On q–Simplicial Posets

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Abstract

Much study of simplicial complexes and the more general simplicial posets has been undertaken. In this thesis we introduce q-analogues (q a prime power) of these two familiar objects, called q–simplicial complexes and q–simplicial posets respectively. A simplicial complex is linked to finite sets and its subsets, the q–analogue is linked to a vector space (over the finite field of q elements) and its subspaces. This extends an idea of Gian-Carlo Rota in a 1971 paper. We build the language to describe these structures and obtain results that form the foundation of a study of these mathematical objects. This thesis, rather than one main result, builds up a string of results on q–simplicial posets. Important examples are described with emphasis on those associated to the subgroup lattice of finite groups.

A focus is on the characterization of q–simplicial complexes, that is determining when q–simplicial posets form q–simplicial complexes. In the ordinary simplicial setting the problem is straightforward, while in the q-analogue setting the problem turns out to be extremely non-trivial. We exhibit two different approaches with the first giving a sufficiency criteria for the important examples associated to groups and giving modular representations of these groups. In the second approach a more generally applicable result is obtained by considering a generalization of the vertex sets of simplicial complexes to the q-analogue setting. We exhibit a class of q–simplicial posets which contain a simplicial complex as a subposet with this containment preserving certain properties. An important example is to be found in the subgroup lattice of the Symmetric Group.

We also apply the ideas of modular homology to sequences of modules associated to the incidence structure of q–simplicial posets. We use these modules to obtain modular representations of finite groups.

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Stuart Alder, September 2010.

"Perhaps the most surprising thing about mathematics is that it is so surprising. The rules which we make up at the beginning seem ordinary and inevitable, but it is impossible to foresee their consequences."

- E.C. Titchmarsh as quoted in 'Mathematical Maxims and Minims' by N. Rose.

Introduction

Simplicial complexes are well known mathematical objects which have received much attention in the literature. In this thesis we begin the exploration of a 'q–analogue' of both simplicial complexes and their close mathematical associates, the so called simplicial posets. The standard simplicial case is represented by the parameter $q = 1$ case and the q–analogue by when the parameter q is a prime power. We then call the analogues respectively *q–simplicial complexes* (*q–complexes* for short) and *q–simplicial posets* (*q–posets* for short).

In the way that a simplicial complex is intrinsically linked to a finite set and a collection of its subsets so our q–analogue (q a prime power) is linked to a vector space (over the finite field of q elements) and a collection of its subspaces. This idea for a q–analogue of a simplicial complex was first suggested (it would appear) in a 1971 paper by Gian-Carlo Rota ([43]). Surprisingly subsequent to that publication we find no explicit mention of these q–analogues. Thus the area remains open to a full description and an investigation of these posets. This text then represents the first step on this path and there remains many fertile areas associated which would suggest the possibility of further study.

We place the standard simplicial case in a setting which allows us to easily describe the combinatorial generalization that we seek. Indeed throughout we try to draw parallels to the standard simplicial case and where possible obtain results analogous to those already know or derived here in the standard case. As such this thesis has as its aim not a single theorem but a string of results which give understanding on the mathematical structures being discussed and where possible tie them into existing results or directly extend such results.

The q–complexes form a subclass of q–posets in an analogous way that simplicial complexes are a subclass of simplicial posets. Characterizing this relation will form an area of focus. Whilst in the standard $q = 1$ a straightforward result can be realized, in the q a prime power case we encounter an extremely knotty problem. We do however make some progress in certain areas. Indeed a theme for later chapters is a search for some insight into the solution to this problem. A major aid throughout the study is the algebraic computer system GAP ([16]) and where this is done we reproduce the scripts used.

A major motivation for this work is that these q–analogues of simplicial posets and simplicial complexes arise within the lattice of subgroups. Specifically they arise from elementary abelian q subgroups (where q is a prime). Such examples from group theory will form a focus of much of our work as they give the potential to explore representations of the groups in question. Additionally it is in this area that we can achieve some progress in determining the characterizations of q–complexes.

Outline of Work

We give here a brief outline of the contents of each of the chapters contained in this text:

Chapter 1

Here we lay the groundwork for the rest of the text. In this Chapter we use the language of partially ordered sets to describe q–posets that form the focal point of the text. We set out our standard notation and terminology and collect some basic results about embeddings which will become important later. The second half of the chapter then contains many examples of q–complexes and q–posets including an interesting example for $Alt(5)$, the alternating group on 5 elements, directly related to one of its irreducible modular representations. We briefly discuss some standard concepts of the study of simplicial complexes as applied to our q-analogues.

Chapter 2

In this chapter we deal with combinatorial results for q–simplicial complexes and q–simplicial posets. We extend a result on the characterization of f–vectors of simplicial posets to show that it holds for all q–posets. We go on to consider the incidence matrices associated to q–posets and obtain a new characterization of simplicial posets in terms of the associated incidence matrices. Furthermore we derive the rank of the incidence matrices of q-posets in the characteristic zero setting and an explicit expression for the Möbius function for q–posets.

Chapter 3

We look here to determine when q–posets are q–complexes. In the $q = 1$ standard case this turns out to only require an easily determined condition on the poset. In the case $q \neq 1$ the position is non-trivial. We do however obtain a positive result in the case that the q–posets are associated to the elementary abelian subgroups of some finite group G. Here we show a sufficiency criteria for such q–posets and exhibit examples where this criteria is met. From these we obtain modular representations of these groups.

Chapter 4

Simplicial complexes are defined on a vertex set and we describe an analogous set for certain q–posets. Where these vertex sets exist with a 'nice' property we show that this indicates the q–poset is a q–complex. Under certain conditions a q–poset, which admits a vertex set, contains a subposet isomorphic to a simplicial complex. This containment preserves properties including shellability. We describe examples of such q–posets and in particular spend sometime on an important example associated to the Symmetric Group. Indeed we prove that when q is a prime then all such q–posets with a 'nice' vertex set are isomorphic to a subposet of that associated to the Symmetric Group.

Chapter 5

From the incidence structure of a q–poset we can derive, under certain conditions, a homological sequence. The homology of interest here is *modular homology*. Much work exists on modular homology for simplicial complexes and we look at homological sequences associated to q–posets. In particular we derive completely the dimension of homology modules for a particular class of q–complexes. We also speculate on a general result for q–posets associated to finite groups.

Appendices

We exhibit here group representations derived from the q–posets associated to certain finite groups; consider various combinatorial aspects of one of the important complexes of Chapter 4; undertake example calculations of the modular homology of Chapter 5; and also include here some of the GAP scripts used in obtaining the results in the main chapters and appendices.

Conventions Used in the Text

Throughout this text we follow the convention that any result or definition that is unattributed is new and is not replicated from any known source, to the best of our knowledge. While we endeavour to maintain conformity with definitions found in existing literature, where appropriate we make clear any divergence from the norm.

Chapter 1

Preliminaries and Groundwork

In this chapter we introduce *q–simplicial posets* and *q–simplicial complexes*, the primary objects of interest in this text. These mathematical structures are best described in the language of *partially ordered sets* and this will be the starting point of our investigations in this chapter. Once we have collected our definitions we will illustrate these by looking at key examples of q–simplicial posets and q–simplicial complexes.

1.1 The Mathematical Language of Posets

As Stanley says in his book ([57] p.96) partially ordered sets (posets for short), play an important role in the theory of enumerative combinatorics and are of interest in their own right. We will in this section briefly recall the definitions, terminology and notation associated to posets and which will be used to describe the complexes in this text.

1.1.1 Poset Notation and Terminology

Let us start our exploration of posets by setting out the standard definition (see [57] p.97–99). Note throughout this text all sets we deal with are finite.

Definition 1.1.1. A *partially ordered set* (*poset* for short) is a set $\mathcal{P}$ together with a binary relation $\leqslant$, denoted $(\mathcal{P}, \leqslant)$, satisfying the following three axioms:

1. For all $x \in \mathcal{P}$, $x \leqslant x$ (reflexivity).

2. If $x \leqslant y$ and $y \leqslant x$, then $x = y$ (antisymmetry).
3. If $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$ (transitivity).

If $x \leqslant y$ and $x \neq y$ then we write $x < y$. Additionally we say that two elements x, y in $\mathcal{P}$ are *comparable* if $x \leqslant y$ or $y \leqslant x$, otherwise they are *incomparable*.

We will normally use $\mathcal{P}$ to signify the poset $(\mathcal{P}, \leqslant)$, if the meaning is clear. Where clarity is required we will use $\leqslant_{\mathcal{P}}$ to indicate the binary relation/order with respect to the poset $\mathcal{P}$. We now gather some basic notation and terminology used throughout this text.

Notation. - For $x, y \in \mathcal{P}$ the *interval* $[x, y]$ is the set $\{z \in \mathcal{P} \mid x \leqslant z \leqslant y\}$. If $x \nleqslant y$ then $[x, y] = \varnothing$. For any pair $x, y \in \mathcal{P}$ we say y *covers* x if $x < y$ and $[x, y] = \{x, y\}$. This is denoted by $x \prec y$.
- The *dual* of $\mathcal{P}$ is the poset $\mathcal{P}^*$ which has the same underlying set as $\mathcal{P}$ but with $x \leqslant y$ in $\mathcal{P}^*$ if and only if $y \leqslant x$ in $\mathcal{P}$.
- A minimal element of a poset $\mathcal{P}$ is any element $x \in \mathcal{P}$ such that there exists no $y \in \mathcal{P}$ with $y < x$. If a poset has a unique minimal element we will denote such an element as 0. In the obvious way we also define a maximal element, which we call a *facet* of the poset.
- For an element x of $\mathcal{P}$ we define $\mathcal{P}_{\leqslant x} := \{y \in \mathcal{P} \mid y \leqslant x\}$ and $\mathcal{P}_{x \leqslant} := \{y \in \mathcal{P} \mid x \leqslant y\}$
- A subset $\mathcal{C}$ of $\mathcal{P}$ is called a *chain* if any two elements of $\mathcal{C}$ are comparable. The chain $\mathcal{C}$ is *saturated* if there does not exist a $z \in \mathcal{P} \setminus \mathcal{C}$ such that $x \leqslant z \leqslant y$ with $x, y \in \mathcal{C}$. The length of a chain $\mathcal{C}$ is denoted $l(\mathcal{C}) = |\mathcal{C}| - 1$.
- Let $\mathcal{P}$ have a unique minimal element 0. If for each $x \in \mathcal{P}$ it holds that every saturated chain of $\mathcal{P}$ with minimal element 0 and maximal element x are of the same length, then $\mathcal{P}$ is a *ranked poset*. Then we have a unique *rank function*, $rk : \mathcal{P} \to \{0, 1, \dots n\}$ such that $rk(x) = 0$ if x is a minimal element of $\mathcal{P}$; $rk(y) = rk(x) + 1$ if $x \prec y$ in $\mathcal{P}$ and $rk(\mathcal{P}) = max\{l(\mathcal{C}) \mid \mathcal{C} \text{ a chain of } \mathcal{P}\} = n$. Where clarity is required we indicate the rank function attached to $\mathcal{P}$ by $rk_{\mathcal{P}}$. If $rk(x) = i$, then x is said to be a *ranked element* with *rank* i. We denote the set of elements of rank i as $\mathcal{P}_i$. If all the facets have the same rank then the poset is said to be *pure*.
- The *f-vector* for a ranked poset $\mathcal{P}$ of rank n is the $(n+1)$-tuple $f(\mathcal{P}) = (|\mathcal{P}_0|, |\mathcal{P}_1|, \dots, |\mathcal{P}_n|)$.
- If $x, y \in \mathcal{P}$, then an *upper bound* of x and y is an element $z \in \mathcal{P}$ such that $z \geqslant x$ and $z \geqslant y$. Therefore a *least upper bound* of x and y is an upper bound z of x and y, such that every upper bound w of x and y satisfies $w \geqslant z$. If a

least upper bound exists it is unique and we denote it by $x \vee y$ (read as x *join* y). In a similar way we define *lower bound* and *greatest lower bound* for elements of $\mathcal{P}$. If a greatest lower bound exists then it is denoted $x \wedge y$ (read as x *meet* y). In Figure 1.1 (a) we see a poset where x, y have two lower bounds z and z' and no greatest lower bound. The poset diagrams are standard *Hasse diagrams*, where the posets are represented by graphs. Here vertices represent elements of the poset and edges represent the cover relations such that if $x \prec y$ then x is joined to y by a upwards line. Thus we may think of posets as a special class of directed graphs, where the cover relation is replaced by directed edges.
- A *lattice* is a poset for which every pair of elements has a join and a meet. An example is shown in Figure 1.1 (b) where we see $a \wedge b = d$ and $a \vee b = c$. A *meet-semilattice* is a poset where every pair of elements has a meet.

