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Wave Motion 33 (2001) 359–377
WAVE MOTION
www.elsevier.com/locate/wavemoti

Effect of molecular relaxation on the propagation of sonic booms through a stratified atmosphere

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Received 24 November 1997; received in revised form 18 September 2000; accepted 25 September 2000

Abstract

Nonlinear acoustic wave propagation through a stratified atmosphere is considered. The initial signal is taken to be an isolated N-wave, which is the disturbance that is generated some distance away from a supersonic body in horizontal flight. The effect of cylindrical spreading and exponential density stratification on the propagation of the disturbance is considered, with the shock structure controlled by molecular relaxation mechanisms and by thermoviscous diffusion. An augmented Burgers equation is obtained and asymptotic solutions are derived based on the limit of small dissipation and dispersion. For a single relaxation mode, the solution depends on whether relaxation alone can support the shock or whether a sub-shock arises controlled by other mechanisms. The resulting shock structures are known as fully dispersed and partly dispersed shocks, respectively. In this paper, the spatial location of the transition between fully dispersed and partly dispersed shocks is identified for shocks propagating above and below the horizontal. This phenomenon is important in understanding the character of sonic booms since the transition to a partly dispersed shock structure leads to the appearance of a shorter scale in the shock rise-time, associated with the embedded sub-shock. © 2001 Published by Elsevier Science B.V.

1. Introduction

Accurate predictions of shock overpressure and shock rise-time are important in determining the subjective annoyance of sonic booms produced by supersonic aircraft. To assess the problems associated with sonic booms at ground level, the propagation of disturbances over long ranges must be investigated for a realistic atmosphere. Account must be taken of nonlinearity, geometric effects, the effect of stratification and molecular mechanisms leading to dissipation and dispersion.

The effect of nonlinearity on acoustic propagation was studied in detail by Lighthill [1]. In particular, the competition between nonlinearity and other effects was described. For typical acoustic waves, nonlinearity is locally small, but the effect is cumulative, leading to significant deformation over long ranges. For one-dimensional propagation, inclusion of thermoviscosity in addition to nonlinearity leads to the well-known Burgers equation, which can be solved exactly using the Cole–Hopf linearising transformation [2,3]. For propagation in three-dimensional space, wave evolution can be described by a nonlinear equation along ray paths. Inclusion of geometric effects due to variation in wave-front area leads to a generalised Burgers equation [4,5] for which no linearising transformations are available [6]. Progress is then only possible using numerical methods [7,8] or asymptotic analysis based on the limit of small diffusivity [6].

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0165-2125/01/$ – see front matter © 2001 Published by Elsevier Science B.V.
PII: S0165-2125(00)00078-0

While most studies of finite-amplitude acoustic propagation have considered the effect of thermoviscosity in controlling shock structure, in the atmosphere other physical effects are present which may influence shock structure. Atmospheric turbulence is one mechanism which has an effect on the structure of a shock wave [9,10]. Molecular relaxation associated with the internal vibration of polyatomic molecules also plays an important role in determining the shock structure. Analysis of the interaction between relaxation and nonlinearity for one-dimensional propagation through a uniform media [1,11–15] shows that for some range of parameter values, relaxation alone is insufficient to support a shock. This leads to the appearance of a narrow sub-shock controlled by other mechanisms. For propagation through the air, relaxation modes associated with the internal vibration of both O_2 and N_2 are significant.

All the theories described so far are based on propagation through uniform media. For propagation through the atmosphere, the effect of variation with altitude must also be considered. Sirovich and Chong [16] derived a governing equation for arbitrary density stratification, including the effect of thermoviscous diffusivity. Solutions were then considered only in the inviscid case and hence there is no analysis of shock width. Rogers and Gardner [17] considered propagation through an atmosphere in which the linear sound speed is taken to be constant up to altitudes of 100 km. Comparison with measurements suggests that this is a good approximation. The advantage of this approximation is that the ray paths are straight, hence simplifying the nonlinear evolution equation along the rays. Although Rogers and Gardner included attenuation in their model, the resulting shock structure was not considered. Crighton and Lee-Bapty [18] considered spherical wave motion with thermoviscous dissipation, for an initially sinusoidal wave form. This study is not directly applicable to the propagation of sonic booms due to the periodicity of the disturbance. In addition, symmetry about the flight path of the body creating the disturbance causes sonic booms to spread cylindrically rather than spherically. However, the asymptotic treatment [18] takes full account of the altitude variation of the dissipation parameter, and considers the resulting shock structure.

The propagation of the sonic boom generated by an axisymmetric supersonic body in a medium characterised by a single relaxation mode was considered by Clarke and Sinai [19,20]. Their analysis takes full account of body shape, describing the development of the N-wave structure in the near-field, followed by the evolution of the shock structure over large ranges. However, the effect of density stratification was not considered. In this paper, we consider the case of an N-wave propagating through a stratified atmosphere. The combined effect on the wave form of nonlinearity, wave-front curvature and stratification of the atmosphere is analysed, together with the effect of molecular relaxation and thermoviscosity. We consider the linear sound speed to be independent of altitude, which is consistent with the mean density decreasing exponentially upwards. The variation with altitude of the relaxation and thermoviscous parameters due to density stratification is also considered. General methods for the derivation of model equations governing finite-amplitude acoustic disturbances are well described elsewhere [5]. In Section 2, we describe the salient points in deriving a model equation for this physical situation. We then obtain asymptotic solutions based on the limit of small thermoviscosity and small relaxation, when molecular mechanisms balance nonlinearity only in narrow shock regions.

2. Governing equation

For a disturbance generated by an axisymmetric body travelling through a uniform non-dissipative medium, linear theory predicts a propagating disturbance of the form

$$u(r,t) = u_0\left(\frac{r_0}{r}\right)^{1/2} f\left(t - \frac{r-r_0}{a_0}\right), \tag{1}$$

where u is the particle velocity, a_0 the equilibrium sound speed, r the propagation distance and u_0 the maximum amplitude at some initial location $r = r_0$. At this level of approximation, the shape of the wave is unchanged with propagation distance, while the amplitude decays like $r^{-1/2}$ due to cylindrical spreading of the axisymmetric disturbance. For finite-amplitude disturbances, nonlinear effects must be considered. Typically the effect of nonlinearity is locally small, but has a cumulative effect. Hence, the method of multiple scales can be used to derive an equation

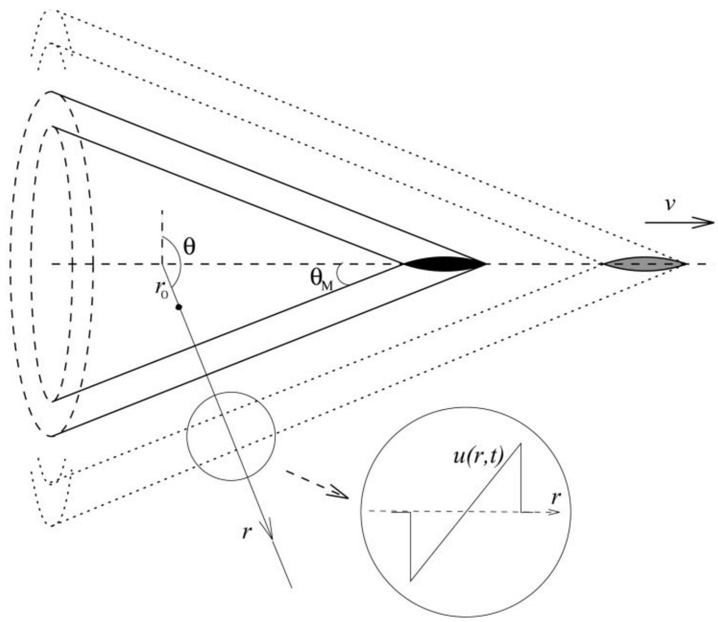

Fig. 1. An illustration of the physical situation of interest: solid lines show the position of the body and the resulting disturbance at some reference time, while the dotted lines show the positions at some later time. The ray angle θ is measured from the vertical.

which determines the evolution of the waveform along a ray, including the correction term due to nonlinearity,

$$\frac{\partial u}{\partial r} - \frac{\gamma+1}{2a_0^2} u \frac{\partial u}{\partial \phi} + \frac{u}{2r} = 0, \tag{2}$$

where γ is the ratio of specific heats and $\phi = t - (r - r_0)/a_0$ the retarded time based on the equilibrium sound speed. The nonlinearity leads to wave steepening as wave crests travel faster than wave troughs.

