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Analysis of a Voting Method for Ranking Network Centrality Measures on a Node-aligned Multiplex Network

Kyle S. Wilkinson

ANALYSIS OF A VOTING METHOD FOR RANKING NETWORK CENTRALITY MEASURES ON A NODE-ALIGNED MULTIPLEX NETWORK

THESIS

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Abstract

In *Joint Concept: Human Aspects of Military Operations* (JC-HAMO), the need for identifying critical actors within a target network is clearly identified as a precursor to successfully influencing decision making and operational outcomes. JC-HAMO seeks methods to identify critical actors within the context of multiple types of networks and over a period of time. This problem can be approached structurally using a time-stamped multilayer network. One method of identifying critical actors in a single layer—fully-aggregated—network involves ranking actors in order of importance by some set of network measures. This thesis explores a method for extending such a ranking of critical actors into a multilayer network context. Specifically, it borrows and applies a methodology from the field of electoral systems to the problem of ranking actors based on a set of rankings for each layer.

The Schulze method—a deterministic voting methodology based on a modified shortest path algorithm—is examined and its performance is assessed through statistical comparison with identified alternative approaches and baseline rankings. Potential advantages and limitations are identified as well as a method for increasing its robustness when the networks of interest contain many isolated components. This is done by adopting a secondary weighting scheme. As a corollary study, an information-theoretic multilayer network layer-reduction heuristic is explored and the resulting rankings on the reduced multilayer network are compared with those of the full multilayer network and those of the corresponding fully-aggregated single layer network. A tertiary effort compares two

distinct multilayer network weighting schemes. Results are based on the study of an open sourced multilayer time-stamped terrorist network.

To my wife: This, as with most things, would have been impossible without you

To my son: You've been more patient and understanding than you should be

To my new daughter: I can't wait to meet you

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Whatever errors exist are of my creation alone.

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Kyle S. Wilkinson

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ANALYSIS OF A VOTING METHOD FOR RANKING NETWORK CENTRALITY MEASURES ON A NODE-ALIGNED MULTIPLEX NETWORK

I. Introduction

1.1 Chapter Overview

This chapter serves as an introduction to the work presented in this thesis. The general topic is discussed along with the underlying motivation for pursuing this research. The specific problems are identified and a brief introduction is given to what will be presented later in the document. This includes the literature review, the methodology of analysis, the analysis of the results, and a conclusion of the research.

1.2 Overview of Thesis Objectives

This thesis is multifaceted. The primary objective is to demonstrate the utility of a social choice theory methodology for ranking network measures on a multilayer network. The secondary objective is to investigate the effects on such rankings of the prior *reduction* (partial aggregation of the layers) of the same multilayer network which is reduced using an information theoretic distance measure and clustering algorithm. The tertiary objective is to investigate the effects of weighting the data within the methodology.

A cross-disciplinary literature review was conducted which successfully bridged distinct domains of study in a novel way to produce a new methodology. This methodology helps to address a fundamental operational problem: finding the most important targets in a multilayer network. Initial results are promising and suggest ample room for future research.

Each objective was met using a mixed approach of statistical and qualitative analyses. Analyses and conclusions were based upon the study of a single multilayer network dataset, the Noordin Top terrorist network. This dataset is described in detail in Chapter III.

1.3 General Issue

In *Joint Concept for Human Aspects of Military Operations* (JC-HAMO), (a publication of future concepts of operations published by the Office of the Joint Chiefs of Staff), it is rightly recognized “that war is fundamentally and primarily a human endeavor” and that “the need to understand relevant actors’ motivations and the underpinnings of their will” continues to be a key challenge within military operations (Office of the Joint Chiefs of Staff, 2016). To this end, it lists as its central idea four action items for the Joint Force as shown in Figure 1.

Central Idea: A Joint Approach to the Human Aspects of Military Operations

To achieve national and military objectives, the Joint Force will develop and adopt an updated mindset and approach that accounts for the human aspects of military operations, recognizing that, even in our technological age, war is primarily a human endeavor. This mindset and approach, which provides the foundation for a core competency, will improve how the force visualizes the environment and interacts with relevant actors within the context of the operational situation. The Joint Force will:

- *Identify* the range of relevant actors and their associated social, cultural, political, economic, and organizational networks.
- *Evaluate* contextual relevant actor behavior.
- *Anticipate* relevant actor decision-making.
- *Influence* the will and decisions of relevant actors.

Figure 1: JC-HAMO Central Idea and Four Actions

(Office of the Joint Chiefs of Staff, 2016)

In order to accomplish all four action items, relevant actors must be identified by considering networks similar to those listed in Figure 1, bullet 1 (Office of the Joint Chiefs of Staff, 2016). A relevant actor is defined as "individuals, groups, and populations whose behavior has the potential to substantially help or hinder the success of a particular campaign, operation, or tactical action" (Office of the Joint Chiefs of Staff, 2016).

Additionally, a relevant actor's "religion, ethnicity, gender, language, tribe, social class, caste, occupation, or geographic area of birth" will contribute to his or her perceptions of interest (Office of the Joint Chiefs of Staff, 2016, p. 18). These aspects can be modeled as network layers. These network layers may be related; some of the layers may be more or less important than others in determining an actor's perceptions and behaviors (Office of the Joint Chiefs of Staff, 2016).

In addition to emphasizing the importance of relevant actor identification, JC-HAMO goes on to identify several required capabilities for the Joint Force. Key among these in the context of this research is section 7.1, *Required Capabilities to Identify the Range of Relevant Actors and Their Associated Networks.* This is mission-specific and can include identification of individuals and any appropriate groupings thereof. The mission-specificity requirement drives the need for continual re-evaluation of key actors based on changing mission objectives (Office of the Joint Chiefs of Staff, 2016, p. 26). This re-evaluation will ideally identify “constraints and enablers of behavior” from the past, in the present, and for the future (Office of the Joint Chiefs of Staff, 2016, p. 19).

JC-HAMO Section 7.1.1, *the ability to understand the evolving operational environment through the human aspects lens* lists several additional networks of interest, including political, religious, and community affiliations, patronage, financial, commercial and logistic relationships, education and social status, informational, and psychological considerations (Office of the Joint Chiefs of Staff, 2016, pp. 26-27). An ensemble of such networks can be modeled using a multilayer network formulation (Kivela, Arenas, Barthelemy, Gleeson, Moreno, and Porter, 2014).

These identified capability needs for the Joint Force align well with the objectives of this research. *Identify the Range of Relevant Actors and their Associated Networks* is directly related to the problem of identifying critical nodes, but with an extension into a multilayer context; the primary objective of this research is to offer a method for identifying relevant actors in terms of some ranking of nodal network measures—measures under which each node is given its own value—on a multilayer network.

Once a measure or set of measures is identified to appropriately measure *relevance*, such measures may be incorporated into this methodology. The methodology gives rise to a list of relevant actors under the chosen measure(s).

Additionally, the temporal perspective is addressed when data includes timestamps, leading to potential identification of key events' impacts on the actors' relevance over time. This allows one to meet the goal of gaining "an appreciation of how behavior evolves over time as a result of various stimuli, including friendly force operations and activities in the environment" (Office of the Joint Chiefs of Staff, 2016, p. 28).

The primary benefit of a multilayer representation is clear: additional information can be recorded to yield new and unique insights not visible when viewing the problem through the lens of a single network, or at a single time (Brummitt, Lee, & Goh, 2012; Kivela, *et al.*, 2014). The cost is, of course, an increased need for data to create the multilayer network in the first place. This is compounded when considering the need to collect data at repeated time intervals to build a *temporal lens*, a desired framework given in JC-HAMO (Office of the Joint Chiefs of Staff, 2016). Such additional need for information can be met in the form of intelligence products. This aligns well with the identified intelligence requirement defined in JC-HAMO and is integral with any potential solution (Office of the Joint Chiefs of Staff, 2016, pp. 27-29).

In fact, as with any analysis, the outcome is largely dependent on the quality of data evaluated. Understanding relevant actors, their changes over time, and their relative importance within various network aspects will depend on reliable information. Thus it is crucial to gain appropriate intelligence to feed any model (Office of the Joint Chiefs of

Staff, 2016, p. 43). Nevertheless, with good information, a multilayer network model can be useful in identifying relevant actors. The methodology proposed in this thesis can be applied to address the problem of identifying relevant actors.

1.2.2 Research Overview.

In this research, a time-stamped terrorist network dataset is analyzed. This dataset is the Noordin Top network consisting of 139 network actors, 12 relationships, and 120 monthly timestamped data frames (Cunningham, Everton, & Murphy, 2016; Everton, 2013). The timestamps represent whether or not a network member was present in the network during a given month in the 120 month period (Everton, 2013).

The data are recorded as square adjacency matrices each having 139 rows and columns corresponding to the 139 actors. These each carry a timestamp value (numbered 1 to 120) and a relationship—or aspect—type (numbered 1 to 12). The matrices are aligned to form a two-dimensional 1440 matrix array with each entry representing a network layer within the multilayer networks. This time-stamped data is investigated for trends in stability of rankings.

The network is also analyzed in its non-time-stamped state. This is a network which consists of 12 layers which represent the aggregation of the time data for each layer. Rankings are computed on this single multiplex using a weight-modified Schulze voting method (Schulze, 2011). The multiplex is next subjected to a Jensen-Shannon distance layer-reduction algorithm (De Domenico, Nicosia, Arenas, & Latora, 2015) and rankings are computed on the reduced network. These rankings are compared to the rankings for the full multiplex network and against the fully-aggregated network. Statistical conclusions are drawn as are qualitative conclusions based on the identities of the nodes ranked in the top 20 positions under each set of conditions.

1.4 Problem Statements

1. Can a select voting method be adopted and demonstrated to effectively produce rankings of nodes for a multilayer network under select network measures?
2. How are such rankings affected by reduction in the number of layers within the multilayer network using a select layer reduction algorithm?
3. How do changes in weight distributions alter ranking outcomes?

1.5 Approach

A method for identifying critical nodes in a multilayer network context is needed. Borgatti (2006) defined the key player problem as being of two types, positive and negative (Borgatti, 2006). He noted that it is an old problem and had been originally approached by identifying critical nodes using network centrality measures, but that this approach suffered from two problems: the goal problem and the ensemble problem. The goal problem states that the solution should reflect more than just finding an optimal cut set, but rather should consider the quality of the resulting cut (Borgatti, 2006). The ensemble problem states that the optimal solution for a set of nodes is not necessarily the same as the set of optimal solutions for single nodes (Borgatti, 2006).

This thesis does not attempt to fully formulate Borgatti's (2006) definitions of the key player problem, positive or negative, within a multilayer network context; rather it seeks to extend the original problem of identifying critical nodes within a network through ranking measures of centrality. Therefore, a method is explored for ranking such measures to determine critical nodes within a multilayer network context.

Traditionally, the first step to computing centrality rankings on a multilayer network is to aggregate the network's layers into one single layer network and then compute the measure and its ranking. This aggregation can cause information loss, partially obviating the benefit of conducting critical node identification on a multilayer data set.

One approach considered in the literature is to reformulate existing single layer network measures of interest to apply within a multilayer network. Some such attempts are reviewed, and it is noted that such an approach is both non-trivial, and requires a separate effort for each measure desired.

In contrast, the proposed approach solves the ranking problem on each layer individually and combines the several rankings to form one composite ranking, which is representative of the multilayer structure. The challenge lies in how to make a meaningful aggregation of rankings which might account for the information contained in the multilayer structure.

Potential approaches to rank aggregation include employing a numerical average, convex combination, linear combination, or some other summative procedure to arrive at a composite ranking given several unique rankings associated with each layer of a multilayer network. Some such approaches are discussed in the literature review.

This thesis instead borrows from the field of social choice theory to propose the use of a deterministic combination method based on a widest path algorithm in a novel way

(Schulze, 2011). Social choice theory methods have previously been applied to other disciplines; in the case of social network analysis, this has been only in the context of single layer networks. Social choice theory's application to multilayer social networks is seemingly a new contribution by this research.

A challenge arises when working with multilayer networks: the computational complexity of network analysis scales with the number of layers being analyzed. For large datasets with a large number of layers, this can pose substantial computational challenges. This motivates the desire for applying layer reduction—selective aggregation—mechanisms, but their effects on network measures need to be better understood (De Domenico, *et al.*, 2015). Thus, a corollary problem in this thesis is the investigation of a layer reduction technique and its effects on resulting network centrality rankings.

Optimal reduction of layers is a combinatorically difficult problem. To ensure an optimal solution, each potential way of partitioning network layers may be considered. Different combinations might result in different overlap of information contained within the aggregated layers, which might then yield different measurement values. The problem can be reduced to the general set partitioning problem and the set of possible partitions scales as the M^{th} Bell number for M layers. Given this, a heuristic is needed for choosing which layers to aggregate and which to maintain as a separate layer within the reduced multilayer network. A prominent heuristic is chosen from the literature and investigated for its effects on rankings.

This work also seeks to make statistical assessments on the resulting measurement rankings and as such, poses a series of specific statistical research questions to be addressed. The rankings are directly compared using the Friedman's test. In this way, the statistical (dis)similarity of ranking outcomes under different subsets of the full multilayer network's information content can be determined. The Spearman's correlation coefficient is used to identify correlations between rankings. Correlation values are observed over time to identify blocks of time where the rankings remain well-correlated, implying stability.

Additional qualitative questions are investigated to establish the utility or benefit of the proposed methodology. The approaches taken to answer the full set of research questions are described in Chapter III.

1.6 Assumptions

Several assumptions were made in the course of this research. A notable assumption is that built-in implementations of algorithms within MATLAB are accurate and precise. It was also generally assumed that claims in peer-reviewed research are accurate excepting minor editing errors. If findings were presented in a paper, those findings were accepted unless testing was specifically conducted during this research which demonstrated otherwise.

The data used are assumed to be accurate; analysis is predicated on the data serving as a ground truth reference. Data of the sort is often collected in a snowballing manner in which a target of interest (or one on whom it is easiest to find information) is observed or investigated further. This creates a snowball effect around the target, so that most data will, by definition, relate to the original target. Snowballing can introduce some bias and thus does limit the final conclusions that can be drawn in general. These conclusions must instead be interpreted under this caveat.

1.7 Implications

The proposed methodology demonstrates use of a new tool for identification of critical nodes in a multilayer network. Critical can be defined in many ways; the methodology discussed is not dependent on any particular definition. This is true so long as the measure of criticality can be computed for each node. It also requires that a numeric ranking of the measure imply an ordinal valuation of the nodes.

If it can be shown that the proposed method is applicable, even with limitations, to the identification of critical nodes on a multilayer network, then a new aperture will be opened between two research domains: the literature of ranking nodes on multilayer networks and the literature of social choice theory and the various voting methods therein.

1.8 Preview

Chapter I described the desire to balance computation costs with information derived from additional layers within a network. It described a need for additional research into identifying critical nodes in a multilayer context. Chapter II will present a review of the relevant literature focusing on social network analysis, multilayer networks, multilayer network reduction methods, multilayer network centrality measures, and social choice theory.

Chapter 3 will list the methodology in detail to include a description of the data and their processing, layer reduction, centrality rankings, statistical comparisons, and qualitative analysis processes used. Chapter IV lists the results and their analysis. Chapter

V presents a final summary of the work along with directions for future research and recommendations for action.

II. Literature Review

2.1 Chapter Overview

This chapter describes the relevant literature reviewed in the course of conducting the studies in this thesis. In this literature review, a very brief review of social network analysis literature is conducted. Next it discusses multilayer social networks and their measurement. A short review of multilayer network reduction techniques that allow for adequate rank comparisons is then produced. Finally, the field of social choice theory and voting theory is explored.

2.2 Description

Judicious application of resources toward operational ends involves identifying targets of highest impact by whatever measures are deemed important. It is natural to first measure a set of possible targets to give each target a values. The possible targets can then be ranked based on their relative values. These rankings then correspond to a list of targets, which are prioritized by the chosen measure. Incorporating multiple network layers and multiple measures of value produces multiple rankings. Aggregation of rankings to produce a final, composited ranking is of interest.

Information gathering and targeting practices involve the judicious use of limited resources. Any ability to gain additional benefits from equivalent resources or identical benefits from fewer resources is of interest to the Department of Defense. For a social network, additional data in the form of a new set of relational ties can be represented as distinct network layers within the multilayer network. Each new layer can contribute different information to the analysis of a social network. However, gathering the information needed to build these layers can be expensive in terms of resource allocation. Any ability to reduce the required number of layers—and thus amount of data—while maintaining statistically equivalent analytic conclusions is of great interest.

2.3 Relevant Research

2.3.1 Overview.

This literature review begins by focusing on basic concepts in social network analysis (SNA). This review then examines extensions of social networks and SNA into multilayer network formulations. Multilayer networks are networks with more than one layer where each layer represents a distinct relationship between nodes (Kivela, *et al*., 2014). Multilayer networks may be able to represent real world systems with greater fidelity since real-world social networks are seldom well-described by a single relation (Boccaletti, *et al*., 2014). The potential benefits of these multilayer formulations are explored through a brief exposition of information theoretic applications and findings. Some attempts at developing multilayer centrality measures are then explored. Finally, a brief review of the field of social choice theory with a focus on voting theory is conducted.

2.3.2 Social Networks.

SNA is the analysis of networks of social relationships between individuals or groups (Wasserman & Faust, 1994) through the use of network and graph theories (Otte & Rousseau, 2002). It involves theoretical concepts, methods and techniques to identify social relations, their structure, and their influence on behavior, attitudes, beliefs, and knowledge (Prell, 2012).

SNA is based in part on an assumption of the importance of relationships among interacting individuals. The unit of analysis is not the individual itself, but rather a system consisting of both a collection of individuals and of the links among them (Wasserman & Faust, 1994). SNA therefore implicitly assumes that information is gained by examining the structure of the network that cannot otherwise be identified considering only the components of the network.

There is a growing awareness of the importance of links or interdependencies in explaining the complexity inherent to social systems (Prell, 2012). These connections may be strong, weak, or absent and their strength can represent time, intensity, intimacy, and reciprocity (Granovetter, 1973). This makes for a rich field of research; a basic review of this research follows.

2.3.3 Social Network Components.

Wasserman and Faust (1994) described certain fundamental components used in modeling a social network. These include actors, relational ties, dyads, triads, subgroups, groups, and relations. According to Wasserman and Faust, actors are represented as nodes on the network and relational ties are the arcs between nodes (Wasserman & Faust, 1994). Dyads and triads refer to sets of 2 and 3 nodes and their inclusive arcs, respectively. Both dyads and triads are also subgroups, which can additionally include any number of nodes and their interconnecting arcs. A group is a finite set of actor nodes between which is a set of interconnecting arcs (Wasserman & Faust, 1994). Finally, relations are defined measurements taken between nodes and are represented with relational ties or arcs (Wasserman & Faust, 1994). These components can be represented mathematically and the history of SNA includes a history of its corresponding mathematical models.

2.3.4 Mathematical Representations of Social Networks.

SNA has its roots in a methodology known as sociometry, or the measurement of interpersonal relations in small groups, developed by Moreno (Moreno, 1953) and Moreno and Jennings (Moreno & Jennings, 1938). Moreno represented social networks using a tool called a sociogram, which resembles a digraph but with additional qualitative information represented by size of nodes, colors, and so forth. The sociogram has been extended and formalized through the application of the field of mathematics known as graph theory (Wasserman & Faust, 1994).

Modern social network analysts most commonly represent networks as graphs (Wasserman & Faust, 1994). Graphs consist of both vertices and edges that can represent entities and their pairwise relationships or links, respectively (Harary, 1969). The introduction of graph theory formalisms to SNA allowed for the development of a robust quantitative framework within the field (Wasserman & Faust, 1994). This was motivated by studies into structural balance and reciprocity in networks, specifically triad systems, pioneered by Cartwright and Harary (1956) and Davis (1967) (Cartwright & Harary, 1956; Davis, 1967).

2.3.5 Social Network Analysis Centrality Measures.

Many measures have been developed for the analysis of social networks, which include node centrality measures, clustering/community and modularity measures, shortest paths and distance measures, and adjacency matrix decompositions, among others (Boccaletti, Bianconi, Criado, del Genio, Gomez-Gardenes, Romance, Sendina-Nadal, Wang, and Zanin, 2014).

Centrality measures are of specific interest here. Centrality concerns finding nodes that have a central structural role within a network and is of broad interest in SNA (Boccaletti, *et al*., 2014). A brief survey of network centrality measures identifies node degree, closeness, betweenness, eigenvector centralities, and PageRank centrality (Boccaletti, *et al*., 2014), as well as stress, load and communicability centrality (Guzman, Deckro, Robbins, Morris, & Ballester, 2014). Each of these includes variations, which makes for a long list of social network centrality measures (Wasserman & Faust, 1994; Boccaletti, *et al*., 2014; Guzman, *et al*., 2014).

Betweenness, closeness, eigenvector, degree and PageRank centralities were chosen for study in this thesis due to their low inter-correlation values and history of study and application (Guzman, *et al*., 2014; Boccaletti, *et al*., 2014). As a testament to the prevalence of these measures, MATLAB also includes built in functions for each. It is these functions that are used to compute the centrality values throughout this research.

The previous sections focused on traditional SNA involving single layer networks. Real social networks often include more than just one relation between individuals, but social network models are traditionally limited to only one relation. In contrast, differing social relations represent distinct layers and their combination results in what is called a multilayer network (Kivela, *et al*., 2014).

It is increasingly apparent that multilayer network models are important across many scientific disciplines (Kivela, *et al*., 2014). The body of knowledge concerned with multilayer networks is commonly known as *complex network theory* and falls within the field of complexity science (Boccaletti, *et al*., 2014). An effective construct for representing complex networks may constitute the “new frontier in many areas of science” (Boccaletti, *et al*., 2014). The following sections describe the formulation and analysis of multilayer networks in greater detail and compare and contrast these with their traditional single layer network counterparts.

2.3.6 Multilayer Networks.

Within a social network, many relations may exist between the same set of nodes representing a wide variety of interpersonal or intergroup relations (Kivela, *et al*., 2014). These might include friend relations, family relations, professional or workplace relations, acquaintance relations, and time-varying relations (Kivela, *et al*., 2014). Each relation can be modeled as a separate set of edges, resident on a separate layer of the network. Layers may alternatively be categorized as similarities, social relations, interactions, and flows (Borgatti, Mehra, Brass, & Labianca, 2009). Further, these relations may represent differing strengths of connections; thus, failure to account for layers individually implicitly assumes interpersonal ties are identical (Hamill, Deckro, Chrissis, & Mills, 2008).

A network that includes more than one layer is most often referred to as a multilayer network but has also been labeled as a network of networks, multiplex network, interdependent network, and many other names (Boccaletti, *et al*., 2014). Subtle differences in meaning across authors and disciplines can be a significant cause of confusion (Kivela, *et al*., 2014). An example of a multilayer network with three layers is shown in Figure 2. Intra-layer edges are represented by solid lines and inter-layer edges by dashed lines. In a general multilayer network, inter-layer edges may connect nodes to different nodes directly, as represented by the diagonal dashed lines in Figure 2.

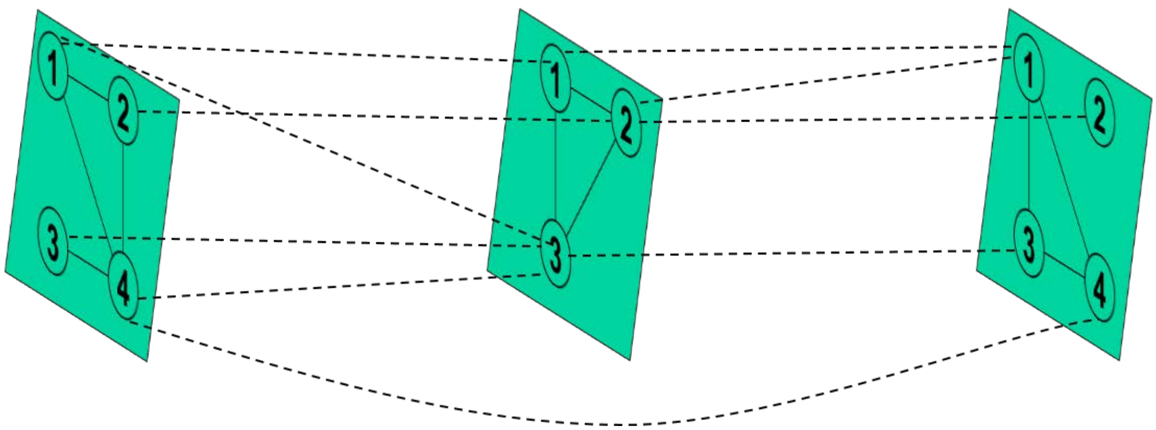

Figure 2: Example Multilayer Network with Three Layers

This study adopts the definition of a multiplex network as a special case of multilayer networks where a node does not connect to another node *across* layers, but only *within* layers; this construct is especially useful within the field of SNA (Boccaletti, *et al*., 2014). Each node in this case might represent a person and each layer contains that person and their connections to other nodes corresponding to that layer's relation only. A three layer multiplex network is shown in Figure 3. This is the same network shown in Figure 2, but the inter-layer edges are now restricted to connecting identical nodes between layers.

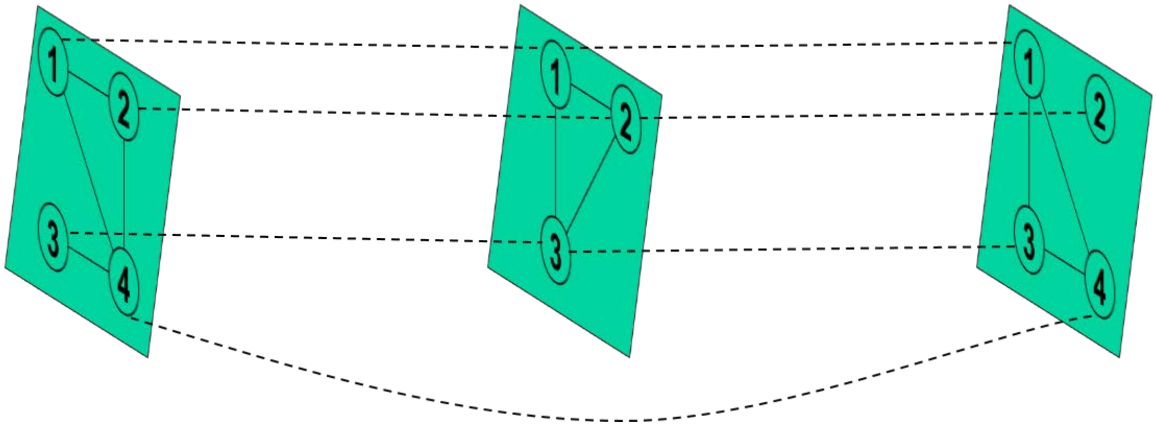

Figure 3: Multiplex Network Example with Three Layers

Node alignment is the existence of the same community of actors on all layers within a multilayer network (Kivela, *et al*., 2014). Node alignment ensures that intra-layer adjacency matrices for each layer are of equal size, by maintaining a constant set of nodes in each network layer. This has the benefit of simplifying mathematical expressions, but can produce many nodes which are represented as being in a layer, without actually having any meaningful connections in that layer. Thus, node-alignment can introduce additional isolated nodes, or nodes which are not connected to any other nodes. Figure 4 illustrates this by including node four on layer two where it was not previously located. Inter-layer edges are drawn to the new node four, but within layer two no additional intra-layer edges are added.