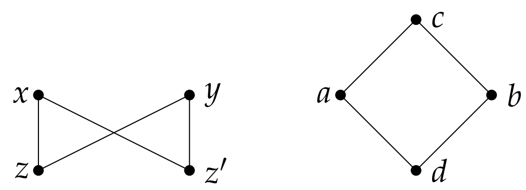

Figure 1.1: Poset Examples: (a) Lower Bounds and (b) Lattice.

1.1.2 Subposets and Order Ideals

Now that we have defined a poset the natural question to ask is what ordering do we attach to a subset that will make this into a *subposet* of the original poset.

Definition 1.1.2. Let $\mathcal{Q}$ be a poset and $\mathcal{P}$ a subset of $\mathcal{Q}$. Then let $\mathcal{P}$ form a poset with the ordering induced by the following cover relations:

for all $x, y \in \mathcal{P}$, we have $x \prec_{\mathcal{P}} y$ if and only if $x \prec_{\mathcal{Q}} y$ (cover-preserving property).

Then with this ordering $\mathcal{P}$ is a *subposet* of $\mathcal{Q}$, which we denote as $\mathcal{P} \sqsubseteq \mathcal{Q}$.

Remarks.
1. Alternatively we can say that given any subset $\mathcal{P}$ of $\mathcal{Q}$ there exists one ordering on $\mathcal{P}$ that gives $\mathcal{P} \sqsubseteq \mathcal{Q}$, namely that induced by the cover relations. This is the *cover-relation induced order* on the subposet $\mathcal{P}$.
2. Stanley in his book ([57] p.98) calls such a subset a *weak subposet*.
3. Let $\mathcal{P} \sqsubseteq \mathcal{Q}$. If $x \leqslant_{\mathcal{P}} y$ this indicates a chain with x, y as end points exists in $\mathcal{P}$, such a chain must also exist in $\mathcal{Q}$. Therefore an immediate consequence of this definition is that if $\mathcal{P} \sqsubseteq \mathcal{Q}$ then $x \leqslant_{\mathcal{P}} y$ implies $x \leqslant_{\mathcal{Q}} y$. However,

the reverse implication will not necessarily hold. Figure 1.2 shows $\mathcal{P} \sqsubseteq \mathcal{Q}$ which illustrates this point, since here $x \leqslant_{\mathcal{Q}} y$ but $x \nleqslant_{\mathcal{P}} y$.

4. Recalling that a poset may be thought of as a directed graph then the definition of a subposet is equivalent to saying that a subposet is an *induced subgraph* of the larger poset.
5. If we have two subposets $\mathcal{P}_1 \sqsubseteq \mathcal{Q}$ and $\mathcal{P}_2 \sqsubseteq \mathcal{Q}$, then we can form the union of these subposets $\mathcal{P}_1 \cup \mathcal{P}_2 \sqsubseteq \mathcal{Q}$. Here the new subposet has an underlying set equal to the union of the underlying sets of $\mathcal{P}_1$ and $\mathcal{P}_2$.

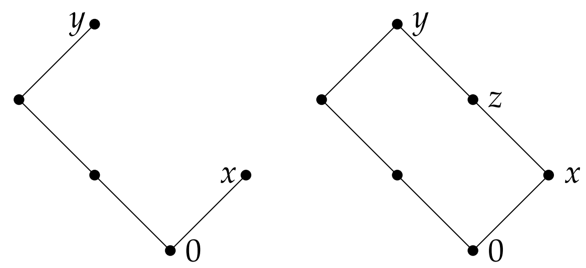

Figure 1.2: Posets $\mathcal{P}$ and $\mathcal{Q}$ where $\mathcal{P} \sqsubseteq \mathcal{Q}$.

Note: In the rest of this text all posets are finite and ranked with a unique minimal element denoted by 0.

Staying on the same track we want to give some attention to an important class of subposets, namely *order ideals*, which will be a recurring theme in this text.

Definition 1.1.3. Let $\mathcal{Q}$ be a poset and $\mathcal{P} \sqsubseteq \mathcal{Q}$. Then $\mathcal{P}$ is an *order ideal* in $\mathcal{Q}$ if for all $y \in \mathcal{P}$ and $x \in \mathcal{Q}$, with $x \leqslant_{\mathcal{Q}} y$, we then have $x \in \mathcal{P}$. Then $\mathcal{P}$ forms a poset with the cover-relation induced order from $\mathcal{Q}$. We denote this by $\mathcal{P} \trianglelefteqslant \mathcal{Q}$.

Remarks. 1. An immediate corollary is that if $\mathcal{P} \trianglelefteqslant \mathcal{Q}$ then for all $x, y \in \mathcal{P}$ we have $x \leqslant_{\mathcal{P}} y$ if and only if $x \leqslant_{\mathcal{Q}} y$. The remarks after Definition 1.1.2 give the forward implication. To see that the reverse holds we note that $x \leqslant_{\mathcal{Q}} y$ implies there is a chain in $\mathcal{Q}$ of the form $x = x_0 \prec x_1 \prec \cdots \prec x_n = y$, for some $n \in \mathbb{N}$ (if $n = 0$ then $x = y$ and result is obvious). By the cover-preserving property of subposets and as $\mathcal{P} \trianglelefteqslant \mathcal{Q}$ it follows that the chain $x = x_0 \prec x_1 \prec \cdots \prec x_n = y$ exists in $\mathcal{P}$ with $x_i \in \mathcal{P}$ for all i. Thus $x \leqslant_{\mathcal{P}} y$.

2. Being an order ideal is a transitive property, that is, if $\mathcal{O} \trianglelefteqslant \mathcal{P} \trianglelefteqslant \mathcal{Q}$ then $\mathcal{O} \trianglelefteqslant \mathcal{Q}$. Take $y' \in \mathcal{O}$ and $x' \leqslant_{\mathcal{Q}} y'$. Since $\mathcal{O} \trianglelefteqslant \mathcal{P}$ this implies $y' \in \mathcal{P}$ and so by $\mathcal{P} \trianglelefteqslant \mathcal{Q}$ and remark (1) this implies that $x' \leqslant_{\mathcal{P}} y'$. But since $\mathcal{O} \trianglelefteqslant \mathcal{P}$ we have $x' \in \mathcal{O}$ as required.

3. If $\mathcal{P}_1 \trianglelefteq \mathcal{Q}$ and $\mathcal{P}_2 \trianglelefteq \mathcal{Q}$ then $\mathcal{P}_1 \cup \mathcal{P}_2 \trianglelefteq \mathcal{Q}$ and $\mathcal{P}_1 \cap \mathcal{P}_2 \trianglelefteq \mathcal{Q}$, where this represents respectively the union and intersection of the underlying sets of $\mathcal{P}_1$ and $\mathcal{P}_2$.
4. For all $x \in \mathcal{P}$ we have $[0, x] = \mathcal{P}_{\leqslant x} \trianglelefteq \mathcal{P}$. Also conversely if $\mathcal{P} \trianglelefteq \mathcal{Q}$ then $\mathcal{P} = \bigcup_{x \in \mathcal{P}} \mathcal{Q}_{\leqslant x}$. Thus $\mathcal{P} \trianglelefteq \mathcal{Q}$ if and only if $\mathcal{P} = \bigcup_{x \in \mathcal{P}} \mathcal{Q}_{\leqslant x}$.

1.1.3 Embeddings and Isomorphisms

Our next logical step in the exploration of posets is to determine when we can say that two arbitrary posets are 'equal' or more formally when two posets are isomorphic.

Definition 1.1.4. Let $\mathcal{P}$ and $\mathcal{Q}$ be posets. A map[1] $\varphi : \mathcal{P} \hookrightarrow \mathcal{Q}$ is an *embedding* if the following hold:

- φ is injective and,
- for all $x, y \in \mathcal{P}$ we have $x \prec_{\mathcal{P}} y$ if and only if $\varphi(x) \prec_{\mathcal{Q}} \varphi(y)$.

Remarks. 1. If φ is bijective then it is easy to see that $\varphi^{-1} : \mathcal{Q} \hookrightarrow \mathcal{P}$ is also an embedding and so φ induces an *isomorphism*, denoted by $\mathcal{P} \cong \mathcal{Q}$. Furthermore in this case where $\mathcal{Q} = \mathcal{P}$ we say that φ is an *automorphism* of $\mathcal{P}$. The set of automorphisms of $\mathcal{P}$ form a group $Aut(\mathcal{P})$.
2. If $\mathcal{P}$ and its dual $\mathcal{P}^*$ are isomorphic then $\mathcal{P}$ is *self-dual*.
3. It is clear that if $\phi : \mathcal{Q} \hookrightarrow \mathcal{P}$ and $\theta : \mathcal{P} \hookrightarrow \mathcal{R}$ are embeddings, then their combination $\theta \circ \phi : \mathcal{Q} \hookrightarrow \mathcal{R}$ is also an embedding.

We have in fact already seen a basic example of an embedding:

Example 1.1.5. If $\mathcal{P}$ are $\mathcal{Q}$ are posets with $\mathcal{P} \sqsubseteq \mathcal{Q}$ then the inclusion given by the identity map is an embedding $\mathcal{P} \hookrightarrow \mathcal{Q}$.

The following are then some immediate consequences of the definition of embedding and isomorphism:

Proposition 1.1.6. *Let $\mathcal{P}$ and $\mathcal{Q}$ be posets for which there exists an embedding $\varphi : \mathcal{P} \hookrightarrow \mathcal{Q}$. Then:*

1. *For all $x, y \in \mathcal{P}$ we have $x \prec_{\mathcal{P}} y$ if and only if $\varphi(x) \prec_{\varphi(\mathcal{P})} \varphi(y)$. Also $\mathcal{P} \cong \varphi(\mathcal{P})$.*

[1] Here φ is strictly a map between the underlying sets of $\mathcal{P}$ and $\mathcal{Q}$.

2. *For all $x, y \in \mathcal{P}$ we have $x \leqslant_{\mathcal{P}} y$ if and only if $\varphi(x) \leqslant_{\varphi(\mathcal{P})} \varphi(y)$.*
3. *If $\varphi(0_{\mathcal{P}}) = 0_{\mathcal{Q}}$, then $rk_{\mathcal{P}}(x) = rk_{\mathcal{Q}}(\varphi(x))$ for all $x \in \mathcal{P}$. That is, the map is rank preserving. Furthermore, if φ gives an isomorphism $\mathcal{P} \cong \mathcal{Q}$ then for all $x' \in \mathcal{Q}$ we have $rk_{\mathcal{Q}}(x') = rk_{\mathcal{P}}(\varphi^{-1}(x'))$.*
4. *If φ gives an isomorphism $\mathcal{P} \cong \mathcal{Q}$ and $\mathcal{R} \trianglelefteqslant \mathcal{P}$ then $\varphi(\mathcal{R}) \trianglelefteqslant \mathcal{Q}$.*