The disturbance generated by a supersonic body is illustrated in Fig. 1. For a body travelling at speed $v > a_0$, a head shock of positive amplitude is generated in the form of a Mach cone with half-angle $\theta_M = \sin^{-1}(1/M)$, where $M = v/a_0$. Similarly a tail shock of equal and opposite amplitude is generated. Close to the body, the exact form of the disturbance depends on the detailed geometry of the body, but over a number of wavelengths, the effect of the quadratic nonlinearity is to form an N-wave. The evolution of the disturbance along ray paths normal to the shock front can then be determined. We consider a ray which makes angle θ with the upward vertical.

Linear terms associated with dissipation and dispersion or other physical processes can then be added to (2), as for a linear system, without repeating the multiple scales analysis. In the present work, we consider the effect of density stratification, together with molecular effects of thermoviscous diffusion and relaxation.

If the linear sound speed, a_0, is independent of altitude, the hydrostatic balance gives the density variation as

$$\rho = \rho_0 \exp\left(\frac{-z}{H}\right) = \rho_0 \exp\left(\frac{-r\cos\theta}{H}\right), \tag{3}$$

where $z = r\cos\theta$ is the vertical coordinate and H the height scale over which significant gravitational effects occur, $H = a_0^2/\gamma g$. Here γ is the ratio of specific heats and g the gravitational acceleration. Inclusion of density variation also leads to an additional linear term in the evolution equation [16]. Thus we consider a governing equation of the form

$$\frac{\partial u}{\partial r} - \frac{\gamma+1}{2a_0^2} u \frac{\partial u}{\partial \phi} + \frac{u}{2r} - \frac{u\cos\theta}{2H} = \mathcal{L}(u), \tag{4}$$

where $\mathcal{L}$ is a *linear* differential (or integro-differential) operator describing the effects of molecular processes. The form of $\mathcal{L}$ for a wide range of physical processes is summarised in the review paper by Makarov and Ochmann [21]. Since the linear sound speed is constant, the ray paths are straight lines and so $\cos\theta$ is constant along each ray.

Thermoviscous diffusion is the process most commonly included when studying the propagation of finite-amplitude waves. The contribution to the right-hand side of (4) due to thermoviscous diffusivity is

$$\mathcal{L}_{\mathrm{TV}}(u) = \frac{\Delta}{2a_0^3}\frac{\partial^2 u}{\partial\phi^2}, \tag{5}$$

where Δ is the diffusivity of sound [1],

$$\Delta = \frac{1}{\rho}\left(\frac{4}{3}\mu + \mu_{\mathrm{B}} + \frac{(\gamma-1)\kappa}{c_p}\right). \tag{6}$$

Here the first term arises from the viscosity of the medium; the second term is the bulk viscosity, usually taken to be $\mu_{\mathrm{B}} = 0.6\mu$ on the basis of experimental work; and the third term gives the thermal contribution to diffusivity, with κ the coefficient of thermal conductivity and c_p the coefficient of specific heat at constant temperature.

For acoustic propagation through the atmosphere, thermoviscous diffusion is inadequate in describing the complex molecular processes present, and relaxation processes must be considered. The essential feature of a relaxing fluid is that the partition of energy among the available modes does not respond instantaneously to changes imposed by a time-dependent flow. Each physical relaxation mode has a characteristic timescale, and if the relaxation time associated with a particular physical process is comparable to the disturbance timescale, then the effect of relaxation must be accounted for in determining the evolution of the disturbance. For a gaseous medium, it is the partition of internal vibration energy within polyatomic molecules that give rise to significant relaxation effects. For air, relaxing modes associated with O_2 and N_2 are dominant. For linear propagation of a harmonic disturbance, with relaxation effects included, the phase velocity increases monotonically with signal frequency from a_0, the equilibrium or low-frequency sound speed to a_∞, which is known as the frozen sound speed. Thus each relaxation mode, ν, is characterised by two parameters, the relaxation time T_ν and the difference between the equilibrium and frozen sound speeds, $(\Delta a)_\nu = a_\infty - a_0$. Each relaxation mode gives a contribution to the governing nonlinear equation,

$$\mathcal{L}_\nu(u) = \frac{(\Delta a)_\nu}{a_0^2}\mathrm{e}^{-\phi/T_\nu}\int_{\phi_0}^{\phi}\mathrm{e}^{\phi'/T_\nu}u_{\phi\phi}(\phi', r)\,\mathrm{d}\phi', \tag{7}$$

where the lower limit of integration ϕ_0 is determined by the boundary conditions on $u(\phi, r)$. As noted earlier, the derivation of the governing equation rests on the assumption that all the effects included have a small effect on the propagation over a distance of one wavelength. Assuming that a typical lengthscale for the disturbance is l, we assume the following:

1. $l/r \ll 1$ for the effect of geometric spreading to be small;
2. $u_0/a_0 \ll 1$ for nonlinear effects to be small;
3. $l/H \ll 1$ for the stratification effects to be small;
4. $\Delta/a_0 l \ll 1$ for thermoviscous effects to be small;
5. $(\Delta a)_\nu/a_0 \ll 1$ for the effect of the ν relaxation mode to be small.

Hence the governing equation in dimensional form for a system with one relaxation mode is given by

$$\frac{\partial u}{\partial r} - \frac{\gamma+1}{2a_0^2}u\frac{\partial u}{\partial\phi} + \frac{u}{2r} - \frac{u\cos\theta}{2H} = \frac{\Delta}{2a_0^3}\frac{\partial^2 u}{\partial\phi^2} + \frac{\Delta a}{a_0^2}\mathrm{e}^{-\phi/T}\int_{\phi_0}^{\phi}\mathrm{e}^{Y/T}u_{YY}(Y, r)\,\mathrm{d}Y, \tag{8}$$

where ϕ is the retarded time based on the equilibrium (low-frequency) sound speed.

The variation of the diffusivity and relaxation parameters with altitude must also be considered. From (6) it is clear that if μ is independent of altitude, which is a good approximation for an isothermal atmosphere, then $\Delta \propto \rho^{-1}$. In addition, it can be shown that the relaxation time for each mode is inversely proportional to density [22], whilst the difference between equilibrium and frozen sound speeds is independent of ρ. Hence

$$\varDelta = \varDelta_0 \exp\left(\frac{(r-r_0)\cos\theta}{H}\right), \qquad T_\nu = (T_\nu)_0 \exp\left(\frac{(r-r_0)\cos\theta}{H}\right), \qquad (\Delta a)_\nu = \text{constant}. \tag{9}$$

It should be noted that the dependence of these parameters on altitude is mutually consistent, since for signal frequencies much less than the relaxation frequency the effect of relaxation is purely diffusive with a diffusivity coefficient

$$\varDelta_\nu = a_0 (\Delta a)_\nu T_\nu, \tag{10}$$

which has the correct dependence on altitude.