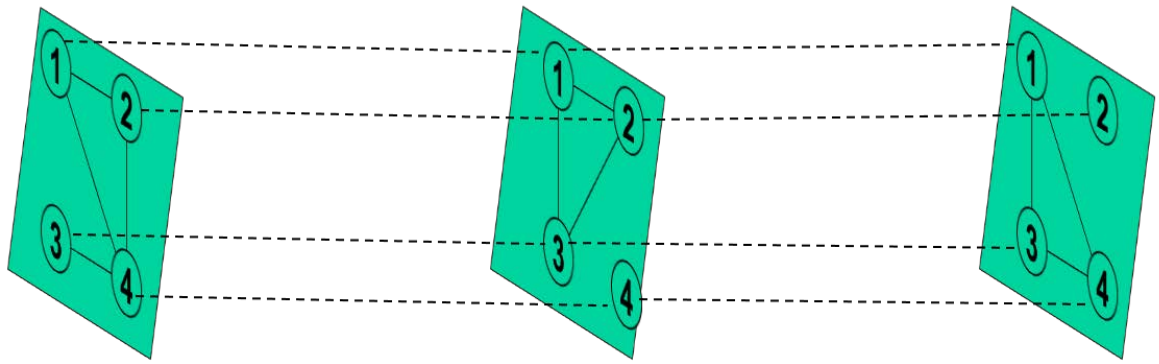

Figure 4: Node-aligned Multiplex Network Example with Three Layers

Despite this potential difficulty of isolated nodes, there is a benefit to the use of a node-aligned multiplex network formulation. Inter-layer edges may be understood to exist uniformly; therefore there is no need to store their edge values (Boccaletti, *et al*., 2014). In the cases where inter-layer edges cannot implicitly be assumed to be uniform, but node-alignment occurs, an additional inter-layer adjacency matrix must be created. In the case of node-alignment this matrix—sometimes referred to as a super-adjacency matrix—is in $\mathbb{R}^{nm \times nm}$ where n is the number of nodes and m is the number of layers (Boccaletti, *et al*., 2014).

2.3.7 Reducibility of Multilayer Networks.

Multilayer network reduction is a concept which has received increasing attention over the past few years beginning with a paper titled *Structural Reducibility of Multilayer Networks* (De Domenico, *et al*., 2015). The stated motivation for reducing the structure of a multilayer network is to reduce the computational complexity when performing network operations and analysis. In this sense, structural reducibility is a proposed method for pre-processing a multilayer network to compress it (De Domenico, *et al*., 2015).

In the case of the Noordin Top dataset considered in this thesis, the computational savings resulting from the reduction of the number of layers is negligible, as the total dataset is relatively small. However, the case is easily made that larger datasets—such as large social media or communications datasets—may see a substantial reduction in subsequent network processing times if a quantity of layers can be removed from consideration while maintaining similar analytic results. The impact of the reduction on further analysis of the network is left as an open area of research by De Domenico *et al.* (2015). This thesis examines the question in the context of centrality rankings under the proposed rank aggregation method.

Other reduction methods have been proposed, both quantitative (Wang & Liu, 2017; Stanley, Shai, Taylor, & Mucha, 2016; Taylor, Shai, Stanley, & Mucha, 2016) and qualitative (Crawford, Gera, Miller, & Shrestha, 2016). Wang and Liu (2017) apply a modified version of the method described by De Domenico *et al.* (2015) to help identify community structures. They suggest an improvement to the heuristic by using simple rules to eliminate certain combinations *a priori* (Wang & Liu, 2017).

Taylor, *et al.*, (2016) first develop a stochastic block modeling approach to selectively aggregating network layers for the purposes of reducing the network and then use their method to help identify communities of nodes within the multilayer network (Taylor, *et al.*, 2016; Stanley, *et al.*, 2016).

Finally, Crawford, *et al.* (2016) apply a subject matter expert binning process to choose which layers to combine based on assumed characteristics of the networks of interest (Crawford, *et al*., 2016). They then investigate the effects on community structures after reducing the network according to this process (Crawford, *et al*., 2016).

The reduction method chosen for study is the method originally discussed by De Domenico *et al.* (2015), referred to as the Jensen Shannon distance (JSD) method. The JSD method consists of applying a distance metric to all pairs of layers of a multilayer network, choosing the smallest pairwise distance, aggregating the associated pair of adjacency matrices, recomputing the pairwise distances, and repeating until all layers have been aggregated into a single layer network. At each iteration, the cardinality of the set of layers decreases by one and a corresponding quality function is evaluated.

The maximum value of the quality function is identified and the corresponding set of layers and their aggregation pattern is adopted as the (sub)-optimal reduction of the multilayer network. The optimal solution can only be guaranteed by a complete enumeration of the problem space, which is an NP-hard problem equivalent to finding all possible partitions of a set and scales as the M^{th} Bell number—or super-exponentially—with the number of layers, *M* (De Domenico, *et al*., 2015).

This heuristic is roughly equivalent to the agglomerative hierarchical clustering heuristic first given by Ward (Ward, 1963). The distance proposed in De Domenico *et al.* (2015) is the JSD. The objective function maximizes the distinguishability between the fully-aggregated network—considered the baseline value—and the reduced network using the relative Von Neumann entropy values for each network (De Domenico, *et al*., 2015).

The Jensen Shannon distance is defined as the square root of the quantum Jensen Shannon divergence value, which is itself related to the Kulback-Liebler divergence. The

Kulback-Liebler divergence has been used to evaluate the similarity between networks; specifically, it was used to compare constructed networks to the exemplar upon which they are constructed (Nystrom, Robbins, Deckro, & Morris, 2015). This allowed for the selection of the most similar network to the exemplar despite deliberate changes to network features, such as overall size (Nystrom, *et al*., 2015). The Jensen Shannon distance, however, meets more of the qualifications to be considered a metric under specific circumstances, though a general proof has not yet been developed (De Domenico, *et al*., 2015).

The Kulback-Liebler divergence (D_{KL}) is given in equation 1 (De Domenico, *et al*., 2015) as:

$$D_{KL}(\rho||\sigma) = Tr[\rho\big(log_2(\rho) - log_2(\sigma)\big)] \qquad 1$$

where ρ and σ represent the combinatorial Laplacian matrices of the two graphs being compared (De Domenico, *et al*., 2015).

The combinatorial Laplacian ($\mathcal{L}$) matrix of a graph is defined by De Domenico *et al.* (2015) as a diagonal matrix of the row sums of the original adjacency matrix less the original adjacency matrix rescaled by one over twice the number of edges. This is given by equation 2 as:

$$\mathcal{L} = c * (D - A) \qquad 2$$

where D is the diagonal matrix of the row sums of the nodes of the original graph, A is the adjacency matrix of the original graph, and c is defined as $\frac{1}{2|E|}$ where $|E|$ is the number of edges in the original graph.

The Jensen Shannon divergence (D_{JS}) is a variation of the Kulback-Liebler divergence involving a mixed state of the density matrices (De Domenico, *et al*., 2015) and is given by equation 3 as:

$$D_{JS}(\rho||\sigma) = \frac{1}{2}D_{KL}(\rho||\mu) + \frac{1}{2}D_{KL}(\sigma||\mu) = h(\mu) - \frac{1}{2}[h(\rho) + h(\sigma)] \qquad 3$$

where μ is the mixture (average) of the two density matrices ρ and σ.

This then yields the Jensen Shannon distance (D_{JD}) which is defined as the square root of the Jensen Shannon divergence (De Domenico, *et al*., 2015) and shown in equation 4 as:

$$D_{JD} = \sqrt{D_{JS}} \qquad 4$$

The Von Neumann entropy (h_A) of a graph is given by equation 5 (De Domenico, *et al*., 2015) as:

$$h_A = -Tr[\mathcal{L}\ log_2\ \mathcal{L}] \qquad 5$$

where Tr is the trace of a matrix and $\mathcal{L}$ is defined in equation 2.

The Von Neumann entropy of a graph can also be formulated as the Shannon entropy of its power spectrum which is given in the following equivalence shown in equation 6 (De Domenico, *et al*., 2015) as:

$$h_A = -Tr[\mathcal{L}\, log_2\, \mathcal{L}] = -\sum_{i=1}^{N} \lambda_i\, log_2(\lambda_i) \qquad 6$$

where N is the number of nodes in the graph and λ_i is the i^{th} eigenvalue of the Laplacian matrix associated with the graph.

The average Von Neumann entropy ($\overline{H}(\mathcal{A})$) across the M layers of the multilayer network can thus be defined as follows in equation 7 (De Domenico, *et al*., 2015) as:

$$\overline{H}(\mathcal{A}) = \frac{\sum_{\alpha=1}^{M} h_{A^{[\alpha]}}}{M} \qquad 7$$

where $\mathcal{A}$ is the set of adjacency matrices representing each layer in the multilayer network, α is the index referring to a given layer within the multilayer network, $A^{[\alpha]}$ is the adjacency matrix for layer α, $h_{A^{[\alpha]}}$ is the Von Neumann entropy of layer α, and M is the number of layers within the multilayer network such that $M = |\mathcal{A}|$.

Let $\mathcal{R}$ be defined as the set of adjacency matrices for the reduced multilayer network where $|\mathcal{R}| \leq M$. Then the average Von Neumann entropy for the reduced multilayer network $\mathcal{R}$, ($\overline{H}(\mathcal{R})$) is given by equation 8 as:

$$\overline{H}(\mathcal{R}) = \frac{\sum_{\alpha=1}^{|\mathcal{R}|} h_{R^{[\alpha]}}}{|\mathcal{R}|} \qquad 8$$

The quality function ($q(\mathcal{R})$) is defined as the unit difference of the ratio of the average Von Neumann entropies on the reduced set of layers of the multilayer network against the Von Neumann entropy of the fully aggregated network. The quality function measures the distinguishability of the baseline fully-aggregated network compared with the reduced multilayer network (De Domenico, *et al*., 2015), and is given by equation 9 as:

$$q(\mathcal{R}) = 1 - \frac{\overline{H}(\mathcal{R})}{h_A}$$ 9

where h_A is the entropy of the fully-aggregated graph corresponding to the linear of combination of the multilayer network's adjacency matrices.

The JSD method given by De Domenico *et al.* (2015) attempts to reduce the number of layers as much as possible while avoiding both spurious reductions and failure to reduce mostly redundant layers. A spurious reduction is a combination of layers which are actually distinct from each other and mostly redundant layers are those which are most highly similar to each other.

The resulting reduction of layers is a structural reduction and the JSD method guarantees nothing about how the reduced network's structure might or might not alter any analytic results (De Domenico, *et al*., 2015). This thesis applies statistical analyses to determine if significant changes occur to rankings of network centrality measurements taken on a multilayer network reduced with this method versus those taken on the original multilayer network.

2.3.8 Information Gain on Multilayer Networks.

The analysis of a multilayer network can take on several general forms: a multilayer network can be analyzed after a single layer is produced by aggregating or projecting the layers together; a multilayer network can be analyzed in its multilayer state directly;

alternatively, a multilayer network can be analyzed as separate layers (Boccaletti, *et al*., 2014).

The first approach—aggregating the layers of a multilayer network into a single layer prior to analysis—requires choosing an appropriate aggregation method. Aggregation refers to the summation of layer edge values to produce a single set of edge values (Boccaletti, *et al*., 2014). This single set of values produces a single network layer and corresponding adjacency matrix. Aggregation can be computed in several ways: binary, summative, and weighted. A binary aggregation is also referred to as a projection and yields a final single layer network with edge values equal to zero or one (Kivela, *et al*., 2014). This is computed by first summing edge weights for all layers and then assigning a value of one if a non-zero edge weight is present; a value of zero is maintained if a zero-valued edge weight is present.

A summative aggregation is the simple linear combination of edge weights for all layers. The summative aggregation of layers thus represents a summative accounting of the occurrence of edges across the layers (Boccaletti, *et al*., 2014). The resulting edge weights will take on natural number values (including zero) between zero and the number of layers in the multilayer network.

Weighted aggregation can be computed by first applying a uniform weight to each edge within a given layer. By similarly applying such layer weights to every layer in the network prior to aggregation, a weighted aggregation becomes a weighted linear combination of edge weights between layers (Kivela, *et al*., 2014). Such edge weights can take on any additive combination of values of weights between zero and the sum of weights for all possible layers.

These different aggregation approaches will usually result in different values when the aggregated network is measured. Such values will also usually differ when compared with measures taken on the original multilayer network; this is similarly true when analyzing individual layers in isolation and combining results.

Such differences suggest emergent or synergistic effects can be present between layers. This creates fundamental limitations on the analysis of a network without full knowledge of its multiplexity and thus motivates the study of multilayer networks *per se* (Brummitt, *et al*., 2012). A multilayer network should ideally be analyzed directly in its multilayer state (Cozzo, Banos, Meloni, & Moreno, 2013). However, doing so can present substantial challenges.

2.3.9 Multilayer Social Network Components.

Multilayer network models include a number of identical or similar components as those of single layer networks (Boccaletti, *et al*., 2014). Nodes, arcs, groups, subgroups, actors, relational ties, dyads, triads, and relations are each present in multilayer social network formulations since a single layer network can be considered a special case of a multilayer network that has only one layer.

In some cases, the meanings of these single layer components are altered or extended to account for additional layers (Boccaletti, *et al*., 2014; Battiston, Nicosia, & Latora, 2014). These concepts are sometimes further altered by interpretations of possible interlayer connections between otherwise identical nodes. Within a multiplex network, interlayer connections exist only between identical nodes on each layer, but within multilayer networks more generally, connections can occur between any nodes both on the same layer and on different layers (Boccaletti, *et al*., 2014).

A key addition to this list of common single layer network components when modeling multilayer networks is the idea of a layer. In its simplest form, a layer includes everything that a single layer network does, but has the distinction within the multilayer network framework of being repeatable (Boccaletti, *et al*., 2014). In other words, while a single layer network consists of only one layer, by definition a multilayer network includes one or more layers.

2.3.10 Mathematical Representations of Multilayer Social Networks.

A method for including multiple relations between nodes on the same networks can involve defining multiple distinct edges between the same pair of nodes, with each edge representing a different relation shared by the pair. As the number of relations being represented on a multiplex network increases, the dimensionality of the necessary mathematical representation increases. An adjacency matrix is often used to represent whether an edge exists between two nodes within a single layer network.

Adjacency matrix representations are limited to describing only a single value between any two nodes (De Domenico, Sole-Ribalta, Cozzo, Kivela, Moreno, Porter, Gomez, and Arenas, 2013). Many researchers have used aggregation to account for all layers using only one adjacency matrix. This process is a surjective mapping—it causes a one-way loss of information—as the vector value corresponding to a set of edges that occurs in a set of layers is reduced to the scalar value corresponding to a single summed edge weight on the single aggregated layer (Kivela, *et al*., 2014; Boccaletti, *et al*., 2014). As illustrated in Figure 5, it is clear that the aggregation of the three layers will result in only one possible single layer network (for a given set of edge weights on each layer); however, any attempt to reverse the process produces several possible layer combinations which could produce the same single layer network.

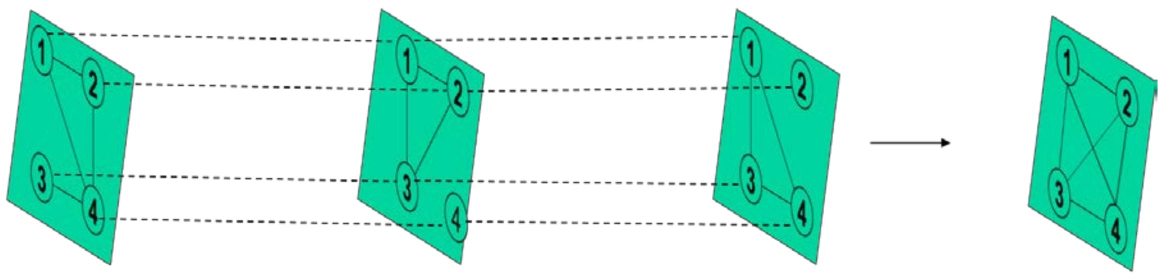

Figure 5: Aggregation of Network Layers

When an additional layer is added to a multilayer network, an additional set of edges is created. Incorporation of additional edge sets to describe additional relations involves increasing the dimensionality of the corresponding adjacency matrix. A tensor is the more general form of a scalar, vector, or matrix. A matrix is a second order tensor, and therefore an increase in dimensionality can be represented more generally with a tensor whose order is greater than two (De Domenico, *et al*., 2013).

Tensor index notation has been suggested as a succinct method of representing higher-dimension social network data (Kivela, *et al*., 2014; De Domenico, *et al*., 2013). To represent a dataset which is both ordinally-coupled along a temporal dimension and categorically-coupled as distinct layers representing different relationships, a sixth order tensor is needed in general (Kivela, *et al*., 2014). Still, vector and matrix representations remain used when possible as perhaps a more intuitive method and will be used in this thesis (Battiston, *et al*., 2014).

2.3.11 Multilayer Social Network Centrality Measures.

Many single layer network centrality measures cannot be applied directly within the context of multilayer networks. For instance, node degree within a multilayer network must be represented in vector form and thus does not present a clear method to construct an ordered list of nodes (Boccaletti, *et al*., 2014). Measures such as closeness and betweenness are based on the structure of the network and can therefore be more easily translated into a multilayer setting, though they may still be complicated by any distinction between intra-layer and inter-layer arcs (Boccaletti, *et al*., 2014).

There is therefore an interest in developing new but analogous multilayer measures (Boccaletti, *et al*., 2014). Some attempts to develop centrality measures specific to multilayer networks are offered by Sole-Ribalta, De Domenico, Gomez and Arenas (2014), Halu, Mondragon, Panzarasa, and Bianconi (2013), and Sola, Romance, Criado, Flores, Garcia del Amo, and Boccaletti (2013) for betweenness, PageRank, and eigenvector centralities, respectively (Sole-Ribalta, *et al*., 2014; Halu, *et al*., 2013; Sola, *et al*., 2013). What follows is a description of these methods.

These three papers attempt to extend a different single layer network centrality measure into a multilayer network context. Each paper describes a qualitative difference between the single layer measure's ranking and its multilayer variant's rankings. Halu, *et al*. (2013) and Sole-Ribalta, *et al*. (2014) stop short of a statistical analysis; however, Sola *et al.* (2013) perform a non-parametric analysis using the Spearman and Kendall rank correlation coefficients between rankings on the eigenvector centralities of the aggregated network and the proposed multiplex eigenvector centrality rankings (Halu, *et al*., 2013; Sole-Ribalta, *et al*., 2014; Sola, *et al*., 2013).

Sole-Ribalta, *et al*. (2014) define their multilayer analogue to betweenness centrality beginning with the standard definition of betweenness centrality for a node ν as the number of shortest paths for all node pairs which contain node ν (Freeman, 1977). Their primary extension is to include interlayer edge links as part of the possible paths (Sole-Ribalta, *et al*., 2014). This accounts for individuals who serve as bridges or hubs between layers to be ranked more highly in relative betweenness scores than they might be in the aggregated network (Sole-Ribalta, *et al*., 2014). They examine only unweighted graphs, but claim edge weights can be incorporated easily with the use of Dijkstra's algorithm as opposed to the breadth-first approach they take in the paper (Sole-Ribalta, *et al*., 2014). Thus, a weighting scheme could be created which weights path edges that exist between networks, though it is left unclear how such weights should be developed.

Halu, *et al.* (2013) created a multilayer extension to the PageRank algorithm under a node-aligned multiplex network structure (Halu, *et al*., 2013). The central idea of the Multiplex PageRank is that a node's PageRank score on one layer should interact with the same node's PageRank score on another layer (Halu, *et al*., 2013). An underlying assumption is that such interaction effects are positive in nature. In other words, if a node is not central in one layer, its overall centrality is only improved by its being central in another layer (Halu, *et al*., 2013). Multiplex PageRank was defined in four versions: additive, multiplicative, combined and neutral (Halu, *et al*., 2013).

The additive definition states that a node's centrality on network A can be augmented by the centrality it has on network B (Halu, *et al*., 2013). The multiplicative definition states that a node's centrality on network A involves an interaction effect between the centrality value on network A and that on network B (Halu, *et al*., 2013). The combined version of Multiplex PageRank centrality simply combines the additive and multiplicative versions. Finally, the neutral version is a reduction to the standard PageRank definition for each layer in isolation. Thus the PageRank for a node on network A has no effect on the PageRank for a node on network B (Halu, *et al*., 2013). With the exception of the neutral variety, the Multiplex PageRank centrality produces a single vector of values (Halu, *et al*., 2013).

Sola *et al.* (2013) define variations on an eigenvector centrality for directed or undirected, unweighted node-aligned multiplex networks. They identify two scalar measures—the *eigenvector centrality of the projection graph* and the *uniform eigenvector-like centrality*—and three vector measures—the *independent layer eigenvector-like centrality*, the *local heterogeneous eigenvector-like centrality*, and the *global heterogeneous eigenvector-like centrality*—corresponding to eigenvector centralities for a node-aligned multiplex network (Sola, *et al*., 2013).

The two scalar-valued measures correspond to a standard interpretation of the eigenvector centrality under different aggregation modes. Both of these scalar-valued measures assign a scalar centrality score to each node. These are produced by summing the *transposed* adjacency matrices of each layer of the multiplex network in two ways: a projection (binary aggregation) and an unweighted aggregation (linear combination) of network edges (Sola, *et al*., 2013).

They define a projection network as the *binary* combination of the *transposed* adjacency matrices for the layers within the multiplex network. This then implies that the *eigenvector centrality of the projection graph* is the usual single-layer eigenvector centrality as measured on a binary aggregated network. This corresponds to an unweighted version of the network were it represented in single layer form (Sola, *et al*., 2013).

In contrast, the *uniform eigenvector-like centrality* is defined similarly on the linear combination or summative aggregation of the *transposed* adjacency matrices of all layers in the multiplex network. In this case, the aggregation network is not a binary projection, but rather a network whose edge weights represent the summative accounting of all edges present on each layer of the multiplex (Sola, *et al*., 2013).

The three vector-valued measures correspond to an eigenvector centrality score on the full multiplex network resulting in a vector-valued centrality score for each node. Defining the eigenvector centrality of a network, denoted by c_k, as the principal eigenvector of the *transpose* of the adjacency matrix for the network's k^{th} layer of M layers, Sola *et al.* (2013) first define the *independent-layer eigenvector-like centrality* as the matrix which is the augmented vector $C = (c_1|c_2| \ldots |c_m)$ (Sola, *et al.*, 2013). Thus the vector $c_{i.}$ (Sola *et al.* (2013) use dot notation: $c_{i.}$ is the row vector corresponding to the i^{th} row of C for all j) is the vector-valued *independent-layer eigenvector-like centrality* for node i.

Next, Sola *et al.* (2013) introduce the concept of a directed, non-negative influence matrix W which defines the level of directional influence that one layer has on another layer. Though they define W to be generally directed, their study focuses on two varieties of W: symmetric and asymmetric. The symmetric variety renders the graph to be undirected. The asymmetric topology chosen maintains a directed graph whose adjacency matrix' lower triangular values are equal to the square of the reflection of the upper triangular values about the diagonal. In this thesis, a set of vectors of weights has been applied to each layer rather than the use of such an influence matrix approach. The influence matrix approach reduces to the vector of weights approach if the matrix W has identical rows $w_{i.}$ all equal to the chosen vector of weights.

A *local heterogeneous eigenvector-like centrality* is defined as the principal eigenvector of the weighted *transposed* adjacency matrix for each layer where the weights are the entries w_{ij}. Thus ${c_1}^*$ is the principal eigenvector for the matrix given by $A_1^* = \sum_{j=1}^{m} w_{1j} A_j$. The *local heterogeneous eigenvector-like centrality* for the entire multiplex is then given as the augmented matrix of these positive, normalized eigenvectors represented by $C^* = ({c_1}^* | {c_2}^* | \dots | {c_m}^*)$ for multiplex layers $1 \dots M$. As before, the vector ${c_{i.}}^*$ (the row vector corresponding to the i^{th} row of C^*) is the vector-valued *local heterogeneous eigenvector-like centrality* for node i.

Finally, the *global heterogeneous eigenvector-like centrality* is defined as the Khatri-Rao product of the influence matrix W and the block matrix consisting of each *transposed* adjacency matrix $A_i \in \mathbb{R}^{n \times n}$ for layer i of layers $1 \dots M$ given by

$(A_1|A_2|\ldots|A_m) \in \mathbb{R}^{n\times nm}$. This product results in a block matrix of the form given by equation 10 as:

$$A^{\otimes} = \begin{pmatrix} [w_{11}A_1] & \cdots & [w_{1m}A_m] \\ \vdots & \ddots & \vdots \\ [w_{m1}A_1] & \cdots & [w_{mm}A_m] \end{pmatrix} \in \mathbb{R}^{(nm)\times(nm)} \quad 10$$

where w_{ij} is an entry in the influence matrix W and $[w_{ij}A_j]$ is a matrix in $\mathbb{R}^{n\times n}$ (Sola, *et al.*, 2013).

Then, the related principal eigenvector of $A^{\otimes}$ is denoted as $c^{\otimes}$ and is a vector in $\mathbb{R}^{nm}$ given by $c^{\otimes} = \left(c_1^{\otimes}\middle|c_2^{\otimes}\middle|\ldots\middle|c_m^{\otimes}\right)^T$ with each $c_j^{\otimes} \in \mathbb{R}^n$. Sola *et al.* (2013) then define the augmented matrix of these positive, normalized eigenvectors as the *global heterogeneous eigenvector-like centrality* and denote it as $C^{\otimes} = \left(c_1^{\otimes}\middle|c_2^{\otimes}\middle|\ldots\middle|c_m^{\otimes}\right) \in \mathbb{R}^{n\times m}$ (Sola, *et al*., 2013). Thus similar to before, the vector $c_{i.}^{\otimes}$, (the row vector corresponding to the *i*th row of $C^{\otimes}$) is the vector-valued *global heterogeneous eigenvector-like centrality* for node i.

Notably, there are challenges in defining an overall ranking of the nodes' centrality when each node's ranking is a vector-value (Sola, *et al*., 2013). Defining a consistent ranking for vectors is not trivial (Boccaletti, *et al*., 2014). Prior to conducting their statistical comparison of rankings using Spearman's rank correlation and Kendall's Tau, Sola *et al*. (2013) present methods for combining their M vector-valued eigenvector rankings into a single scalar-valued ranking. They apply both a convex combination and a simple summation of values (Sola, *et al*., 2013).

The convex combination is applied on each column for both the *independent-layer eigenvector-like centrality* matrix and the *local heterogeneous eigenvector-like centrality* matrix. The weights of the convex combination must sum to one and the weights chosen for each layer were uniform values for the M layers resulting in weights of $\frac{1}{M}$ for all layers, effectively producing an average eigenvector centrality score; however, these could be varied to represent a relative weighting scheme for the eigenvector centrality rankings (Sola, *et al*., 2013).

A simple summation is applied to the *global heterogeneous eigenvector-like centrality* since the sum of all entries in the matrix $C^{\otimes}$ is one, consequentially making the sum of a column vector $c_j^{\otimes}$ equivalent to the percentage of influence resident within the corresponding layer *j* within the multiplex (Sola, *et al*., 2013).

The forgoing examples are notable attempts to translate single layer network centrality measures into a node-aligned multiplex setting. The methodology explored in this thesis does not depend on any particular mathematical extension of a centrality (or any other) measure. This allows it to be applied more generally.

Each of the discussed centrality measure extensions is an individually-tailored attempt to define a centrality measure within the context of multiplex networks. This is a potentially fruitful approach, but ultimately the measure of interest is really the *ranking* of the centrality measures rather than the *values* of the measures. In the case of the betweenness centrality and PageRank centrality extensions, a single vector of measured values is produced, resulting in a single ranking (Sole-Ribalta, *et al*., 2014; Halu, *et al*., 2013). In contrast, the eigenvector centrality extension results in a set of vectors, which must then be aggregated despite the set of rankings being representative of the larger multiplex structure (Sola, *et al*., 2013). A more broadly applicable approach may be to apply a general procedure for ranking centrality measures in such a way that network layer information is incorporated into the final centrality ranking, irrespective of the centrality measure used.

This can also be useful because centrality scores are not always comparable between layers; the values of the scores depend on the network topology. This in turn creates a need for normalized versions of each measure prior to combing them. In contrast, when comparing the ranking of centrality values, normalization is not needed.

As was shown in Sola *et al.* (2013), a simple linear combination or convex combination is a possible method; however, there are known shortcomings with such a method if one looks to the field of social choice theory. Some of these shortcomings can be overcome with the use of other methods to combine rankings. Such methods are reviewed next.