Proof. (1.) $\varphi(\mathcal{P})$ is a subset of $\mathcal{Q}$, therefore with the cover-relations induced order $\varphi(\mathcal{P}) \sqsubseteq \mathcal{Q}$. So by definition of a embedding and subposet respectively, if $x, y \in \mathcal{P}$ we have $x \prec_{\mathcal{P}} y$ if and only if $\varphi(x) \prec_{\mathcal{Q}} \varphi(y)$ if and only if $\varphi(x) \prec_{\varphi(\mathcal{P})} \varphi(y)$. Then since φ is injective, it follows that $\varphi : \mathcal{P} \hookrightarrow \varphi(\mathcal{P})$ is bijective and so gives the isomorphism $\mathcal{P} \cong \varphi(\mathcal{P})$.
(2.) Take $x \leqslant_{\mathcal{P}} y$. We can assume $x \neq y$ as result is obvious if $x = y$. There exists a chain in $\mathcal{P}$ of the form $x = x_0 \prec x_1 \prec \cdots \prec x_n = y$, for some $n \in \mathbb{N}$. By definition of an embedding we have the chain $\varphi(x) = \varphi(x_0) \prec \varphi(x_1) \prec \cdots \prec \varphi(x_n) = \varphi(y)$ in $\varphi(\mathcal{P}) \sqsubseteq \mathcal{Q}$ and thus $\varphi(x) \leqslant_{\varphi(\mathcal{P})} \varphi(y)$. Since $\varphi^{-1} : \varphi(\mathcal{P}) \hookrightarrow \mathcal{P}$ is an embedding, by the same argument the converse result holds.
(3.) We know that for all $x \in \mathcal{P}$ there exists a chain $0_{\mathcal{P}} = x_0 \prec x_1 \prec \cdots \prec x_n = x$ of length $n = rk_{\mathcal{P}}(x)$. Since φ is cover-preserving, $\varphi(0_{\mathcal{P}}) = 0_{\mathcal{Q}}$, and φ is injective the image of this chain in $\mathcal{Q}$ is $0_{\mathcal{Q}} = \varphi(x_0) \prec \varphi(x_1) \prec \cdots \prec \varphi(x_n) = \varphi(x)$, also a chain of length n. This then implies that $rk_{\mathcal{Q}}(\varphi(x)) = n = rk_{\mathcal{P}}(x)$ as required. If φ gives an isomorphism then φ^{-1} is an embedding with $\varphi^{-1}(0_{\mathcal{Q}}) = 0_{\mathcal{P}}$. So the final result follows by same argument.
(4.) Let $x \in \varphi(\mathcal{R})$ and $y \in \mathcal{Q}$, with $y \leqslant_{\mathcal{Q}} x$. Then by (ii) we have $\varphi^{-1}(y) \leqslant_{\mathcal{P}} \varphi^{-1}(x) \in \mathcal{R}$. Since $\mathcal{R} \trianglelefteqslant \mathcal{P}$ it follows that $\varphi^{-1}(y) \in \mathcal{R}$ which implies $y \in \varphi(\mathcal{R})$. Thus $\varphi(\mathcal{R}) \trianglelefteqslant \mathcal{Q}$. □

1.2 q–Simplicial Posets and q–Simplicial Complexes

We now turn attention to defining the focus of this text, that is *q–simplicial posets* and *q–simplicial complexes*. To do this we form a distinguished poset $\mathcal{P}(V)$ for differing values of a parameter q.

Definition 1.2.1. Let n be a natural number and $q = 1$ or a prime power. Then:

- For $q = 1$ let V be a finite set of size n. Then $\mathcal{P}(V)$ is the poset of all subsets of V ordered by containment. This is then the *power set*, also denoted 2^n or B_n and called the *Boolean Algebra of rank n*, where the poset rank is determined by cardinality.

- For q a prime power let V be a vector space of dimension n over $\mathbb{F}_q := GF(q)$. Then $\mathcal{P}(V)$ is the poset of all subspaces of V ordered by containment. Thus $\mathcal{P}(V)$ is the *projective space* associated to V. In this poset, the rank is determined by the dimension.

Notation. In the rest of the text we now reserve the symbol q so that either $q = 1$ or $q = p^a$, where p is some prime and $a \in \mathbb{N}$. Furthermore by $q \geqslant 1$ we mean that either $q = 1$ or $q = p^a$ and by $q > 1$ we mean that $q = p^a$.

For all values of q the poset $\mathcal{P}(V)$ is of course a well known mathematical object. It is easy to see that it forms a lattice and it is self-dual for all $q \geqslant 1$. If $x, y \in \mathcal{P}(V)$ then $x \wedge y = x \cap y$ is the intersection of sets or subspaces (for $q = 1$ and q a prime power respectively). Additionally $x \vee y = x \cup y$ if $q = 1$ and $x \vee y = x + y$ (the sumspace) if q is a prime power. More importantly we have the property that if we take any interval $[x, y] \sqsubseteq \mathcal{P}(V)$ then it is isomorphic to $\mathcal{P}(V')$ for a 'smaller' set/vector space V'. Formally V' is the difference set $y \setminus x$ if $q = 1$ or V' is the quotient space y/x if $q > 1$. This interval property is then the motivation for the first of our two main definitions:

Definition 1.2.2. Let $q \geqslant 1$ and let $(\mathcal{Q}, \leqslant)$ be a ranked poset with a unique minimal element. Then $\mathcal{Q}$ is a *q–simplicial poset* (*q–poset* for short) if for all $x, y \in \mathcal{Q}$ with $x \leqslant y$ in $\mathcal{Q}$, we have $[x, y] \cong \mathcal{P}(U)$ for some U such that:
- U is a finite set if $q = 1$;
- U is a finite dimensional vector space over $\mathbb{F}_q$ if $q > 1$.

Remarks. 1. Let $x, y, \mathcal{Q}$ and U be as in the definition above. Then an immediate consequence is that if $q = 1$ then $|U| = rk_{\mathcal{Q}}(y) - rk_{\mathcal{Q}}(x)$ is uniquely defined. Similarly if $q > 1$ for $\dim(U) = rk_{\mathcal{Q}}(y) - rk_{\mathcal{Q}}(x)$.

2. Equally we could make the definition that $\mathcal{Q}$ is a q–poset if for all $y \in \mathcal{Q}$ we have $[0, y] \cong \mathcal{P}(U)$ for some U, with the conditions on U as above depending on q. Then for any $x, y \in \mathcal{Q}$ and $x \leqslant_{\mathcal{Q}} y$ it follows that $[x, y] \cong \mathcal{P}(U')$ for some U', by the interval property referred to earlier. Conversely if we take $x = 0$ in Definition 1.2.2 this implies $[0, y] \cong \mathcal{P}(U)$ for some U . Thus the two definitions are equivalent.
3. In the $q = 1$ case this definition has been previously given in Stanley's book ([57] p.135), where he calls such a poset a *simplicial poset* and this is reflected in the choice of terminology. As a consequence it is entirely natural that we use the terms 1–poset and simplicial poset interchangeably. However, the definition in the $q > 1$ case has not appeared previously in literature, to the best of our knowledge.

4. It should be noted that there exist non-Desarguesian projective planes which are not of the form $\mathcal{P}(V)$ for $V = \mathbb{F}_q^3$. They consist of equal number of points and lines which form a ranked poset by inclusion, when we add unique maximal and minimal elements. This ranked poset is such that each interval of length $\leqslant 2$ is a projective space, as defined above, but the interval of length 3 is not.
5. To keep a consistent use of notation for all values of q we will use the notation $|x|$ and $rk(x)$ interchangeably for the rank of an element x of a q–poset for $q \geqslant 1$.

Following on from this definition we next consider an important subclass of q–simplicial posets, which is the second of the main definitions. Once again the definition in the $q = 1$ and the $q > 1$ cases are completely analogous.

Definition 1.2.3. Let $q \geqslant 1$ and correspondingly let V be a finite set or a finite dimensional vector space over $\mathbb{F}_q$. Then $\mathcal{Q}$ is a *q–simplicial complex* (*q–complex* for short) on V if $\mathcal{Q} \trianglelefteq \mathcal{P}(V)$. In addition, if $\mathcal{R}$ is an arbitrary q–poset such that $\mathcal{R} \cong \mathcal{Q}$, then $\mathcal{R}$ is also a *q–simplicial complex*.

Remarks. 1. For all $q \geqslant 1$ it is clear that all q–complexes are q–posets.
2. If we consider the $q = 1$ case, with V a vertex set, then the definition almost coincides with the standard definition of a simplicial complex in algebraic topology (see for example [56] p.19). We differ here slightly from the usual definition by removing the requirement that a simplicial complex contains *all* vertices in the vertex set, that is the condition $\mathcal{Q}_1 = V$ (where $\mathcal{Q}$ is as in the above definition). We will use the terms 1–complex and simplicial complex interchangeably. When we talk of a simplicial complex we are considering the poset formed from the set of elements of the complex with the binary relation of containment, in much of the literature this is referred to as the *face poset* of a simplicial complex.
3. In the $q > 1$ case this definition is not completely new. In his 1971 paper ([43]) Gian-Carlo Rota defined a q–complex as an order ideal in $\mathcal{P}(V)$, where V is a finite dimensional vector space over $\mathbb{F}_q$. Historically this appears to have been the first time that this definition was given and no explicit reference to this concept in literature has appeared since. The idea was recently revived by Valery Mnukhin ([34]). We have extended the previous definition to include posets isomorphic to such order ideals.
4. Any order ideal of a q–complex $\mathcal{Q}$ is it itself a q–complex by the transitivity property of order ideals. Such an order ideal of $\mathcal{Q}$ is a *subcomplex* of $\mathcal{Q}$.

Notation. For the rest of text we will use the symbols $\subseteq$ and $\leqslant$ to represent the binary relation in q–posets interchangeably. As mentioned above, for any x, y in $\mathcal{P}(V)$ we have $x \wedge y = x \cap y$. Therefore we will use the more usual $\cap$ symbol to indicate the meet for elements of the q–posets, if such a meet exists.

The relationship between q–posets and q–complexes is straightforward in the $q = 1$ case, but in the $q > 1$ case the situation becomes more intricate. We will show in Chapter 3 that for the $q = 1$ case being a meet-semilattice is a necessary and sufficient condition for a simplicial poset to be a simplicial complex. However, in the $q > 1$ case finding such a characterization presents us with a non-trivial problem. Being a meet-semilattice is no longer sufficient in the $q > 1$. However as the following proposition shows it is still a necessary condition.

Proposition 1.2.4. *A q–complex is a meet-semilattice for any $q \geqslant 1$.*

Proof. Let $\mathcal{Q} \trianglelefteq \mathcal{P}(V)$ and let $x, y \in \mathcal{Q}$. So $x, y \in \mathcal{P}(V)$ and there exists $z := x \cap y$ in $\mathcal{P}(V)$. Since $\mathcal{Q}$ is an order ideal we have $z \in \mathcal{Q}$. Let $z^* \leqslant_{\mathcal{Q}} x, y$. Then $z^* \in \mathcal{P}(V)$ and this implies, by remarks (1.) after Definition 1.1.3, that $z^* \leqslant_{\mathcal{P}(V)} x, y$ and thus $z^* \leqslant_{\mathcal{P}(V)} z$. The same remarks also imply that $z^* \leqslant_{\mathcal{Q}} z$. Finally by Proposition 1.1.6 the result also holds for any $\mathcal{R} \cong \mathcal{Q}$. □

In determining whether a q–poset $\mathcal{Q}$ is a q–complex we will need to show that we can define an embedding of $\mathcal{Q}$ into a suitable $\mathcal{P}(V)$. In this regard we will need to know when a map between q–posets is an embedding and we will make use later of the following elementary result:

Proposition 1.2.5. *Let $\phi : \mathcal{P} \to \mathcal{Q}$ be a map where $\mathcal{P}, \mathcal{Q}$ are both q–posets ($q \geqslant 1$). Then if the following conditions all hold ϕ is an embedding :*

1. *ϕ is injective.*
2. *For all $x \in \mathcal{P}$ we have $rk_{\mathcal{P}}(x) = rk_{\mathcal{Q}}(\phi(x))$ (rank preserving).*
3. *For all $x \leqslant_{\mathcal{P}} y$ we have $\phi(x) \leqslant_{\mathcal{Q}} \phi(y)$ (order preserving).*

Proof. Let $x, y \in \mathcal{P}$. If $x \prec_{\mathcal{P}} y$ then $x \leqslant_{\mathcal{P}} y$ and $rk_{\mathcal{P}}(x) = rk_{\mathcal{P}}(y) - 1$. But by (2.) and (3.) it follows that $\phi(x) \leqslant_{\mathcal{Q}} \phi(y)$ and $rk_{\mathcal{Q}}(\phi(x)) = rk_{\mathcal{Q}}(\phi(y)) - 1$. Thus $\phi(x) \prec_{\mathcal{Q}} \phi(y)$. Now let $x, y \in \mathcal{P}$ with $rk_{\mathcal{P}}(y) = k + 1$. By (2.) and (3.) it follows that $\{\phi(z) \mid z \in [0, y]_k\} \subseteq [0, \phi(y)]_k$. But as $rk_{\mathcal{P}}(y) = rk_{\mathcal{Q}}(\phi(y))$ and $\mathcal{P}, \mathcal{Q}$ are both q–posets we have $|[0, y]_k| = |[0, \phi(y)]_k|$. Thus by condition (1.)

we have $\{\phi(z) \mid z \in [0,y]_k\} = [0,\phi(y)]_k$. Then if $\phi(x) \prec_{\mathcal{Q}} \phi(y)$ it follows that $\phi(x) \in [0,\phi(y)]_k$ and thus by (1.) we must have $x \in [0,y]_k$. Therefore $x \prec_{\mathcal{P}} y$. By Definition 1.1.4 we conclude that ϕ is an embedding. □

1.3 Examples of q–Complexes and q–Posets

From the remarks after Definition 1.2.3 it is evident that 1–complexes are nothing more than *simplicial complexes*, these of course being standard and well known objects of algebraic topology. Therefore we concentrate the exposition of examples here on the less well known q–complexes for $q > 1$ and on q–posets for $q \geqslant 1$.