In the present paper, we do not address the generation of the sonic boom by the body. Close to the body, cylindrical spreading is insufficient to describe the geometric effect on the disturbance. However, as noted earlier, nonlinearity leads to wave steepening and so at some distance from the source an N-wave has been formed. We then consider the evolution of this wave as it propagates over long distances. Hence we consider as a boundary condition an N-wave of duration t_0 and amplitude u_0

$$u(r=r_0, t) = \begin{cases} -u_0 \dfrac{t}{t_0}, & |t| < t_0, \\ 0, & |t| > t_0. \end{cases} \tag{11}$$

The governing equation is nondimensionalised, with the effect of stratification and geometric spreading scaled out by setting

$$V = \left(\frac{r}{r_0}\right)^{1/2} \exp\left(-\frac{(r-r_0)\cos\theta}{2H}\right)\left(\frac{u}{u_0}\right), \qquad X = \left(\frac{(\gamma+1)u_0}{2a_0^2 t_0}\right) r \tag{12}$$

and $\tau = \phi/t_0$. For the present time, we consider just one relaxation mode, in which case (8) gives

$$\begin{aligned} &V_X - \left(\frac{X_0}{X}\right)^{1/2} \exp\left(\frac{\alpha\cos\theta(X-X_0)}{2}\right) V V_\tau \\ &\quad = \epsilon_0 \exp(\alpha\cos\theta(X-X_0)) V_{\tau\tau} + K\,\mathrm{e}^{-\tau/\varOmega} \int^\tau \mathrm{e}^{y/\varOmega} V_{yy}(y, X)\,\mathrm{d}y, \end{aligned} \tag{13}$$

where $\varOmega = \varOmega_0 \exp(\alpha\cos\theta(X-X_0))$, with

$$\alpha = \frac{2a_0^2 t_0}{(\gamma+1)u_0 H}, \qquad \epsilon_0 = \frac{\varDelta_0}{(\gamma+1)t_0 u_0 a_0}, \qquad \varOmega_0 = \frac{T_0}{t_0}, \qquad K = \frac{2(\Delta a)}{(\gamma+1)u_0} \tag{14}$$

and

$$X_0 = \frac{(\gamma+1)u_0 r_0}{2a_0^2 t_0}.$$

This equation can then be converted to a more recognisable form by setting $x = X/X_0$, $\lambda = \alpha X_0 \cos\theta/2$, and then rescaling the spatial variable

$$Z = 1 + X_0\,\mathrm{e}^{-\lambda} \int_1^x \frac{\mathrm{e}^{\lambda x'}}{x'^{1/2}}\,\mathrm{d}x' \tag{15}$$

to give

$$V_Z - V V_\tau = \epsilon_0 Q_1(Z) V_{\tau\tau} + K Q_2(Z)\,\mathrm{e}^{-\tau/\varOmega} \int^\tau \mathrm{e}^{y/\varOmega} V_{yy}\,\mathrm{d}y. \tag{16}$$

Here the range-dependent parameters characterising the effective diffusivity and relaxation terms are given implicitly as

$$Q_1(Z) = x^{1/2}\,\mathrm{e}^{\lambda(x-1)}, \qquad Q_2(Z) = x^{1/2}\,\mathrm{e}^{-\lambda(x-1)}, \qquad \Omega(Z) = \Omega_0\,\mathrm{e}^{2\lambda(x-1)}. \tag{17}$$

Using the terminology of Pierce [22], Eq. (16) may be considered as an augmented Burgers equation. Clearly (16) can be generalised to take account of multiple relaxation modes

$$V_Z - VV_\tau = \epsilon_0 Q_1(Z) V_{\tau\tau} + Q_2(Z)\sum_\nu K_\nu \mathcal{L}_\nu(V). \tag{18}$$

Alternatively, for one relaxation mode the integro-differential equation can be converted to the form

$$\left(1 + \Omega_0 E(Z)\frac{\partial}{\partial\tau}\right)(V_Z - VV_\tau - \epsilon_0 Q_1(Z) V_{\tau\tau}) = \Gamma_0 Q_1(Z) V_{\tau\tau}, \tag{19}$$

where

$$\Gamma_0 = K\Omega_0, \qquad E(Z) = \mathrm{e}^{2\lambda(x-1)}. \tag{20}$$

The governing equations must then be solved subject to the boundary condition

$$V(Z=1,\tau) = \begin{cases} -\tau, & |\tau| < 1, \\ 0, & |\tau| > 1. \end{cases} \tag{21}$$

In deriving the model equation, assumptions were made about the relative sizes of certain parameters. However, these assumptions place no restriction on the sizes of the dimensionless parameters Ω_0 and Γ_0 characterising each relaxation mode, nor on the parameter Δ_0 describing the magnitude of the thermoviscous diffusivity. In Section 3, we assume that Ω_0 and Γ_0 are small and comparable in magnitude, and that diffusivity may be neglected. The waveform then evolves as a spreading N-wave together with narrow relaxation dominated shock regions. The asymptotic analysis is developed in Section 3.

3. Asymptotic theory

We now consider the case when a single relaxation mode is present, with Ω_0 and Γ_0 both $\mathrm{O}(\delta)$, with diffusivity $\mathrm{o}(\delta)$. In this limit, the outer solution of (19) is given by

$$V(Z,\tau) = \begin{cases} -\dfrac{\tau}{Z}, & |\tau| < Z^{1/2}, \\ 0, & |\tau| > Z^{1/2}. \end{cases} \tag{22}$$

Shocks controlled by the relaxation mechanism must then be inserted at $\tau = \pm Z^{1/2}$. Rescaling this shock region, once the amplitude governed by nonlinear spreading is factored out, the leading-order solution is the familiar steady-state relaxing shock profile [12–14], once the amplitude governed by nonlinear spreading is factored out. Numerical solutions can be obtained for more than one relaxation mode [11]. Here we consider only one relaxation mode since an analytic (albeit implicit) leading-order solution is available. We then focus on when this composite description of lossless outer with inserted relaxation shocks breaks down as the leading-order wave solution. This can come about in the following three ways:

1. the shock becomes relatively wide;
2. the relaxation shock is no longer the leading-order solution in the shock region;
3. the shock centre is displaced by an $\mathrm{O}(1)$ amount.

Hence the next order term in the shock description must be calculated. The analysis follows the methods of Crighton and Scott [6], who consider a one-dimensional wave through a homogeneous medium. One difference in the approach taken here is that new variables are introduced so that the lossless outer solution is independent of time. This has the added advantage of making the numerical solution easier to obtain accurately. Defining

$$w = VZ^{1/2}, \qquad y = \tau Z^{-1/2}, \tag{23}$$

(19) becomes

$$\left(1 + \frac{\Omega_0 E}{Z^{1/2}}\frac{\partial}{\partial y}\right)(2Zw_Z - 2ww_y - w - yw_y - 2\epsilon_0 Q_1 w_{yy}) = 2\Gamma_0 Q_1 w_{yy}. \tag{24}$$

Since we are concerned with Ω_0 and Γ_0 small, with the diffusive effects smaller than the relaxation effects, we set

$$\Omega_0 E Z^{-1/2} = A\delta, \qquad \Gamma_0 Q_1 = \tfrac{1}{2}B\delta, \qquad \epsilon_0 Q_1 = \tfrac{1}{2}C\epsilon, \tag{25}$$

with $\epsilon = \mathrm{o}(\delta)$. This gives

$$\left(1 + \delta A\frac{\partial}{\partial y}\right)(2Zw_Z - 2ww_y - w - yw_y - \epsilon C w_{yy}) = \delta B w_{yy}, \tag{26}$$

where the range-dependent parameters are given implicitly as

$$A = A_0 Z^{-1/2}\,\mathrm{e}^{2\lambda(x-1)}, \qquad B = B_0 x^{1/2}\,\mathrm{e}^{\lambda(x-1)}, \qquad C = C_0 x^{1/2}\,\mathrm{e}^{\lambda(x-1)}. \tag{27}$$

The outer solution is given by

$$w = -y + \mathrm{O}(\delta^n) \quad \text{for all}\, n > 0 \tag{28}$$

for $|y| < 1$, with shocks inserted at $y = \pm 1$. In the following analysis we consider the shock structure at $y = -1$, the leading shock, though corresponding results for the other shock readily follow. Rescaling this shock,

$$Y = \frac{y+1}{\delta} \tag{29}$$

gives

$$\left(1 + A\frac{\partial}{\partial Y}\right)(2ww_Y - w_Y) + Bw_{YY} = \delta\left\{\left(1 + A\frac{\partial}{\partial Y}\right)\left(2Zw_Z - w - Yw_Y - C\frac{\epsilon}{\delta^2}w_{YY}\right)\right\}. \tag{30}$$

Writing $w = W + \delta w_1 + \mathrm{O}(\delta^2)$, matching conditions to the outer are given by

$$W, w_1 \to 0 \quad \text{as} \quad Y \to -\infty, \qquad W \to 1, w_1 \to -Y \quad \text{as} \quad Y \to \infty. \tag{31}$$

3.1. *Leading-order shock solution*

Solving the leading-order equation, we obtain

$$W_Y = \frac{W(1-W)}{2AW + (B-A)} \tag{32}$$

and hence

$$Y - S(Z) = (B-A)\log W - (A+B)\log(1-W). \tag{33}$$

Here $S(Z)$ gives the position of the shock centre and is undetermined at this order, instead being calculated from solvability conditions at the next order. It follows from (33) that the relative width of the relaxation shock, i.e. the actual shock width compared to the overall disturbance scale is $\mathrm{O}(\delta \max(A, B))$.