2.3.12 Rankings.

Rank-ordered nodes are broadly interesting within the context of SNA as they can contribute to an understanding of key nodes, though by themselves, they do not necessarily answer the key player problem (Borgatti, 2006). Still, determining the overall rank of importance (by some measure or combination of measures) of nodes within a network can be of interest to decision makers seeking to target a portion of the network either for influence in the case of marketing or influence operations, or for direct action in the case of military kinetic or law enforcement operations. Additionally, this initial evaluation of ranks on a multilayer network is a potential first step to an extension into a more general solution methodology of the key player problem on multilayer networks.

A ranking is an ordinal set of numbers signifying relative importance. Rankings can be analyzed using non-parametric methods and have successfully been used to compare social network measures for correlation and computational times in an attempt to identify the best measure for a given task (Guzman, *et al*., 2014). Within the context of this research, a ranking refers to a 1 to N ordinal list of a given measure which is determined at each of N nodes. Thus, it is a method which is typically defined for a single layer network. For a ranking method to be useful in the context of multilayer networks, an overall ranking is needed which accounts for the rankings which exist on each layer: a composite ranking.

2.3.13 Social Choice Theory.

The concept of the aggregation of rankings is not new. A literature search for the term *ranking aggregation* yields several aggregation approaches applied across a variety of fields (Lin, 2010). Rank aggregation has been used to inform website rankings and search results (Dwork, Kumar, Naor, & Sivakumar, 2001; Renda & Straccia, 2003). Rank aggregation has been used to build complete rankings of genetic information in bioinformatics studies (DeConde, *et al*., 2006; Pihur, Datta, & Datta, 2008). Rank aggregation has also been used successfully to build a combined ranking of features for use in neural networks (Prati, 2012). It has also been applied to decide on how to merge propositional logic knowledge bases (Yue, Liu, & Hunter, 2007). This thesis, in contrast, seeks to apply rank aggregation to a multilayer social network. It does so by surveying the domain which is concerned with determining exact overall rankings based on a set of independent rankings. This domain is known as social choice theory.

Social choice theory deals explicitly with identifying collective choices using approaches which consider general social welfare and utility. The sub-field of voting theory (or electoral systems theory) is the study of methods for determining winners of an election (Pacuit, 2017). Aggregation of rankings from each voter should ideally produce an overall ranking which clearly and unambiguously results in a fair assignment of the winner, first runner up, second runner up, and so forth (Stahl & Johnson, 2006). There is a great deal of literature on the topic and only a brief exposition is included in the following section.

One broad area of interest within electoral systems is the study of preferential voting (Pacuit, 2017). Preferential voting, or rank-ordered voting, is a method in which each voter assigns a preference value to each candidate (Pacuit, 2017). This yields a separate rank-ordered candidate list for each voter. For the purposes of this study, the methods considered are restricted to preferential voting methods since it matches the structure of the problem: finding a rank-ordered value of critical nodes in a multilayer network. There is more than one way to structure a preference voting method. The two primary ways are through use of cardinal values and ordinal values (Stahl & Johnson, 2006). This thesis focuses on ordinal-valued voting rather than cardinal-value voting. This was done primarily for two reasons.

First, the literature on cardinal voting methods is much sparser. Second, when analyzing cardinally-valued rankings, the need for normalization between voters' rankings arises to ensure each voter's ranking is weighted equally (Stahl & Johnson, 2006). This is a daunting problem when considering voting preferences between people (Stahl & Johnson, 2006). It can still introduce challenges when the cardinal values represent network centrality measures. Each measure must first be normalized for comparison. Normalization techniques can vary depending on whether disconnected components are considered. Nevertheless, the cardinal-valued rankings contain more information than their corresponding ordinal-value rankings; a measure of magnitude in preference relations between candidates is maintained (Stahl & Johnson, 2006). Therefore, application of cardinal-valued preferential voting methods to SNA is likely to be of interest in future works.

Within the domain of ordinal-valued rank-ordered voting, many different methods have been developed (Pacuit, 2017). Methods within voting theory are evaluated based on a series of criteria considered to be important to the idea of a free and fair election (Stahl & Johnson, 2006). This section briefly surveys methods within voting theory and which criteria they satisfy or fail. The count of satisfied or failed criteria for a given method is then used as a proxy to determine applicability of the method to the specific problem of ranking critical nodes on a multilayer network. The first criterion considered is the monotonicity criterion.

The monotonicity criterion requires that the addition of a worse (better) vote for a candidate should not be able to improve (harm) their outcome (Pacuit, 2017). Any method which fails the monotonicity criterion was immediately discarded as being inappropriate for modeling the problem at hand. If a network layer is added to the multilayer network in which a node is ranked more highly than it is in other layers *all other things being equal*, then it is undesirable for that node to move down in the overall ranking.

Computational complexity is an important consideration. Most algorithms in voting theory can be computed in polynomial time. One exception is the Kemeny Young method, which has complexity $O(N!)$ where N is the number of candidates (Young & Levenglick, 1978). This study treats nodes as candidates, and this method would therefore require on the order of 139! Evaluations to compute. This is a computational feat that is clearly impossible with modern computing methods and so this method was discarded for this study.

Nevertheless, it should be noted that the study of the Kemeny Young method is important theoretically because it represents the maximum likelihood estimator for an aggregated ranking by determining an overall ranking which is at a minimum distance from all input rankings (Young & Levenglick, 1978) (Young, 1988). Thus there is extensive interest in this method and several approximation techniques have been developed as will be discussed further.

Having thus discarded a number of possible methods, five methods remained for consideration which do satisfy a majority of established voting criteria. These are the Borda, Copeland, Minimax, Schulze, and Tideman methods. Each is now considered in turn.

The Copeland method fails the resolvability criterion. The resolvability criterion requires that a tie between two candidates should be decidable with the addition of a single tie-breaking vote in favor of one of the candidates (Schulze, 2011). This is a desirable criterion for a ranking method as it can help eliminate tied values, or at least help decide the winner of a tie when it arises. Decidability is important when considering the context of building a list of critical nodes, potentially for targeting purposes under resource limitations.

The Borda method fails the Condorcet criterion, and the majority criterion (Pacuit, 2017; Stahl & Johnson, 2006). Failure of the majority criterion also implies failure of the mutual majority criterion. The Condorcet criterion requires that if a node exists which is considered better than any other node in a head-to-head comparison, that node must be the overall winner (Condorcet, 1785). The majority criterion similarly requires that if a candidate is preferred by a majority of voters, that candidate must win (Schulze, 2011). The mutual majority criterion is a stronger version of the majority criterion which states that if there is a set of candidates which is item-wise preferred by a majority of voters to all candidates outside of the set, then the overall winner must come from the winning set (Schulze, 2011). Each of these is a meaningful criterion in the context of computing rankings for a multilayer network. Violation of any would call into question the validity of an overall ranking of nodes.

The Minimax method fails the Smith criterion, the mutual majority criterion, and the Condorcet loser criterion (Smith, 1973). The Smith criterion states that any winner must come from the Smith set (Smith, 1973). The Smith set is defined as follows: partition the set of candidates into two disjoint subsets such that any node from set one will always pairwise defeat any node from set two. If such a partition is possible, then the set of pairwise winners is the Smith set (Smith, 1973). The Condorcet loser criterion requires that any candidate which is pairwise defeated by each other candidate cannot be the overall winner (Schulze, 2011). This is a logically desirable characteristic in the context of identifying critical nodes.

Thus we are left with the Schulze and Tideman methods, both of which satisfy all previous criteria described (Schulze, 2011; Tideman, 1987). Both the Schulze and the

Tideman methods additionally satisfy Independence of Smith-dominated alternatives ISDA) (Schulze, 2011). The ISDA criterion requires that an added candidate will not change the winner as long as the added candidate is not a member of the Smith set.

The Schulze method fails the local independence of irrelevant alternatives (LIIA) criterion while Tideman's ranked-pairs method does satisfy LIIA. LIIA is the only criterion for which the Schulze and Tideman methods differ in performance. The LIIA criterion guarantees that the rankings will remain consistent if a node is removed from the network. Despite the Schulze method being slightly less generally applicable in this regard, it is directly related to the common network problem of identifying all shortest paths (Schulze, 2011; Pollack, 1960). Thus it was chosen as the primary voting method under study.

Establishing the applicability of voting theory criteria to the problem of ranking critical nodes on multilayer networks is not necessarily straightforward. It may be argued, alternatively to the arguments presented here, that a criterion which is important in the context of voting on political candidates does not matter in the context of ranking centrality measures. In this case, some methods which were removed from consideration may still yield useful rankings. Acknowledging this, the Borda count method was considered for additional comparison due to its longtime use and ease of implementation: the Borda method is a linear combination of rankings (Stahl & Johnson, 2006; de Borda, 1770/1781). Thus, researchers may apply the Borda method without acknowledging (or perhaps knowing) that it is named as such (Sola, *et al*., 2013).

This thesis borrowed from the social choice theory literature to apply a deterministic method for aggregating vector-valued node rankings to produce a scalar-valued node ranking for each node on a node-aligned multiplex network. A literature review revealed few previous applications of social choice theory to the analysis of a social network. One application sought to improve rankings of influential nodes on a Twitter data set by aggregating rankings from individual measures (Subbian & Melville, 2011). Another sought to predict links in a dynamic network over time by incorporating ranking information on topological measures (Pujari & Kanawati, 2012). Both used weighted (supervised) variants of the Kemeny Young method and the Borda count method.

Pujari and Kanawati (2012) applied a similar method but with the goal of predicting changes in links on a dynamic co-authorship network. However, once again their study was limited to an investigation of a single layer network (Pujari & Kanawati, 2012).

Subbian and Melville (2011) chose weights based on an objective measure of performance for each network measure considered against a ground truth data set. Thus their rankings served as an interesting approach to aggregation of rankings of network measures according to how *good* the measure is. However, their study was limited to a single layer Twitter network dataset (Subbian & Melville, 2011). Since the ultimate goal of identifying influential network members aligns well with this thesis, Subbian and Melville's approach is discussed in more detail.

Subbian and Melville (2011) restricted their study to rankings with no tied values; thus, they broke all ties prior to computation of rankings and did so randomly (Subbian & Melville, 2011). However, in social networks, especially *dark networks*, there may exist disconnected components and isolated nodes (Morris & Deckro, 2013). Such disconnected components and nodes often result in tied values when computing centralities, which then result in tied rankings. Rather than immediately attempt to remove such tied values, this thesis investigates ways to work with them and still yield useful results.

Subbian and Melville apply the Kemeny Young method to their problem in part because it has been proven to represent a maximum likelihood estimator (Subbian & Melville, 2011; Young & Levenglick, 1978). Unfortunately, finding a Kemeny Young ranking is an NP hard problem for $N \geq 4$, making it impractical without applying approximation methods. (Bartholdi, Tovey, & Trick, 1989; Dwork, *et al*., 2001). Approximation techniques have been developed and were applied by Subbian and Melville (2011) (Schalekamp & van Zuylen, 2009; Ailon, Charikar, & Newman, 2008; Subbian & Melville, 2011). However, approximation techniques may not yield optimal solutions.

Voting theory seeks a precise, deterministic winner in an election, as credibility of elections depends on such a result (Stahl & Johnson, 2006). Applying a heuristic to a voting method can arguably negate this original intent. This may be argued to be a sound approach when extending voting theory into social network analysis, but this thesis attempts to maintain such original intent by considering only methods which yield exact solutions in polynomial time.

Both Subbian and Melville (2011) and Pujari and Kanawati (2012) considered the aggregation of rankings based on separate measures, but not the aggregation of rankings based on separate networks—i.e. an application to the domain of multilayer networks. This thesis is apparently the first such application in the literature. It thus provides a significant contribution to the field of social network analysis.

2.3.14 Schulze Method.

Not all criteria which are relevant in the context of an electoral system are necessarily relevant to the idea of this thesis: that a layer can cast a vote (weighted or unweighted) on the ranking of a set of nodes under some measure. The desired outcome is solely a ranking of multilayer network nodes of interest for further analysis or action. This is not a political outcome which might desirably be held to some ideal of democratic fairness. Nevertheless, as a point of entry, this methodology adopts the Schulze method—which satisfies the greatest number of criteria while being solvable in polynomial time using a modification of a common network algorithm—and applies it to the problem of compositing rankings of measures on a multilayer network.

The Schulze method is a rank-ordered voting method which produces a self-consistent composite ranking of candidates based on input rankings from each voter (Schulze, 2011). Voters must consistently use either an ascending or a descending number line ordering of the candidates, but it allows for voters to assign any value to the candidates, including tied values (Schulze, 2011). If tied values are assigned, they are treated as equal vote preferences.

In the case of a descending order ranking, larger (smaller) numbers on the number line imply a larger (smaller) preference; the value of the number is ignored and only the relative ordering of preferences is considered (Schulze, 2011). A tie represents a voter's indifference between the candidates. If a candidate is not given a value, this is interpreted as the voter strictly preferring all marked candidates above any unmarked candidate. All unmarked candidates are considered to be tied (Schulze, 2011).

Once a complete listing of votes is received, the Schulze method counts all preference relationships and stores the values in an asymmetric adjacency matrix representing a directional graph whose weights are the count of the directional preferences of each candidate to all others (Schulze, 2011). A directional preference is defined as a node being ranked as strictly better than another node. For example, if candidate A is preferred to candidate B seven times while candidate B is preferred to candidate A four times, the adjacency entry (A,B) is set to 7 and entry (B,A) is set to 4.

This preference matrix is then subjected to a strongest path algorithm (Schulze, 2011). The strongest path algorithm is a variation of the shortest path algorithm that instead of a shortest path, identifies the best path which allows for maximum path size between two nodes where the path sizes are given by the weights on each arc (Pollack, 1960). Thus, the strongest path from node A to node B is the path with the maximum minimum edge weight. Once all strongest paths are calculated, the number of strongest paths a node belongs to is interpreted as the overall score for that node (Schulze, 2011).

Within every possible pair, the winner is determined by comparing their two entries in a matrix of strongest path values. Each pairwise winner is indicated in a binary winner matrix. The row sum of the winner matrix represents the overall scores for each candidate. The candidates are then ranked according to this overall score, yielding a composite ranking (Schulze, 2011). This entire process is summarized in Figure 6.

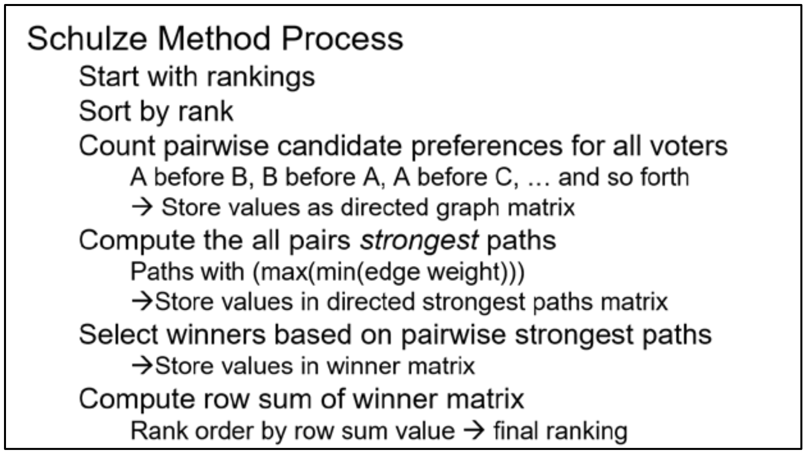

Figure 6: Overview of Schulze Voting Method

By using this approach, measures which are more difficult to extend into a multilayer network context—such as degree centrality (Boccaletti, *et al*., 2014)—can be computed to form composite rankings which are representative of the multilayer network structure and are ultimately of more interest than the scores themselves. Beyond individual centrality measures, any measure that can be computed for each node within a multilayer network can be ranked using this method. Thus, this methodology is in theory not limited to individual centrality measures, but could also be based on any function of centrality measures. It could alternatively be applied to other qualitative or quantitative nodal measures considered to be important to the network analysis. This is true as long as the measure can be used to produce a ranking for each node on each layer. Although it is more broadly applicable, this thesis' scope is limited to network centrality rankings, demonstrating that results can be computed on each layer individually and then combined in such a way as to arrive at a meaningful overall ranking for the multilayer network.

2.4 Summary

SNA is a well-studied field which continues to yield important applications and theoretical advances. It is based on the application of graph theory to sociograms which represent people and their relational links (Wasserman & Faust, 1994).

The extension of network analysis to include multidimensional, multilayer, or multiplex networks has shown significant advances in application, as real world networks can seldom be well-represented by only one layer (Kivela, *et al*., 2014). Indeed the multilayer network framework yields additional degrees of freedom which give rise to new phenomena which cannot occur under the previous framework of ordinary single layer networks (Kivela, *et al*., 2014).

The addition of layers within large network datasets can create computational difficulties and thus methods have been sought to reduce the number of layers while maintaining similarity to the original dataset (De Domenico, *et al*., 2015).

Many SNA measures exist and several of the classic single layer measures have been extended to fit multilayer formulations as well (Martin & Porter, 2012; Sola, *et al*., 2013; Sole-Ribalta, *et al*., 2014; Voros & Snijders, 2017). Analytic techniques from linear algebra have been extended to include not only matrix representations of traditional single layer networks, but also tensor representations of multilayer networks (De Domenico, *et al*., 2013; Kolda & Bader, 2009).

Network centrality measures can be used to identify a prioritized list of nodes of interest within a network and some attempts have been made to extend centrality measures to multilayer networks (Boccaletti, *et al*., 2014; Halu, *et al*., 2013; Iacovacci & Bianconi, 2016; Sola, *et al*., 2013; Sole-Ribalta, *et al*., 2014; Kivela, *et al*., 2014).

Voting theory provides possible tools for generating a generic multilayer network ranking of nodal measures independent of the mathematical extensions of particular centrality measures. The Schulze voting method is one tool that produces a unique and meaningful list of rankings when applied to several lists of rankings (Schulze, 2011). In this thesis it is applied in the context of multilayer SNA to create a composite ranking of nodes for a multilayer network.

This chapter discussed a review of literature relevant to an understanding of background material and tools applied in this thesis. Next, a detailed methodology of this research is described.

III. Methodology

3.1 Chapter Overview

As Chapter II detailed, multilayer networks can be used to contain more information than single-layer networks as each layer represents a distinct connection pattern for a given connection type (Kivela, *et al*., 2014; Boccaletti, *et al*., 2014). Each layer also increases network size and brings with it additional computational costs. The ability to aggregate, or combine, layers in such a way that information is not lost while the final number of required layers is minimized is a desirable goal as the size of the networks under consideration increases.

With the addition of these new network layers, however, more computational difficulties can arise when performing calculations on the full network (De Domenico, *et al*., 2015). Each new layer adds another set of nodes which needs to be measured. Depending on the construction of the multilayer network, inter-layer connections may also be present and these would require additional data storage and processing (Boccaletti, *et al*., 2014). Methods for combining—or aggregating—redundant layers while simultaneously maintaining as much useful information gained from the additional network layers as possible is thus desirable. However, the concept of useful information needs to be studied further. This methodology therefore investigates the effects of this reduction process on resulting network centrality rankings under various conditions.

To do this, the quantum Jensen Shannon distance reduction method proposed by De Domenico *et al.* (2015) was applied to the studied dataset on individual time stamps and upon the network which was first aggregated along the time dimension. Composite network centrality rankings were computed on the resulting networks and these composite rankings were statistically compared to composite rankings computed with the full multiplex network and the fully-aggregated single layer network.

Additionally, the Schulze composite methodology was varied by applying vote weight distributions and by computing an overall composite of the five composited rankings, one for each centrality measure. This was done on the full multiplex and the reduced multiplex. This same process was similarly used on the fully-aggregated network centrality rankings under the same vote weight distributions. These rankings were then compared for statistical differences and correlations.

Further, correlations were computed for each time stamp for each composite centrality ranking for the full and reduced multiplexes. These were computed for the aggregated rankings as well and for each of the five centrality measures. The forgoing was accomplished under unweighted layer aggregation and under a layer weight distribution. Changes to implied stability—high time series correlation—of these rankings over time were investigated and compared. Finally, rankings were compared qualitatively to assess relative inclusion of known key members of the network within the top 20 ranked positions.

3.2 Research Questions

3.2.1 Comparative Statistical Questions.

For the full multiplex network with no timestamps (a single unweighted 12 layer multiplex):

1. Do the Schulze method's composite rankings significantly differ from the fully-aggregated single layer network's rankings using the Friedman test at the $\alpha = 0.05$ level?
    a. Does this hold for each centrality measure?
2. Do the Schulze method's composite rankings for the full multiplex network differ significantly from the Schulze composite rankings for the reduced multiplex network using the Friedman test at the $\alpha = 0.05$ level?
    a. Does this hold for each centrality measure?
3. By applying the three vote weight distributions shown in Table 2 to the Schulze method, are significant changes to the Schulze rankings observed using the Friedman test at the $\alpha = 0.05$ level?
    a. Does this hold for each centrality measure?
4. If an overall composite ranking is defined as the Schulze composite ranking of the individual Schulze rankings for each network measure, do the unweighted overall composite rankings significantly differ from the overall composite rankings under each weight distribution in Table 2 using the Friedman test at the $\alpha = 0.05$ level?
    a. Do they differ from the overall aggregated rankings built by running the Schulze method on the fully-aggregated single layer network's rankings for all centrality measures using the Friedman test at the $\alpha = 0.05$ level?

3.2.2 Correlative Statistical Questions.

For the time-stamped network array of 120, 12 layer multiplex networks:

5. How are changes in the implied stability of the five centrality measures' correlations over time impacted by use of the Schulze composite rankings, as measured by the Spearman rank correlation coefficient and sign test at the $\alpha = 0.05$ level?
    a. Does this hold for the both the full and the reduced multiplex networks?

For the full multiplex network with no timestamps (a single, unweighted 12 layer multiplex):

6. By applying the three vote weight distributions in Table 2 to the Schulze method and comparing results, are significant correlations between rankings for each weight observed using the Spearman rank correlation at the $\alpha = 0.05$ level?
    a. Does this hold for each centrality measure?
7. If an overall composite ranking of all centrality measures is compared against the composite rankings for each centrality measure, are significant correlations among the rankings observed using the Spearman rank correlation at the $\alpha = 0.05$ level?
    a. What correlations occur if the overall composite is produced from composites for each measure computed under the three weight distributions?

3.2.3 Qualitative Questions.

For the full multiplex network with no timestamps (a single, unweighted 12 layer multiplex):

8. How do the final lists of network members in the top 20 ranked positions compare for each centrality measure?

a. Between the Schulze method's rankings and aggregated method's rankings?
b. Between the overall composite rankings and the aggregated network's rankings?
c. When considering the three weight distributions?
d. Between the Schulze method and the Borda count method?

3.3 Materials and Equipment

All work was completed on a HP Z840 desktop computer with 64 gigabytes of RAM. MATLAB R2016a was used for all computations (MATLAB, 2016).

3.4 Data Description

The data used for this study represent a terrorist network located largely in Indonesia. Noordin Mohammed Top was the leader of the conglomeration of terrorist groups operating in the area and the network is eponymously referred to as the Noordin Top network, or simply as the Noordin network. The datasets are published as appendices in books authored by faculty of the Naval Postgraduate School, compiled from open source data and relying heavily on a report issued by the International Crisis Group which details the network's operational and personnel history (International Crisis Group, 2007; Cunningham, *et al*., 2016; Everton, 2013).

The data were received originally as two separate databases. The first is a timeline of 120 timestamps each representing one month beginning on January 1, 2001 and ending on December 31, 2010 (Everton, 2013). The ten year span of timestamp data represent which of a possible 139 network actors were known to be actively present in the network during each month. Activity begins with the first known mention of the actor as a part of the Noordin Top network and ends either when the actor is killed or captured and thus removed from active participation. If the actor is captured and subsequently released, they are included once again within the network (if they are known to have reentered into participation). Thus, there may exist several starts and stops for a given actor which may result in gaps within their activity timeline (Everton, 2013).

The second database consists of relational information between the actors within the Noordin network consisting of both one-mode and two-mode networks (Cunningham, *et al*., 2016). A one-mode network is a network wherein the set of nodes is compared against itself creating a square adjacency matrix. In the case of the Noordin network, a one-mode network is a 139 by 139 node adjacency matrix which identifies whether a particular actor is adjacent to another actor for a given relationship.

A two-mode network, in contrast, is a network wherein the set of nodes is compared against some other set of features. This may result in a non-square adjacency matrix. For example, within the dataset, the actors are compared against a set of 14 named operations. In this case, the adjacency matrix is a two-mode 139 by 14 matrix with an adjacency entry representing whether a particular actor is known to have participated in a particular operation (Cunningham, *et al*., 2016).

One-mode networks within the dataset were used as given to represent a layer within the final multilayer dataset and two-mode networks were pre-processed to become one-mode networks prior to inclusion. Pre-processing the above example consisted of identifying actors who participated in the same named operations. These actors were then inter-linked in a new one-mode adjacency matrix where each entry represents whether actors were co-participants in any given operation.

During data processing, 12 one-mode adjacency matrices representing 12 different relationship types were ultimately compiled. These relations include business, classmates, communications, friendship, kinship, logistical function, logistical location, meetings, operations, organizations, soulmates, and training. Maintaining alphabetical order, these layers are also referred to as layers 1 through 12, respectively.

The *business* layer was originally a two-mode network and represents whether actors were engaged in the same business activities. The *classmates* layer was originally a one-mode network and indicates whether actors attended school together. The *communications* layer was originally a one-mode network and captures whether an actor communicated directly with another actor. The *friendship* layer was originally one-mode and is simply whether any two actors were considered to be friends. Similarly, the *kinship* layer was one-mode and indicates familial relations between actors. The *logistical function* layer was a two-mode network detailing logistical roles that each actor is known to have played. The *logistical place* layer was a two-mode network which describes logistical locations with which a given actor was involved. The *meetings* layer was a two-mode network of known significant meeting participation by the actors. The *operations* layer was a two-mode network representing in which operations an actor participated. The *organizations* layer was a two-mode network listing to which sub-organizations within the Noordin Top network each actor belonged. The *soulmates* layer was a one-mode network identifying whether an actor attended the same religious institution as another actor.

Finally, the *training* layer was a two-mode network which labels actor participation in a series of identified training events. This is summarized in Table 1.

Table 1: List of Network Layers

Layer Number	Name	Original Mode
1	business	2
2	classmates	1
3	communications	1
4	friendship	1
5	kinship	1
6	logistical function	2
7	logistical place	2
8	meetings	2
9	operations	2
10	oganizations	2
11	soulmates	1
12	training	2

This 12 layer multilayer network can be thought of as an aggregation over the entire recorded time period of the Noordin network's operations. To transform it into a time-stamped multilayer network, the 12 layers are combined with the timestamped data. This combination is accomplished by treating the timestamps as an indicator variable and item-wise multiplying the 12 adjacency matrices by the indicator values. If a node's indicator value at a given timestamp is 1, then that node's adjacency values are included in the multilayer network at that timestamp. If a node's indicator value at a given timestamp is 0, then that node's adjacency values are not included.

Effectively, because adjacency matrix entries represent a relation between two nodes, a logical AND operation is used. If both node A and B are present in that timestamp, then the adjacency entry for pair (A,B) is allowed to exist (it may still turn out that A and

B are not linked and will thus have an entry of zero). If either A or B are not present in that timestamp, then the adjacency entry for pair (A,B) will not exist; it is forced to zero. This is summarized by equation 11 as:

$$A^m(n_a, n_b), t = A^m(n_a, n_b) * (I_{n_a t} * I_{n_b t}) \qquad 11$$

where A^m is the m^{th} adjacency matrix for the M layers of the multilayer network, n is a node in the network, t is the timestamp being considered, and I_{nt} is the indicator variable for whether node n is in the network at time t.

Once this item-wise logical multiplication is accomplished, the result is 120 separate 12-layer multilayer networks. For each timestamp, the multilayer network consists of entries for all 12 relationship types (layers) which allow non-zero edge weights for only those actors who were an active part of the Noordin network during that timestamp.