1.3.1 q–Complexes

We start by giving the most generic description of q–complexes for $q \geqslant 1$. Let X be any subset of $\mathcal{P}(V)$, where V is a finite set if $q = 1$ or V is a vector space over $\mathbb{F}_q$ if $q > 1$. Form the set $\mathcal{Q}_X := \{y \in \mathcal{P}(V) \mid y \leqslant x \text{ for some } x \in X\} = \bigcup_{x \in X} \mathcal{P}_{\leqslant x}$. (Note: we will use this notation equally for any subset X of a q–poset $\mathcal{Q}$). With the cover-relation induced order it is clear that $\mathcal{Q}_X \trianglelefteq \mathcal{P}(V)$, so it is a q–complex on V and is by construction a subcomplex of any q–complex containing X. Thus $\mathcal{Q}_X$ is the smallest q–complex containing X. In essence this describes up to isomorphism all possible q–complexes.

However, it is perhaps more enlightening to describe more explicit examples of complexes, especially those that appear in a non-trivial or unexpected manner. For historical completeness we start with the first example of a q–complex (for $q > 1$) explicitly described in the literature:

Example 1.3.1. The *q–sphere* $[S_n]_q$ is the poset of all n-dimensional spaces (and their subspaces) in an $(n+1)$-dimensional vector space over $\mathbb{F}_q$. Thus a q-sphere is $\mathcal{P}(V)$ less its sole facet, namely V itself. Evidently this is a q–complex on V. This description of the structure as a q–complex was made first by Rota in his 1971 paper [43], alongside his definition of a q–complex which was cited earlier.

We turn now to another interesting example associated with the q–analogue of designs (here $q > 1$):

Example 1.3.2. In the standard sense a $t-(v,k,\lambda)$ *design* is a pair $(X,\mathcal{B})$ where X is a v-element set of *points* and $\mathcal{B}$ is a collection of k-element subsets of X (*blocks*) with the property that every t-elements subset of X is contained in exactly λ blocks (see, for example, [11] p.47). There exists a generalization of this to finite fields (described fully in [60]): given a finite field $\mathbb{F}_q$, a $t-(n,k,\lambda)$ *design* over $\mathbb{F}_q$ is a collection $\mathcal{B}$ of k-dimensional subspaces of an n-dimensional vector space over $\mathbb{F}_q$, called *blocks*, with the property that any t-dimensional subspace is contained in exactly λ blocks. If $\mathcal{B}$ is the set of all k-dimensional subspaces, the design is said to be *trivial*.

If we then consider all the blocks in such a design along with all their subspaces they will naturally form a q–complex. Thus the q–complex we describe is $\mathcal{Q}_{\mathcal{B}}$. This is only of real interest if there exist non-trivial designs. Literature exists on the search for non-trivial designs when $t>1$. The first such example can be seen in Thomas' 1987 paper [60] for $t=2$. It is shown there that if $\gcd(n,6)=1$, then there exists a $2-(n,3,7)$ design over $\mathbb{F}_2$. Furthermore if $n>5$, then the design is nontrivial.

Further 2-designs are described in Suzuki's papers of 1990 [58] and 1992 [59] and also in a paper of 1998 by Itoh [19]. The first design for $t>3$ can be found in a 2005 paper by Braun, Kerber and Laue [7], along with a method for systematic construction of such designs.

If a standard design has parameter $\lambda=1$, then it forms a *Steiner system*. In the same manner as described above we also have a q–analogue of a Steiner system, which is called a *Steiner structure*. Again if we take the blocks and all their subspaces then it forms a q–complex. Non-trivial examples do exist and there is some exploration of these in Schwartz and Etzion's 2002 paper [46].

We mentioned at the start of this section that 1–complexes are just simplicial complexes. However, occasionally simplicial posets turn out to be simplicial complexes in an unexpected and interesting way. Here we give an example of just such a complex:

Example 1.3.3. We form a poset from the words over the finite alphabet $A=\{0,a,b,c,\ldots\}$, where 0 is a special symbol. Fix n and define a binary relation on A^n by $x=(x_1,x_2,\ldots,x_n)\leqslant y=(y_1,y_2,\ldots,y_n)$ if and only if for all $1\leqslant i\leqslant n$ we have $x_i=y_i$ or $x_i=0$, while y_i is arbitrary. We can understand this as a deletion order. Then $(A^n,\leqslant)$ is a simplicial poset. To see this we note that the element $\underline{0}=(0,0,\ldots,0)$ is the unique minimal element of $(A^n,\leqslant)$. Then for any element $x \in A^n$ we have $[\underline{0}, x] \cong \mathcal{P}(U)$, where U is the set of indices i such that $x_i \neq 0$.

We can show that $(A^n, \leqslant)$ is a simplicial complex in general. Let $V = \{(a, i), (b, i), \ldots \mid i = 1, \ldots, n\}$ and let $\mathcal{P}(V)$ be the associated power set with the order given as usual by inclusion and where $\mathcal{P}(V) \cong 2^{n(|A|-1)}$. We next define a map $\varphi : A^n \to \mathcal{P}(V)$ by:

$$(x_1, \ldots, x_n) \longrightarrow \{(x_i, i) \mid x_i \neq 0 \text{ and } 1 \leqslant i \leqslant n\}.$$

By this construction φ is injective and maintains the cover-relations, thus φ is an embedding and so $A^n \cong \varphi(A^n)$. We then claim that the image of A^n is an order ideal in $\mathcal{P}(V)$ and thus A^n is a simplicial complex. To see this claim take any $x' \in \varphi(A^n)$, where $x' = \{(x_{i_1}, i_1), \ldots, (x_{i_k}, i_k)\}$ and $x' = \varphi(x)$ for $x = (x_{i_1}, \ldots, x_{i_k}) \in A^n$. Then take any $y' \leqslant_{\mathcal{P}(V)} x'$ and $|y'| = j \leqslant k$. Take $y = (y_1, \ldots, y_j)$ such that $y_i = 0$ if $(x_i, i) \notin y'$ and $y_i = x_i$ if $(x_i, i) \in y'$. Then $y \in A^n$, $y \leqslant_{A^n} x$ and $\varphi(y) = y' \in \varphi(A^n)$. Thus $\varphi(A^n) \trianglelefteq \mathcal{P}(V)$.

As an example let $A = \{0, a, b\}$ and $n = 3$ along with the set $V = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$. Define $\varphi : A^3 \hookrightarrow \mathcal{P}(V)$ as above. The image of $(A^3, \leqslant)$ under φ is a simplicial complex in the standard sense. So with V as the vertex set, $\varphi\left((A^3, \leqslant)\right)$ can be considered as an octahedron (Figure 1.3) via the usual pictorial representation of simplicial complexes (see [40] p.4).

For $|A| \geqslant 2$ we have that $(A^n, \leqslant)$ forms a meet-semilattice, since if $x, y \in A^n$ their unique lower bound is formed by the element z with $z_i = x_i$ if $x_i = y_i$ and 0 otherwise. As we will see in Chapter 3 this is actually a sufficient condition for a simplicial poset to form a simplicial complex.

1.3.2 Simplicial Posets : The $q = 1$ Case

One can find/construct simplicial posets that do not form simplicial complexes. We set out below such an example.

Example 1.3.4. Let X be a set with $|X| \geqslant 4$ and let $\mathcal{G}$ be the set of labelled graphs on subsets of X. Then we define an order by saying $G' \leqslant_{\mathcal{G}} G$ if and only if $V(G')$ is a subset of $V(G)$ and $G' := G[V']$, that is G' is the graph induced by a subset $V' = V(G')$ of vertices $V = V(G)$. So $(\mathcal{G}, \leqslant_{\mathcal{G}})$ has a unique minimal element given by the empty graph with no vertices, denote this as 0. Furthermore, $\mathcal{G}$

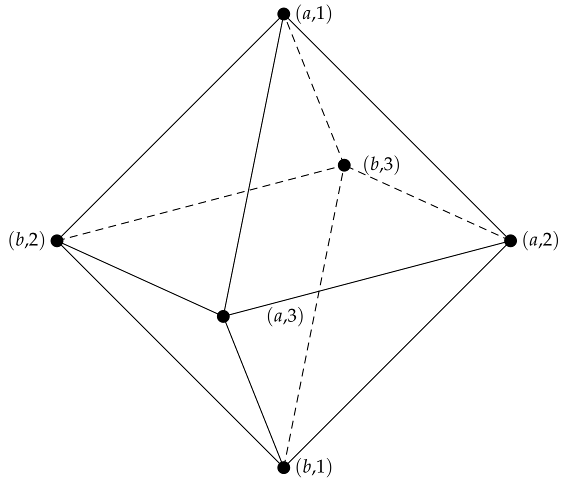

Figure 1.3: The Simplicial Complex $(A^3, \leqslant)$ via the map φ.

forms a simplicial poset since if $G = (V, E) \in \mathcal{G}$ we have that $[0, G] \cong \mathcal{P}(V)$ (that is the power set on $V = V(G)$). To see this take any $G \in \mathcal{G}$. Then $G^* \in [0, G]$ is the induced subgraph of G with $V(G^*) \subseteq V(G)$. The induced subgraphs of G in $[0, G]$ are in bijection with the set of subsets of $V(G)$. In particular by definition of the order on $\mathcal{G}$ for $G', G^* \in [0, G]$ we have $G' \leqslant G^*$ if and only if $V(G') \subseteq V(G^*)$.

Assume (without loss of generality) that $X = \{1, 2, 3, 4, \ldots\}$. Then the fact that $\mathcal{G}$ is not a meet-semilattice is illustrated by considering the following two distinct graphs of rank 4 in $\mathcal{G}$:

Then we note that they both contain the following two graphs of rank 3:

Therefore they do not have a unique meet and so by Proposition 1.2.4 it follows that $\mathcal{G}$ is not a simplicial complex.

Note: This example can be generalized to hypergraphs on a fixed set X.

1.3.3 q–Posets : The $q > 1$ Case

We finish the exposition of examples by turning attention to q–posets when $q > 1$. It is straightforward to draw a Hasse diagram of a q–poset which is not a q–complex, for example consider Figure 1.4. Here the elements x, y do not have a meet and thus this q–poset is not a meet-semilattice. So by Proposition 1.2.4 this figure illustrates a 2–poset which is not a 2–complex.

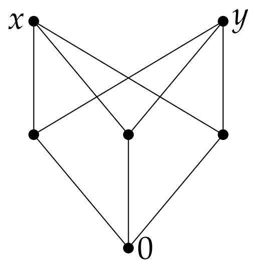

Figure 1.4: Example of a q–Poset that is not a q–complex, with $q = 2$.

A general and important class of q–posets comes from group theory. Recall an elementary abelian q–group X (q here is strictly a prime) is a vector space over $\mathbb{F}_q$ and the lattice of the subgroups of X is the projective space $\mathcal{P}(X)$.

Notation. Let G be a finite group and let q be a prime dividing the order of G. Then the poset of all elementary abelian q–subgroups ordered by inclusion is denoted $\mathcal{A}_q(G)$.