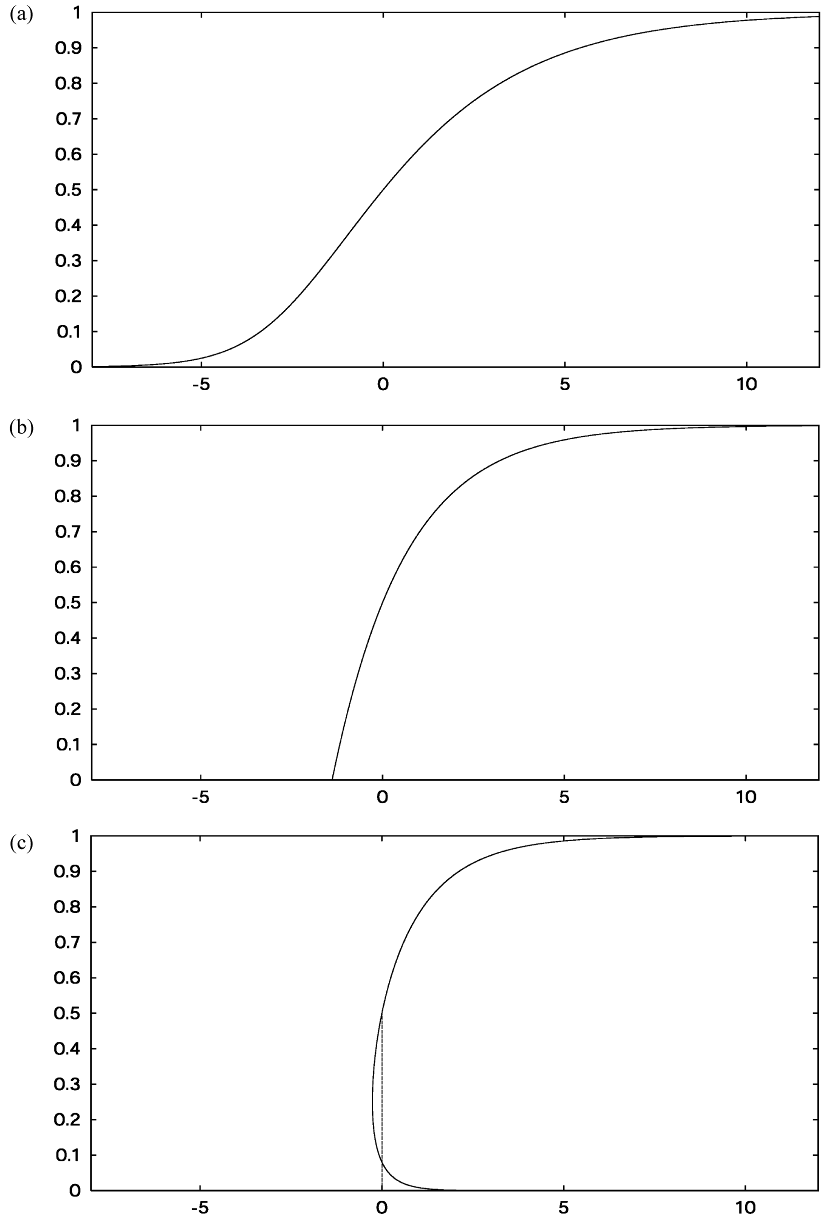

Fig. 2. Illustration of the leading-order shock profiles given by (33) for $A = 1.0$ and (a) $B = 2.0$; (b) $B = 1.0$; (c) $B = 0.5$.

The shock profile $W(Y, Z)$ is shown in Fig. 2 for $A = 1.0$ and different values of B in order to illustrate the different types of shock structure possible. If $B > A$, $W(Y, Z)$ given by (33) satisfies the required matching conditions to the outer shock. Thus the shock structure is entirely determined by a balance between relaxation mechanisms and nonlinearity and is said to be fully dispersed [1,13]. The shock profile for $A = 1.0$, $B = 2.0$ is illustrated in Fig. 2a. When $A = B$, there is a discontinuity in W_Y at $W = 0$, which is illustrated in Fig. 2b. However, if $B < A$, the boundary condition as $Y \to -\infty$ is not satisfied by (33). Hence, relaxation alone cannot support the shock and a sub-shock controlled by other physical mechanisms must be inserted. This is discussed further in Section 3.4. Fig. 2c shows a partly dispersed shock profile for $B = 0.5$. The dotted line shows the position of the inserted sub-shock using methods described later in this paper. At this stage all that should be noted is that transition of the shock from fully dispersed to partly dispersed leads to the appearance of a narrower length scale, or equivalently a shorter timescale, due to the sub-shock.

It may be noted that (33) does not match back to the initial condition (21) as $Z \to 1$. Formally, an embryo-shock region should be inserted which describes the evolution of the discontinuous initial condition to the relaxing shock structure (33). However, this region is of purely academic interest since the initial discontinuous profile was chosen as an approximation of the disturbance at the specified initial location. In practice, a finite-width shock develops during the initial propagation away from the source when the current theory is not valid due to geometric effects more complicated than cylindrical spreading.

Since the coefficients A and B are known functions of propagation distance X, the nature of the relaxation shock at different ranges can be determined. Substituting the expressions for A and B, the condition for a fully dispersed solution becomes

$$f(X) \equiv \frac{e^{\lambda(x-1)}}{x^{1/2}Z^{1/2}} < \frac{B_0}{A_0} = 2K_0. \tag{34}$$

Since $f(X = X_0) = 1$, if $K_0 < \frac{1}{2}$ the shock will initially be partly dispersed. For horizontal ray paths ($\lambda = 0$) and rays below the horizontal ($\lambda < 0$), $f(X)$ is a decreasing function and hence the shock will always become fully dispersed. The large-X forms for the range-dependent parameters A, B and f are given in Appendix A. Above the horizontal, the analysis is more complicated. At large-X, $f(X)$ is an increasing function for all α, but if $\lambda < \frac{1}{2}(1 + X_0)$ then $f(X)$ initially decreases. This is true if

$$\cos\theta < \frac{1 + X_0}{\alpha X_0} \tag{35}$$

in which case we can show that $f(X)$ decreases to some minimum f_m at X_m, then increases monotonically for all $X > X_m$. Otherwise $f(X)$ is a monotonic function of X. The variation of f as a function of $x = X/X_0$ is illustrated in Fig. 3 for different values of λ. Thus if (35) is satisfied and the relaxation parameters satisfy

$$1 > 2K_0 > f_m \tag{36}$$

from Fig. 3c it appears that the wave will be initially partly dispersed, then become fully dispersed, before becoming partly dispersed again. Hence we expect to see a thin inner shock controlled by viscosity, which disappears at some finite range, then reappears at some greater range. The change in the shock structure for the different parameter ranges is summarised in Table 1. In Fig. 4, the value of $f_m(\lambda)$ is plotted for $0 < \lambda < \frac{1}{2}(1 + X_0)$. Thus it is seen that for rays slightly above the horizontal ($0 < \lambda \ll 1$) there is a range of values K_0 for which we see a transition from partly dispersed to fully dispersed and back to partly dispersed, with the associated change in shock rise-time. Moreover, for rays close to the horizontal, the fully dispersed phase extends to large ranges. The location, $X = X_2$, at which fully dispersed shocks become partly dispersed is given by

$$\frac{e^{\lambda(x_2-1)}}{\sqrt{x_2 Z(x_2)}} = 2K_0, \quad x_2 = \frac{X_2}{X_0}. \tag{37}$$

