm(.025,lower.tail=F)*sqrt(VETA)
HighConf=EETA+qnorm(.025,lower.tail=F)*sqrt(VETA)
```

F.7 Sequential Generalized Method, BP Strategy

```
#Calculate and Set Parameters
p1<#Set Prevalence Class 1
p2<#Set Prevalence Class 2
w21<# Set cost 2|1
w12<# Set cost 1|2
YA=#Class 1 for Test 1
XA=#Class 2 for Test 1
Y=#Class 1 for Test 2
X=#Class 2 for Test 2
n1=#Size of Class 1 (both tests)
n2=#Size of Class 2 (both tests)
start=c(0,.1)
L=c(-1000,-1000)
U=c(1000,1000)
K=2500
ybar1=c(mean(XA),mean(X))
ybar2=c(mean(YA),mean(Y))
corx=cor(XA,X)
cory=cor(YA,Y)
covx=cov(XA,X)
covy=cov(YA,Y)
Sigma1=matrix(c(var(XA),covx,covx,var(X)),nrow=2)
Sigma2=matrix(c(var(YA),covy,covy,var(Y)),nrow=2)
sigma1=matrix(c(1,corx,corx,1),nrow=2)
```

```
sigma2=matrix(c(1,cory,cory,1),nrow=2)
#Draw from multivariate T distribution
t1=rmvt(K,sigma=sigma1*((n1-1)/(n1-3)),df=n1-1)
t2=rmvt(K,sigma=sigma2*((n2-1)/(n2-3)),df=n2-1)
#Calculate Pivots
rm1a=c(rep(ybar1[1],K))-(t1[,1]*sqrt(var(XA)/n1))
rm1b=c(rep(ybar1[2],K))-(t1[,2]*sqrt(var(X)/n1))
Rm1=cbind(rm1a,rm1b)
rm2a=rep(ybar2[1],K)-t2[,1]*sqrt(var(YA)/n2)
rm2b=rep(ybar2[2],K)-t2[,2]*sqrt(var(Y)/n2)
Rm2=cbind(rm2a,rm2b)
#Draw from Wishart (already a pivot)
Rs1=matrix(rWishart(K,df=n1-1,Sigma=Sigma1)/(n1-1),
           ncol=4,byrow=T)
Rs2=matrix(rWishart(K,df=n2-1,Sigma=Sigma2)/(n2-1),ncol=4,
           byrow=T)
f=function(x){
  hun2=function(par){
    (1-pmvnorm(lower=-Inf,upper=par,mean=c(x[1],x[2]),
               sigma=matrix(c(x[3:6]),nrow=2)))*(p1*w21)+
      (pmvnorm(lower=-Inf,upper=par,mean=c(x[7],x[8]),
               sigma=matrix(c(x[9:12]),nrow=2)))*(p2*w12)}
  y=nlminb(start,hun2,lower=L,upper=U)
  BC=round(y$objective,6)
  return(BC)
}
ap1=cbind(Rm1,Rs1,Rm2,Rs2)
RBC=apply(ap1,1,FUN=f)
#Confidence Intervals
lowconf=quantile(RBC,.025)
hiconf=quantile(RBC,.975)
```

F.8 Sequential Generalized Method, BN Strategy

```
#Calculate and Set Parameters
p1<#Set Prevalence Class 1
p2<#Set Prevalence Class 2
w21<# Set cost 2|1
w12<# Set cost 1|2
YA=#Class 1 for Test 1
XA=#Class 2 for Test 1
Y=#Class 1 for Test 2
X=#Class 2 for Test 2
n1=#Size of Class 1 (both tests)
n2=#Size of Class 2 (both tests)
start=c(0,.1)
L=c(-1000,-1000)
U=c(1000,1000)
K=2500
ybar1=c(mean(XA),mean(X))
ybar2=c(mean(YA),mean(Y))
corx=cor(XA,X)
```

```
cory=cor(YA,Y)
covx=cov(XA,X)
covy=cov(YA,Y)
Sigma1=matrix(c(var(XA),covx,covx,var(X)),nrow=2)
Sigma2=matrix(c(var(YA),covy,covy,var(Y)),nrow=2)
sigma1=matrix(c(1,corx,corx,1),nrow=2)
sigma2=matrix(c(1,cory,cory,1),nrow=2)
#Draw from multivariate T
t1=rmvt(K,sigma=sigma1*((n1-1)/(n1-3)),df=n1-1)
t2=rmvt(K,sigma=sigma2*((n2-1)/(n2-3)),df=n2-1)
#Create Pivotal Quantities
rm1a=c(rep(ybar1[1],K))-(t1[,1]*sqrt(var(XA)/n1))
rm1b=c(rep(ybar1[2],K))-(t1[,2]*sqrt(var(X)/n1))
Rm1=cbind(rm1a,rm1b)
rm2a=rep(ybar2[1],K)-t2[,1]*sqrt(var(YA)/n2)
rm2b=rep(ybar2[2],K)-t2[,2]*sqrt(var(Y)/n2)
Rm2=cbind(rm2a,rm2b)
#Draw from Wishart (Already Pivots)
Rs1=matrix(rWishart(K,df=n1-1,Sigma=Sigma1)/
             (n1-1),ncol=4,byrow=T)
Rs2=matrix(rWishart(K,df=n2-1,Sigma=Sigma2)/
             (n2-1),ncol=4,byrow=T)
#BC Function for RBC
f=function(x){
  fp=function(par){1-pnorm(par[1],mean=x[1],sd=sqrt(x[3]))-
      pnorm(par[2],mean=x[2],sd=sqrt(x[6]))+
      (pmvnorm(lower=-Inf,upper=par,mean=c(x[1],x[2]),
               sigma=matrix(c(x[3:6]),nrow=2)))}
  tp=function(par){1-pnorm(par[1],mean=x[7],sd=sqrt(x[9]))-
      pnorm(par[2],mean=x[8],sd=sqrt(x[12]))+
      (pmvnorm(lower=-Inf,upper=par,mean=c(x[7],x[8]),
               sigma=matrix(c(x[9:12]),nrow=2)))}
  hun2=function(par){p2*w12*(1-tp(par))+p1*w21*(fp(par))}
  y=nlminb(start,hun2,lower=L,upper=U)
  BC=round(y$objective,6)
  return(BC)
}
ap1=cbind(Rm1,Rs1,Rm2,Rs2)
RBC=apply(ap1,1,FUN=f)
#Confidence Intervals
lowconf=quantile(RBC,.025)
hiconf=quantile(RBC,.975)
```

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13. SUPPLEMENTARY NOTES

14. ABSTRACT

The current emphasis on including correlation when comparing diagnostic test performance is quite important, however, there are cases in which correlation effects may be negligible with respect to inference. This proposed work examines the impact of including correlation between classification systems with continuous features by comparing the optimal performance of two diagnostic tests with multiple outcomes as well as providing inference for a sequence of tests. We define the optimal point using Bayes Cost. Through simulation, we quantify the impact of correlation on standard errors comparing two tests and evaluate the resulting errors with respect to CI coverage and width under varying diagnostic test settings. When formulas are required for better inference to include correlation, we provide updated techniques that properly extend the Delta and Generalized method. Additionally, to date, no methods have been applied to quantify the performance of a sequence of tests. Therefore, the inference methods derived in this work are extended to sequenced tests where feature correlation is unavoidable and must be accounted for when developing inference on tests.

15. SUBJECT TERMS

Bayes Cost, optimal point, Youden Index, diagnostic test, correlation, confidence interval, sequenced tests

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