Remarks. We denote the identity element and trivial subgroup in a group by $\mathbb{1}$ and this is an element of $\mathcal{A}_q(G)$. Our definition of $\mathcal{A}_q(G)$ differs slightly from that given in Benson's book [4] p217 where he defines $\mathcal{A}_q(G)$ to be the poset of *non-trivial* elementary abelian q–subgroups of G, whereas we include the trivial subgroup $\mathbb{1}$ in $\mathcal{A}_q(G)$.

From the property of an elementary abelian q-group being a vector space we can say more about $\mathcal{A}_q(G)$:

Proposition 1.3.5. *Let G be a finite group and let q be a prime dividing the order of G. Then the poset $\mathcal{A}_q(G)$ forms a q–poset.*

Proof. Clearly $(\mathcal{A}_q(G), \subseteq)$ forms a ranked poset and each element $x \in \mathcal{A}_q(G))$ is of the form $x \cong C_q^n$ for some n, where C_q is the cyclic group of order q. The minimal element is given by the trivial group $\mathbb{1}$. By the correspondence theorem we have $[x, y] \cong [\mathbb{1}, y/x]$, where y/x is an elementary abelian q-group. Then by the standard result ([44] p.42) we have that $[\mathbb{1}, y/x]$ is isomorphic to $\mathcal{P}(V')$ with $V' = \mathbb{F}_q^{(|y|-|x|)}$ and thus the required result follows. $\square$

Remarks.

1. We can go further and take any subset X of groups in $\mathcal{A}_q(G)$. Then $\mathcal{Q}_X$ forms a q-subposet of $\mathcal{A}_q(G)$. This is of particular interest when we take X to be a union of conjugacy classes of subgroups of a finite group G, this is an example we return to repeatedly.
2. If a q–poset is of the form $\mathcal{Q}_X \trianglelefteq \mathcal{A}_q(G)$ with X a union of conjugacy classes of subgroups then G acts on the q–poset by conjugation.
3. A cogent question to ask at this point is whether $\mathcal{A}_q(G)$ forms a q–complex or whether it contains order ideals which are q–complexes. One thing we can say is that such order ideals are not always q–complex. In this regard we recall from Ivanov and Shpectorov's book ([21]) that the finite simple group the Monster is the universal embedding group of its geometry of singular subgroups (which are some pure 2B elementary abelian subgroups of rank up to 5). This fact implies that this 2–poset is not a 2–complex. Countering this we can find examples (See Example 1.3.6 below) where the identification as a q–complex can be completed in a straightforward manner. However, in general we do not have an immediate answer and we will consider this problem in detail in Chapter 3.

Example 1.3.6. Let $G := Alt(5)$ be the alternating group on the set $\{1, 2, 3, 4, 5\}$. The facets of $\mathcal{A}_2(G)$ are subgroups of the shape $\langle (j\,k)(l\,m), (j\,l)(k\,m)\rangle$, that is the Klein-4 groups. There are 5 such groups with mutually trivial intersection. Label the maximal groups as $K_i = \langle x_i, y_i\rangle$, with $i = 1, \ldots, 5$ and let $V = \langle v_i, \ldots, v_5, w_1, \ldots, w_5\rangle$ be a 10-dimensional vector space over $\mathbb{F}_2$. We define a map $\phi : \mathcal{A}_2(G) \to \mathcal{P}(V)$ given by $\phi(\langle x_i\rangle) = \langle v_i\rangle$, $\phi(\langle y_i\rangle) = \langle w_i\rangle$, $\phi(\langle (x_i)\cdot(y_i)\rangle) = \langle v_i + w_i\rangle$, $\phi(\mathbb{1}) = 0$, and by extension $\phi(K_i) = \langle v_i, w_i\rangle$. This map is injective and cover-preserving in both directions. Let $X = \{\langle v_i, w_i\rangle | 1 \leqslant i \leqslant 5\}$ and form the order ideal $\mathcal{Q}_X$ in $\mathcal{P}(\mathbb{F}_2^{10})$. Then we see that ϕ induces an isomorphism of $\mathcal{A}_2(G)$ with $\mathcal{Q}_X$. Thus $\mathcal{A}_2(G)$ is a q–complex. A Hasse diagram of this complex can be seen in Figure 1.5.

The above then describes the most natural way to form an embedding of this complex. However, embeddings may not be unique and we perhaps would like to describe one that is minimal in terms of the dimension of the projective space. For our example $\mathcal{A}_2(G)$ we can construct an embedding into a projective space of a 4-dimensional vector space over $\mathbb{F}_2$. This will be minimal since any pair of 2-dimensional spaces in a 3-dimensional vector space (for any $q > 1$)

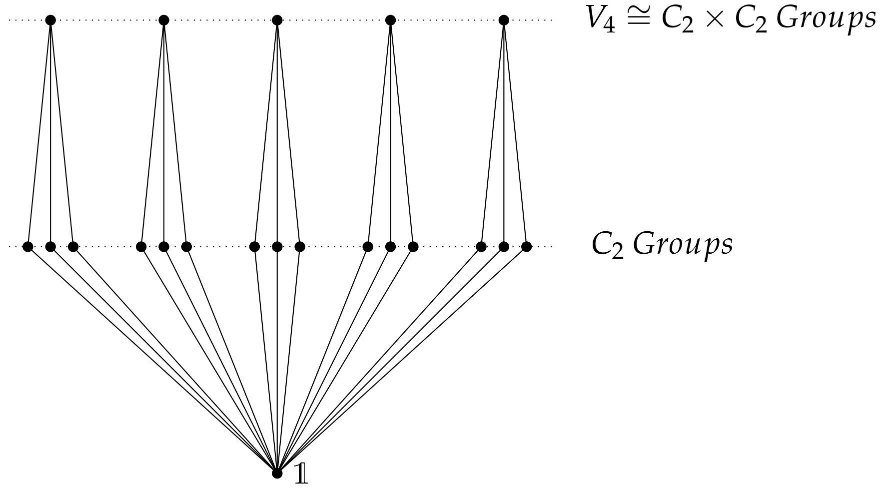

Figure 1.5: 2–Complex of the Klein-4 groups in A_5.

always intersect in a 1-dimensional space and thus no embedding of $\mathcal{A}_2(G)$ into a 3-dimensional space can exist.

A minimal embedding comes from considering an irreducible modular representation of G. The matrix representations of G are available at the online source *ATLAS of Finite Group Representations* [66]. In the list for G we take the second representation of degree 4 over $\mathbb{F}_2$, which has associated distinguishing letter b in the source. Since this is a faithful representation we can describe G as a subgroup of $GL(V)$, where $V = \mathbb{F}_2^4$. Then using the algebraic computer system GAP [16] it is easy to check that G (considered as a subgroup of $GL(V)$) has an orbit $\mathcal{O}$ of size 5 on the subspaces of dimension 2 contained in V. This orbit contains five 2-dimensional spaces, which have mutually trivial intersections. Thus $\mathcal{A}_2(G)$ is isomorphic to $\mathcal{Q}_{\mathcal{O}}$ and therefore by previous comments we have described a minimal embedding.

This representation as described is an example of a *good representation*:

Definition 1.3.7. Let G be a group and let $X = \{X_1, X_2, \ldots, X_k\}$ be a conjugacy class of elementary abelian subgroups of rank m contained in $\mathcal{Q} := \mathcal{A}_q(G)$. Then if $V := \mathbb{F}_q^n$ for some $n \geqslant m$ we say a faithful representation $\rho : G \to GL(V)$ is a *good representation* for $\mathcal{Q}_X$ if the following both hold:

1. There exists an orbit $\mathcal{O} = \{O_1, O_2, \ldots O_k\}$ in $\rho(G)$'s action on $\mathcal{P}(V)_m$ along with an isomorphism $\varphi : \mathcal{Q}_X \xrightarrow{\cong} \mathcal{Q}_{\mathcal{O}}$ such that $\varphi(X_i) = O_i$ for all i.
2. For all $g \in G$ and for all i we have that $\varphi((X_i)^g)) = \rho(g)(O_i)$.

Remarks. 1. The first condition tells us that $\mathcal{Q}_X$ is a q–complex.

2. It is straightforward to check using GAP that the modular representation as described previously for $G = A_5$ is a good representation for $\mathcal{A}_2(G)$ (see Appendix D for script used).
3. There is an alternative method of determining this minimal embedding. It is a standard result that $G = A_5$ is isomorphic to $SL_2(4)$ (a special linear group) and so has a natural 2–dimensional module over $\mathbb{F}_4$, which can be viewed as a 4–dimensional module over $\mathbb{F}_2$. In this module the five 1–dimensional $\mathbb{F}_4$ subspaces (which are 2–dimensional over $\mathbb{F}_2$) have pairwise trivial intersections, and so provide an embedding for $\mathcal{A}_2(G)$.
4. Another example of a good representation is seen when we consider modular representations of $H = Alt(6)$. Here the modular representation of degree 4 over $\mathbb{F}_3$ (as noted in [66]) is a good representation for $\mathcal{A}_3(H)$. Again this is easily checked using GAP. By contrast using a similar argument as for $Alt(5)$ we can calculate that the dimension of the free embedding for $\mathcal{A}_3(H)$ is 20.

Example 1.3.8. As a final example consider $\mathcal{A}(V)$ the *Affine Geometry of* V for a vector space $V = \mathbb{F}_q^n$ for some $q > 1$ (see for example [42] Chp. 16). Here the elements of $\mathcal{A}(V)$ are called *flats* with the elements being cosets of subspaces of V, that is $\mathcal{A}(V) = \{a + U \mid a \in V, U \text{ a subspace of V}\}$. We call a the *flat representative* and U the *base*. The rank in $\mathcal{A}(V)$ is given by dimension of the base, that is $rk(a+U) = \dim(U)$. We say that $(a+U) \subseteq_{\mathcal{A}(V)} (b+W)$ if $U \subseteq_{\mathcal{P}(V)} W$ and $a = b + w$ for some $w \in W$. The vector space V is then a unique maximal element in $\mathcal{A}(V)$ with rank equal to $\dim(V)$. To see this first pick a vector $v \in V$ as the flat representative for V (clearly $(v+V) = V$ for all $v \in V$). Then take any element $(a+U) \in \mathcal{A}(V)$. Since $U \subseteq_{\mathcal{P}(V)} V$ and $a = v + (a-v)$, where $a - v \in V$, it follows that $(a+U) \subseteq_{\mathcal{A}(V)} (v+V) = V$. So it follows that $[a+U, v+V] = [a+U, a+V] \cong \mathcal{P}(V/U) \cong \mathcal{P}(V')$ for $V' \cong \mathbb{F}_q^m$, with $m = \dim(V) - \dim(U)$. However $\mathcal{A}(V)$ is not a q–poset since for every vector $v \in V$ we have that $(v + \langle 0_{\mathcal{P}(V)} \rangle)$ is a distinct element of rank 0. But, as $\mathcal{P}(V')$ is a self-dual poset the facts above tell us (by remark 2. after definition 1.2.3) that the dual poset $(\mathcal{A}(V))^*$ is a q–poset with unique minimal element V.

1.4 Order Complexes and $\mathcal{A}_q(G)$

To any poset $\mathcal{P}$ we can associate a simplicial complex $\Delta(\mathcal{P})$ called the *order complex of* $\mathcal{P}$. The elements of $\Delta(\mathcal{P})$ are the chains of the original poset $\mathcal{P}$, where the chains are of the form $x_1 <_{\mathcal{P}} x_2 <_{\mathcal{P}} \cdots <_{\mathcal{P}} x_r$ for $x_i \in \mathcal{P}$. To describe the order on this complex consider $x, y \in \Delta(\mathcal{P})$. Then $x \leqslant_{\Delta(\mathcal{P})} y$ if the underlying set of the chain x is contained in the underlying set of the chain y.