For rays just above the horizontal, $\lambda \to 0^+$, and it is clear that $x_2 \to \infty$ with $\lambda x_2 \to \infty$. Hence

$$Z(x_2) \to \frac{X_0}{\lambda^{1/2}} \frac{e^{\lambda x_2}}{(\lambda x_2)^{1/2}}, \tag{38}$$

which leads to an implicit equation for $x_2(\lambda)$

$$\frac{e^{2\lambda x_2}}{\lambda x_2} = 4K_0^2 X_0 \frac{1}{\lambda^3}. \tag{39}$$

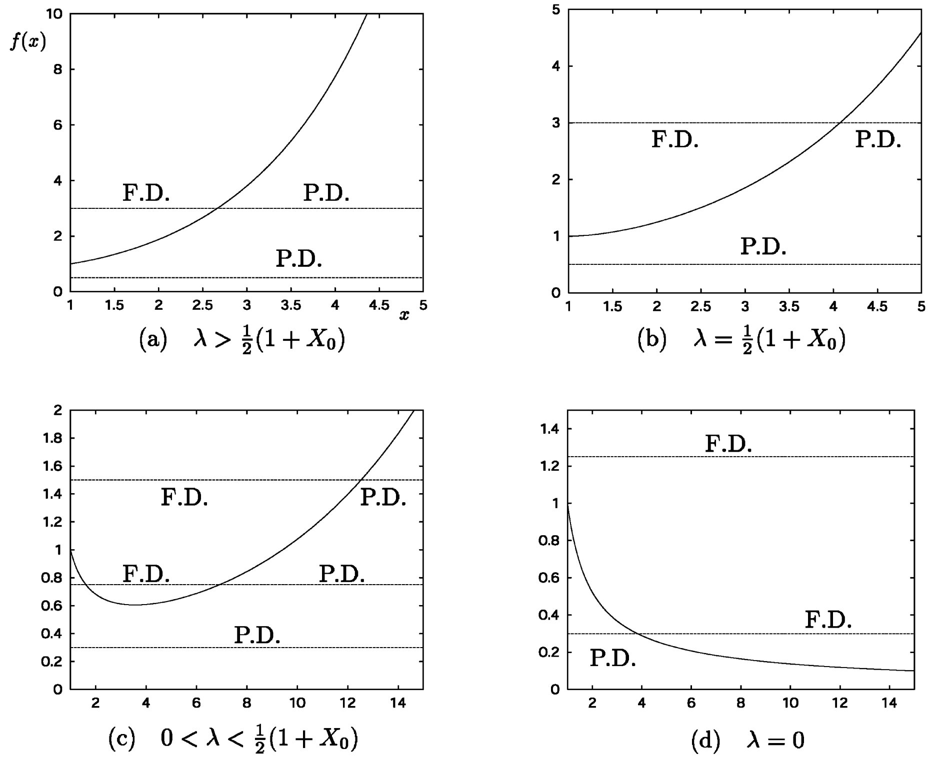

Fig. 3. Variation of f as a function of x for $X_0 = 1$ with: (a) $\lambda = 1.5$; (b) $\lambda = 1.0$; (c) $\lambda = 0.5$; (d) $\lambda = 0.0$. Regions where the shock is fully dispersed (F.D.) and partly dispersed (P.D.) are marked for various values of K_0.

Table 1
Classification of shock structure for all ray paths, dependent on initial conditions[a]

Case I. Above horizontal with $\cos\theta > (1 + X_0)/\alpha X_0$		
$K_0 < \frac{1}{2}$	Partly dispersed	
$K_0 > \frac{1}{2}$	Fully dispersed	$X < X_2$
	Partly dispersed	$X > X_2$
Case II. Above horizontal with $\cos\theta < (1 + X_0)/\alpha X_0$		
$K_0 < \frac{1}{2} f_{\mathrm{m}}$	Partly dispersed	
$\frac{1}{2} f_{\mathrm{m}} < K_0 < \frac{1}{2}$	Partly dispersed	$X < X_1$
	Fully dispersed	$X_1 < X < X_2$
	Partly dispersed	$X_2 < X$
$\frac{1}{2} < K_0$	Fully dispersed	$X < X_2$
	Partly dispersed	$X_2 < X$
Case III. Horizontal rays and rays below the horizontal		
$K_0 < \frac{1}{2}$	Partly dispersed	$X < X_1$
	Fully dispersed	$X > X_1$
$K_0 > \frac{1}{2}$	Fully dispersed	

[a] Here the ranges $X_1(\lambda)$ and $X_2(\lambda)$ denote the roots of $f(X) = 2K_0$.

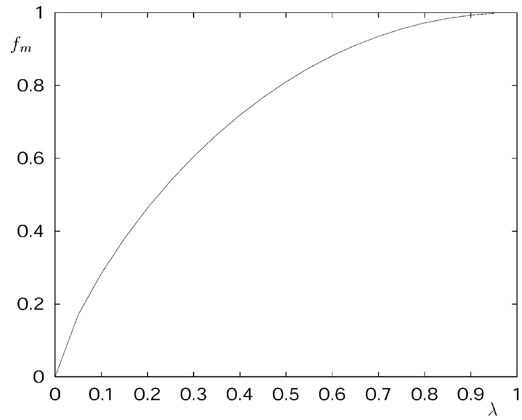

Fig. 4. The minimum value, $f_{\rm m}$, of the function $f(x)$, defined in (34), as a function of λ.

Solving for small λ gives the perturbation solution

$$X_2 = \frac{1}{2}X_0\frac{1}{\lambda}\left(3\log\left(\frac{1}{\lambda}\right) + \log\left(\log\left(\frac{1}{\lambda}\right)\right) + \log(6K_0^2X_0) + {\rm o}(1)\right). \tag{40}$$

The regions of occurrence of fully dispersed and partly dispersed shocks is illustrated more clearly in Section 4 when the shock structure in the vertical plane containing the ray path is analysed.

3.2. First-order correction term for fully dispersed shock

From (30), ignoring the thermoviscous term, the O(δ) correction in the shock region satisfies

$$\left(1 + A\frac{\partial}{\partial Y}\right)((2W-1)w_1)_Y + B(w_1)_{YY} = \left(1 + A\frac{\partial}{\partial Y}\right)(2ZW_Z - W - YW_Y), \tag{41}$$

where $W(Y, Z)$ is the leading-order shock solution defined by (33). Integrating once with respect to Y gives

$$\left(1 + A\frac{\partial}{\partial Y}\right)((2W-1)w_1) + B(w_1)_Y = 2ZI_Z - YW + A(2ZW_Z - W - YW_Y) + G(Z), \tag{42}$$

where $I = \int W\,{\rm d}Y$ and $G(Z)$ is a function which will ultimately be determined by matching to the outer solution. The prospect of solving this equation looks unlikely, since although it is linear and first order, the coefficients are defined implicitly. However, progress can be made. New functions $C_1(Z) = A(Z) + B(Z)$ and $C_2(Z) = A(Z) - B(Z)$ are introduced. With this notation, the leading-order solution is given by

$$Y - S = -C_2\log W - C_1\log(1 - W), \tag{43}$$

with derivatives

$$W_Z = \frac{W(1-W)}{2AW - C_2}\{C_2'\log W + C_1'\log(1-W) - S'\}, \tag{44}$$

$$W_Y = \frac{W(1-W)}{2AW - C_2}. \tag{45}$$

Substituting these expressions into (42) gives

$$(2AW - C_2)(w_1)_Y + (2AW_Y + 2W - 1)w_1 = f(W, Z), \tag{46}$$

where

$$f=\bar{Q}+C_1\log(1-W)-(2\bar{A}-A+\bar{S})W+A\bar{q}W_Y,$$
$$\bar{Q}=\bar{C}_2W\log W-\bar{C}_1(1-W)\log(1-W),\quad \bar{q}=\bar{C}_2\log W+\bar{C}_1\log(1-W)-\bar{S}. \tag{47}$$