3.5 Data Processing

To produce the 120 layer time-stamped multiplex network, the full network array was aggregated only along the aspect dimension for each timestamp. This was done in two ways. The first method performed a simple summative aggregation—a unit-weighted linear combination of adjacency matrices—resulting in edge weights which represent the count of identical edges among all 12 layers for each timestamp. The second method performed a weighted aggregation—a non-unit-weighted linear combination of adjacency matrices—resulting in edge weights which represent the weighted count of identical edges among the 12 layers at each timestamp. This was done for three weight distributions as described in section 3.8.

The weighted and unweighted aggregation methods were applied to build aggregated data for comparison against layer-reduced networks under different layer weight distributions. The layer-reduction process is described next.

The Jensen Shannon distance reduction method was employed on each of the timestamps of the 120 layer time-stamped multiplex network to reduce the 12 layer multiplex to an R-layered *reduced multiplex* where $|R| \leq 12$. Thus the cardinality of the set of layers for the reduced multiplex was allowed to vary with each timestamp. This was done for both the non-unit-weighted and unit-weighted multiplex layers.

A non-timestamped network was also analyzed. This consisted simply of the original 12 layer multilayer network, with no changes.

3.6 Node Alignment and Isolates

The data process described above creates a node-aligned multiplex network where each adjacency matrix is a 139 by 139 matrix, thereby representing all possible actors on the network. This size is invariant regardless of whether the actor exists in that timestamp or not. In other words, the multilayer network is fully node-aligned. This allows for a simpler representation of the network, but can introduce artificial isolates. A node which has no incoming or outgoing edges within the network is an isolate. Thus for a given layer at a given timestamp, there may exist both induced or artificial isolates in addition to natural or true isolates.

Here a true isolate is considered to be a node which is part of the network at the given timestamp but does not have any recorded relationship with any other node coincident at that timestamp. An artificial isolate, in contrast, does not truly exist at the given timestamp, but remains represented within the node-aligned adjacency matrices. Thus true and artificial isolates are indistinguishable within this representation and are treated as equivalent under this methodology. This introduces some error into the methodology and its impacts and potential solutions will be discussed in chapters IV and V.

3.7 Ranking Methods

Once the data have been processed to produce the eight by three arrays summarized in Table 3, statistical comparisons were made both between methods of aggregation—weighted versus unweighted—and between use of the full multiplex, JSD-reduced multiplex, and the fully-aggregated network. The values chosen for comparison are five rank-ordered centrality measures comprising betweenness, closeness, eigenvector, degree, and PageRank centralities. This list of centrality measures was informed by the analysis by Guzman *et al.* (2014) which indicates that these five measures are not highly correlated among themselves (Guzman, *et al*., 2014).

Comparisons were conducted along both dimensions of the data: the time dimension and the relation dimension. Each 139 by 139 adjacency matrix was reduced to a single 1 to 139 node ranking for each of the five centrality measures. Thus along the relational dimension, 12 rankings of length 139 were computed for the full multiplex. For the reduced multiplex, anywhere from 1 to 12 rankings were potentially computed for each of the five centrality measures. For the aggregated network, a single 1 to 139 nodal ranking was computed for each of the five considered measures.

For the full multiplex data, a 120 by 12 multiplex array of rank vectors of length 139 was produced. The comparisons along the aspect dimension were thus achieved by comparing rankings between columns for each time stamp. These rankings were computed across all layers for all timestamps and thus a comparison along the time dimension of the 120 by 12 multiplex array is achieved by similarly comparing the rankings between rows for each layer.

Composite ranks were computed for each of the five measures within each timestamp for both the full multiplex data array and the reduced multiplex data array. These time-stamped ranks were investigated for correlation patterns.

3.8 Weighting Methods

When conducting an analysis of a social network, edge weights are often defined to have unit values; however, different edge weights can alter the resulting value of the measure. This is true when both measuring a single layer network as well as when considering the relative importance of each layer of a multilayer network. The aggregation of a unit-weighted multilayer network is the unit-weighted sum—linear combination—of associated adjacency matrices. The aggregation of a weighted multilayer network is similarly the weighted linear combination of the associated adjacency matrices.

The weights representing the relative importance of each layer to the analysis might be drawn from analytic expertise, either as the output of some other analysis, or as the result of elicitation of a subject matter expert's assessment.

3.8.1 Vote Weighting.

Weights were applied to each layer's *ranking*, effectively giving each layer a number of votes based on its weight. This is referred to as vote weighting. In this second case, the weights must be integer-valued. In the case of assigning layer weights as edge weights on the adjacency matrix prior to aggregation, there is no such integer restriction. The Schulze method is a proportional method, however, meaning that the weights are only important on a relative scale, and not in terms of their actual values. In other words, any non-integer weights can be scaled to produce a common set of integer weights as a series of least common multiples. This maintains the same relative proportions—yielding the same results—and satisfies the integer requirement under the Schulze voting method methodology for layer weighting.

To emphasize the utility of applying weights in such a manner, three distinct distributions of relative weights were selected and applied. Each corresponds to an emphasis on a certain conceptual grouping of aspect types, borrowing from a similar idea proposed by (Crawford, *et al*., 2016). Instead of partitioning the layers into disjoint sets representing trust, lines of communication, and knowledge as do Crawford *et al.* (2016), three different groupings were produced which were assigned distributions of weights and retained all layers within each distribution. The first emphasizes location—or actual physical proximity—of actors within the network. The second emphasizes operations—participation in the same operations—of actors within the network. The third emphasizes personal ties between actors within the network.

The range of weights assigned for each of the three distributions was limited to integer values between one and three for comparison. These represent a truly relative and additive scale and can be derived using any weighting method which assures such a scale is produced. Thus, layers with a value of two and three were weighted 100% and 200% more heavily than the baseline, respectively. Weights were applied with consideration of how each layer's aspect might contribute to the distribution of interest. In other words, the contextual meaning of each layer was used to scale each layer's weight in this study.

The first distribution is the location distribution which is given by the vector [2 2 1 1 1 1 3 3 2 2 3 3]. Thus, layers 7, 8, 11, and 12 were given the largest weight. These are *logistic place*, *meetings*, *soulmates*, and *training*. Each refers directly to actors being recorded in the same physical location at the same time. Layers 1, 2, 9, and 10 were given the intermediate weight. These are *business*, *classmates*, *operations*, and *organizations*. These include aspects which encourage location but don't explicitly refer to it. Layers 3, 4, 5, and 6 were allocated the baseline weight. These are *communications*, *friendship*, *kinship*, and *logistic function*. Though friendship and kinship do often imply location at some point in time, they do not necessarily imply location during the timeframe of the dataset. Likewise communications and logistic function are not necessarily related to location.

The second distribution is the *operations* distribution which is given by the vector [1 1 2 1 1 3 3 3 3 2 1 3]. Thus layers 6, 7, 8, 9, and 12 were given the largest weight. These are *logistic function*, *logistic place*, *meetings*, *operations*, and *training*. Operations and training are directly related to operations. Meetings in this context are meetings for planning operations and are thus included. Logistics functions are likewise crucial to operations. Layers 3 and 10 were assigned the intermediate weight. These are *communications* and *organizations*. All those who are communicating are not necessarily sinvolved in operations, though they may likely be tangentially involved; likewise for relevant organizations within the dataset. Layers 1, 2, 4, 5, and 11 were given the baseline weight. These are *business*, *classmates*, *friendship*, *kinship* and *soulmates*. While an argument could be made that business fronts are related to the operations in terms of money-laundering in support of operations, they are considered here to be more loosely connected. Friendship, kinship, and soulmates are personal relationships which are considered to transcend particular operations.

The third distribution is the personal ties distribution and is given by the vector [1 3 1 3 3 1 1 1 1 1 2 2]. Thus layers 2, 4, and 5 are given the largest weight. These are *classmates*, *friendship*, and *kinship*. These three were deemed the most personal or intimate ties recorded within the dataset as they refer to long-standing or very personal relationships. Layers 11 and 12 were given the intermediate weight. These are *soulmates* and *training*. Co-participation in religious and operational training can build deep bonds through small group dynamics. Layers 1, 3, 6, 7, 8, 9, and 10 were assigned the baseline weight. These are *business*, *communications*, *logistic function*, *logistic place*, *meetings*, *operations*, and *organizations*. These are likely only tangentially related to any personal relationships. The three weight distributions are summarized in Table 2.

Table 2: Table of Weight Distributions

LAYER	Weight Distribution		
	Operations	Location	Personal Ties
Business	1	2	1
Classmates	1	2	3
Communications	2	1	1
Friendship	1	1	3
Kinship	1	1	3
Logistics Function	3	1	1
Logistics Place	3	3	1
Meetings	3	3	1
Operations	3	2	1
Organizations	2	2	1
Soulmates	1	3	2
Training	3	3	2

The point of the use of these weight distributions was not to establish an ironclad method for establishing the distributions themselves, but rather to demonstrate the method and the effects of the distributions on the outcomes of the rankings under this methodology. Specifically of interest was whether there were statistically significant differences of rankings under each weight distribution and whether there were meaningfully different outcomes of which nodes were ranked highest under the different emphases.

These outcomes were once again measured statistically using the Friedman test statistic with a Wilcoxon-Nemenyi-McDonald-Thompson (WNMT) multiple comparison correction. The null hypothesis was that the distribution of differences of the two ranks is centered on zero, implying that both ranks come from the same distribution. All of the preceding comparisons were also checked for correlations using the Spearman rank correlation coefficient. These tests are described in more detail in section 3.9.

3.8.2 Layer Weighting.

The preceding weighting scheme is a unique approach to a weighting scheme for a multilayer network under this voting methodology. Traditionally, in social network analysis, edge weights are used to provide information on relative values of edges within a single network layer. To extend this concept to a multilayer network, layer weights were used to weight edges uniformly for a given layer. In other words, if a layer is given a layer weight of three, then each edge weight within that layer will have a value of three.

To maintain consistency, the same three weight distributions discussed in the vote weighting section were also applied as layer weights. Such weight vectors may represent any relative scale of interest, but could intuitively be thought of as the relative perceived importance of each network layer to the overall analysis of the multilayer network. This contrasts slightly with the interpretation of vote weights as the number of times each layer's ranking is counted in the Schulze method.

The resulting edge weights on the aggregated network can take on the value of any additive combination of the layer weights used. For example, if three layers weighted 1, 2, and 3, respectively are aggregated in this manner, the resulting edge weights may have a range of values to include 1, 2, 3, 5, and 6.

Additionally, unit weights were applied to each layer prior to simple aggregation yielding a weighted—summative—aggregation. This resulted in a final edge weight equal to the number of times the edge occurred in all of the layers. Thus, layer weights were effectively generated: *operations*, *location*, *personal ties*, and *summative*.

The ranks were compared between the full multiplex in its weighted states versus its unweighted state, the reduced multiplex in its weighted states versus its unweighted state, and the aggregated network in its weighted states versus its unweighted states. All comparisons were conducted using the Friedman test statistic with a WNMT multiple comparison adjustment.

3.9 Statistical Methods

3.9.1 Overview.

Rankings are ordinal sets of values that can be statistically compared using non-parametric methods. Specifically, Spearman's rank correlation statistic, ρ, was chosen for determining rank correlations. Friedman's test statistic with a WNMT multiple comparison adjustment was chosen to test for significant differences. All tests were computed using MATLAB's built-in functions. For Spearman, the MATLAB command *corr()* was used with the *type* specified as s*pearman*. For the Friedman statistic, the MATLAB command *friedman()* was used.

3.9.3 Testing for Significant Differences.

All tests for significant differences between ranks described in this section refer to the non-timestamped multiplex network. Thus comparisons are between the full 12 layer multiplex network, the reduced (5 layer) multiplex network, and the fully-aggregated single layer network.

The Friedman's rank test is a method for examining all ranks across a set of voters simultaneously using a block matrix design and is given in equation 12 as:

$$Q = \frac{n\sum_{j=1}^{k}\left(\left(\frac{1}{n}\sum_{i=1}^{n} r_{ij}\right) - \frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k} r_{ij}\right)^2}{\frac{1}{n(k-1)}\sum_{i=1}^{n}\sum_{j=1}^{k}\left(r_{ij} - \frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k} r_{ij}\right)^2} \tag{12}$$

where r_{ij} is the rank of the entry in the block matrix at location (i,j), n is the number of rows in the matrix and k is the number of columns in the matrix.

Friedman's rank test uses the null hypothesis that all groups have been chosen from a population having equal median values (Berenson, Levine, & Krehbiel, 2012). It takes as assumptions:

1. "The [n] blocks are independent so that the values in one block have no influence on the values in any other block.
2. The underlying variable is continuous.
3. The data constitute at least an ordinal scale of measurement within each of the [n] blocks.
4. There is no interaction between the [n] blocks and the [k] treatment levels.
5. The [k] populations have the same variability.
6. The [k] populations have the same [CDF]." (Berenson, Levine, & Krehbiel, 2012)

The blocks (rows) were the listing of nodes and the treatments (columns) were the sets of rankings. Thus the Friedman test as implemented determines whether any set of rankings is significantly different from any other set; however, the Friedman test does not identify which set of rankings is different or whether more than one set is different. To determine this, each pairwise comparison must be computed individually using a multiple comparison method.

A Type I error is the probability of incorrectly rejecting a null hypothesis. It is described by the alpha value. For example, given an $\alpha = 0.05$ level a Type I error is expected in one out of 20 tests. When considering pairwise comparisons, the number of actual tests may quickly exceed 20 (this is true for greater than 7 compared items) making a Type I error more likely to occur by chance alone. Therefore, a correction should be made to account for the likelihood of committing a Type I error on account of the number of tests computed. This is known as a multiple comparison correction.

There exist several possible correction methods, but the correction method chosen was the WNMT correction, as it is a conservative correction. It sets the family-wise error rate to be equal to the individual test alpha level and then divides the original alpha level by the number of tests. Thus given an $\alpha = 0.05$ and 20 tests, Bonferroni's correction yields $\frac{0.05}{20} = 0.025$ as the alpha levels to be used for each test. This ensures a family-wise error rate equal to 0.05, the original alpha level chosen.

Tests for significance were conducted to answer research questions 1 through 4. The block design had to be constructed in such a way that each block was independent of each other. Nodes are assumed to be independent of other nodes in the block structure. Interaction effects between layers and node rankings, if they exist, are considered a structural feature inherent to network data and are not being tested.

For question 1, the Schulze rankings were computed for each centrality measure by compositing the rankings generated for each layer of the 12 layer multiplex network which was aggregated along the time dimension. Each centrality measure was also computed and ranked for the unweighted, fully-aggregated single layer network.

To answer question 2, the Schulze composite rankings were first computed for each of the five centrality measures. These were computed using both the full 12 layer multiplex network which was first aggregated along the time dimension. The same composite rankings were also computed for the time-aggregated 12 layer multiplex after it was reduced using the Jensen Shannon reduction method.

The approach to question 3 was to compute the Schulze composite rankings for each centrality measure under four vote weight conditions: the three vote weight distributions and the unweighted composite.

Question 4 required a second application of the Schulze method algorithm using the first Schulze method application's outputs. This produced a composite of composites for the rankings, essentially creating an overall ranking which is a function of the rankings under each separate centrality measure. This process was repeated using no vote weights and using the three vote weight distributions. The vote weights were applied during the first run of the algorithm. No weights were applied to the composite centrality rankings for the second run. Additionally, the Schulze algorithm was applied to the unweighted centrality rankings produced from the fully-aggregated network to produce an overall aggregated network ranking for comparison.

3.9.2 Testing for Correlations.

This section refers to the time-stamped multiplex data, either full, reduced, or fully-aggregated to a single layer. Thus there are 120 rankings computed under each condition.

Spearman's rank correlation coefficient was used to answer research questions five through seven. Spearman's rank correlation coefficient is defined in equation 13 as:

$$\rho = \frac{cov(R_1, R_2)}{\sigma_{R_1}\sigma_{R_2}} \qquad 13$$

where R_1 represents the rankings of the first group (population) to be compared, R_2 represents the rankings of the second group (population) to be compared, σ is the standard deviation of the ranks and $cov(R_1, R_2)$ is the covariance of the ranks for the first and second groups (populations).

To answer question 5, a Spearman rank correlation at the $\alpha = 0.05$ level was computed using pairwise values of the ranks between all pairs of timestamps for each measure. This was repeated for the full multiplex, the reduced multiplex, and the fully-aggregated network. The baseline was considered the unweighted (binary) versions of the full, reduced, and fully-aggregated networks, respectively.

Four aggregations were computed using the three weight distributions as layer weights and a summative (unit-weighted) approach. The Schulze composite method was applied to the weighted full and weighted reduced multiplex with no vote weights. The Schulze method was then applied to the unweighted full multiplex network using the three weight distributions as vote weights. This is summarized in Table 3.

In Table 3, SUM refers to the unit weights, OPS refers to weights determined by the *operations* weight distribution, LOC refers to weights determined by the location weight distribution, and PERS refers to weights determined by the personal ties weight distribution. Layer refers to the method of layer weighting and Vote refers to the method of vote weighting. Thus, for example the *Layer SUM, no Vote* implies the multiplex was unit-weighted by layer with no vote weights applied during the Schulze method and *No Layer*, *Vote PERS* means that the layers were binary-weighted, but the personal ties weight distribution was applied using the vote weight methodology. The baseline networks were chosen to be the completely unweighted (binary) networks. These are listed in the first row for each of the three columns.

Table 3: Summary of Weighting Methods Applied

Full Multiplex	**Reduced Multiplex**	**Aggregated Network**
No Layer, No Vote	No Layer, No Vote	No Layer
Layer SUM, No vote	Layer SUM, No Vote	Layer SUM
Layer OPS, No Vote	Layer OPS, No Vote	Layer OPS
Layer LOC, No Vote	Layer LOC, No Vote	Layer LOC
Layer PERS, No Vote	Layer PERS, No Vote	Layer PERS
No Layer, Vote OPS		
No Layer, Vote LOC		
No Layer, Vote PERS		

The full list of eight by three arrays of correlation matrices for each centrality measure are included in Appendix A for visual reference. To assess how different each correlation matrix is from the baseline (binary network matrix), the following methodology was adopted. Each weighted correlation matrix was compared against its respective binary-weighted baseline value using MATLAB's *signtest()* function at the $\alpha = 0.05$ level. The matrices were first arrayed as a vector and then compared. The sign test was used to determine if the samples come from the same distribution. The results are tabulated in Chapter IV.

For question 6, the composite rankings under each of the three vote weight distributions for each of the five centrality measures were computed. These were computed using the unweighted full 12 layer multiplex network which had first been aggregated along the time dimension. This resulted in 5 groups of 3 rankings.

For question 7, each ranking computed for question 6 was fed into the Schulze algorithm again to retrieve an overall composite ranking as a function of all five of the centrality measures considered simultaneously. This resulted in 3 overall rankings, one for each vote weight distribution from Table *2*.

The 18 rankings computed for questions 6 and 7 were then combined with the unweighted vote rankings and a Spearman's correlation coefficient was computed for each pairwise comparison.

3.10 Qualitative Methods

3.10.1 Comparing Ranked Nodes.

To answer question 9, several rankings were compiled for comparison. First, Schulze composite rankings were computed using the unweighted full 12 layer multiplex network which had first been aggregated along the time dimension. This was done for each of the five centrality measures. Next the overall Schulze composite was computed using the Schulze composites for each of the five measures. This resulted in 6 rankings.

These 6 rankings were then re-computed under each of the vote weight distributions and finally an aggregate ranking was also computed on the single layer aggregation of the original dataset for each centrality measure. This resulted in a total of 23 rankings.

Comparing this set of rankings qualitatively depends on first arranging the ranks in a directly comparable way. All rankings were sorted in descending order; however, tied values exist which can make direct comparisons problematic. Instead, they must first be processed by assigning the average of the tied rank positions as the true ranking. In this way, ties are made explicit and a direct comparison becomes possible. This is illustrated through an example given in Table 4.

Table 4: Ranking Comparison Example

	Rank Set 1			Rank Set 2		
Absolute Position	Tied Rank	Raw Rank	Node ID	Node ID	Raw Rank	Tied Rank
139	138.5	137	43	105	139	139
138	138.5	137	105	43	138	138
137	137	136	80	23	137	137
136	136	135	23	82	136	136

In Table 4's example, the nodes are first sorted in descending order according to their raw rankings. Their identification numbers (Node ID) are recorded corresponding to their absolute positions seen in the first column. When ties exist between raw ranks, a direct comparison between positions of Node IDs for each set can be misleading. For example, Rank Set 1 shows Node ID 43 in the first position and Node ID 105 in the second position. In contrast, Rank Set 2 shows the reverse. When using the raw rank values, it might be concluded that Rank Set 2 has ranked node 43 higher (value of 138) than Rank Set 1 ranked node 43 (value of 137). Rank Set 2 also ranked node 105 higher (value of 139) than did Rank Set 1 (value of 137).

In fact, Rank Set 1 ranked nodes 43 and 105 as tied, producing uncertainty in the comparison, as they could just as easily be listed in reverse order (in fact they are listed in numeric order by default). To account for such possible discrepancies, the absolute positions of the tied nodes 43 and 105 within Rank Set 1 are averaged, and the average value is assigned as the tied rank value. Thus Rank Set 1 contains a tied rank value of 138.5 for both node 43 and node 105. Rank Set 2 has no tied values and so the tied rank assumes the same values as the absolute position of the nodes. When the tied rank values for nodes 43 and 105 are then compared between the sets, Rank Set 2 has still ranked node 105 higher (value of 139) than did Rank Set 1 (value of 138.5), but it now ranked node 43 *lower* (value of 138) than did Rank Set 1 (value of 138.5). Thus the result is reversed when the scales are aligned to account for ties in this manner.

This same process described in the example from Table 4 was used to produce all subsequent ranking comparisons and to conduct qualitative assessments based on each actor's relative ranking within the top 20 nodes. In this way it was assessed whether a given method had identified qualitatively important actors (as determined using identifiers within the dataset and open source information) and how their relative standings compared.

3.11 Summary

Chapter III described a series of research questions and the respective methodologies used to answer them. Detailed statistical and qualitative processes were discussed and examples were provided for clarity. Once the methodologies were applied and results collected, analysis was conducted on these results. These analyses and results are discussed in depth in Chapter IV.

IV. Analysis and Results

4.1 Chapter Overview

The preceding chapters have detailed the background, motivation, related literature, and methodology of this thesis. This chapter describes the results achieved through use of this methodology and discusses their implications and relevance. It lists the analysis and results in order of the research questions given in Chapter III, section 3.10.

4.2 Research Questions Answered

This section lists results and discussion which are relevant to each research question listed in section 3.10. It is organized by type of question with comparative statistical questions listed first, followed by correlative statistical questions, followed by qualitative questions. A detailed explanation of each question and its answer is discussed.

4.2.1 Comparative Statistical Questions.

The first set of questions are comparative statistical questions which seek to answer whether or not rankings are significantly different from each other. To answer these questions, the Friedman test was used along with a WNMT multiple comparison correction. All comparative tests were conducted at the $\alpha = 0.05$ level. The outputs are tabulated to list the identifiers for the two sets of rankings being compared, the lower bound on the estimate, the estimate itself, the upper bound, and the p-value. Significant p-values are colored red and green.

All tests were also conducted using the time-aggregated 12 layer unweighted multiplex network, which was then either reduced using the Jensen Shannon distance method, or aggregated to produce an unweighted fully-aggregated single layer network. A complete listing of all Friedman test block designs are shown in Appendix B, Table 26 and results of the corresponding tests are listed in Appendix B, Table 27. A complete listing of all WNMT multiple comparison correction test results is given in Appendix B, Table 28 through Table 31.

These results answer very specific statistical questions for this multilayer network dataset in particular. They should not be considered general results and by themselves they do not provide a qualitative assessment of the utility of the Schulze method. Still, they are useful to show that significantly different results are possible under its application.

4.2.1.1 Question 1.

Question 1 asks if the Schulze composite rankings are significantly different from the fully-aggregated network rankings. Rankings were computed and compared for each centrality measure. The results are listed in Table 5 and show that when the Schulze composite rankings are compared against the rankings for the fully-aggregated network, most centrality measures show no significant differences. Degree centrality is the only measure which did produce a statistically significantly different ranking between the Schulze composite and the aggregated rankings ($p < 0.01$).

This suggests that for the Noordin Top network, the Schulze method of compositing the set of rankings for each network layer tends to produce statistically comparable results when compared to the more standard approach of first aggregating the network and then computing a single ranking, for most network centrality measures except degree centrality.

Table 5: Schulze Composite versus Aggregated Rankings WNMT Test Results

Aggregated	Composited	Reject Null at α= 0.05?	p Value Bound
Betweenness	Betweenness	0	> 0.10
Closeness	Closeness	0	> 0.20
Degree	Degree	1	< 0.01
Eigenvector	Eigenvector	0	> 0.20
PageRank	PageRank	0	> 0.20

4.2.1.2 Question 2.

Question 2 asks if Schulze composite rankings are significantly different from the composite rankings computed on the reduced multiplex network. Rankings were computed and compared for each centrality measure. The results are listed in Table 6 and show that all comparisons between Schulze composite rankings on the full and reduced multiplex are significantly different. This suggests that layer reduction has a significant effect on the behavior of the Schulze composite method, regardless of centrality measure used.

Table 6: Schulze Composite Rankings on Full Multiplex versus Composite Rankings on Reduced Multiplex - WNMT Test Results

Full Composite	Reduced Composite	Reject Null at α= 0.05?	p Value Bound
Betweenness	Betweenness	1	< 0.0001
Closeness	Closeness	1	< 0.0005
Degree	Degree	1	< 0.0001
Eigenvector	Eigenvector	1	< 0.0001
PageRank	PageRank	1	< 0.025

4.2.1.3 Question 3.

Question 3 asks if the Schulze composite rankings are significantly different from each other under each of the vote weight distributions. Rankings were computed and compared for each centrality measure. The results are listed in Table 7 and show that 19 out of the 30 pairwise comparisons are significantly different.

Notably, the location vote weight distribution did not produce statistically significantly different composite rankings when compared to the unweighted composite rankings for all measures. In contrast, the *personal ties* vote weight distribution produced significantly different composite rankings compared to the unweighted rankings in all cases; *operations* produced significant differences in all cases except eigenvector centrality ($p > 0.20$).

In addition, the *location* and *personal ties* weight distributions produced rankings which were significantly different for all five centrality measures. The *operations* weights' rankings, however, were statistically different from those of the location weights except for degree ($p > 0.20$) and eigenvector ($p > 0.20$) centralities. Similarly, the *operations* weights produced significantly different rankings compared to the *personal ties* weights only for eigenvector ($p < 0.005$) and betweenness (< 0.0001) centralities.

Table 7: Schulze Composite Unweighted Rankings versus Three Weight Distributions - WNMT Test Results

	Composited Rankings		Reject Null at α= 0.05?	p Value Bound
Betweenness	Unweighted	Operations Weights	1	< 0.0001
	Unweighted	Location Weights	0	> 0.20
	Unweighted	Personal Ties Weights	1	< 0.0005
	Operations Weights	Location Weights	1	< 0.0001
	Operations Weights	Personal Ties Weights	1	< 0.0001
	Location Weights	Personal Ties Weights	1	< 0.0005
Closeness	Unweighted	Operations Weights	1	< 0.0001
	Unweighted	Location Weights	0	> 0.05
	Unweighted	Personal Ties Weights	1	< 0.0001
	Operations Weights	Location Weights	1	< 0.0005
	Operations Weights	Personal Ties Weights	0	> 0.20
	Location Weights	Personal Ties Weights	1	< 0.01
Degree	Unweighted	Operations Weights	1	< 0.01
	Unweighted	Location Weights	0	> 0.20
	Unweighted	Personal Ties Weights	1	< 0.0001
	Operations Weights	Location Weights	0	> 0.20
	Operations Weights	Personal Ties Weights	0	> 0.20
	Location Weights	Personal Ties Weights	1	< 0.01
Eigenvector	Unweighted	Operations Weights	0	> 0.20
	Unweighted	Location Weights	0	> 0.20
	Unweighted	Personal Ties Weights	1	< 0.0001
	Operations Weights	Location Weights	0	> 0.20
	Operations Weights	Personal Ties Weights	1	< 0.005
	Location Weights	Personal Ties Weights	1	< 0.0001
PageRank	Unweighted	Operations Weights	1	< 0.0001
	Unweighted	Location Weights	0	> 0.20
	Unweighted	Personal Ties Weights	1	< 0.0001
	Operations Weights	Location Weights	1	< 0.01
	Operations Weights	Personal Ties Weights	0	> 0.10
	Location Weights	Personal Ties Weights	1	< 0.0001

4.2.1.4 Question 4.

Question 4 asks if the overall Schulze composite rankings are significantly different from the overall weighted Schulze composite rankings and aggregated rankings. Rankings were computed and compared for each centrality measure. There were significant differences identified by the Friedman tests. The results are listed in Table 8 and indicate that only the *operations* weight distribution produced statistically significantly different overall Schulze composite rankings.