While we will not consider order complexes, we mention here briefly some work of others on the order complex associated to particular q–posets. Recall the q–poset $\mathcal{A}_q(G)$ (q here a prime) for a finite group G defined in the previous section. Let us now temporarily revert to Benson's definition of this poset so that it includes only non-trivial subgroups. In this guise the order complex $\Delta(\mathcal{A}_q(G))$ (often called the *Quillen Complex*) is a well studied object, with particular emphasis on the topology and homology of this complex. In an important result Quillen showed in his 1978 paper ([41]) that it is homotopy equivalent to $\Delta(\mathcal{S}_q(G))$, where $\mathcal{S}_q(G)$ is the poset of all non-trivial q–subgroups of G (i.e those of order a power of q). Naturally here $\mathcal{A}_q(G) \sqsubseteq \mathcal{S}_q(G)$, in fact $\mathcal{A}_q(G) \trianglelefteq \mathcal{S}_q(G)$. Further discussion of this topic can be found in Chapter 6 of Benson's book [4] and in papers by Webb [64], Aschbacher [2] and Smith [54].

On a more specific note there have been a number of papers concentrating on $\Delta(\mathcal{A}_q(G))$ in the case where G is the Symmetric Group. See papers by Bouc [6]; Ksontini [28] and [29]; Shareshian [47], [48] and [49] with Wachs. We will spend a little time in Chapter 4 considering further an order ideal of the q–poset $\mathcal{A}_q(G)$ when G is the Symmetric Group.

1.5 Shellability of q–Complexes

A frequently discussed property of simplicial complexes is that of *shellability*. We consider here the extension of this property to q–complexes in general. There are a number of equivalent definitions when working in the $q = 1$ case (see [8] p.214), however these do not all extend satisfactorily to the $q > 1$ case. Therefore we restrict ourselves to the following definition which makes sense for all values of $q \geqslant 1$. Recall that a ranked poset is pure if all its facets have the same rank.

Definition 1.5.1. Let $\mathcal{Q}$ be a pure q-complex of rank d. Then $\mathcal{Q}$ is shellable if its facets can be enumerated $x_1, x_2, \ldots, x_n$ such that for all $2 \leqslant i \leqslant n$ we have that[2] $\left(\bigcup_{j=1}^{i-1}[0, x_j]\right) \cap [0, x_i]$ is pure of rank $d-1$. In this case $x_1, x_2, \ldots, x_n$ is then called a *shelling order* or a *shelling* of $\mathcal{Q}$.

[2] Equivalently $\left(\bigcup_{j=1}^{i-1} \mathcal{Q}_{\leqslant x_j}\right) \cap \mathcal{Q}_{\leqslant x_i}$

We have already met in Section 1.3 a q–complex ($q > 1$) which is shellable by this definition:

Example 1.5.2. The q-sphere $[S_n]_q$(for $n \geqslant 1$) is shellable for any linear ordering of its facets. Recall that the facets are the n dimensional subspaces of an $n + 1$ dimensional vector space over $\mathbb{F}_q$. Then the conclusion that it is shellable follows immediately from the fact that any two such n-dimensional spaces always intersect in an $(n - 1)$- dimensional subspace.

There is much literature on shellable simplicial complexes (see for example Chapter III of Stanley's book [56] and also the book [17]). Whilst we will not be considering shellable q–complexes in generality, in Chapter 4 we do consider a set of shellable q–complexes which have a direct connection to shellable simplicial complexes.

Chapter 2

Describing the Combinatorics of q–Posets

We turn attention in this chapter to various combinatorial aspects of q–posets. We start by introducing q-binomial coefficients, a q-analogue of binomial coefficients. These are used to describe the combinatorics of general q–posets in the same way binomial coefficients are used in the $q = 1$ case. We next consider what can be said about the f-vector of q–posets. Much is known about f-vectors in the $q = 1$ case. We recall the well known *Kruskal-Katona Theorem* and consider a similar result for the $q > 1$ case. Following this we go onto to review the incidence structures apparent in q–posets, in particular the associated incidence matrices and their properties. We finish with a look at Möbius inversion.

2.1 Vector Spaces and the q–Binomial Coefficient

Binomial coefficients arise naturally in the combinatorics of simplicial complexes. When we consider $q > 1$ we need the q–analogue of the binomial coefficients. Many results for standard binomial coefficients then have direct q-analogues. For $i \in \mathbb{N}$ and $n, k \in \mathbb{Z}_{\geqslant 0}$ let:

$$[i]_q := 1 + q + q^2 + \ldots + q^{i-1} = \sum_{j=0}^{i-1} q^j = \frac{q^i - 1}{q - 1} \qquad \text{if } q \neq 1,$$

$$(i!)_q := [1]_q \cdot [2]_q \ldots [i]_q \qquad \text{and,} \quad [0]_q := 1 \qquad (0!)_q := 1 \qquad ,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{(n!)_q}{(k!)_q((n-k)!)_q} & \text{if } 0 \leqslant k \leqslant n; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ is the *q–binomial coefficient* (also called the *Gaussian Polynomial*, see for example [10]).

Remarks. 1. For $1 \leqslant k \leqslant n$ we have that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = \prod_{j=0}^{k-1} \frac{q^{n-j}-1}{q^{j+1}-1}$.

2. If we fix n we can think of the q–binomial as a function in k. From the previous comment we have that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = \left(\frac{q^{n-k+1}-1}{q^k-1}\right) \cdot \left[\begin{smallmatrix} n \\ k-1 \end{smallmatrix}\right]_q$ and from this it is then easy to see that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ is an increasing function if $k < \frac{n+1}{2}$, a decreasing function if $k > \frac{n+1}{2}$ and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} n \\ k-1 \end{smallmatrix}\right]_q$ if $k = \frac{n+1}{2}$. Thus if n is even $\left[\begin{smallmatrix} n \\ (\frac{n}{2}) \end{smallmatrix}\right]_q$ is a unique maximum for $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ and if n is odd it has two equal maximal values $\left[\begin{smallmatrix} n \\ (\frac{n-1}{2}) \end{smallmatrix}\right]_q$ and $\left[\begin{smallmatrix} n \\ (\frac{n+1}{2}) \end{smallmatrix}\right]_q$.

3. The connection to standard binomial coefficients is quickly apparent if we let $q \to 1$ in the expressions above:

$$\begin{aligned} \lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \lim_{q\to 1} \left(\frac{[n]_q \cdot [n-1]_q \ldots [n-k+1]q}{[k]_q \cdot [k-1]_q \ldots [1]q} \right) \\ &= \frac{n \cdot (n-1) \ldots (n-k+1)}{(k) \cdot (k-1) \ldots (1)} = \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k}, \quad \text{the standard binomial coefficient.} \end{aligned}$$

From this we give the following identity as an extension to cover all values of q:

$$\begin{bmatrix} n \\ k \end{bmatrix}_1 := \binom{n}{k} \quad .$$

These coefficients have applications in the study of partitions (see Chp.3 [1] and Chp.1 [57]), however we are interested in their connection to vector spaces and this is made evident in the following theorem.

Theorem 2.1.1. *([22] p.18) Let V be a finite dimensional vector space of dimension n over $\mathbb{F}_q$ and let x be a b-dimensional subspace of V with $0 \leqslant b \leqslant k \leqslant n$. Then the number of k-dimensional subspaces y of V such that $y \supseteq x$ is given by:*

$$\begin{bmatrix} n-b \\ k-b \end{bmatrix}_q = \begin{bmatrix} n-b \\ n-k \end{bmatrix}_q \quad .$$

Proof. Given a basis of x the number of ways of choosing $k-b$ further linearly independent vectors from V that do not lie in x is given by $(q^n - q^b) \cdot (q^n -$

$q^{b+1})\cdots(q^n - q^{k-1})$. However any k-dimensional subspace which contains x has the following number of bases which extend the given basis of x : $(q^k - q^b)\cdot(q^k - q^{b+1})\cdots(q^k - q^{k-1})$. Thus the number of k-dimensional spaces containing x is given by the quotient of these two numbers:

$$
\begin{aligned}
&\frac{(q^n - q^b)(q^n - q^{b+1})\cdots(q^n - q^{k-1})}{(q^k - q^b)(q^k - q^{b+1})\cdots(q^k - q^{k-1})}\\
=&\frac{(q^{n-b}-1)(q^{(n-b)-1}-1)\cdots(q^{(n-b)-(k-b-1)}-1)}{(q^1-1)(q^2-1)\cdots(q^{k-b}-1)}\\
=&\begin{bmatrix} n-b \\ k-b \end{bmatrix}_q = \begin{bmatrix} n-b \\ n-k \end{bmatrix}_q \qquad \text{by standard result ([1] p.35).}
\end{aligned}
$$

□

Remarks. If we set $b = 0$ in the above theorem this will give us the standard result for the number of k-dimensional subspaces of V. Since $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_1 = \binom{n}{k}$, we may replace V in the above result by a finite set of n elements and still have a valid statement.

2.2 The f-Vector Explored

We begin with some background on what is known on f-vectors for the $q = 1$ case. Recall first that if $\mathcal{P}$ is a poset of rank n then its *f-vector* is the $(n+1)$-tuple $f(\mathcal{P}) = (|\mathcal{P}_0|, |\mathcal{P}_1|, \ldots, |\mathcal{P}_n|)$.

Given the definition of an f-vector of a q–poset perhaps an obvious question then to ask is: when does an $(n+1)$-tuple $f \in \mathbb{N}^{n+1}$ form the f-vector of a q–poset? A characterization of all $(n+1)$-tuples that are the f-vector of a simplicial complex was found independently by Schützenberger, Kruskal and Katona and is described in the well known *Kruskal-Katona Theorem* (original sources [45], [27] and [26]).

To state this theorem we recall the *i-binomial expansion* of a natural number (see [56] p.55). Given two positive integers $n, i \,:\, n \geqslant i > 0$ we write

$$n = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \ldots + \binom{m_j}{j}, \text{ where } m_i > m_{i-1} > \ldots m_j \geqslant j \geqslant 1 \quad .$$

It is a fact that such an expansion always exists for any $n, i \in \mathbb{N}$ and is always unique. For such an expansion of n we then define

$$n^{(i)} := \binom{m_i}{i+1} + \binom{m_{i-1}}{i} + \ldots + \binom{m_j}{j+1} \quad .$$

Using this notation we state the The Kruskal-Katona Theorem which completely characterizes f-vectors of simplicial complexes (statement of theorem as [56], p.55):

Theorem 2.2.1 (The Kruskal-Katona Theorem)**.** *An n-tuple $(f_0, f_1, \ldots f_n) \in \mathbb{N}^{n+1}$ is the f-vector of some simplicial complex (equivalently a* 1*–complex) if and only if* $0 < f_{i+1} < f_i^{(i+1)}$*, for* $1 \leqslant i \leqslant n-1$.

A result of similar ilk for the more general setting of simplicial posets (equivalently 1–posets) appears in Stanley's 1991 paper:

Theorem 2.2.2. *([55]) Let $f = (f_0, f_1, \ldots, f_d) \in \mathbb{Z}^{d+1}_{>0}$. The following two conditions are then equivalent:*

1. *There exists a simplicial poset $\mathcal{Q}$ of rank d with f-vector equal to f.*
2. $f_i \geqslant \binom{d}{i}$ *for* $i \in \{1, \ldots, d\}$ *and* $f_0 = 1$.

In the next section we will show that the latter theorem actually extends to all q–posets ($q \geqslant 1$).