Here $\bar{A}=A(Z)+2ZA'(Z)$, etc. The function $G(Z)$ has been substituted using the matching conditions that $W, w_1\to 0$ as $Y\to-\infty$. The other matching condition is that $w_1\to -Y$ and $W\to 1$ as $Y\to\infty$, and this condition fixes $S(Z)$,

$$S(Z)=S_0-2A-\int\frac{B}{2Z}\,\mathrm{d}Z. \tag{48}$$

Solving (46) then gives

$$\frac{(2AW-C_2)w_1}{W(1-W)}=\int\frac{(2AW-C_2)f}{W^2(1-W)^2}\,\mathrm{d}W. \tag{49}$$

Substituting for $f(W,Y)$, this finally gives $w_1=\tilde{w}_1-W_YT(Z)$, where $T(Z)$ is an undetermined function and

$$(2AW-C_2)\tilde{w}_1=\sum_{i=1}^{6}a_i(Z)J_i(W). \tag{50}$$

The functions $a_i(Z)$ and $J_i(W)$ are given in Appendix B. The function $T(Z)$ is fixed at the next order of the asymptotic analysis (i.e. $\mathrm{O}(\delta^2)$), but this is unnecessary for the present purposes since

$$w=W(Y-S)+\delta(\tilde{w}_1-W_YT(Z))+\mathrm{O}(\delta^2)=W(Y-S-\delta T)+\delta\tilde{w}_1+\mathrm{O}(\delta^2), \tag{51}$$

and so the term involving $T(Z)$ corresponds to an $\mathrm{O}(\delta)$ shift in the centre of the relaxation shock.

3.3. Breakdown of asymptotic structure for a fully dispersed shock

The reason for considering the $\mathrm{O}(\delta)$ correction to the relaxation shock was in order to study the possible breakdown of the composite asymptotic structure of outer solution with inserted relaxation shock. Restricting attention to rays below the horizontal, when the shock is fully dispersed at large ranges, we can then consider whether the shock remains relatively narrow, whether the displacement of the shock centre becomes large and whether (33) remains valid as a leading-order approximation of the shock.

An estimate of the shock width is given by

$$\left(\frac{W}{W_y}\right)_{y=\mathrm{O}(1)}=\delta\frac{C_1-2AW}{1-W}\sim\delta x^{1/2}\,\mathrm{e}^{-|\lambda|x}\quad\text{as}\quad X\to\infty, \tag{52}$$

using the large-X form of $A(Z)$ and $B(Z)$ given in Appendix A. Thus the shock does not become wide, but becomes thinner at large ranges.

As $X\to\infty$, the shock centre $S\sim\mathrm{e}^{-2|\lambda|x}$ and hence the displacement of the shock centre never becomes O(1) on the outer scaling. Finally, using the large-X forms of the functions $a_i(Z)$ given in Appendix B,

$$\frac{\delta\tilde{w}_1}{W}\sim\delta X\quad\text{as}\quad X\to\infty. \tag{53}$$

Hence the leading-order shock solution (33) becomes invalid when $X=\mathrm{O}(\delta^{-1})$. To complete the asymptotic analysis, the shock region should be rescaled at this range in order to obtain the new leading-order shock description.

However, at this range $B \gg A$ and hence the leading-order shock solution has approached the Taylor shock profile. Thus the subsequent asymptotic analysis follows that for a stratified atmosphere with no relaxation mechanisms present [6].

3.4. Embedded sub-shocks in partly dispersed relaxing shocks

When $B < A$, the leading-order solution (33) does not satisfy the required boundary condition as $Y \to -\infty$. If we assume that thermoviscosity controls the sub-shock, the governing equation in the shock region is

$$\left(1 + A\frac{\partial}{\partial Y}\right)\left(2ww_Y - w_Y - C\frac{\epsilon}{\delta}w_{YY}\right) + Bw_{YY} = 0. \tag{54}$$

Here we assume $\epsilon \ll \delta$, otherwise the whole shock would be controlled by thermoviscous effects. Formally, we also assume that $\delta^2 \ll \epsilon$, so that the first correction term to the shock solution is controlled by viscosity rather than by the relaxation mode. However, our investigation of the $\mathrm{O}(\delta)$ correction to a fully dispersed shock suggests that inclusion of higher-order terms does not resolve the transition to a partly dispersed structure.

In Fig. 2c, we illustrated how for $B < A$, a sub-shock is embedded in the relaxation shock at some position $Y = Y_1(Z)$, where Y_1 is still to be determined. The partly dispersed shock solution is then given by $W = 0$ for $Y < Y_1$, while for $Y > Y_1$, W is given implicitly by (33) with the solution branch chosen such that $W \to 1$ as $Y \to \infty$. A viscous sub-shock resolves the discontinuity in W at $Y = Y_1(Z)$. Rescaling this sub-shock by setting $\tilde{Y} = (Y - Y_1)\delta/\epsilon$, gives

$$\tilde{w} = \frac{A-B}{2A}\left(\tanh\left(\frac{\tilde{Y}}{D}\right) + 1\right), \qquad D(Z) = \frac{2AC}{A-B}. \tag{55}$$

Thus the sub-shock has amplitude $W_s = (A - B)/A$, width $\epsilon D(Z)$ (based on the original y-scale) and the location of the centre of the sub-shock is given by

$$Y_s = S + (B - A)\log W_s - (A + B)\log(1 - W_s). \tag{56}$$

Finally the position of the centre of the relaxation shock must be calculated. Instead of applying a solvability condition at higher order as in Section 3.2, it proves easier to calculate $S(Z)$ by integral methods.

The asymptotic description of the relaxing shock previously determined can be written in composite form for $y < 0$

$$w = \mu(Y)\{W - H(Y) + \delta(w_1 + YH(Y)) - yH(Y)\}, \tag{57}$$

where $H(Y)$ is the Heaviside function, and $\mu = 1$ for a fully dispersed shock and $\mu = H(Y - Y_s)$ for a partly dispersed shock. This composite description is uniformly valid in $(-\infty, 0]$, except in the vicinity of the viscous sub-shock, and can readily be checked. For $Y = \mathrm{O}(1)$, substituting for y gives $w = \mu(W + \delta w_1)$, the inner solution; while for $Y \to \infty$, we obtain $w \to -y$, the outer solution (28).

Integrating (26) with respect to y, between $-\infty$ and 0, and using the conditions $w(0) = 0$, $w_y(0) = -1$ and $w_{yy}(0) = 0$ from (28), gives

$$2ZI_Z = \delta B + \epsilon C, \tag{58}$$

where $I = \int_{-\infty}^{0} w\,\mathrm{d}y$. Using the composite description for a fully dispersed shock, and recalling that $Y = (y+1)/\delta$, defining $W_1(Z) = W(Y = 0, Z)$, we have

$$\begin{aligned} I &= -\int_{-1}^{0} y\,\mathrm{d}y + \delta\left(\int_{-\infty}^{0} W\,\mathrm{d}Y + \int_{0}^{\infty}(W-1)\,\mathrm{d}Y\right) + \mathrm{O}(\delta^2) \\ &= \frac{1}{2} + \delta\left(\int_{0}^{W_1}\frac{W}{W_Y}\mathrm{d}W + \int_{W_1}^{1}\frac{W-1}{W_Y}\mathrm{d}W\right) + \mathrm{O}(\delta^2), \\ &= \frac{1}{2} + \delta\left(\int_{0}^{W_1}\frac{2AW+(B-A)}{1-W}\mathrm{d}W - \int_{W_1}^{1}\frac{2AW+(B-A)}{W}\mathrm{d}W\right) \\ &= \frac{1}{2} + \delta(-2A + (B-A)\log W_1 - (B+A)\log(1-W_1)) + \mathrm{O}(\delta^2) = \frac{1}{2} + \delta(-2A - S) + \mathrm{O}(\delta^2). \end{aligned}$$

Hence, using (58) and letting $\epsilon/\delta \to 0$, we obtain

$$S(Z) = S_0 - 2A - \int \frac{B}{2Z}\mathrm{d}Z \tag{59}$$

in agreement with (48), which was obtained using the $\mathrm{O}(\delta)$ correction term to the shock structure.