Table 8: Schulze Overall Composite Unweighted Rankings versus Three Weight Distributions and Aggregated Composite - WNMT Test Results

Overall Composited Rankings		Reject Null at α= 0.05?	p Value Bound
Unweighted	Operations Weights	1	< 0.0005
Unweighted	Location Weights	0	> 0.20
Unweighted	Personal Ties Weights	0	> 0.20
Unweighted	Aggregated	0	> 0.20
Operations Weights	Location Weights	1	< 0.0001
Operations Weights	Personal Ties Weights	1	< 0.05
Operations Weights	Aggregated	1	< 0.025
Location Weights	Personal Ties Weights	0	> 0.20
Location Weights	Aggregated	0	> 0.20
Personal Ties Weights	Aggregated	0	> 0.20

The previous tables listed select test results. The complete listing of test results is located in Appendix B, in Table 27 through Table 30. These results are also summarized in Table 9.

Table 9: Overall Summary of Multiple Comparison Test Results

	Reduction	Composite	OPS	LOC	PERS	Any weight
% significant	0.643	0.167	0.867	0.143	0.571	0.535
% insignificant	0.357	0.833	0.133	0.857	0.429	0.465

Having answered the questions posed regarding any significant differences in rankings under the variety of measures, reduction, and weights described, correlations between rankings are explored next.

4.2.2 Correlative Statistical Questions.

4.2.2.1 Question 5.

Spearman rank correlations were computed for each timestamp under each of the conditions from Table 3 resulting in five figures (Figure 7 through Figure 16, which refer to betweenness, closeness, eigenvector, degree, and PageRank centrality, respectively) each with an eight by three array of correlation matrices as seen in Appendix A. Correlations were computed using Spearman's correlation coefficient and were colored red for low correlations and green for high correlations. The diagonals are colored black for clarity. All correlation tests were conducted at the $\alpha = 0.05$ level.

The first row for each of the arrays represents the Schulze composite rankings on the full 12-layer unweighted multiplex at each time stamp, the Schulze composite rankings on the reduced-layer unweighted multiplex at each time stamp and the standard rankings on the fully-aggregated unweighted multiplex network at each time stamp.

The second row shows the Schulze composite rankings on the full 12-layer layer-unit-weighted multiplex network at each time stamp, the Schulze composite rankings on the summative reduced-layer layer-weighted multiplex network at each time stamp, and the standard rankings on the summative fully-aggregated multiplex network at each time stamp.

The third row shows the *operations* weight distribution applied as layer weights for the full, reduced, and aggregated networks. The fourth row shows the location weight distribution applied as layer weights and the fifth row similarly shows the personal ties weight distribution.

The sixth, seventh, and eighth rows show the unweighted full, reduced, and aggregated networks with the *operations*, *location*, and *personal ties* vote weights, respectively.

Each weighted correlation matrix was compared against its respective binary-weighted baseline value using MATLAB's *signtest()* function at the $alpha = 0.05$ level. This was used to determine if the samples come from the same distribution. These results are listed in Table 10 through Table 14.

Table 10: Betweenness Centrality Rankings Correlation Comparisons

Betweenness					
Schulze				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
OPS	None	0.0039	0.0001	Reject	< 0.0001
LOC	None	0.0039	0.0001	Reject	< 0.0001
PERS	None	0.0039	0.0001	Reject	< 0.0001
None	OPS	0.0434	0.0087	Reject	< 0.0001
None	LOC	0.0313	0.0029	Reject	< 0.0001
None	PERS	0.0215	0.0013	Reject	< 0.0001
Reduced Schulze				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
SUM	None	0.0736	0.0047	Reject	< 0.0001
OPS	None	0.0786	0.0061	Reject	< 0.0001
LOC	None	0.1200	0.0100	Reject	< 0.0001
PERS	None	0.0739	0.0044	Reject	< 0.0001
Aggregated				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
SUM	N/A	0.0145	0.0002	Reject	< 0.0001
OPS	N/A	0.0118	0.0001	Reject	0.0386
LOC	N/A	0.0171	0.0002	Reject	0.0386
PERS	N/A	0.0167	0.0002	Reject	< 0.0001

As can be seen in Table 10, for betweenness centrality, the mean absolute differences vary under vote weighting, but not under layer weighting for the Schulze method on the full multiplex network. However, the mean differences were not known to be significant until the sign test was applied to the correlation values. The sign test computed this by taking the difference between the correlation coefficients and checking that the resulting distribution had a median of zero.

Once it was applied, it showed that each of the correlation arrays are significantly different from the baseline unweighted multiplex. This is also true for the comparisons made on the reduced multiplex network and the fully-aggregated network. In fact, these same results hold for all centrality measures.

Table 11: Closeness Centrality Rankings Correlation Comparisons

Closeness					
Schulze				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
OPS	None	0.0094	0.0002	Reject	< 0.0001
LOC	None	0.0094	0.0002	Reject	< 0.0001
PERS	None	0.0094	0.0002	Reject	< 0.0001
None	OPS	0.0066	0.0000	Reject	< 0.0001
None	LOC	0.0079	0.0001	Reject	< 0.0001
None	PERS	0.0092	0.0001	Reject	< 0.0001
Reduced Schulze				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
SUM	None	0.0099	0.0004	Reject	< 0.0001
OPS	None	0.0112	0.0004	Reject	< 0.0001
LOC	None	0.0116	0.0004	Reject	< 0.0001
PERS	None	0.0097	0.0004	Reject	< 0.0001
Aggregated				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
SUM	N/A	0.0001	0.0000	Reject	< 0.0001
OPS	N/A	0.0001	0.0000	Reject	< 0.0001
LOC	N/A	0.0001	0.0000	Reject	< 0.0001
PERS	N/A	0.0001	0.0000	Reject	< 0.0001

Table 12: Degree Centrality Rankings Correlation Comparisons

Degree					
Schulze				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
OPS	None	0.0002	0.0000	Reject	< 0.0001
LOC	None	0.0002	0.0000	Reject	< 0.0001
PERS	None	0.0002	0.0000	Reject	< 0.0001
None	OPS	0.0124	0.0002	Reject	< 0.0001
None	LOC	0.0124	0.0002	Reject	< 0.0001
None	PERS	0.0127	0.0002	Reject	< 0.0001
Reduced Schulze				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
SUM	None	0.0177	0.0005	Reject	< 0.0001
OPS	None	0.0224	0.0006	Reject	< 0.0001
LOC	None	0.0166	0.0005	Reject	< 0.0001
PERS	None	0.0155	0.0005	Reject	0.0027
Aggregated				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
SUM	N/A	0.0006	0.0000	Reject	< 0.0001
OPS	N/A	0.0006	0.0000	Reject	< 0.0001
LOC	N/A	0.0005	0.0000	Reject	< 0.0001
PERS	N/A	0.0007	0.0000	Reject	< 0.0001

Table 13: Eigenvector Centrality Rankings Correlation Comparisons

Eigenvector					
Schulze				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
OPS	None	2.97E-06	1.22E-11	Reject	< 0.0001
LOC	None	4.01E-06	1.79E-11	Reject	< 0.0001
PERS	None	3.42E-06	1.23E-11	Reject	0.0242
None	OPS	7.04E-06	3.20E-11	Reject	< 0.0001
None	LOC	7.29E-06	3.65E-11	Reject	< 0.0001
None	PERS	7.21E-06	3.75E-11	Reject	< 0.0001
Reduced Schulze				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
SUM	None	3.39E-05	2.02E-09	Reject	< 0.0001
OPS	None	4.72E-05	1.80E-09	Reject	< 0.0001
LOC	None	4.25E-05	2.16E-09	Reject	< 0.0001
PERS	None	4.03E-05	3.07E-09	Reject	< 0.0001
Aggregated				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
SUM	N/A	0.0058	0.0001	Reject	< 0.0001
OPS	N/A	0.0086	0.0002	Reject	< 0.0001
LOC	N/A	0.0104	0.0002	Reject	< 0.0001
PERS	N/A	0.0072	0.0001	Reject	< 0.0001

Table 14: PageRank Centrality Rankings Correlation Comparisons

PageRank					
Schulze				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
OPS	None	1.58E-06	9.00E-12	Reject	< 0.0001
LOC	None	1.37E-06	8.00E-12	Reject	< 0.0001
PERS	None	1.10E-06	6.38E-12	Reject	< 0.0001
None	OPS	8.44E-06	6.11E-11	Reject	< 0.0001
None	LOC	1.43E-05	1.07E-10	Reject	< 0.0001
None	PERS	1.42E-05	1.37E-10	Reject	< 0.0001
Reduced Schulze				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
SUM	None	2.84E-05	6.81E-10	Reject	< 0.0001
OPS	None	3.39E-05	8.05E-10	Reject	< 0.0001
LOC	None	3.14E-05	6.49E-10	Reject	< 0.0001
PERS	None	2.99E-05	8.49E-10	Reject	< 0.0001
Aggregated				Paired Sign Test (alpha = 0.05) H0: median = 0	
Layer weight	Vote weight	Mean Absolute Difference	Variance	Decision	P value
SUM	N/A	4.46E-05	4.68E-09	Reject	< 0.0001
OPS	N/A	4.38E-05	4.74E-09	Reject	< 0.0001
LOC	N/A	4.39E-05	4.85E-09	Reject	< 0.0001
PERS	N/A	5.40E-05	5.00E-09	Reject	< 0.0001

4.2.2.2 Question 6.

Next, Spearman correlations were computed for pairs of rankings to test whether significant correlations exist between the unweighted Schulze composite ranking and the composite rankings under each of the three vote weight distributions. This comparison was computed for each of the five centrality measures and the results were tabulated and are shown in Table 15 using the same coloring described in the results for Question 5.

The high correlation values within the betweenness centrality block suggested that some assessment of differences in correlation should be produced. To investigate which measures are most susceptible to changes in rankings under different weight distributions, the average group auto-correlation for each centrality measure was computed. This was done by first sorting the values into blocks for each centrality measure and then averaging the upper triangular correlation coefficients for each diagonal block in Table 15. The results are presented in Table 16.

The average values show that betweenness centrality was the least susceptible to changes in rankings under the three vote weight distributions (*location*, *operations*, and *personal ties* weights are abbreviated *Loc*, *Ops*, and *Pers*, respectively in the table) for the time-aggregated network.

Table 15: Correlation Chart for Unweighted and Weighted Schulze Composite Rankings

	Betweenness	Betweenness Loc	Betweenness Ops	Betweenness Pers	Closeness	Closeness Loc	Closeness Ops	Closeness Pers	Eigenvector	Eigenvector Loc	Eigenvector Ops	Eigenvector Pers	Degree	Degree Loc	Degree Ops	Degree Pers	PageRank	PageRank Loc	PageRank Ops	PageRank Pers
Betweenness		0.99	0.97	0.98	0.88	0.88	0.77	0.86	0.80	0.78	0.77	0.82	0.80	0.82	0.78	0.85	0.82	0.84	0.78	0.81
Betweenness Loc	0.99		0.97	0.98	0.88	0.89	0.78	0.86	0.82	0.80	0.78	0.82	0.82	0.83	0.80	0.85	0.84	0.86	0.80	0.82
Betweenness Ops	0.97	0.97		0.95	0.84	0.85	0.76	0.83	0.77	0.75	0.73	0.79	0.77	0.79	0.75	0.81	0.79	0.81	0.75	0.78
Betweenness Pers	0.98	0.98	0.95		0.87	0.87	0.79	0.87	0.81	0.80	0.78	0.85	0.81	0.84	0.80	0.85	0.84	0.88	0.80	0.83
Closeness	0.88	0.88	0.84	0.87		0.82	0.75	0.76	0.77	0.73	0.77	0.78	0.82	0.80	0.79	0.78	0.77	0.81	0.80	0.78
Closeness Loc	0.88	0.89	0.85	0.87	0.82		0.80	0.76	0.79	0.84	0.79	0.74	0.82	0.84	0.80	0.77	0.80	0.80	0.77	0.78
Closeness Ops	0.77	0.78	0.76	0.79	0.75	0.80		0.65	0.81	0.81	0.75	0.72	0.81	0.74	0.72	0.75	0.72	0.80	0.71	0.70
Closeness Pers	0.86	0.86	0.83	0.87	0.76	0.76	0.65		0.68	0.70	0.65	0.75	0.68	0.73	0.68	0.75	0.73	0.74	0.67	0.73
Eigenvector	0.80	0.82	0.77	0.81	0.77	0.79	0.81	0.68		0.87	0.79	0.80	0.91	0.76	0.83	0.81	0.84	0.85	0.84	0.77
Eigenvector Loc	0.78	0.80	0.75	0.80	0.73	0.84	0.81	0.70	0.87		0.86	0.74	0.89	0.83	0.84	0.80	0.90	0.81	0.85	0.76
Eigenvector Ops	0.77	0.78	0.73	0.78	0.77	0.79	0.75	0.65	0.79	0.86		0.70	0.81	0.91	0.92	0.76	0.83	0.83	0.83	0.72
Eigenvector Pers	0.82	0.82	0.79	0.85	0.78	0.74	0.72	0.75	0.80	0.74	0.70		0.83	0.73	0.74	0.78	0.80	0.81	0.77	0.78
Degree	0.80	0.82	0.77	0.81	0.82	0.82	0.81	0.68	0.91	0.89	0.81	0.83		0.79	0.85	0.82	0.88	0.82	0.82	0.81
Degree Loc	0.82	0.83	0.79	0.84	0.80	0.84	0.74	0.73	0.76	0.83	0.91	0.73	0.79		0.88	0.75	0.86	0.83	0.81	0.73
Degree Ops	0.78	0.80	0.75	0.80	0.79	0.80	0.72	0.68	0.83	0.84	0.92	0.74	0.85	0.88		0.80	0.89	0.83	0.86	0.80
Degree Pers	0.85	0.85	0.81	0.85	0.78	0.77	0.75	0.75	0.81	0.80	0.76	0.78	0.82	0.75	0.80		0.81	0.86	0.84	0.85
PageRank	0.82	0.84	0.79	0.84	0.77	0.80	0.72	0.73	0.84	0.90	0.83	0.80	0.88	0.86	0.89	0.81		0.85	0.89	0.80
PageRank Loc	0.84	0.86	0.81	0.88	0.81	0.80	0.80	0.74	0.85	0.81	0.83	0.81	0.82	0.83	0.83	0.86	0.85		0.82	0.76
PageRank Ops	0.78	0.80	0.75	0.80	0.80	0.77	0.71	0.67	0.84	0.85	0.83	0.77	0.82	0.81	0.86	0.84	0.89	0.82		0.81
PageRank Pers	0.81	0.82	0.78	0.83	0.78	0.78	0.70	0.73	0.77	0.76	0.72	0.78	0.81	0.73	0.80	0.85	0.80	0.76	0.81	

Table 16: Average Group Auto-Correlations for Five Centrality Measures

Centrality Measure	Average Group Auto-Correlation
Betweenness	0.97
Closeness	0.76
Eigenvector	0.79
Degree	0.82
PageRank	0.82

The next least-susceptible measures are degree and PageRank centralities, which are tied with an average group auto-correlation of 0.82. Thus while it is clear that significant correlations were observed between unweighted Schulze composite rankings and the composite rankings under each vote weight distribution, there is a distinct difference in the level of correlation between betweenness centrality and the other four centrality measures.

This notably higher resistance to changes in rankings found in the betweenness centrality composite rankings led to a modification of this methodology. Betweenness centrality was removed from consideration and the correlations were recomputed. The results with betweenness centrality excluded were then compared with the original tests which do contain betweenness centrality to note the differences, if any.

4.2.2.3 Question 7.

Next, the question was asked whether an overall Schulze composite ranking correlates with its component composite rankings for each centrality measure. The correlation chart depicted in Table 17 shows that all overall composite rankings are correlated regardless of vote weight distribution used. Additionally, the overall composite rankings under the *operations* weight distribution is least susceptible to inclusion and exclusion of the betweenness centrality. In contrast, the unweighted overall composite rankings are most susceptible to changes when betweenness centrality is added or removed.

Table 17: Correlations of Overall Schulze Composite Rankings with and without Betweenness Centrality

	Overall	Overall Loc	Overall Ops	Overall Pers	Overall w/o bet	Overall Loc w/o Bet	Overall Ops w/o Bet	Overall Pers w/o Bet
Overall		0.91	0.79	0.80	0.76	0.82	0.79	0.78
Overall Loc	0.91		0.83	0.81	0.83	0.81	0.83	0.81
Overall Ops	0.79	0.83		0.82	0.89	0.87	0.96	0.80
Overall Pers	0.80	0.81	0.82		0.83	0.79	0.81	0.90
Overall w/o bet	0.76	0.83	0.89	0.83		0.90	0.90	0.82
Overall Loc w/o Bet	0.82	0.81	0.87	0.79	0.90		0.87	0.80
Overall Ops w/o Bet	0.79	0.83	0.96	0.81	0.90	0.87		0.79
Overall Pers w/o Bet	0.78	0.81	0.80	0.90	0.82	0.80	0.79	

The preceding sections addressed the specific statistical questions of interest. It demonstrated that significant differences exist between rankings using the Schulze composite method and other methods under the variety of tested conditions. These results say little about the actual utility of the Schulze composite method. They are different, but that does not necessarily mean they are as good or better. To help to answer whether the Schulze composite method is as good or better than the typical aggregated network approach—among other questions—qualitative comparisons are conducted and discussed in the following section.

4.2.3 Qualitative Questions.

4.2.3.2 Question 9.

Finally, the question was asked whether the sets of nodes placed in the top 20 ranking positions under the various ranking composite methodologies are similar and whether their relative standings are similar. This qualitative assessment was conducted on the time-aggregated full 12 layer multiplex network and the fully-aggregated network. It was done by first arraying the top 20 node identifiers along with their tied rank values as described in section 3.10.

The first comparison is between the Schulze composite rankings and the fully-aggregated rankings for each of the five centrality measures. This is summarized in Table 18.

Table 18: Comparison of Top 20 Ranked Nodes - Composite versus Aggregated

	Betweenness				Closeness				Eigenvector				Degree				PageRank			
	Composite		Aggregated		Composite		Aggregated		Composite		Aggregated		Composite		Aggregated		Composite		Aggregated	
Absolute Rank	Tied Rank	Node ID	Node ID	Tied Rank	Tied Rank	Node ID	Node ID	Tied Rank	Tied Rank	Node ID	Node ID	Tied Rank	Tied Rank	Node ID	Node ID	Tied Rank	Tied Rank	Node ID	Node ID	Tied Rank
139	139	105	105	139	*138.5*	43	43	139	139	43	123	139	*138.5*	43	43	139	*138.5*	43	105	139
138	138	1	43	138	*138.5*	105	105	138	138	105	45	138	*138.5*	105	105	138	*138.5*	105	43	138
137	137	2	80	137	137	23	45	*136.5*	137	23	43	137	137	23	45	137	137	80	45	137
136	136	3	18	136	136	80	123	*136.5*	136	1	124	136	136	80	123	136	136	23	123	136
135	135	4	45	135	135	125	23	*133.5*	*134.5*	2	115	135	135	119	80	135	135	82	80	135
134	134	5	82	134	134	6	80	*133.5*	*134.5*	82	23	134	134	6	23	*133*	134	10	125	134
133	133	6	8	133	133	10	124	*133.5*	133	3	125	133	133	10	124	*133*	133	125	23	133
132	132	7	19	132	131	15	125	*133.5*	132	4	78	132	*131.5*	82	125	*133*	*131.5*	6	7	132
131	131	8	28	131	131	82	7	*130.5*	131	7	105	131	*131.5*	125	7	131	*131.5*	119	124	131
130	130	9	7	130	131	107	115	*130.5*	*129.5*	5	7	130	*129.5*	15	115	130	*128.5*	15	115	130
129	129	10	125	129	127	1	78	129	*129.5*	131	128	129	*129.5*	78	78	129	*128.5*	18	19	129
128	128	11	131	128	127	19	19	128	*127*	8	80	128	*127*	19	19	128	*128.5*	78	78	128
127	127	12	6	127	127	78	128	127	*127*	10	63	127	*127*	107	82	127	*128.5*	131	82	127
126	126	13	23	126	127	119	28	*125*	*127*	80	92	126	*127*	131	77	126	126	1	77	126
125	125	14	121	125	127	131	77	*125*	125	9	74	125	125	1	121	*124.5*	125	7	18	125
124	124	15	77	124	124	2	82	*125*	124	11	82	124	124	2	128	*124.5*	124	2	121	124
123	123	16	123	123	123	3	63	123	123	119	121	123	123	3	63	123	123	8	28	123
122	122	17	21	122	122	5	18	122	122	15	19	122	122	5	28	122	122	9	128	122
121	121	18	14	121	121	7	121	121	121	17	126	121	121	7	18	121	*120.5*	3	63	121
120	120	19	69	120	120	4	119	120	*119.5*	6	52	120	120	18	74	*119.5*	*120.5*	19	8	120

As seen in Table 18, the rankings produced by the Schulze composite method differ from those produced using the aggregated network, for each centrality measure. Betweenness centrality under the Schulze composite method produces essentially a listing in rank order by Node ID, except for node 105 at the top rank position. This is unexpected, especially given the conclusion that the aggregated betweenness ranking and the Schulze composite betweenness ranking are not statistically significantly different as was reported in Table 5. This anomaly also informed the decision to alter the methodology to compute rankings both including and excluding betweenness centrality for comparison.

The remaining four centrality measures illustrated some notable differences and similarities. The Schulze method placed nodes 43 and 105 in the top positions for all centrality measures. The aggregated method did so for all measures except eigenvector centrality where node 105 (Noordin Top himself) is listed in the ninth position. Assuming that Noordin Top is likely to be highly connected to people who are also highly connected—the measurement pertaining to eigenvector centrality—the Schulze composite method has arguably outperformed the standard aggregated method in this case.

Another noticeable difference was the inclusion of node 131 in the top 20 nodes list for all measures when the Schulze composite method is used. Node 131 is Usman Bin Sef and was listed as a key enabler for the network within the dataset; he was also mentioned by the International Crisis Group as a leader of the East Java Wakalah, and the one who helped hide Noordin Mohammed Top (International Crisis Group, 2007).

The aggregated method placed node 45 consistently in the top five positions, whereas the Schulze composite method excluded node 45 from the top 20 positions in all cases. Node 45 is Chandra and no additional information was found on his importance to the network, but he was listed as a fighter in the dataset.

The next comparison was between the overall Schulze composite rankings on the full multiplex, the overall Schulze composite rankings on the reduced multiplex and the composited fully-aggregated rankings. Results are shown in Table 19 both with and without betweenness centrality.

Table 19: Comparison of Top 20 Ranked Nodes - Overall Full, Reduced, Aggregated with and without Betweenness

	All Measures						Without Betweenness					
	Overall Composite Full		Overall Composite Aggregate		Overall Composite Reduced		Overall Composite Full		Overall Composite Aggregate		Overall Composite Reduced	
Absolute Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank	Nod e ID	Tied Rank	Node ID	Tied Rank
1	43	139	43	139	80	139	43	139	43	139	105	139
2	105	138	105	138	105	138	105	138	105	138	80	138
3	23	137	45	137	15	137	23	137	45	137	15	137
4	80	136	123	136	43	136	80	136	123	136	43	136
5	6	135	80	135	125	135	10	135	23	*134.5*	125	135
6	10	134	23	134	1	134	82	134	80	*134.5*	1	134
7	82	133	124	133	2	133	6	133	124	133	2	133
8	1	132	125	132	3	132	119	132	125	132	3	132
9	15	131	7	131	4	131	125	131	7	131	4	131
10	119	130	115	130	5	130	15	130	115	130	5	130
11	125	129	78	129	7	129	1	129	78	129	7	129
12	2	128	19	128	8	128	78	128	19	128	8	128
13	3	127	82	127	6	127	131	127	82	127	6	127
14	78	126	77	126	9	126	2	126	77	126	9	*125*
15	5	125	128	125	10	125	3	125	128	125	10	*125*
16	7	124	18	*122.5*	18	124	19	124	121	124	18	*125*
17	4	123	28	*122.5*	19	123	7	123	28	*122.5*	19	*125*
18	131	122	63	*122.5*	11	122	5	122	63	*122.5*	23	122
19	8	121	121	*122.5*	23	121	4	121	18	121	11	121
20	9	120	119	120	12	120	8	120	119	120	21	*120*

As can be seen in Table 19, the reduced layer network composite ranking reverts to a nearly-ordered list of Node IDs after only the first five top-ranked nodes and node 43 drops by four positions, though it is listed as number one in all other cases. This suggests that a substantial amount of relevant information was lost during the layer reduction process resulting in a subsequent loss of relevant rankings for the nodes. In addition, this result for the reduced multiplex network is nearly the same both with and without betweenness centrality rankings included.

In contrast, many more relevant actors are included in the top 20 ranks for both the overall Schulze composite and overall Schulze aggregated rankings. Further, the distribution of nodes is qualitatively somewhat similar. Both list nodes 43 and 105—Azahari Husein and Noordin Mohammad Top—as the two top nodes. These were two of the most prominent figures within the Noordin Top network. This result is true both with and without betweenness centrality.

Owing to the behavior of the Schulze method on the betweenness centrality rankings, inclusion of betweenness centrality may be skewing the overall composite. Thus the overall composite was computed without considering composite betweenness centrality ranking as a component. This does alter the final list of top 20 nodes. Notably, node 131 (Usman Bin Sef) moves up in the list by five positions.

Next, a comparison is shown between the unweighted composite rankings, the composite rankings for the three vote weight distributions, the standard aggregated rankings, and the composite reduced rankings. This is repeated for each of the five centrality measures and the results are given in Table 20 through Table 24, respectively.

Table 20: Comparison of Top 20 Ranked Nodes - Composite of Full Unweighted and Weighted, Aggregated and Composite Reduced – Betweenness Centrality

	Unweighted		Weighted						Unweighted			
	Composite Full		Composite Full OPS		Composite Full LOC		Composite Full PERS		Aggregate		Composite Reduced	
Absolute Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank
1	105	139	105	139	105	139	105	139	105	139	1	139
2	1	138	1	138	43	138	23	138	43	138	2	138
3	2	137	2	137	1	137	80	137	80	137	3	137
4	3	136	3	136	2	136	43	136	18	136	4	136
5	4	135	4	135	3	135	1	135	45	135	5	*134.5*
6	5	134	5	134	4	134	2	134	82	134	6	*134.5*
7	6	133	6	133	5	133	3	133	8	133	7	133
8	7	132	7	132	6	132	4	132	19	132	8	132
9	8	131	8	131	7	131	5	131	28	131	9	131
10	9	130	9	130	8	130	6	130	7	130	10	130
11	10	129	10	129	9	129	7	129	125	129	11	129
12	11	128	11	128	10	128	8	128	131	128	12	128
13	12	127	12	127	11	127	9	127	6	127	13	127
14	13	126	13	126	12	126	10	126	23	126	14	126
15	14	125	14	125	13	125	11	125	121	125	15	125
16	15	124	15	124	14	124	12	124	77	124	16	124
17	16	123	16	123	15	123	13	123	123	123	17	123
18	17	122	17	122	16	122	14	122	21	122	18	122
19	18	121	18	121	17	121	15	121	14	121	19	121
20	19	120	19	120	18	*119.5*	16	120	69	120	20	120

Table 20 lists the weighted versus unweighted composite rankings for betweenness centrality. As before, the Schulze method suffers in performance when considering the betweenness centrality rankings. As can be seen, the rankings are essentially ordered lists of the node IDs, with the most diversity seen under the personal ties weight distribution. The Schulze composite rankings for the reduced multiplex is a node ID-ordered list with the exception of the tied values for nodes five and six. Thus it seems once again that the Schulze method has failed to produce meaningful results for betweenness centrality.