2.2.1 The f-Vector of q–Posets

We now turn to the more general setting of q–posets and q–complexes. It is natural to expect that a similar result to the Kruskal-Katona Theorem may be achievable for q-complexes. But as Greene and Kleitman say in [18] p.74 "*. . . apparently nothing is known about analogs of the Kruskal-Katona Theorem for subspaces of a finite vector space*". In a 1999 paper by Bezrukov and Blokhuis [5] they do obtain a restricted Kruskal-Katona type theorem for $\mathcal{P}(V)$ when $V = \mathbb{F}_2^n$, but this does not appear to extend to a general case even in the $q = 2$ setting. Nevertheless, as we indicated previously, Theorem 2.2.2 of the previous section can be extended to the general q–poset case. Here we use the convention that a simplicial poset is a 1-poset. The proof of the extended theorem then relies on the following result:

Lemma 2.2.3. *Let $q \geqslant 1$ and let $\mathcal{Q}$ be a q–poset of rank d with f-vector equal to $(1, f_1, \ldots, f_d) \in \mathbb{Z}^{d+1}_{>0}$. Choose j with $0 < j \leqslant d$. Then there is a q–poset $\mathcal{Q}^*$ with f-vector equal to $(1, f_1, \ldots, f_{j-1}, f_j + 1, f_{j+1}, \ldots, f_d) \in \mathbb{Z}^{d+1}_{>0}$.*

Proof. Let w be a new element and form the enlarged poset $\mathcal{Q}^* := \mathcal{Q}\dot{\cup}\{w\}$ (in terms of sets). Fix a $u \in \mathcal{Q}_j$ and define for any $x \in \mathcal{Q}^*$ that $x <_{\mathcal{Q}^*} w$ if and only if $x <_{\mathcal{Q}} u$. Also $w \leqslant_{\mathcal{Q}^*} x$ if and only if $x = w$. For all other $x, y \in \mathcal{Q}^* \setminus w$ we inherit the order from $\mathcal{Q}$. We note that as a result w is a facet of $\mathcal{Q}^*$. Then $[x, u] \cong [x, w]$ for all $x \leqslant_{\mathcal{Q}^*} w$ and $[w, x] = \varnothing$ unless $x = w$. All other intervals in $\mathcal{Q}^*$ (those that do not contain w) are isomorphic to the corresponding interval in $\mathcal{Q}$. So $\mathcal{Q}^*$ is a q–poset and since by this construction $rk(w) = j$ it is evident that $f(\mathcal{Q}^*) = (1, f_1, \ldots, f_{j-1}, f_j + 1, f_{j+1}, \ldots, f_d)$. $\square$

We now use this to obtain the proof of the extended version of Theorem 2.2.2:

Theorem 2.2.4. *Let $f = (f_0, f_1, \ldots, f_d) \in \mathbb{Z}^{d+1}_{>0}$ and $q \geqslant 1$. The following two conditions are then equivalent:*

1. *There exists a q-poset $\mathcal{Q}$ of rank d with f-vector $f(\mathcal{Q}) = f$.*
2. *$f_i \geqslant \begin{bmatrix} d \\ i \end{bmatrix}_q$ for $i \in \{1, \ldots, d\}$ and $f_0 = 1$.*

Proof. (1) implies (2). Let $v \in \mathcal{Q}_d$. Then by definition for all $1 \leqslant i \leqslant d$ we have $|[0, v]_i| = \begin{bmatrix} d \\ i \end{bmatrix}_q$ and since $[0, v]_i \subseteq \mathcal{Q}_i$ it follows that $f_i \geqslant \begin{bmatrix} d \\ i \end{bmatrix}_q$. As a q–poset always has a unique minimal element by definition it follows that $f_0 = 1$.
(2) implies (1). For all q we have that if $|V| = d$ then $\mathcal{P}(V)$ is a q–poset of rank d with f-vector $f(\mathcal{P}(V)) = (1, \begin{bmatrix} d \\ 1 \end{bmatrix}_q, \ldots, \begin{bmatrix} d \\ d \end{bmatrix}_q)$. So we take this f-vector as our base step. Then repeatedly apply the result of Lemma 2.2.3 for varying choices of j (where j is the parameter in the statement of the Lemma). From this we see that if $f = (1, f_1, \ldots, f_d) \in \mathbb{Z}^d_{>0}$ with $f_i \geqslant \begin{bmatrix} d \\ i \end{bmatrix}_q$ for $i \in \{1, \ldots, d\}$, then it is an f-vector of some q–poset of rank d. Thus condition (i) is met. $\square$

2.2.2 A Combinatorial Result for f-Vectors

Later we will see q–posets where the f-vectors of subposets formed by certain intervals in the poset are known. We ask the question now as to whether it is possible to reconstruct the f-vector of the full poset from such knowledge of the f-vectors of these subposets. The following proposition will allow us to do such a calculation in particular settings:

Notation. If $\mathcal{P}$ is a ranked poset, let $f_i^{\mathcal{P}} := |\mathcal{P}_i|$. It follows that if $\mathcal{P}$ is of rank n it has f-vector $f(\mathcal{P}) = (f_0^{\mathcal{P}}, f_1^{\mathcal{P}}, \ldots, f_n^{\mathcal{P}})$.

Proposition 2.2.5. *Let $\mathcal{Q}$ be a q–poset of rank n ($q \geqslant 1$). Then for all $i, j \,:\, 0 \leqslant i \leqslant j \leqslant n$ we have:*

$$f_j^{\mathcal{Q}} = \frac{1}{\left[{j \atop i}\right]_q} \sum_{x \in \mathcal{Q}_i} f_{j-i}^{\mathcal{Q}_{x\leqslant}} \quad .$$

Proof. We count the set of pairs $\mathcal{S} = \{(x, y) \mid x \in \mathcal{Q}_i,\ y \in \mathcal{Q}_j \text{ with } x \leqslant y\}$ in two ways. Firstly fix a $y \in \mathcal{Q}_j$. By the definition of a q–poset $|[0, y]_i| = \left[{j \atop i}\right]_q$, which equals the number of $x \leqslant y$ with $x \in \mathcal{Q}_j$. Since this value is the same for all $y \in \mathcal{Q}_j$ and $|\mathcal{Q}_j| = f_j^{\mathcal{Q}}$, summing over all choices of y we have that:

$$|\mathcal{S}| = \begin{bmatrix} j \\ i \end{bmatrix}_q \cdot f_j^{\mathcal{Q}} \quad . \tag{2.1}$$

Secondly fix an $x \in \mathcal{Q}_i$. Then the number of $y \in \mathcal{Q}_j$ with $x \leqslant y$ is given by $f_{(j-i)}^{\mathcal{Q}_{x\leqslant}}$. As this expression holds for all $x \in \mathcal{Q}_i$, if we sum over all choices for x we have that:

$$|\mathcal{S}| = \sum_{x \in \mathcal{Q}_i} f_{j-i}^{\mathcal{Q}_{x\leqslant}} \quad . \tag{2.2}$$

Equating the expressions (2.1) and (2.2) and rearranging gives us the required result. □

2.2.3 The h-Vector of a q–Poset

We make brief mention at this point of a companion to the f–vector which is a standard concept in the combinatorics of the $q = 1$ setting. Let $\mathcal{Q}$ be a shellable q–complex (as per Definition 1.5.1 for $q \geqslant 1$) of rank d and let $x_1, x_2, \ldots, x_n$ be a shelling order for $\mathcal{Q}$. For each $i \in \{1, \ldots, n\}$ let $\mathcal{I}_i = \left(\bigcup_{j=1}^{i-1} \mathcal{Q}_{\leqslant x_j}\right) \cap \mathcal{Q}_{\leqslant x_i}$. By definition $\mathcal{I}_i \trianglelefteq [0, x_i]$ is a pure q–complex of rank $d - 1$.

For each i let $c(i)$ be the number of distinct co-dimension 1 subspaces (co-rank 1 subsets if $q = 1$) of x_i that are the facets of $\mathcal{I}_i$. Form the list $c(1), c(2), \ldots$ of these values. The frequency with which a particular number c appears in this list (as we look at $i = 1, 2, 3, \ldots$) is the c^{th} component of the *h-vector* for our shellable q–complex $\mathcal{Q}$. Thus we may have zero entries in the h-vector and it will be of the form $h(\mathcal{Q}) = (h_0, h_1, h_2, \ldots, h_{\left[{d \atop d-1}\right]_q})$.

Remarks. 1. For $q = 1$ we have a formula for the entries based on the entries of the f-vector. Let $\mathcal{Q}$ be a simplicial complex of rank d with f-vector $f(\mathcal{Q}) = (f_0, f_1, f_2, \ldots, f_d)$. Then the *h-vector* $h(\mathcal{Q}) = (h_0, h_1, \ldots, h_d)$ is given by the following relation, if $\mathcal{Q}$ is shellable:

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_i \qquad 0 \leqslant k \leqslant d \quad . \tag{2.3}$$

This result is independent of the shelling order used (see p.80 of Stanley's book [56]) and gives the same values as the definition above. From (2.3) it is clear that for simplicial complexes knowing the h-vector is equivalent to knowing the f-vector. This expression does arise in the study of Cohen-Macaulay Rings (see [8])).
Of course this expression (2.3) can be calculated for any simplicial complex. In this way it can be used in certain cases to tell us if a complex is not shellable. In particular a given complex is not shellable if $h_k < 0$ for any k.

2. In the $q > 1$ setting the above construction makes sense, however unlike in the $q = 1$ case the h-vector we obtain is not uniquely defined. For example let $\mathcal{Q}$ be the q-sphere $[S_2]_2$, which has f-vector $f(\mathcal{Q}) = (1,7,7)$. We know that this has many different shelling orders, but the resulting h-vector depends on the shelling order we use. For example it is possible to obtain $h(\mathcal{Q}) = (1,1,2,3)$ and $h(\mathcal{Q}) = (1,2,0,4)$. As a result there does not appear to be an analogy of the expression (2.3) for this definition of a h–vector when $q > 1$.

3. We can conclude from the above discussion that it may make no sense to talk about h-vectors for the $q > 1$ case, even for the best behaved q–complexes.

2.3 Incidence Structure and Incidence Matrices for q–Posets

We now investigate incidence structures related to a q-poset $\mathcal{Q}$ and in particular incidence matrices associated to $\mathcal{Q}$. In doing this we recall that q-posets are ranked, so we can consider the interaction (in terms of the order $\leqslant$ in $\mathcal{Q}$) between the set of elements of different ranks. In analysing incidence structures one of the avenues available to us is to encode the information in the form of an incidence matrix. We start by defining such matrices in general for ranked posets and then go onto consider their properties in the setting of q–posets.

2.3.1 Incidence Matrices for q–Posets

Let $\mathcal{P}$ be a ranked poset. Then for $0 \leqslant k \leqslant l \leqslant rk(\mathcal{P})$ we have an *incidence structure* $(\mathcal{P}_k, \mathcal{P}_l)$, by saying that $y \in \mathcal{P}_l$ is *incident with* $x \in \mathcal{P}_k$ if and only if $x \leqslant y$. We next form the associated *incidence matrix* ${}_k[\mathcal{P}]_l$. The rows of ${}_k[\mathcal{P}]_l$ are indexed by the elements of $\mathcal{P}_k$ and the columns indexed by the elements of $\mathcal{P}_l$, both under some ordering. If we have $\mathcal{P}_k = \{x_1, \ldots, x_i, \ldots\}$ and $\mathcal{P}_l = \{y_1, \ldots, y_j, \ldots\}$ then the ij^{th} entry of this incidence matrix is given by:

$$({}_k[\mathcal{P}]_l)_{(ij)} = \begin{cases} 1 & \text{if } x_i \leqslant y_j; \\ 0 & \text{otherwise.} \end{cases}$$

Remark. A different ordering of the elements $\mathcal{P}_k$ and $\mathcal{P}_l$ will of course lead to different matrices, but these are conjugate under some permutation matrices. For the rest of this text given a poset $\mathcal{P}$, we assume that we have selected and fixed the ordering on each set of ranked elements $\mathcal{P}_i$.

We now move on to consider incidence matrices in the setting of q–posets. Here we have the following result, which is a direct extension of the known results in the boolean algebra ($q = 1$) case and in the projective space $\mathcal{P}(V)$ ($q > 1$) case. Here the binomial coefficients of the original theorem are replaced by q-binomial coefficients. (Recall that we have the identity $\left[{n \atop k}\right]_1 := \binom{n}{k}$).