For a partly dispersed shock, where the shock centre cannot easily be determined by considering higher-order terms,

$$\begin{aligned} I &= -\int_{-1}^{0} y\,\mathrm{d}y + \delta\left(\int_{W_\mathrm{s}}^{W_1}\frac{W}{W_Y}\mathrm{d}W + \int_{W_1}^{1}\frac{W-1}{W_Y}\mathrm{d}W\right) + \mathrm{O}(\delta^2) \\ &= \frac{1}{2} + \delta\left(-2B - S + (B+A)\log\left(\frac{B}{A}\right)\right) + \mathrm{O}(\delta^2), \end{aligned}$$

where $W_\mathrm{s} = (A-B)/A$, the amplitude of the viscous sub-shock. Hence the location of the partly dispersed shock is given by

$$S(Z) = S_0 - 2B + (B+A)\log\left(\frac{B}{A}\right) - \int \frac{B}{2Z}\mathrm{d}Z. \tag{60}$$

Letting $B/A \to 1$, the condition for the transition between fully dispersed and partly dispersed shock structure, we see that (59) and (60) are consistent.

4. Summary

In Section 3, the asymptotic structure of relaxing shocks was determined along rays. The change in shock structure can be seen more clearly if results are plotted in the vertical plane containing the ray. Defining $\hat{z}$ to be the non-dimensional vertical coordinate and $\hat{x}$ to be the non-dimensional horizontal coordinate in the plane of the ray, the $(\hat{x}, \hat{z})$ plane is divided into regions of partly dispersed and fully dispersed relaxing shocks. Choosing $X_0 = 1$ and $\alpha = 2\sqrt{2}$, results are plotted for $K_0 = 0.4$ and 0.8. In Fig. 5a, where $K_0 < \frac{1}{2}$, it is seen that for $0 \le \theta \le \theta_\mathrm{c}$ the shock is partly dispersed at all ranges (ray I), while for $\theta_\mathrm{c} < \theta < \pi/2$ the shock is partly dispersed, then becomes fully dispersed before returning to have a partly dispersed structure (ray II). Here θ_c satisfies

$$f_\mathrm{m}(\tfrac{1}{2}\alpha X_0 \cos\theta_\mathrm{c}) = 2K_0. \tag{61}$$

On any ray above the horizontal the shock eventually becomes partly dispersed, though this will be at a large distance from the source location for rays just above the horizontal. Using (39), the transition to partly dispersed structure

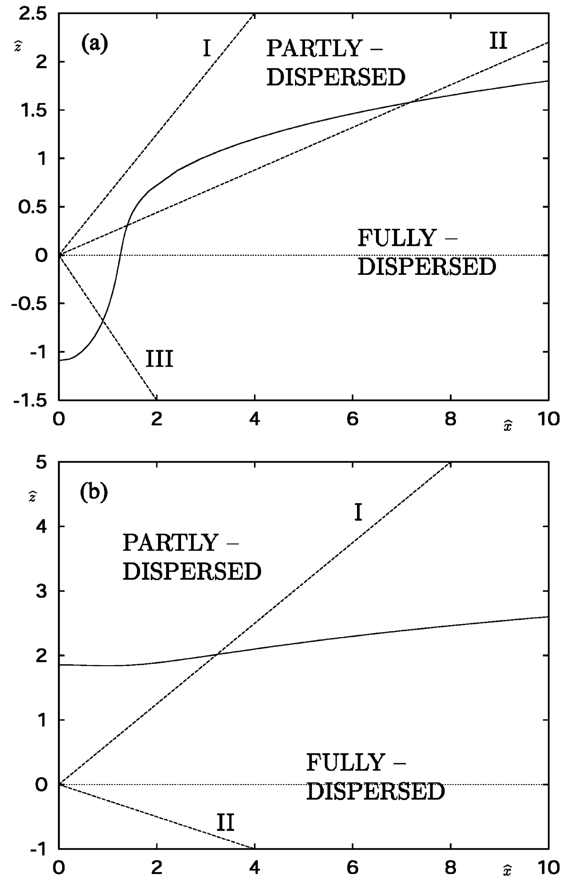

Fig. 5. Illustration of the regions of space where the relaxation shock structure is partly dispersed and fully dispersed for (a) $K_0 < \frac{1}{2}$ and (b) $K_0 > \frac{1}{2}$. The marked ray-paths illustrate the different cases of wave evolution described in the text.

occurs at

$$\hat{x} \sim \left(\frac{\alpha X_0}{2}\right)^{1/2} \left(\frac{1}{2K_0^2 X_0 \alpha}\right)^{1/3} \mathrm{e}^{\alpha \hat{z}/3} \hat{z}^{2/3} \tag{62}$$

as $\hat{x} \to \infty$. This agrees with the numerical results presented in Fig. 5. For $\pi/2 < \theta$ the shock is initially partly dispersed, but then becomes fully dispersed (ray III).

In Fig. 5b, where $K_0 > \frac{1}{2}$, for ray paths above the horizontal the shock is fully dispersed before eventually becoming partly dispersed (ray I), while all rays below the horizontal have fully dispersed shocks throughout their evolution (ray II).

The asymptotic form of the solution at large ranges can also be deduced. Below the horizontal, the shock is fully dispersed, but since $A/B \to 0$ as $X \to \infty$, the shock approaches a Taylor profile which narrows with range. Subsequently the Taylor solution becomes invalid as a leading-order description of the shock region. However, since the effect of the relaxation is purely viscous at this range, the subsequent asymptotic analysis follows theories for media with only thermoviscosity present.

Above the horizontal, the shock is partly dispersed, but since $A/B \to \infty$, the amplitude of the viscous sub-shock $W_\mathrm{s} = (A - B)/A$ tends to unity, and hence the sub-shock subsumes the relaxation shock and viscosity controls the entire shock region. So, as for the case of propagation above the horizontal, the long-range asymptotic development can be determined from an analysis of propagation in a medium without relaxation. From the results of Section 3.4, using (55) we see that the sub-shock becomes relatively wide on the overall disturbance scale when $\epsilon B = \mathrm{O}(1)$, i.e.

$$\epsilon x^{1/2}\,\mathrm{e}^{\lambda x} = \mathrm{O}(1), \tag{63}$$

but becomes wide on the scale of the relaxing shock earlier, when $\delta B = \mathrm{O}(1)$. Combining (56) and (60), the centre of the viscous sub-shock is given by

$$Y_{\mathrm{s}} = S_0 - 2B + (B - A)\log\left(\frac{A - B}{A}\right) - \int \frac{B}{2Z}\,\mathrm{d}Z \tag{64}$$

and $Y_{\mathrm{s}} = \mathrm{O}(B)$ as $X \to \infty$. Hence the displacement of the shock centre becomes O(1) at the same range as the shock becomes relatively wide. However, in the present work we have not considered whether the Taylor solution becomes invalid as a leading-order description before the shock becomes wide. This analysis would be covered by the case when relaxation effects are omitted.