Table 21: Comparison of Top 20 Ranked Nodes - Composite of Full Unweighted and Weighted, Aggregated and Composite Reduced – Closeness Centrality

	Unweighted		Weighted						Unweighted			
	Composite Full		Composite Full OPS		Composite Full LOC		Composite Full PERS		Aggregate		Composite Reduced	
Absolute Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank
1	43	138.5	43	139	43	138.5	105	139	43	139	80	138.5
2	105	138.5	105	138	105	138.5	23	138	105	138	105	138.5
3	23	137	23	137	23	137	125	137	45	136.5	15	137
4	80	136	80	136	80	136	43	136	123	136.5	43	136
5	125	135	131	135	125	135	80	135	23	133.5	125	135
6	6	134	10	134	6	133	82	134	80	133.5	1	133.5
7	10	133	125	133	119	133	6	133	124	133.5	2	133.5
8	15	131	82	132	131	133	10	131.5	125	133.5	3	132
9	82	131	6	130	107	131	44	131.5	7	130.5	4	131
10	107	131	78	130	1	130	15	130	115	130.5	5	130
11	1	127	119	130	10	129	19	129	78	129	7	129
12	19	127	107	128	15	127.5	30	127	19	128	19	128
13	78	127	1	127	82	127.5	78	127	128	127	23	127
14	119	127	15	126	19	125.5	119	127	28	125	8	126
15	131	127	19	124.5	78	125.5	1	124.5	77	125	6	124.5
16	2	124	28	124.5	2	124	31	124.5	82	125	41	124.5
17	3	123	18	123	3	123	2	123	63	123	9	122.5
18	5	122	2	122	4	122	3	122	18	122	88	122.5
19	7	121	3	121	5	121	5	121	121	121	10	120
20	4	120	4	120	7	120	9	120	119	120	18	120

The difficulties noted for betweenness centrality do not exist with the remaining centrality measures. In fact, the rankings appear to have included many key figures from the network. Table 21 lists the results for closeness centrality and it can be seen that the three vote weight distributions produce notable changes in the rankings. When operations and location are emphasized, we see that nodes 43 and 105 remain in the top positions.

When personal ties are emphasized, however, node 43 moves down 3 positions. Node 43 (Azahari Husein) was the head of operations and the chief bomb-maker for the network. It makes sense that *operations* and *location* weights would not alter his position while the *personal ties* weights might.

Also in the case of the personal ties weight distribution, node 125 rises to the third highest position and is Ubeid, Noordin Mohammed Top's courier. Node 23 rises to second position and is Ahmad Rofiq Ridho, though very little additional information was found on his identity or importance except that he is also listed as a courier in the dataset.

Similar results hold for the remainder of the network centrality measures shown in Table 22, Table 23, and Table 24. This shows that vote weight distributions can greatly influence the final ranking of nodes within a multiplex network. It strongly suggests that meaningful changes occur based on the actual distribution of weights applied. This makes sense as the integer weights can be thought of as a number of additional voters who agree on the node rankings. The more weight that is given, the more the composite ranking will tend toward the weighted ranking preference.

Table 22: Comparison of Top 20 Ranked Nodes - Composite of Full Unweighted and Weighted, Aggregated and Composite Reduced – Eigenvector Centrality

	Unweighted		Weighted						Unweighted			
	Composite Full		Composite Full OPS		Composite Full LOC		Composite Full PERS		Aggregate		Composite Reduced	
Absolute Rank	**Node ID**	**Tied Rank**	**Node ID**	**Tied Rank**	**Node ID**	**Tied Rank**	**Node ID**	**Tied Rank**	**Node ID**	**Tied Rank**	**Node ID**	**Tied Rank**
1	43	139	43	139	43	139	105	139	123	139	43	139
2	105	138	105	138	105	138	43	138	45	138	105	138
3	23	137	23	137	23	137	82	137	43	137	1	*136.5*
4	1	136	131	136	1	*135.5*	1	136	124	136	2	*136.5*
5	2	*134.5*	80	135	82	*135.5*	2	135	115	135	3	135
6	82	*134.5*	82	134	2	134	3	134	23	134	4	134
7	3	133	10	133	3	*132.5*	4	133	125	133	5	*132.5*
8	4	132	119	132	131	*132.5*	5	*131.5*	78	132	6	*132.5*
9	7	131	2	131	4	131	7	*131.5*	105	131	7	131
10	5	*129.5*	3	*128.5*	5	130	8	*129.5*	7	130	8	130
11	131	*129.5*	6	*128.5*	7	129	23	*129.5*	128	129	9	129
12	8	*127*	7	*128.5*	10	128	9	128	80	128	10	128
13	10	*127*	15	*128.5*	6	*126.5*	11	127	63	127	11	*126.5*
14	80	*127*	79	126	8	*126.5*	6	*125.5*	92	126	15	*126.5*
15	9	125	83	125	9	125	12	*125.5*	74	125	13	*124.5*
16	11	124	4	*123.5*	11	124	10	*123.5*	82	124	18	*124.5*
17	119	123	5	*123.5*	12	*122.5*	13	*123.5*	121	123	14	123
18	15	122	8	122	119	*122.5*	14	122	19	122	17	122
19	17	121	9	121	13	121	17	121	126	121	19	121
20	6	*119.5*	11	120	14	*119.5*	30	*118.5*	52	120	23	120

Table 23: Comparison of Top 20 Ranked Nodes - Composite of Full Unweighted and Weighted, Aggregated and Composite Reduced – Degree Centrality

	Unweighted		Weighted						Unweighted			
	Composite Full		Composite Full OPS		Composite Full LOC		Composite Full PERS		Aggregate		Composite Reduced	
Absolute Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank
1	43	*138.5*	43	139	43	*138.5*	105	139	43	139	80	139
2	105	*138.5*	105	138	105	*138.5*	43	138	105	138	105	138
3	23	137	23	137	80	137	23	137	45	137	15	*136.5*
4	80	136	80	136	6	*135.5*	80	136	123	136	43	*136.5*
5	119	135	119	135	23	*135.5*	82	135	80	135	125	135
6	6	134	10	134	10	134	125	134	23	*133*	1	*133.5*
7	10	133	125	133	119	*132.5*	6	133	124	*133*	2	*133.5*
8	82	*131.5*	6	*130.5*	125	*132.5*	10	*131.5*	125	*133*	3	*131.5*
9	125	*131.5*	78	*130.5*	15	131	119	*131.5*	7	131	19	*131.5*
10	15	*129.5*	82	*130.5*	78	*129*	78	130	115	130	4	130
11	78	*129.5*	131	*130.5*	82	*129*	15	129	78	129	88	129
12	19	*127*	107	128	131	*129*	19	128	19	128	110	128
13	107	*127*	15	127	1	127	1	127	82	127	5	127
14	131	*127*	18	*125.5*	2	126	2	126	77	126	7	126
15	1	125	19	*125.5*	3	125	3	125	121	*124.5*	8	125
16	2	124	28	124	4	124	5	124	128	*124.5*	6	124
17	3	123	1	*122.5*	5	123	7	123	63	123	9	123
18	5	122	79	*122.5*	7	122	9	*121*	28	122	10	122
19	7	121	2	121	9	121	30	*121*	18	121	18	121
20	18	120	3	120	18	120	31	*121*	74	*119.5*	11	119

Table 24: Comparison of Top 20 Ranked Nodes - Composite of Full Unweighted and Weighted, Aggregated and Composite Reduced – PageRank Centrality

	Unweighted		Weighted						Unweighted			
	Composite Full		Composite Full OPS		Composite Full LOC		Composite Full PERS		Aggregate		Composite Reduced	
Absolute Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank	Node ID	Tied Rank
1	43	138.5	43	139	43	138.5	105	139	105	139	105	139
2	105	138.5	105	138	105	138.5	43	137.5	43	138	80	138
3	80	137	80	137	80	137	80	137.5	45	137	15	136.5
4	23	136	23	136	125	136	23	136	123	136	43	136.5
5	82	135	119	135	23	135	125	135	80	135	125	135
6	10	134	82	133.5	82	134	82	134	125	134	1	134
7	125	133	125	133.5	10	133	6	133	23	133	2	133
8	6	131.5	10	132	6	131	10	131.5	7	132	3	132
9	119	131.5	131	131	15	131	119	131.5	124	131	4	131
10	15	128.5	78	130	119	131	7	129.5	115	130	5	130
11	18	128.5	6	128.5	19	128	78	129.5	19	129	7	129
12	78	128.5	15	128.5	39	128	8	128	78	128	18	128
13	131	128.5	18	126.5	78	128	9	125.5	82	127	23	127
14	1	126	19	126.5	1	126	15	125.5	77	126	8	125.5
15	7	125	28	125	2	125	18	125.5	18	125	19	125.5
16	2	124	79	123.5	3	124	28	125.5	121	124	6	123.5
17	8	123	107	123.5	4	123	19	122	28	123	28	123.5
18	9	122	1	122	5	122	58	122	128	122	39	121
19	3	120.5	2	121	7	121	122	122	63	121	63	121
20	19	120.5	3	119.5	9	120	1	120	8	120	82	121

By looking at the previous tables, it was observed that tied values seem to cause problems with the quality of the rankings when the Schulze method is used. This was first thought to be limited to betweenness centrality, but in fact was present in all measures to varying degrees.

In the case of this methodology, node-alignment is required, and for this data set, this causes a large number of isolated nodes to be forced into many of the layers. These isolated nodes are given the same value within each of the centrality measures, which results in tied ranking values. To check if such isolated nodes were indeed the problem, random networks were generated for testing.

The prescribed node degree connected graph (PNDCG) generation algorithm was used to create 12 new network layers using the degree distributions of the original 12 networks (Morris, O'Neal, & Deckro, 2014). The PNDCG algorithm produces fully connected networks. These 12 new networks were processed using the Schulze method and the resulting rankings were observed. It appeared that the tied value effects were eliminated as there was no observable numeric ordering dominance. Thus, isolated nodes seem to be the source of the problem.

Unfortunately, the outputs of the PNDCG algorithm, while representative of the original networks, are still random networks; the algorithm was used to verify the problem, but cannot be used to fix the problem. Instead, a different method was needed which would adjust for the isolated node effects on the original data. Additional weighting methods were explored as a potential mitigating solution.

First, the vote weights were adjusted according to the density of the network layers. Density is defined as the ratio of the number of edges in a network to the total possible number of edges were the network fully connected. The rationale was that a greater proportion of overall density contained within a layer implies a greater proportion of information contributed to the multilayer network by that layer. The total density for the multilayer network was computed by adding the densities of all 12 layers. The density proportion for each layer was calculated as the ratio of the density for a layer to the total density of the multilayer network.

100 votes were then allocated to each layer according to these proportions. This new assignment of vote weights was treated as the baseline weight values. For comparison purposes, the original three weight distributions were then each added to this baseline to produce three augmented weights. This appeared to improve the performance of the Schulze algorithm, but a second weighting adjustment was also attempted.

The second set of weights were applied in an attempt to adjust for the number of isolated nodes directly. The vote weight for each layer was given the value of the number of connected nodes within that layer. This new distribution of vote weights was once again treated as a baseline weighting and was augmented by each of the original three layer weight distributions for comparison. This appeared to improve the performance of the Schulze method even further.

To better assess the following comparisons, a list of important actors for the Noordin Top network was compiled. This list was based primarily on a reading of one of the sources used to build the data sets: the report on the state of the Noordin network by the International Crisis Group (International Crisis Group, 2007). This resulted in the qualitative identification of 24 important actors in the network, based on mentions of their involvement in key operations, leadership status, or direct links to Noordin Mohammed Top. This list is given in Table 25.

Table 25: List of Identified Important Actors in Noordin Network

1	Abdul Aziz
7	Abdullah Sungkar
8	Abu Bakar Ba'asyir
9	Abu Dujanah
13	Abu Rusdan
21	Agus Suryanto
23	Ahmad Rofiq Ridho
31	Ali Imron
32	Amrozi
43	**Azhari Husin**
44	Baharudin Soleh
52	Dulmatin
62	Hambali
63	Hari Kuncoro
74	Imam Samudra
80	Iwan Dharmawan
82	Jabir
105	**Noordin Mohammed Top**
119	Son Hadi
121	Subur Sugiarto
125	Ubeid
128	Umar Patek
131	Usman bin Sef
139	Zulkarnaen

Noordin Mohammed Top and Azahari Husein were two of the most prominent members of the Noordin network and are bolded. The numbers beside the names are the identifiers used in Figure 7 and Figure 8, which follow, and should not be mistaken for the ranking of each actor in the network.

	Connected Weights						Density Weights						Normal Schulze						Measures on Aggregate					
	All Measures	Betweenness	Closeness	Degree	Eigenvector	PageRank	All Measures	Betweenness	Closeness	Degree	Eigenvector	PageRank	All Measures	Betweenness	Closeness	Degree	Eigenvector	PageRank	All Measures	Betweenness	Closeness	Degree	Eigenvector	PageRank
1	105	105	105	105	43	105	80	80	80	43	43	80	43	105	43	43	43	43	105	105	43	43	123	105
2	43	80	43	43	105	43	43	23	43	80	105	43	105	1	105	105	105	105	43	43	105	105	45	43
3	23	23	23	23	23	23	105	131	125	105	80	119	23	2	23	23	23	80	80	80	45	45	43	45
4	80	131	80	80	131	80	23	1	105	23	131	105	80	3	80	80	1	23	18	18	123	123	124	123
5	131	1	82	131	80	131	119	2	23	119	18	10	6	4	125	119	2	82	45	45	23	80	115	80
6	6	2	131	119	82	119	131	3	131	131	10	18	10	5	6	6	82	10	82	82	80	23	23	125
7	82	3	125	6	7	125	10	4	18	125	23	23	82	6	10	10	3	125	8	8	124	124	125	23
8	119	4	6	10	10	6	125	5	119	6	15	131	1	7	15	82	4	6	19	19	125	125	78	7
9	10	5	119	18	18	82	15	6	6	10	79	125	15	8	82	125	7	119	28	28	7	7	105	124
10	7	6	10	82	119	10	18	7	7	18	7	6	119	9	107	15	5	15	7	7	115	115	7	115
11	18	7	28	125	15	18	1	8	82	15	83	7	125	10	1	78	131	18	125	125	78	78	128	19
12	28	8	107	28	79	28	2	9	10	7	2	28	2	11	19	19	8	78	131	131	19	19	80	78
13	125	9	7	7	1	107	6	10	15	28	3	130	3	12	78	107	10	131	6	6	128	82	63	82
14	1	10	9	15	2	7	7	11	79	79	1	78	78	13	119	131	80	1	23	23	28	77	92	77
15	2	28	19	78	3	9	28	12	130	83	4	15	5	14	131	1	9	7	121	121	77	121	74	18
16	9	43	18	107	4	62	3	13	1	82	78	79	7	15	2	2	11	2	77	77	82	128	82	121
17	78	11	30	9	5	78	4	14	19	130	8	83	4	16	3	3	119	8	123	123	63	63	121	28
18	3	12	15	17	8	15	5	15	28	78	9	1	131	17	5	5	15	9	21	21	18	28	19	128
19	4	13	44	34	9	17	8	16	83	1	82	2	8	18	7	7	17	3	14	14	121	18	126	63
20	15	14	130	36	11	19	9	17	2	2	5	3	9	19	4	18	6	19	69	69	119	74	52	8

Figure 7: Identification of Important Nodes: Patterns by Ranking Approach

Figure 7 shows four columns each with six sub-columns of rankings. The rankings which are colored yellow correspond to node identifiers listed in Table 25. The ranking position can be seen on the left of the figure, valued one through 20. The right-most column shows the rankings for each centrality measure measured on the fully-aggregated binary (unweighted) network. The *All Measures* column under the *Measures on Aggregate* heading refers to the Schulze method overall composite ranking of the five centrality measures' rankings on the fully-aggregated binary network.

The *Normal Schulze* heading refers to the Schulze composite rankings as computed on the full, unweighted multiplex network. The subheadings refer to the rankings by the five centrality measures considered, and the *All Measures* subheading is the overall Schulze composite of the five centrality measures' rankings.

The *Density Weights* heading refers to the Schulze composite rankings when the baseline weights were built according to the proportion of overall density found in each layer. Finally, the *Connected Weights* heading refers to the Schulze composite rankings when the baseline weights were adjusted according to the number of isolated components in the network.

As can be seen, the concentration of the yellow node identifiers within the top of the rankings increases as one views the columns from left to right. This suggests that the use of the *Connected Weights* provides a better assessment of node importance than do the others. Additionally, the numeric ordering observed with the bolded, boxed node identifiers decreases in size. This suggests that by emphasizing the layers with fewer isolated nodes, the Schulze method is able to better identify a quality ranking of nodes. It can also be observed that the rankings computed on the aggregated networks show the least concentration of the important node identifiers toward the top of the ranking lists.

The *Connected Weights* were then augmented by the original three weight distributions from Table 3 and the comparison is depicted in Figure 8. It shows that the relative concentration of the yellow node identifiers (important nodes in the network) does not vary much with the additional use of the three original weight distributions. However, some changes do occur. For example, under *Connected PERS* (the adjustment for the *personal ties* weight distribution), node 6 is moved up the list. Upon seeing such movement, one could look more deeply at why such a move might occur when the layers are emphasized according to the *personal ties* weights.

	Connected Weights						Connected OPS						Connected LOC						Connected PERS					
	All Measures	Betweenness	Closeness	Degree	Eigenvector	PageRank	All Measures	Betweenness	Closeness	Degree	Eigenvector	PageRank	All Measures	Betweenness	Closeness	Degree	Eigenvector	PageRank	All Measures	Betweenness	Closeness	Degree	Eigenvector	PageRank
1	105	105	105	105	43	105	43	105	43	43	43	43	43	105	43	43	43	43	105	105	105	105	105	105
2	43	80	43	43	105	43	105	80	105	105	105	105	105	80	105	105	105	105	43	23	23	23	43	80
3	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	80	23	80	82	80	23	43
4	80	131	80	80	131	80	80	131	80	80	131	80	80	131	80	80	131	23	80	6	125	43	82	23
5	131	1	82	131	80	131	131	1	131	131	80	131	131	1	131	131	80	119	82	43	43	6	1	119
6	6	2	131	119	82	119	119	2	119	119	82	119	119	2	119	119	82	131	125	1	80	119	2	125
7	82	3	125	6	7	125	6	3	82	125	10	125	82	3	125	125	10	125	119	2	6	10	3	6
8	119	4	6	10	10	6	10	4	125	6	18	6	125	4	10	6	119	82	6	10	10	82	4	82
9	10	5	119	18	18	82	18	5	6	10	7	82	1	5	82	10	15	10	10	9	30	125	7	10
10	7	6	10	82	119	10	28	6	10	82	119	10	2	6	6	82	79	6	7	3	9	78	8	18
11	18	7	28	125	15	18	82	7	28	18	79	18	3	43	107	18	83	18	131	**4**	96	9	94	131
12	28	8	107	28	79	28	7	8	107	28	15	28	4	**7**	15	15	1	7	1	**5**	107	18	131	7
13	125	9	7	7	1	107	107	9	18	107	125	107	6	**8**	19	28	2	9	2	**7**	131	17	17	28
14	1	10	9	15	2	7	125	10	7	15	2	7	5	**9**	28	107	3	130	9	**8**	44	28	80	8
15	2	28	19	78	3	9	1	28	19	78	6	9	7	**10**	78	78	4	28	18	**11**	18	34	10	9
16	9	43	18	107	4	62	2	43	9	7	45	78	8	**11**	1	79	5	78	15	**12**	119	36	30	62
17	78	**11**	30	9	5	78	15	**11**	30	19	78	15	9	**12**	18	83	7	15	28	**13**	28	53	5	78
18	3	**12**	15	17	8	15	19	**12**	78	9	83	19	10	**13**	130	1	6	79	3	**14**	15	73	11	17
19	4	**13**	44	34	9	17	78	**13**	130	79	3	79	15	**14**	2	130	8	83	4	**15**	73	15	12	34
20	15	**14**	130	36	11	19	9	**14**	1	30	4	30	18	**15**	3	2	9	1	78	**16**	19	131	13	31

Figure 8: Identification of Important Nodes: Patterns with Additional Weights

4.3 Summary

The preceding chapter discussed results of the analysis of the Noordin Top dataset using the methodologies proposed in this thesis. The next chapter sums up the findings and draws some conclusions, and offers items for future research efforts.

V. Conclusions and Recommendations

5.1 Chapter Overview

This chapter presents conclusions, recommendations for action, a review of the significance of the findings of this research, and an identification of any methodological shortcomings and/or ideas for future research.

5.2 Conclusions of Research

Application of the Schulze voting method to the ranking of nodes on a multiplex network yields meaningful rankings that are different from rankings derived through the standard practice of aggregating the network layers prior to computing centrality measures. Both statistically significant differences and correlations were observed under a variety of conditions and across all centrality measures.

Vote weight distributions are potentially a useful way to elicit information from a multiplex network pertaining to rankings of critical nodes. When certain layers are emphasized—given a greater number of votes—different ranking outcomes are produced which seem to align with which layers were emphasized.

Betweenness centrality produced several unexpected and even contradictory results when the Schulze composite method was applied. In fact, the method is sensitive to tied values resulting from isolated nodes in the network. This was ameliorated to some degree through the use of weight adjustments. These adjustments seemed to produce final rankings which listed important actors at the top of the ranking list.

When rankings were computed using the Schulze composite method for the time-stamped network, the time blocks of highly correlated rankings seemed to be more well-defined than for the case of the aggregated time-stamped network. These blocks aligned reasonably well with known events. Firm conclusions are difficult to draw on the quality of alignment as lag effects are unknown.

At least in the case of the network data set considered (and using the proposed Schulze composite method), reducing the data using the Jensen Shannon reduction method proposed in De Domenico *et al.* (2015) resulted in significantly different ranking results under a variety of network centrality measures and with and without layer weights. In some cases these rankings appeared to be less meaningful than rankings derived from the multiplex network without reduction and those derived under the standard approach of fully aggregating the network.

5.3 Significance of Research

This research demonstrated a new methodology for determining nodes of critical importance—in ranked order of importance—for a node-aligned multiplex network. It provided both statistical and qualitative analyses on the differences of ranking outcomes under this methodology. The application of the methodology detailed in this thesis will allow meaningful lists of critical nodes to be produced for a multiplex network. These can also now be produced using the same methodology for any measure of interest or functions thereof. This helps to answer the questions identified within the Joint Concept – *Human Aspects of Military Operations* (Office of the Joint Chiefs of Staff, 2016).

This methodology demonstrated the applicability of the Schulze voting method to network data. However, it is not limited to extensions within social network analysis. It could also be used to aggregate rankings for any set of targets. Thus, for example, it could be used to compile a final target list based on inputs from each service or other agencies during a target review process in a theater of operations.

Additionally, this research demonstrated possible limitations to using the Jensen Shannon layer reduction process. This suggests that reduction of a data set should be considered only when necessary, or only after additional assessments are conducted to determine its appropriateness on a case-specific basis.

5.4 Recommendations for Action

This thesis demonstrated statistical and qualitative differences between the proposed methodology and standard single layer network approaches to ranking nodes within a multiplex network. As a next step, subject matter experts on the Noordin Top network should ideally be consulted to determine if the final rankings produced using this methodology were better than those produced using the standard approach of aggregating the network and then computing rankings.

It is also suggested that this methodology be applied to another dataset for comparison purposes and also with the intent of generating a more thorough qualitative assessment of the outcome by using subject matter expertise.

5.5 Recommendations for Future Research

The methodology presented was based on node-aligned multiplex data. In a multiplex network, the inter-layer connection represents a self-loop, while in a multilayer network, the inter-layer connections are not necessarily self-loops, but could be. The second case allows for a more general class of networks to be represented. A generalization to multilayer models rather than a multiplex model is therefore a useful area of future research.

The node-alignment used in this methodology forced all nodes to be present on each network layer. This resulted in a large number of isolates, especially when considering the already sparse matrices at each time stamp. This led to long tails of tied values within the rankings. The Schulze composite method depends only on relative rankings of a set of nodes. If a node is not given a rank—which would occur if the node were not present on a layer—then it is ranked as zero by default. Thus the Schulze method should be applicable to networks whose layers are not node-aligned and should produce similar results without the need to create node-aligned layers. Verification of this assertion would be a useful future study.

The nature of the method proposed here allows it to function independently of any particular measure chosen. However, some interesting results were observed for betweenness centrality rankings. A future study of interest is whether this effect is an artifact of this data set in particular, or more generally a result of this methodology when it is applied to betweenness centrality rankings.

There are many other ordinal voting methods besides the Schulze method which might be tested within the context of multilayer SNA and production of critical node rankings. There also exist cardinal methods which consider not just the relative ranking of nodes, but also the magnitude of the underlying scores. Application of such cardinal voting methods might prove even more useful when considering network centrality measures, as the additional information contained in the relative score magnitudes can be incorporated into the outputs.

The development of a general method for computing node rankings on a node-aligned multiplex network allows a list of critical nodes to be developed within a multiplex network context. However, to truly answer the key player problem as defined by Borgatti (2006), an adjustment is needed which produces an ensemble ranking. Extending this methodology to produce optimal top groupings of nodes as opposed to an ordered list of nodes is a worthwhile improvement if it is possible.

Finally, a more detailed analysis of the time series data and changes in relative position of critical nodes over time under each ranking procedure would be a very interesting study for a future researcher.

5.6 Summary

This chapter concludes this thesis. It discussed the findings and their interpretation for significance and relevance as well as ideas for future related research efforts.

Appendix A

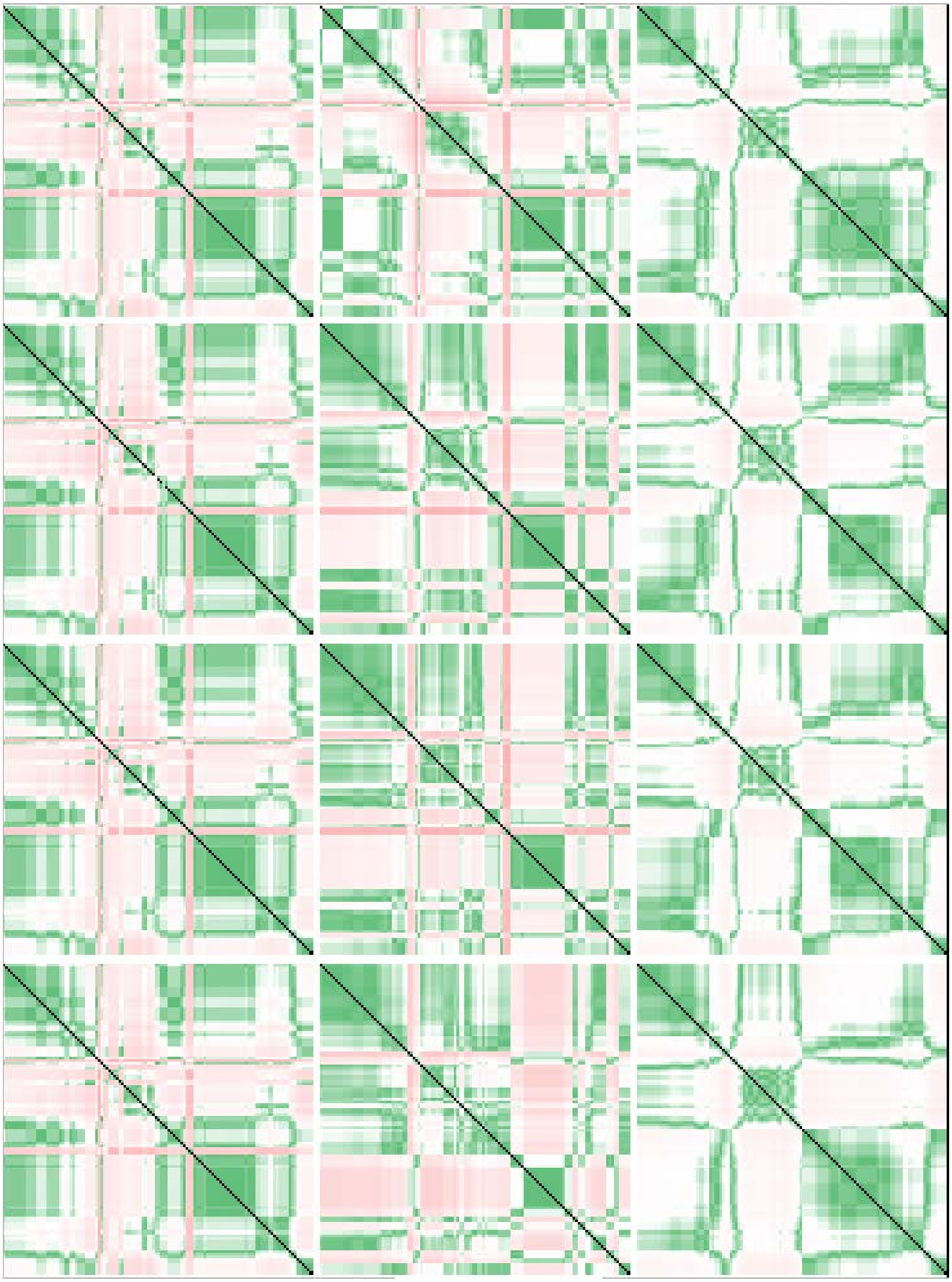

Figure 9: Betweenness Centrality

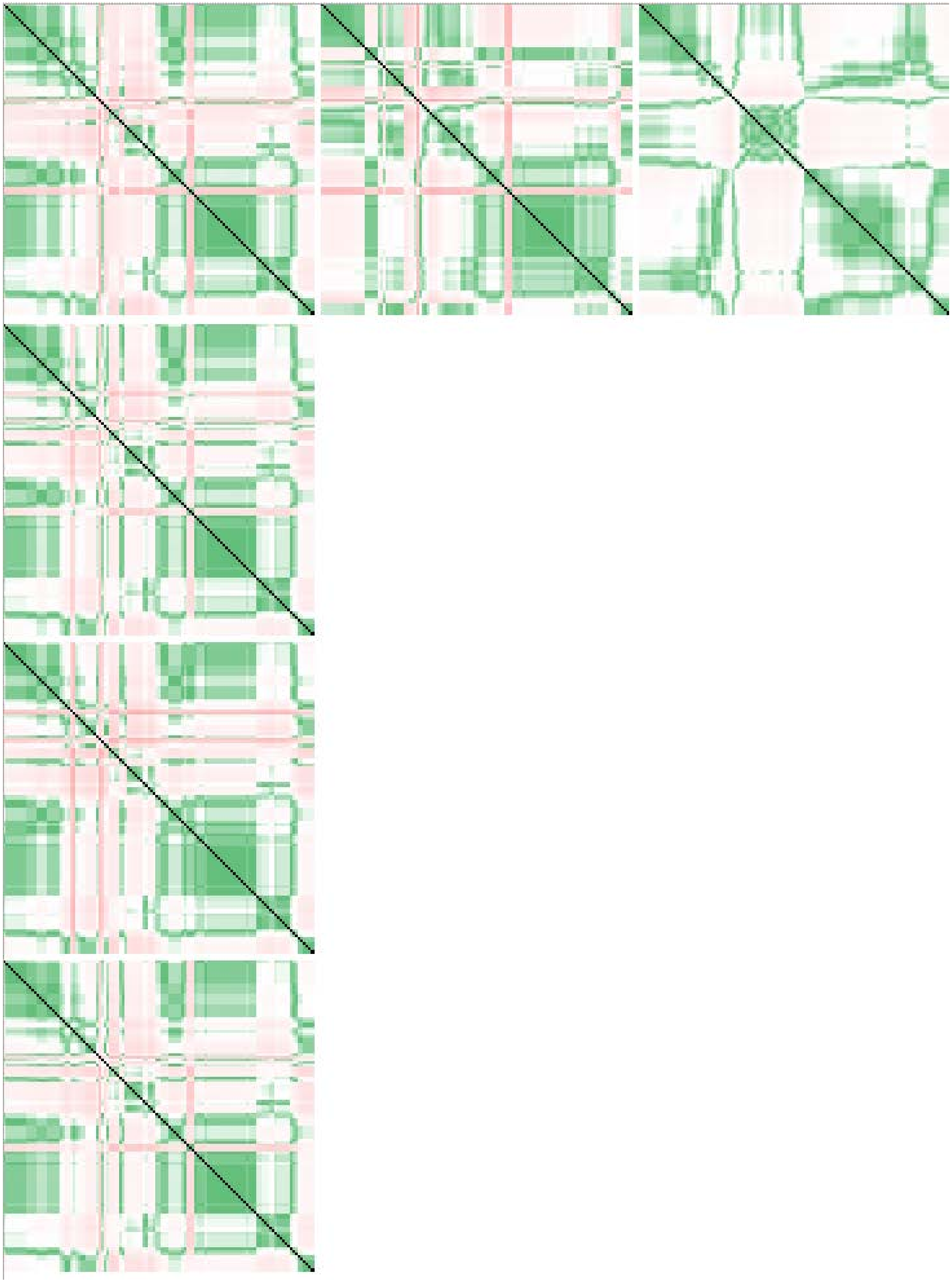

Figure 10: Betweenness Centrality, cont.

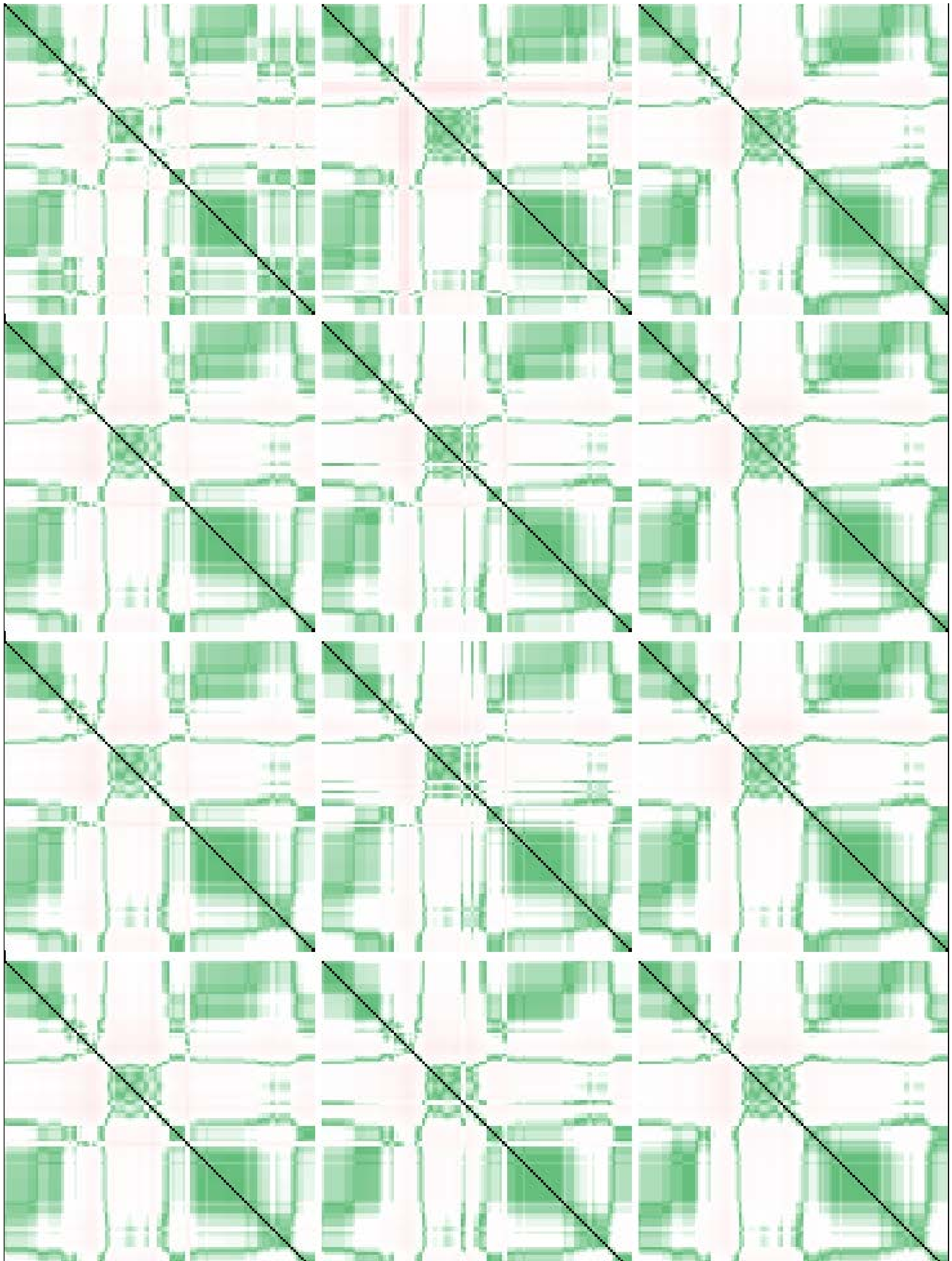

Figure 11: Closeness Centrality

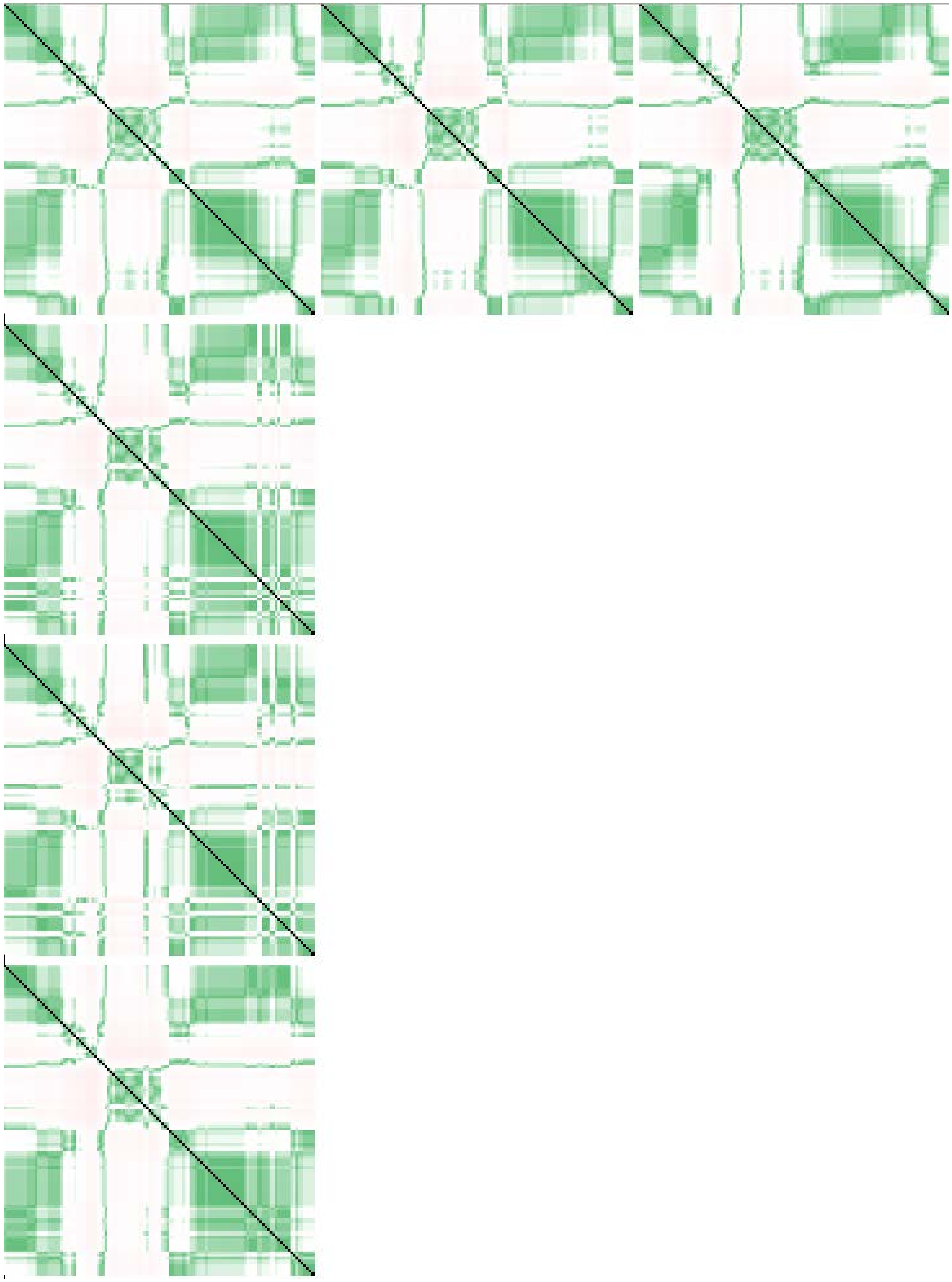

Figure 12: Closeness Centrality, cont.

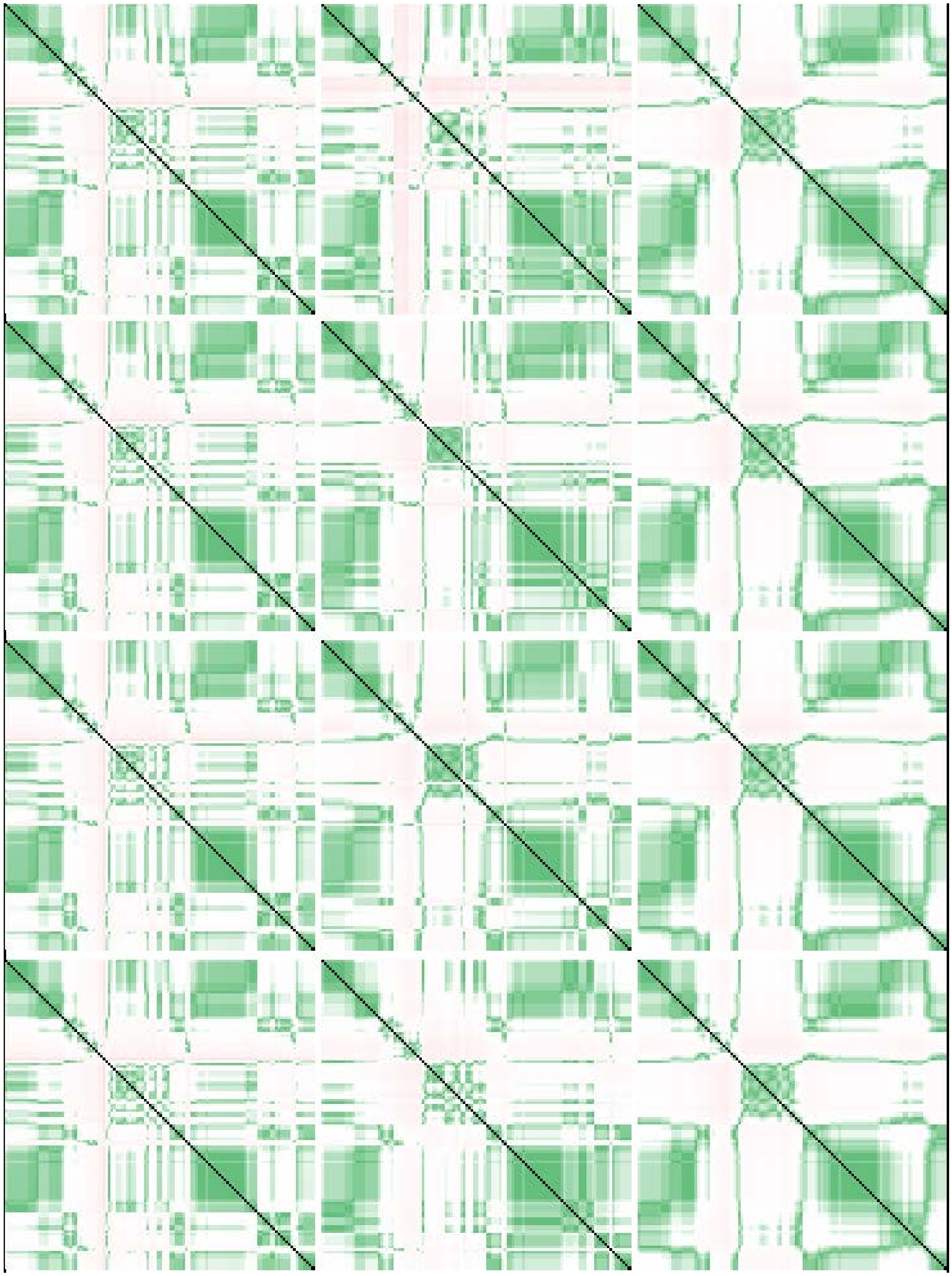

Figure 13: Degree Centrality

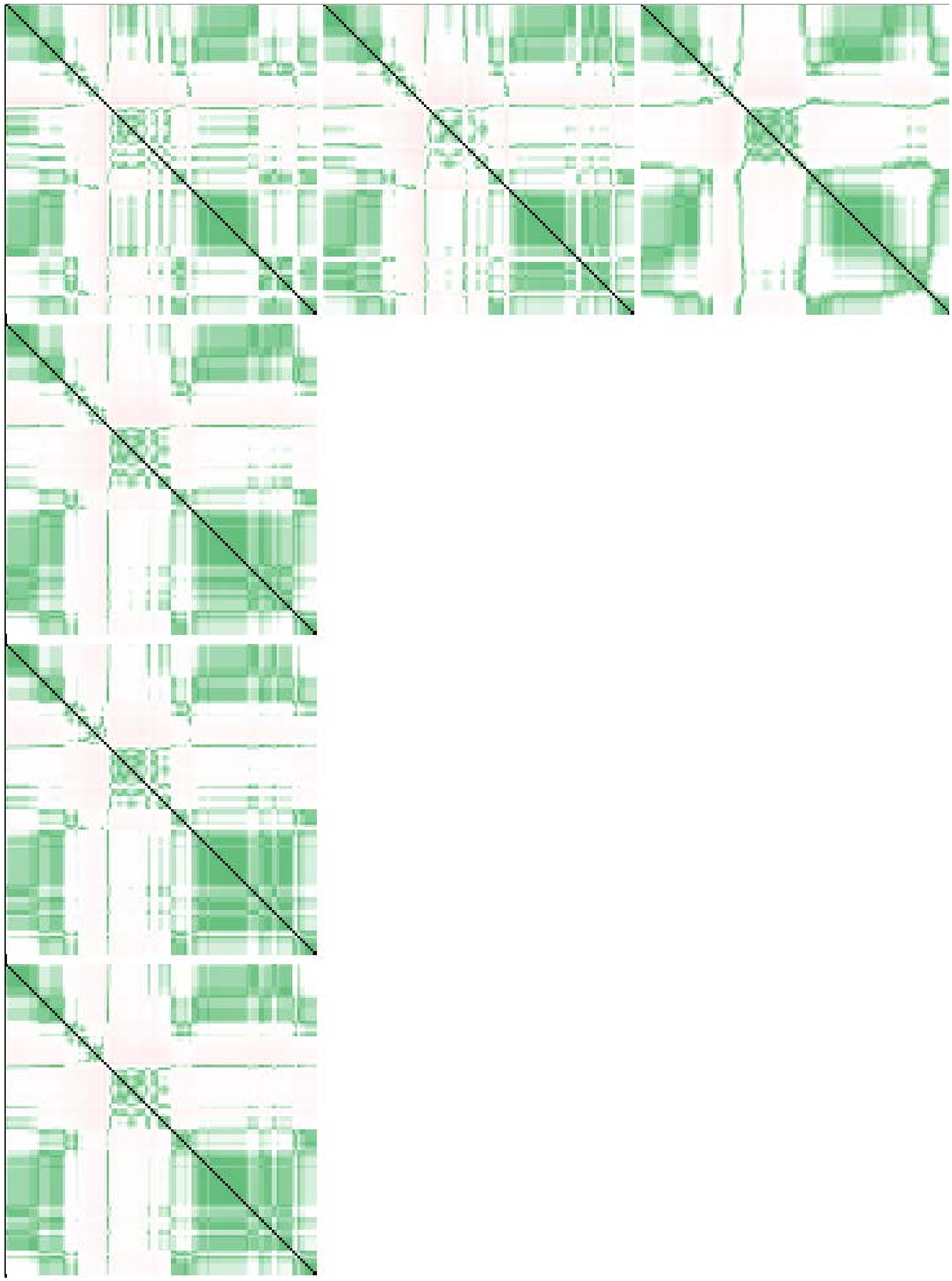

Figure 14: Degree Centrality, cont.

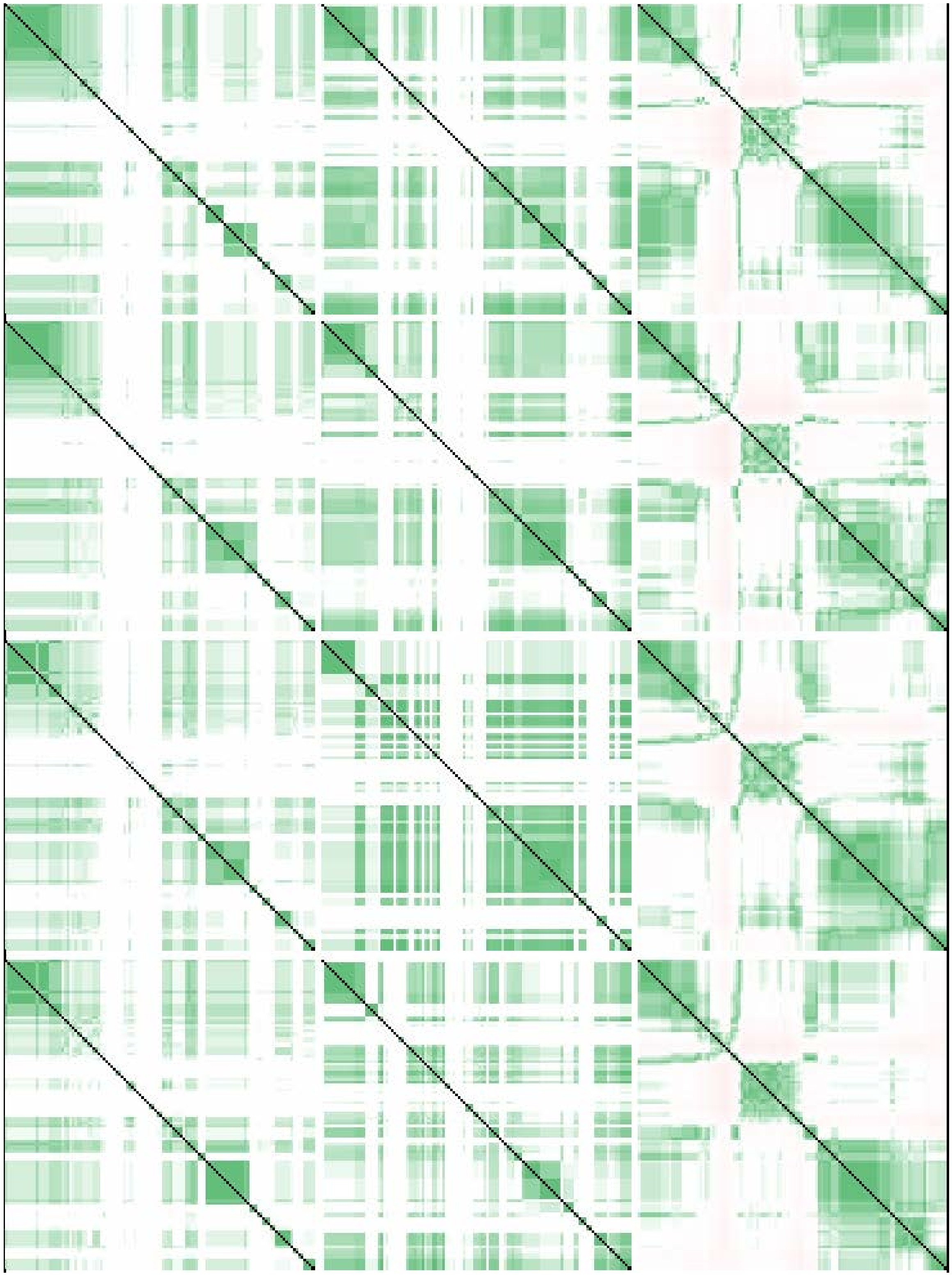

Figure 15: Eigenvector Centrality

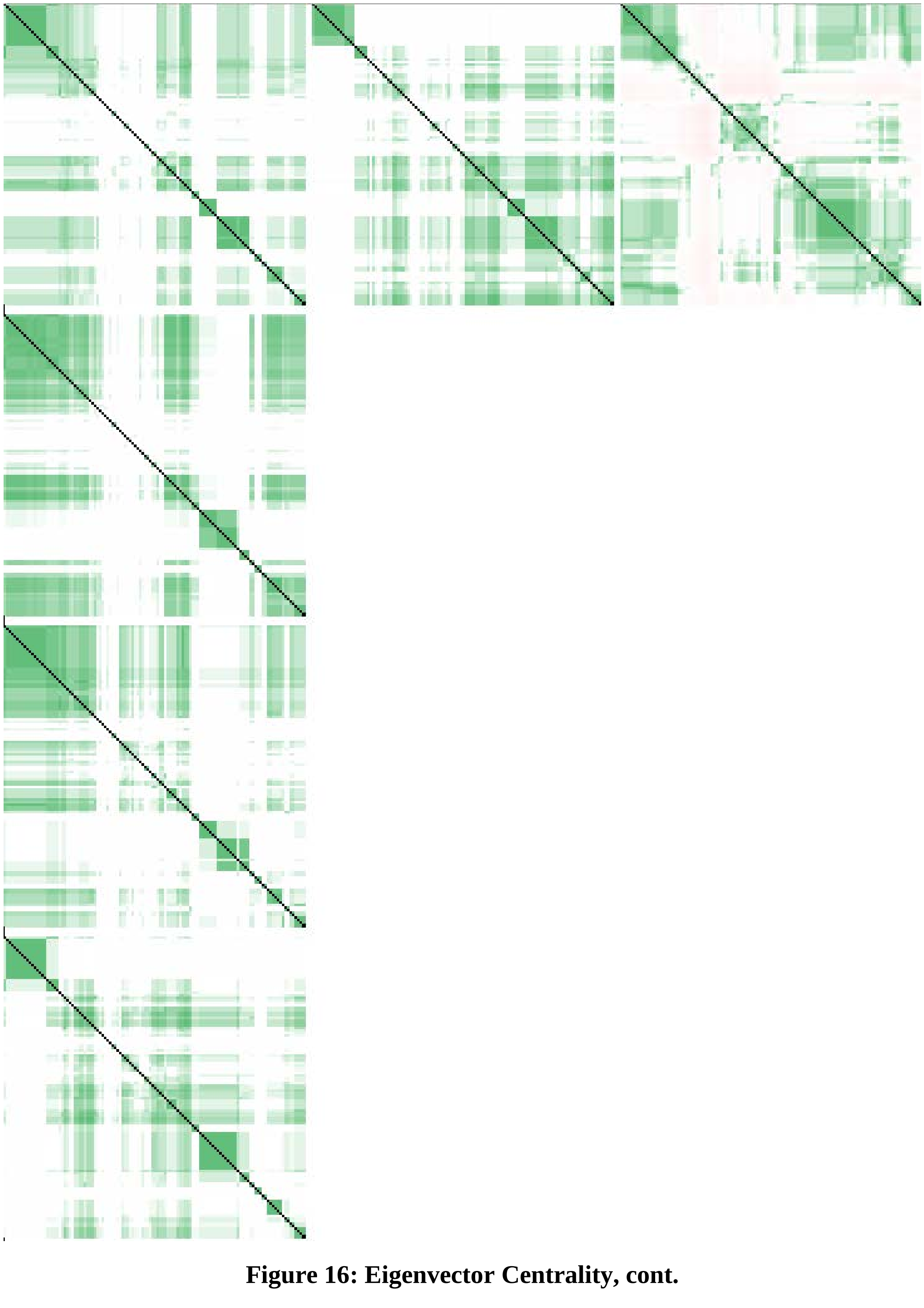

Figure 16: Eigenvector Centrality, cont.