The main conclusion of this paper is that shock waves propagating through a stratified relaxing atmosphere can undergo changes in their internal structure leading to rapid changes in shock width (or equivalently, shock rise-time). Some numerical results for propagation of sonic booms through a stratified relaxing atmosphere are summarised by Cleveland et al. [23]. However, the altitude dependence of the relaxation parameters used in these calculations is not made clear. Also, change in the internal structure of the shocks as range increases is not reported. Hence the asymptotic results of the present paper cannot be compared directly with existing numerical results.

Taking $\lambda = 0$, the conclusions of the present paper can be compared with the analysis of Clarke and Sinai [19,20] for an unstratified medium. If the thickness to length ratio of the supersonic body is $\epsilon_{\mathrm{Cl}} \ll 1$, and $r_0 \gg L$, where L is the length of the body, then the disturbance amplitude at $r = r_0$ is

$$u_0 = k\epsilon^2\left(\frac{L}{r_0}\right)^{1/2} a_0, \tag{65}$$

where $k = \mathrm{O}(1)$, a coefficient determined by the shape of the body. Assuming that $(M^2 - 1)^{1/2}$ is O(1), so the motion of the body lies outside the transonic regime, it then follows that

$$\frac{X_0}{K} = \mathrm{O}\left(\frac{\epsilon_{\mathrm{Cl}}^4}{\delta_{\mathrm{Cl}}}\right), \quad \delta_{\mathrm{Cl}} = \frac{\Delta a}{a_0}, \tag{66}$$

where $\delta_{\mathrm{Cl}} \ll 1$ from the conditions given in Section 2 for the validity of the model equation. Clarke and Sinai identified $\delta_{\mathrm{Cl}} = \mathrm{O}(\epsilon_{\mathrm{Cl}}^4)$ as the scaling of most interest, noting that for this parameter range the shock structure is either fully dispersed at all ranges or is initially partly dispersed and becomes fully dispersed at some critical range. This is in agreement with the present analysis where X_0 is taken to be O(1) and the two cases described by Clarke and Sinai occur depending on the order 1 value of K.

A numerical investigation of (26) will be reported elsewhere. However, a sample calculation is included here to illustrate the changes in shock structure predicted by asymptotic analysis. Using an implicit finite-difference scheme, with mesh points concentrated in the shock regions, (26) was solved for

$$\delta = 0.005, \qquad \epsilon = 0.0001, \qquad A_0 = 1, \qquad B_0 = 2, \qquad \lambda = 1.5, \tag{67}$$

with $X_0 = 1$. Hence $K_0 = B_0/2A_0 = 1$ and $\lambda > \frac{1}{2}(1 + X_0)$, so asymptotic theory predicts that the relaxing shock is initially fully dispersed, then becomes partly dispersed (see Fig. 3a). Two values of the diffusivity parameter were chosen, $C_0 = 1$ and 5, in order to clarify the appearance of viscous sub-shocks. The quantity

$$d(x) = \min_y\left(\frac{1}{|W_y|}\right) \tag{68}$$

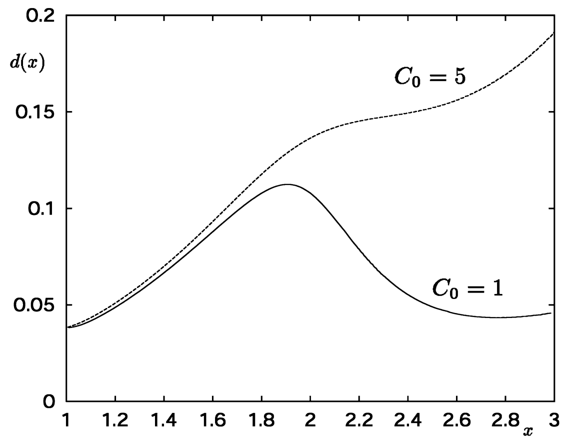

Fig. 6. Numerical results for shock width, $d(x)$, along a ray above the horizontal, illustrating the transition from fully dispersed to partly dispersed structure. Results for two values of thermoviscous diffusivity are plotted.

can be considered as a measure of shock width. In Fig. 6, $d(x)$ is plotted for the parameter values given above. For the parameters chosen, the ray is above the horizontal, and the asymptotic results obtained in Section 3 predict that the shock will become partly dispersed at $x = 2.09$. Hence we expect the appearance of a narrower shock at this range associated with the viscous sub-shock. When $C_0 = 5$, little change in shock width is noted at this range. If the diffusivity is reduced, $C_0 = 1$, the shock width is almost unchanged for $x < 2$ when the shock is fully dispersed, but a sharp reduction of shock width is seen for $x > 2$. Thus these numerical results appear to confirm the predictions of the asymptotic analysis.

Appendix A

In Section 3, the large-X forms of A, B and f are required. The asymptotic results are given here, in terms of $x = X/X_0$, with multiplicative constants omitted.

1. Above horizontal ($\lambda > 0$):

$$Z \sim x^{-1/2}\,\mathrm{e}^{\lambda x}, \qquad A \sim x^{1/4}\,\mathrm{e}^{3\lambda x/2}, \qquad B \sim x^{1/2}\,\mathrm{e}^{\lambda x}, \qquad f \sim x^{-1/4}\,\mathrm{e}^{\lambda x/2}.$$

2. Horizontal ($\lambda = 0$):

$$Z \sim x^{1/2}, \qquad A \sim x^{-1/4}, \qquad B \sim x^{1/2}, \qquad f \sim x^{-3/4}.$$

3. Below horizontal ($\lambda < 0$):

$$Z \sim Z_\infty, \qquad A \sim \mathrm{e}^{-2|\lambda|x}, \qquad B \sim x^{1/2}\,\mathrm{e}^{-|\lambda|x}, \qquad f \sim x^{-1/2}\,\mathrm{e}^{-|\lambda|x}.$$

Appendix B

The $\mathrm{O}(\delta)$ correction to the relaxing shock solution takes the form $w_1 = \tilde{w}_1 - W_Y T(Z)$, where

$$(2AW - C_2)\tilde{w}_1 = \sum_{i=1}^{7} a_i(Z) J_i(W). \tag{B.1}$$

Here the functions $a_i(Z)$ are given by

$$a_1 = \bar{C}_2 C_1, \qquad a_2 = \bar{C}_1(A - C_1) + C_1(C_1 - C_2), \qquad a_3 = \bar{C}_2(A - C_2), \qquad a_4 = C_2(\bar{C}_1 - C_1),$$
$$a_5 = -C_1 S, \qquad a_6 = C_2(C_2 + S) - A\bar{S}, \qquad a_7 = C_1^2,$$

and the functions $J_i(W)$ are given by

$$J_1 = W((1 - W)\log(1 - W) + W\log W), \qquad J_2 = -W(1 - W)(\tfrac{1}{2}(\log(1 - W))^2 + \mathrm{dilog}(1 - W)),$$
$$J_3 = W(1 - W)(\tfrac{1}{2}(\log W)^2 + \mathrm{dilog}(W)), \qquad J_4 = -(1 - W)(W\log W + (1 - W)\log(1 - W)),$$
$$J_5 = W, \qquad J_6 = W(1 - W)(\log W - \log(1 - W)), \qquad J_7 = W\log(1 - W),$$

where dilog is the dilogarithm function defined by [24]

$$\mathrm{dilog}(x) = \int_1^x \frac{\log y}{1 - y}\,\mathrm{d}y.$$

An alternative notation for the dilogarithm function is $\mathrm{Li}_2(x)$ [25] with $\mathrm{Li}_2(x) = \mathrm{dilog}(1 - x)$.

In the limit $W \to 0$, $J_i \to 0$ for all i and hence $w_1 \to 0$ as required. For $W \to 1$, J_1, J_2, J_3, J_4, $J_6 \to 0$, with $J_5 \to 1$ and

$$J_7 \sim \log(1 - W) = \frac{S - Y}{C_1}$$

using (43). Hence in this limit, $w_1 \to -Y$ as required.

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