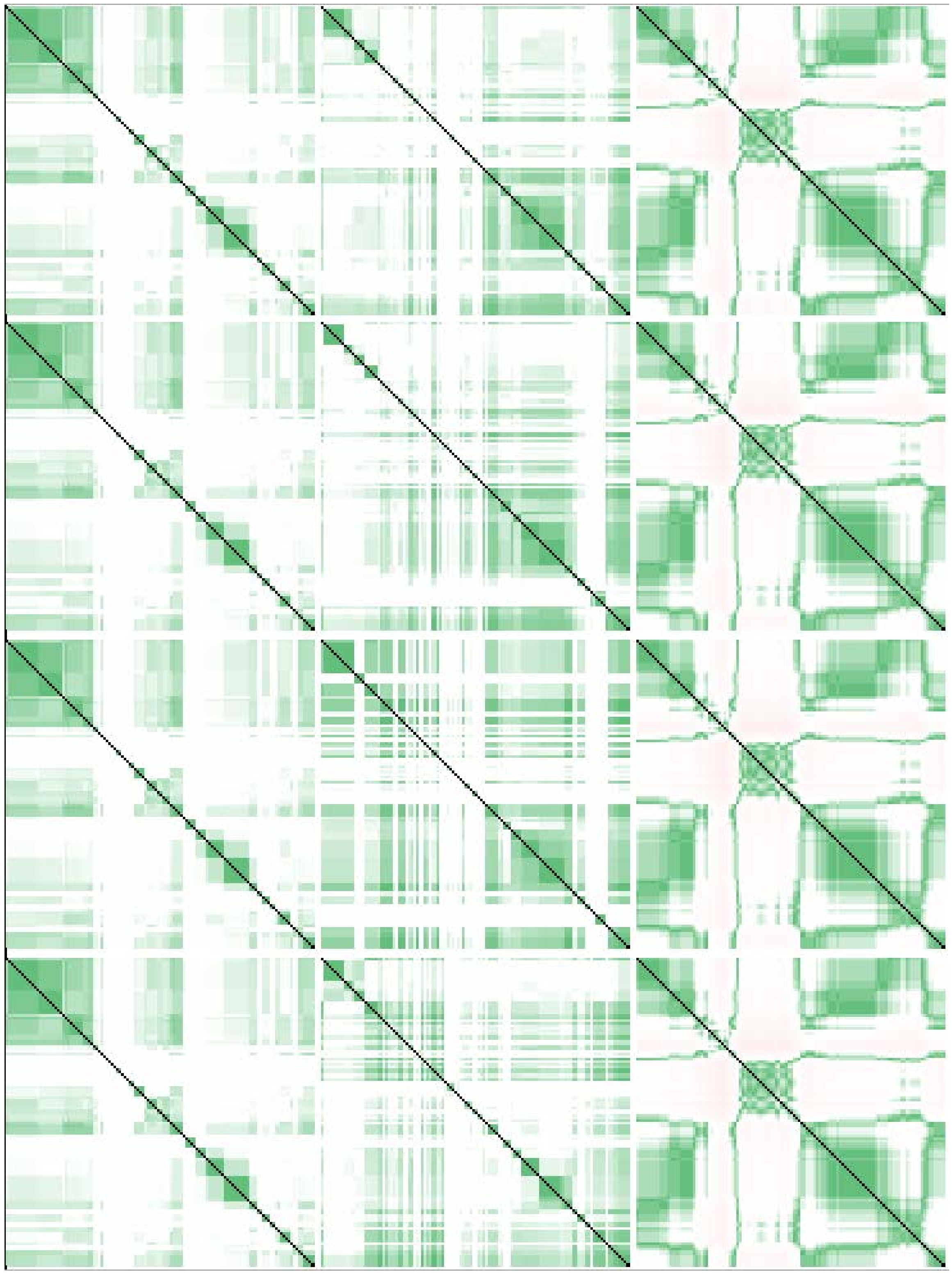

Figure 17: PageRank Centrality

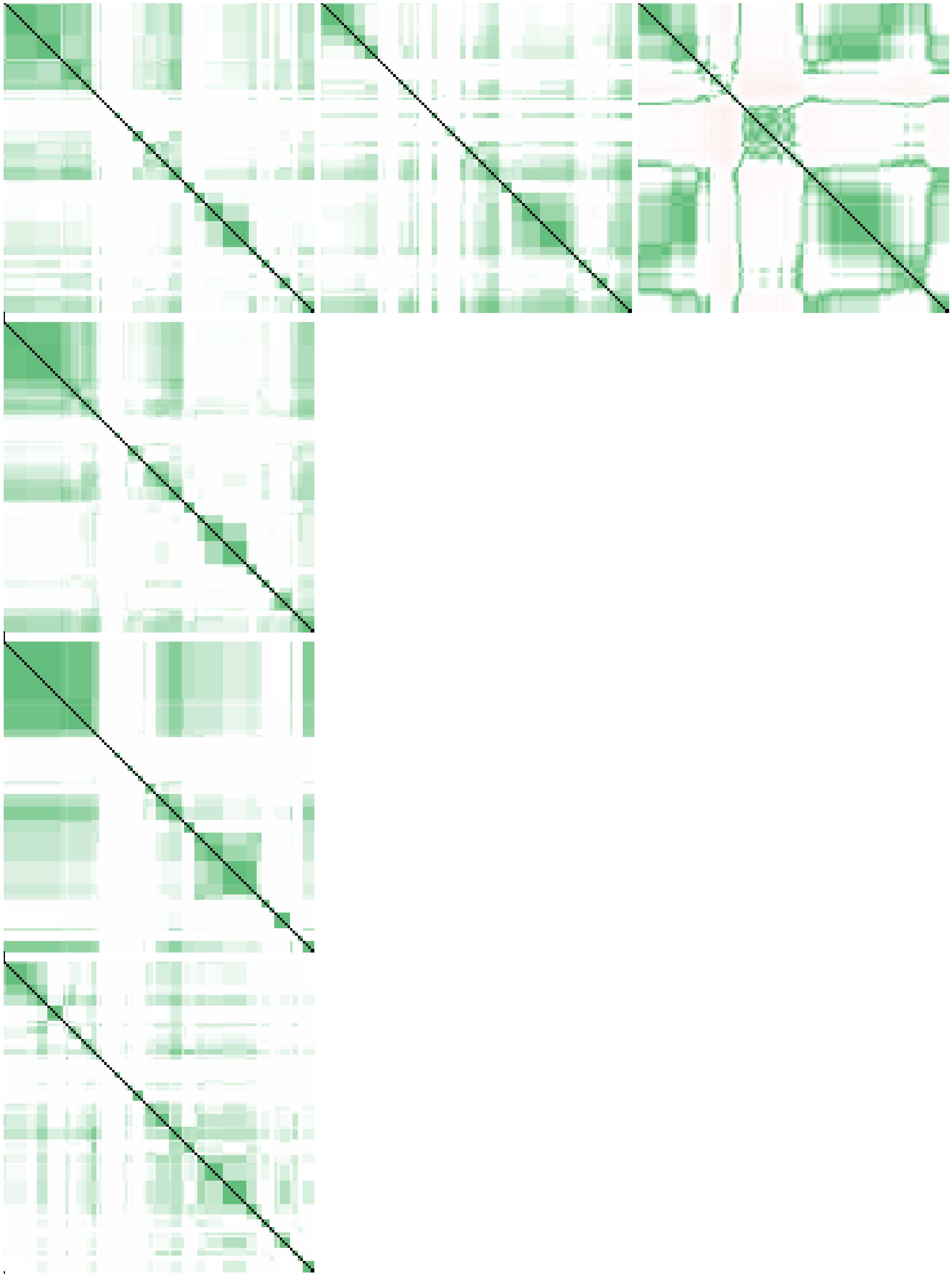

Figure 18: PageRank Centrality, cont.

Appendix B

Table 26: List of Independent Block Designs for Friedman Test

Test Number	Component Description
1	Aggregated - Betweenness
	Composite - Betweenness
	Composite - Betweenness - Reduced
2	Aggregated - Closeness
	Composite - Closeness
	Composite - Closeness - Reduced
3	Aggregated - Degree
	Composite - Degree
	Composite - Degree - Reduced
4	Aggregated - Eigenvector
	Composite - Eigenvector
	Composite - Eigenvector - Reduced
5	Aggregated - PageRank
	Composite - PageRank
	Composite - PageRank - Reduced
6	Composite - Betweenness
	Composite - Betweenness - OPS
	Composite - Betweenness - LOC
	Composite - Betweenness - PERS
7	Composite - Betweenness - Reduced
	Composite - Betweenness - OPS - Reduced
	Composite - Betweenness - LOC - Reduced
	Composite - Betweenness - PERS - Reduced
8	Composite - Closeness
	Composite - Closeness - OPS
	Composite - Closeness - LOC
	Composite - Closeness - PERS
9	Composite - Closeness - Reduced
	Composite - Closeness - OPS - Reduced
	Composite - Closeness - LOC - Reduced
	Composite - Closeness - PERS - Reduced
10	Composite - Degree
	Composite - Degree - OPS
	Composite - Degree - LOC
	Composite - Degree - PERS
11	Composite - Degree - Reduced
	Composite - Degree - OPS - Reduced
	Composite - Degree - LOC - Reduced
	Composite - Degree - PERS - Reduced
12	Composite - Eigenvector
	Composite - Eigenvector - OPS
	Composite - Eigenvector - LOC
	Composite - Eigenvector - PERS
13	Composite - Eigenvector - Reduced
	Composite - Eigenvector - OPS - Reduced
	Composite - Eigenvector - LOC - Reduced
	Composite - Eigenvector - PERS - Reduced
14	Composite - PageRank
	Composite - PageRank - OPS
	Composite - PageRank - LOC
	Composite - PageRank - PERS
15	Composite - PageRank - Reduced
	Composite - PageRank - OPS - Reduced
	Composite - PageRank - LOC - Reduced
	Composite - PageRank - PERS - Reduced
16	Overall Composite
	Overall Composite - OPS
	Overall Composite - LOC
	Overall Composite -PERS
	Aggregated Composite
17	Overall Composite - No Betweenness
	Overall Composite - OPS - No Betweenness
	Overall Composite - LOC - No Betweenness
	Overall Composite -PERS - No Betweenness
	Aggregated Composite - No Betweenness
18	Overall Composite - Reduced
	Overall Composite - OPS - Reduced
	Overall Composite - LOC - Reduced
	Overall Composite -PERS - Reduced
19	Overall Composite - Reduced - No Betweenness
	Overall Composite - OPS - Reduced - No Betweenness
	Overall Composite - LOC - Reduced - No Betweenness
	Overall Composite -PERS - Reduced - No Betweenness
20	Overall Composite
	Aggregated Composite
	Overall Composite - Reduced
21	Overall Composite - No Betweenness
	Aggregated Composite - No Betweenness
	Overall Composite - Reduced - No Betweenness
22	Overall Composite - OPS
	Overall Composite - LOC
	Overall Composite -PERS
	Overall Composite - OPS - Reduced
	Overall Composite - LOC - Reduced
	Overall Composite -PERS - Reduced

Legend:

Aggregated - Fully-aggregated single layer network
Composite - Ranking produced through Schulze method
LOC - Colocation vote weight distribution
Measure - The measure used to produce the node rankings
No Betweenness - Betweenness centrality was removed from the data
OPS - Operational vote weight distribution
Overall Composite - Rankings produced by Schulze method using original composite rankings as inputs
PERS - Personal ties vote weight distribution
Reduced - Multilayer network resulting from JSD reduction algorithm
Aggregated Composite - composite of 5 centralities measured on fully-aggregated single layer network

Table 27: Friedman Test Results

Test Number	Table Outputs					
1	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	30.87	2	15.43	33.85	4.46E-08
	'Error'	222.63	276	0.81		
	'Total'	253.50	416			
2	Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	15.94	2	7.97	16.78	2.27E-04
	'Error'	248.06	276	0.90		
	'Total'	264	416			
3	Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	30.18	2	15.09	31.66	1.33E-07
	'Error'	234.82	276	0.85		
	'Total'	265	416			
4	Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	18.56	2	9.28	18.80	8.27E-05
	'Error'	255.94	276	0.93		
	'Total'	274.50	416			
5	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	7.06	2	3.53	7.41	0.02
	'Error'	257.94	276	0.93		
	'Total'	265	416			
6	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	175.00	3	58.33	141.84	1.52E-30
	'Error'	339.50	414	0.82		
	'Total'	514.5	555			
7	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	183.87	3	61.29	174.86	1.14E-37
	'Error'	254.63	414	0.62		
	'Total'	438.5	555			
8	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	92.81	3	30.94	66.50	2.40E-14
	'Error'	489.19	414	1.18		
	'Total'	582	555			
9	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	85.26	3	28.42	55.95	4.31E-12
	'Error'	550.24	414	1.33		
	'Total'	635.5	555			
10	Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	42.85251799	3	14.28417266	30.8626943	9.09E-07
	'Error'	536.147482	414	1.295042227		
	'Total'	579	555			
11	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	69.35	3	23.12	45.65	6.74E-10
	'Error'	564.15	414	1.36		
	'Total'	633.5	555			

Test Number	Table Outputs					
12	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	52.94	3	17.65	34.79	1.35E-07
	'Error'	581.56	414	1.40		
	'Total'	634.5	555			
13	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	32.22	3	10.74	20.08	1.63E-04
	'Error'	636.78	414	1.54		
	'Total'	669	555			
14	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	90.12	3	30.04	62.37	1.83E-13
	'Error'	512.38	414	1.24		
	'Total'	602.5	555			
15	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	80.41	3	26.80	52.55	2.28E-11
	'Error'	557.59	414	1.35		
	'Total'	638	555			
16	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	68.68	4	17.17	30.61	3.67E-06
	'Error'	1178.82	552	2.14		
	'Total'	1247.5	694			
17	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	83.72	4	20.93	36.41	2.39E-07
	'Error'	1194.78	552	2.16		
	'Total'	1278.5	694			
18	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	101.18	3	33.73	66.19	2.80E-14
	'Error'	536.32	414	1.30		
	'Total'	637.5	555			
19	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	70.10	3	23.37	45.39	7.65E-10
	'Error'	573.90	414	1.39		
	'Total'	644	555			
20	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	35.68	2	17.84	37.22	8.28E-09
	'Error'	230.82	276	0.84		
	'Total'	266.5	416			
21	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	12.33	2	6.17	12.72	1.73E-03
	'Error'	257.17	276	0.93		
	'Total'	269.5	416			
22	'Source'	'SS'	'df'	'MS'	'Chi-sq'	'Prob>Chi-sq'
	'Columns'	485.18	5	97.04	150.64	9.78E-31
	'Error'	1753.32	690	2.54		
	'Total'	2238.5	833			

Table 28: WNMT Multiple Comparison Correction Results

Test Number	Set 1	Set 2	Reject Null at α= 0.05?	p Value Bound
1	Aggregated - Betweenness	Composite - Betweenness	0	> 0.10
	Aggregated - Betweenness	Composite - Betweenness - Reduced	1	< 0.001
	Composite - Betweenness	Composite - Betweenness - Reduced	1	< 0.0001
2	Aggregated - Closeness	Composite - Closeness	0	> 0.20
	Aggregated - Closeness	Composite - Closeness - Reduced	1	< 0.05
	Composite - Closeness	Composite - Closeness - Reduced	1	< 0.0005
3	Aggregated - Degree	Composite - Degree	1	< 0.01
	Aggregated - Degree	Composite - Degree - Reduced	1	< 0.05
	Composite - Degree	Composite - Degree - Reduced	1	< 0.0001
4	Aggregated - Eigenvector	Composite - Eigenvector	0	> 0.20
	Aggregated - Eigenvector	Composite - Eigenvector - Reduced	1	< 0.01
	Composite - Eigenvector	Composite - Eigenvector - Reduced	1	< 0.0001
5	Aggregated - PageRank	Composite - PageRank	0	> 0.20
	Aggregated - PageRank	Composite - PageRank - Reduced	0	> 0.20
	Composite - PageRank	Composite - PageRank - Reduced	1	< 0.025
6	Composite - Betweenness	Composite - Betweenness - OPS	1	< 0.0001
	Composite - Betweenness	Composite - Betweenness - LOC	0	> 0.20
	Composite - Betweenness	Composite - Betweenness - PERS	1	< 0.0005
	Composite - Betweenness - OPS	Composite - Betweenness - LOC	1	< 0.0001
	Composite - Betweenness - OPS	Composite - Betweenness - PERS	1	< 0.0001
	Composite - Betweenness - LOC	Composite - Betweenness - PERS	1	< 0.0005
7	Composite - Betweenness - Reduced	Composite - Betweenness - OPS - Reduced	1	< 0.0001
	Composite - Betweenness - Reduced	Composite - Betweenness - LOC - Reduced	1	< 0.0001
	Composite - Betweenness - Reduced	Composite - Betweenness - PERS - Reduced	1	< 0.0001
	Composite - Betweenness - OPS - Reduced	Composite - Betweenness - LOC - Reduced	1	< 0.0001
	Composite - Betweenness - OPS - Reduced	Composite - Betweenness - PERS - Reduced	1	< 0.005
	Composite - Betweenness - LOC - Reduced	Composite - Betweenness - PERS - Reduced	0	> 0.20
8	Composite - Closeness	Composite - Closeness - OPS	1	< 0.0001
	Composite - Closeness	Composite - Closeness - LOC	0	> 0.05
	Composite - Closeness	Composite - Closeness - PERS	1	< 0.0001
	Composite - Closeness - OPS	Composite - Closeness - LOC	1	< 0.0005
	Composite - Closeness - OPS	Composite - Closeness - PERS	0	> 0.20
	Composite - Closeness - LOC	Composite - Closeness - PERS	1	< 0.01

Table 29: WNMT Multiple Comparison Correction Results, *cont. 1*

Test Number	Set 1	Set 2	Reject Null at α= 0.05?	p Value Bound
9	Composite - Closeness - Reduced	Composite - Closeness - OPS - Reduced	1	< 0.0001
	Composite - Closeness - Reduced	Composite - Closeness - LOC - Reduced	0	> 0.20
	Composite - Closeness - Reduced	Composite - Closeness - PERS - Reduced	0	> 0.20
	Composite - Closeness - OPS - Reduced	Composite - Closeness - LOC - Reduced	1	< 0.0001
	Composite - Closeness - OPS - Reduced	Composite - Closeness - PERS - Reduced	1	< 0.0001
	Composite - Closeness - LOC - Reduced	Composite - Closeness - PERS - Reduced	0	> 0.20
10	Composite - Degree	Composite - Degree - OPS	1	< 0.01
	Composite - Degree	Composite - Degree - LOC	0	> 0.20
	Composite - Degree	Composite - Degree - PERS	1	< 0.0001
	Composite - Degree - OPS	Composite - Degree - LOC	0	> 0.20
	Composite - Degree - OPS	Composite - Degree - PERS	0	> 0.20
	Composite - Degree - LOC	Composite - Degree - PERS	1	< 0.01
11	Composite - Degree - Reduced	Composite - Degree - OPS - Reduced	1	< 0.0001
	Composite - Degree - Reduced	Composite - Degree - LOC - Reduced	0	> 0.20
	Composite - Degree - Reduced	Composite - Degree - PERS - Reduced	0	> 0.20
	Composite - Degree - OPS - Reduced	Composite - Degree - LOC - Reduced	1	< 0.0001
	Composite - Degree - OPS - Reduced	Composite - Degree - PERS - Reduced	1	< 0.005
	Composite - Degree - LOC - Reduced	Composite - Degree - PERS - Reduced	0	> 0.10
12	Composite - Eigenvector	Composite - Eigenvector - OPS	0	> 0.20
	Composite - Eigenvector	Composite - Eigenvector - LOC	0	> 0.20
	Composite - Eigenvector	Composite - Eigenvector - PERS	1	< 0.0001
	Composite - Eigenvector - OPS	Composite - Eigenvector - LOC	0	> 0.20
	Composite - Eigenvector - OPS	Composite - Eigenvector - PERS	1	< 0.005
	Composite - Eigenvector - LOC	Composite - Eigenvector - PERS	1	< 0.0001
13	Composite - Eigenvector - Reduced	Composite - Eigenvector - OPS - Reduced	1	< 0.01
	Composite - Eigenvector - Reduced	Composite - Eigenvector - LOC - Reduced	0	> 0.20
	Composite - Eigenvector - Reduced	Composite - Eigenvector - PERS - Reduced	1	< 0.005
	Composite - Eigenvector - OPS - Reduced	Composite - Eigenvector - LOC - Reduced	1	< 0.05
	Composite - Eigenvector - OPS - Reduced	Composite - Eigenvector - PERS - Reduced	0	> 0.20
	Composite - Eigenvector - LOC - Reduced	Composite - Eigenvector - PERS - Reduced	1	< 0.025
14	Composite - PageRank	Composite - PageRank - OPS	1	< 0.0001
	Composite - PageRank	Composite - PageRank - LOC	0	> 0.20
	Composite - PageRank	Composite - PageRank - PERS	1	< 0.0001
	Composite - PageRank - OPS	Composite - PageRank - LOC	1	< 0.01
	Composite - PageRank - OPS	Composite - PageRank - PERS	0	> 0.10
	Composite - PageRank - LOC	Composite - PageRank - PERS	1	< 0.0001

Table 30: WNMT Multiple Comparison Correction Results, *cont. 2*

Test Number	Set 1	Set 2	Reject Null at α= 0.05?	p Value Bound
15	Composite - PageRank - Reduced	Composite - PageRank - OPS - Reduced	1	< 0.0001
	Composite - PageRank - Reduced	Composite - PageRank - LOC - Reduced	0	> 0.20
	Composite - PageRank - Reduced	Composite - PageRank - PERS - Reduced	0	> 0.10
	Composite - PageRank - OPS - Reduced	Composite - PageRank - LOC - Reduced	1	< 0.0001
	Composite - PageRank - OPS - Reduced	Composite - PageRank - PERS - Reduced	1	< 0.005
	Composite - PageRank - LOC - Reduced	Composite - PageRank - PERS - Reduced	1	< 0.025
16	Overall Composite	Overall Composite - OPS	1	< 0.0005
	Overall Composite	Overall Composite - LOC	0	> 0.20
	Overall Composite	Overall Composite -PERS	0	> 0.20
	Overall Composite	Aggregated Composite	0	> 0.20
	Overall Composite - OPS	Overall Composite - LOC	1	< 0.0001
	Overall Composite - OPS	Overall Composite -PERS	1	< 0.05
	Overall Composite - OPS	Aggregated Composite	1	< 0.025
	Overall Composite - LOC	Overall Composite -PERS	0	> 0.20
	Overall Composite - LOC	Aggregated Composite	0	> 0.20
	Overall Composite -PERS	Aggregated Composite	0	> 0.20
17	Overall Composite - No Betweenness	Overall Composite - OPS - No Betweenness	1	< 0.0001
	Overall Composite - No Betweenness	Overall Composite - LOC - No Betweenness	1	< 0.025
	Overall Composite - No Betweenness	Overall Composite -PERS - No Betweenness	1	< 0.0001
	Overall Composite - No Betweenness	Aggregated Composite - No Betweenness	1	< 0.05
	Overall Composite - OPS - No Betweenness	Overall Composite - LOC - No Betweenness	0	> 0.20
	Overall Composite - OPS - No Betweenness	Overall Composite -PERS - No Betweenness	0	> 0.20
	Overall Composite - OPS - No Betweenness	Aggregated Composite - No Betweenness	0	> 0.20
	Overall Composite - LOC - No Betweenness	Overall Composite -PERS - No Betweenness	0	> 0.20
	Overall Composite - LOC - No Betweenness	Aggregated Composite - No Betweenness	0	> 0.20
	Overall Composite -PERS - No Betweenness	Aggregated Composite - No Betweenness	0	> 0.10
18	Overall Composite - Reduced	Overall Composite - OPS - Reduced	1	< 0.0001
	Overall Composite - Reduced	Overall Composite - LOC - Reduced	0	> 0.20
	Overall Composite - Reduced	Overall Composite -PERS - Reduced	0	> 0.20
	Overall Composite - OPS - Reduced	Overall Composite - LOC - Reduced	1	< 0.0001
	Overall Composite - OPS - Reduced	Overall Composite -PERS - Reduced	1	< 0.0001
	Overall Composite - LOC - Reduced	Overall Composite -PERS - Reduced	1	< 0.025

Table 31: WNMT Multiple Comparison Correction Results, *cont. 3*

Test Number	Set 1	Set 2	Reject Null at α= 0.05?	p Value Bound
19	Overall Composite - Reduced - No Betweenness	Overall Composite - OPS - Reduced - No Betweenness	1	< 0.0001
	Overall Composite - Reduced - No Betweenness	Overall Composite - LOC - Reduced - No Betweenness	0	> 0.20
	Overall Composite - Reduced - No Betweenness	Overall Composite -PERS - Reduced - No Betweenness	0	> 0.20
	Overall Composite - OPS - Reduced - No Betweenness	Overall Composite - LOC - Reduced - No Betweenness	1	< 0.0001
	Overall Composite - OPS - Reduced - No Betweenness	Overall Composite -PERS - Reduced - No Betweenness	1	< 0.0001
	Overall Composite - LOC - Reduced - No Betweenness	Overall Composite -PERS - Reduced - No Betweenness	0	> 0.20
20	Overall Composite	Aggregated Composite	1	< 0.005
	Overall Composite	Overall Composite - Reduced	1	< 0.0001
	Aggregated Composite	Overall Composite - Reduced	1	< 0.025
21	Overall Composite - No Betweenness	Aggregated Composite - No Betweenness	0	> 0.20
	Overall Composite - No Betweenness	Overall Composite - Reduced - No Betweenness	1	< 0.005
	Aggregated Composite - No Betweenness	Overall Composite - Reduced - No Betweenness	0	> 0.10
22	Overall Composite - OPS	Overall Composite - LOC	1	< 0.005
	Overall Composite - OPS	Overall Composite -PERS	0	> 0.20
	Overall Composite - OPS	Overall Composite - OPS - Reduced	0	> 0.20
	Overall Composite - OPS	Overall Composite - LOC - Reduced	1	< 0.0001
	Overall Composite - OPS	Overall Composite -PERS - Reduced	1	< 0.0001
	Overall Composite - LOC	Overall Composite -PERS	0	> 0.20
	Overall Composite - LOC	Overall Composite - OPS - Reduced	1	< 0.0001
	Overall Composite - LOC	Overall Composite - LOC - Reduced	1	< 0.001
	Overall Composite - LOC	Overall Composite -PERS - Reduced	0	> 0.10
	Overall Composite -PERS	Overall Composite - OPS - Reduced	1	< 0.0005
	Overall Composite -PERS	Overall Composite - LOC - Reduced	1	< 0.0001
	Overall Composite -PERS	Overall Composite -PERS - Reduced	1	< 0.001
	Overall Composite - OPS - Reduced	Overall Composite - LOC - Reduced	1	< 0.0001
	Overall Composite - OPS - Reduced	Overall Composite -PERS - Reduced	1	< 0.0001
	Overall Composite - LOC - Reduced	Overall Composite -PERS - Reduced	0	> 0.20

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14. ABSTRACT

Identifying relevant actors using information gleaned from multiple networks is a key goal within the context of human aspects of military operations. The application of a voting theory methodology for determining nodes of critical importance—in ranked order of importance—for a node-aligned multiplex network is demonstrated. Both statistical and qualitative analyses on the differences of ranking outcomes under this methodology is provided. As a corollary, a multilayer network reduction algorithm is investigated within the context of the proposed ranking methodology. The application of the methodology detailed in this thesis will allow meaningful rankings of relevant actors to be produced on a multiplex network.

15. SUBJECT TERMS

Networks, Multilayer, Multiplex, Centrality, Rankings, Social Choice Theory, Electoral Systems, Schulze, Friedman, Non-Parametric, Noordin Top, Weighted Networks, Critical Nodes

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