Theorem 2.3.1. *Let $\mathcal{Q}$ be a q-poset of rank n, where $q \geqslant 1$. For all k, l, m with $0 \leqslant k \leqslant l \leqslant m \leqslant n$ we have:*

$$({}_k[\mathcal{Q}]_l) \cdot ({}_l[\mathcal{Q}]_m) = \begin{bmatrix} m-k \\ m-l \end{bmatrix}_q \cdot ({}_k[\mathcal{Q}]_m) \qquad .$$

Proof. The entry $(({}_k[\mathcal{Q}]_l) \cdot ({}_l[\mathcal{Q}]_m))_{(ij)}$ represents the result of multiplying the i^{th} row of ${}_k[\mathcal{Q}]_l$ by the j^{th} column of ${}_l[\mathcal{Q}]_m$. Let x_i be the element of $\mathcal{Q}_k$ associated to the i^{th} row of ${}_k[\mathcal{Q}]_l$ and z_j be the element of $\mathcal{Q}_m$ associated to the j^{th} column of ${}_l[\mathcal{Q}]_m$. Then, by Definition 1.2.2 and Theorem 2.1.1, we have:

$$((_k[\mathcal{Q}]_l)\cdot(_l[\mathcal{Q}]_m))_{(ij)} = \begin{cases} \left|\{y \in \mathcal{Q}_l \mid x_i \leqslant y \leqslant z_j\}\right| & \text{if } x_i \leqslant z_j; \\ \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} \left|[x_i, z_j]_{(l-k)}\right| = \begin{bmatrix} m-k \\ m-l \end{bmatrix}_q & \text{if } x_i \leqslant z_j; \\ \\ 0 & \text{otherwise.} \end{cases}$$

Then compare this with the corresponding entry of the matrix on the right hand side of our expression:

$$(_k[\mathcal{Q}]_m)_{(ij)} = \begin{cases} 1 & \text{if } x_i \leqslant z_j; \\ \\ 0 & \text{otherwise.} \end{cases}$$

Since the above holds for any (ij) we have the required result. □

In the case $q = 1$ we can derive a converse to Theorem 2.3.1, which gives a characterization of simplicial posets in terms of their incidence matrices.

Theorem 2.3.2. *Let $\mathcal{P}$ be a ranked poset with a unique minimal element. If the following two conditions hold then $\mathcal{P}$ is a simplicial poset (a 1-poset):*

1. *For all comparable pairs $x \leqslant y$ in $\mathcal{P}$, the interval $[x, y]$ forms a meet-semilattice.*
2. *For all k, l, m such that $0 \leqslant k \leqslant l \leqslant m \leqslant rk(\mathcal{P})$ we have $(_k[\mathcal{P}]_l)\cdot(_l[\mathcal{P}]_m) = \binom{m-k}{m-l}\cdot(_k[\mathcal{P}]_m)$.*

Proof. Let $x \leqslant_{\mathcal{P}} y$, with say $|x| = l$ and $|y| = l + n$. By condition 2 we have for $0 \leqslant i \leqslant n$ that $(_l[\mathcal{P}]_{l+i})\cdot(_{l+i}[\mathcal{P}]_{l+n}) = \binom{n}{n-i}\cdot(_l[\mathcal{P}]_{l+n}) = \binom{n}{i}\cdot(_l[\mathcal{P}]_{l+n})$. Thus $|[x, y]_i| = \binom{n}{i}$ and $[x, y]$ has f-vector $f([x, y]) = \left(1, \binom{n}{1}, \ldots, \binom{n}{i}, \ldots, \binom{n}{n}\right)$. We now show by induction that $[x, y] \cong 2^n$. The result is true for any interval of length $n \leqslant 2$, since f-vectors $f = (1), f = (1, 1)$ and $f = (1, 2, 1)$ identify a ranked poset as the boolean algebra 2^n for $n = 0, 1, 2$ respectively.

Assume that any interval of length $r : 0 \leqslant r \leqslant (n-1)$ in $[x, y]$ is isomorphic to 2^r. Let $v_1, \ldots, v_n$ be the elements of $[x, y]_1$ and define a map $\phi : [x, y] \to 2^{\{1,\ldots,n\}} := 2^n$ by

$$\phi(\alpha) = \bigcup_{v_i \in [x,\alpha]_1} \{i\} \quad .$$

Let $\alpha, \beta \in [x,y]$. Now if $\alpha \leqslant_{[x,y]} \beta$ then $[x,\alpha]_1 \subseteq [x,\beta]_1$ and so $\phi(\alpha) \leqslant_{2^n} \phi(\beta)$. Thus ϕ is order preserving. Now take $\alpha \in [x,y] \setminus \{y\}$. By assumption $[x,\alpha] \cong 2^r$, where $r = rk_{[x,y]}(\alpha) = \binom{r}{1} = |[x,\alpha]_1| = rk_{2^n}(\phi(\alpha))$, by definition of ϕ. Also note that $rk_{[x,y]}(y) = n = |[x,y]_1| = rk_{2^n}(\{1,\ldots,n\}) = rk_{2^n}(\phi(y))$. Thus ϕ is rank-preserving.

The map ϕ is also injective. Suppose[1] for $k : 1 \leqslant k \leqslant n-1$ that we have $\alpha, \beta \in [x,y]_k$ with $\phi(\alpha) = \phi(\beta)$ but $\alpha \neq \beta$. This implies $[x,\alpha]_1 = [x,\beta]_1$. By condition 1 the interval $[x,y]$ is a meet-semilattice so it follows that $\gamma = \alpha \cap \beta \in [x,y]$. Since $\alpha \neq \beta$ this implies $rk_{[x,y]}(\gamma) \lneqq k$. But we must also have $[x,\gamma]_1 = [x,\alpha]_1 = [x,\beta]_1$ and this implies $rk_{[x,y]}(\gamma) = |[x,\alpha]_1| = rk_{[x,y]}(\alpha) = k$. We have a contradiction and so $\alpha = \beta$. Thus ϕ is injective and as the f-vectors of $[x,y]$ and 2^n are equal it follows that ϕ is a bijective map.

Proposition 1.2.5 then tells us that ϕ is an embedding and so induces the isomorphism $[x,y] \cong 2^n$. By induction we are done. □

Remarks. We should note that this result does not extend to the $q > 1$ case. An immediate counterexample would be the non-Desarguesian projective planes described in Chapter 1.

2.3.2 The Linear Rank of Incidence Matrices

We now consider the linear rank of the incidence matrices of the previous section over characteristic 0. This question has been fully answered previously in the case of $\mathcal{P}(V)$. We consider here a result for q–posets in general.

We first note that an incidence matrix ${}_k[\mathcal{P}]_l$ associated to a poset $\mathcal{P}$ can be thought of as representing a map between the sets $\mathcal{P}_l$ and $\mathcal{P}_k$. Consider the $\mathbb{R}$-modules with $\mathcal{P}$ and $\mathcal{P}_k$ (for $0 \leqslant k \leqslant rk(\mathcal{P})$) as basis, that is $\mathbb{R}\mathcal{P}$ and $\mathbb{R}\mathcal{P}_k$ respectively with $\mathbb{R}\mathcal{P}_k = 0$ if $k < 0$ or $k > rk(\mathcal{P})$. Then, if $rk(\mathcal{P}) = n$ we have that $\mathbb{R}\mathcal{P} = \oplus_{0 \leqslant k \leqslant n} \mathbb{R}\mathcal{P}_k$. We then define the *inclusion map* $\partial : \mathbb{R}\mathcal{P} \to \mathbb{R}\mathcal{P}$ as the $\mathbb{R}$-homomorphism given by $\partial(x) := \sum y$ for $x \in \mathcal{P}$ where the sum runs over all elements y of co-rank 1 in x. Thus ∂ restricts to maps $\partial : \mathbb{R}\mathcal{P}_k \to \mathbb{R}\mathcal{P}_{k-1}$. Additionally we can consider powers of ∂ of the form $\partial^i : \mathbb{R}\mathcal{P}_k \to \mathbb{R}\mathcal{P}_{k-i}$.

It is a straightforward result that the incidence matrix ${}_k[\mathcal{P}]_l$ represents (up to multiplication by a scalar) the inclusion map ∂^{l-k} between $\mathbb{R}\mathcal{P}_l$ and $\mathbb{R}\mathcal{P}_k$. Thus

[1] Note we can ignore $k = 0$ or $k = n$ as $|[x,y]_0| = |[x,y]_n| = 1$.

the rank of $_k[\mathcal{P}]_l$ can tell us when the inclusion map is injective or surjective.

Note: A matrix M has *full linear rank* if *linear rank*$(M) =$ *min*$\{$ number of rows of M, number of columns of $M\}$. Linear rank here is over characteristic 0.

Theorem 2.3.3. 1. *Let $\mathcal{Q}$ be a q–poset ($q \geqslant 1$) which is pure of rank m. Then for $k \leqslant l$ and $k + l \leqslant m$ we have that the linear rank of $_k[\mathcal{Q}]_l$ is equal to $|\mathcal{Q}_k|$. Hence in this case $_k[\mathcal{Q}]_l$ has full linear rank and in particular $|\mathcal{Q}_k| \leqslant |\mathcal{Q}_l|$.*

2. *Let $\mathcal{Q}$ be a pure q–complex ($q \geqslant 1$) of rank m with $\mathcal{Q} \cong \mathcal{R} \trianglelefteq \mathcal{P}(V)$ and where $|V| = n \geqslant m$. If $\frac{n}{2} \leqslant k \leqslant l \leqslant m$ then $_k[\mathcal{Q}]_l$ attains full rank $|\mathcal{Q}_l|$. In particular $|\mathcal{Q}_l| \leqslant |\mathcal{Q}_k|$.*

Proof. (1.) Assume first that $\mathcal{Q} = \mathcal{P}(V)$ (equivalently $\mathcal{Q}$ has only one facet) for V a finite set if $q = 1$ and V a vector space over $\mathbb{F}_q$ if $q > 1$. Then for all $q \geqslant 1$ and $k \leqslant l$ and $k + l \leqslant m$ the linear rank of $_k[\mathcal{P}(V)]_l$ is $\left[{m \atop k}\right]_q$. Thus $_k[\mathcal{P}(V)_l$ represents a surjective inclusion map. This is the main theorem of Kantor's 1972 paper [25]. Now consider the general case that $\mathcal{Q}$ is a q–poset of rank m. Then in the restriction to a facet the inclusion map represented by $_k[\mathcal{Q}]_l$ is surjective. Therefore in the whole q–poset it follows that the inclusion map represented by $_k[\mathcal{Q}]_l$ is also surjective. Thus $_k[\mathcal{Q}]_l$ has full linear row rank, that is the linear rank of $_k[\mathcal{Q}]_l = |\mathcal{Q}_k|$. Finally since the inclusion map between $\mathcal{Q}_l$ and $\mathcal{Q}_k$ is surjective it follows immediately that $|\mathcal{Q}_k| \leqslant |\mathcal{Q}_l|$.
(2.) Consider again $\mathcal{P}(V)$ with $|V| = n$. Then by part (a) if we take $k' \leqslant l' \leqslant \frac{n}{2}$ we have that $_{k'}[\mathcal{P}(V)]_{l'}$ has full rank $|(\mathcal{P}(V))_{k'}| = \left[{n \atop k'}\right]_q$ and represents an surjective map. However then note that $\mathcal{P}(V)$ is a self-dual poset so in $(\mathcal{P}(V))$ it follows that $_{n-l'}[\mathcal{P}(V)]_{n-k'}$ has full rank $|(\mathcal{P}(V))_{n-l'}| = \left[{n \atop n-l'}\right]_q = \left[{n \atop l'}\right]_q$ and represents an injective map. Since this holds for all $k', l' \leqslant \frac{n}{2}$, we conclude for $\frac{n}{2} \leqslant k \leqslant l$ that $_k[\mathcal{P}(V)]_l$ has full rank $|(\mathcal{P}(V))_l| = \left[{n \atop l}\right]$ and represents an injective map. Therefore in restriction to $\mathcal{R} \trianglelefteq \mathcal{P}(V)$ the result follows immediately. Since $\mathcal{Q} \cong \mathcal{R}$ the result also follows directly for the q–complex $\mathcal{Q}$. □

Remarks. 1. Let $\mathcal{Q}$ be a q–poset as in statement (1.) of Theorem 2.3.3. If $k < \frac{m}{2}$ then $k + (k+1) \leqslant m$ and so $_k[\mathcal{Q}]_{k+1}$ attains full linear rank. Now if $\mathcal{Q}$ is also a q–complex as in statement (2.) of the theorem we have a *'corridor of uncertainty'* for the linear rank of $_k[\mathcal{Q}]_{k+1}$ when k is in the interval $\frac{m}{2} \leqslant k < \frac{n}{2}$. This 'corridor of uncertainty' is minimized if we choose the minimal value for n, that is the minimal value of $|V|$ for which the statement still holds. That is the V for which such an embedding $\varphi : \mathcal{Q} \hookrightarrow \mathcal{P}(V)$ exists and V is the smallest obtainable in terms of rank.

2. Let $\mathcal{Q}$ be a meet-semilattice q–poset of rank m. If it is true that for any two facets of $\mathcal{Q}$ they have an intersection which is of rank below $\frac{m}{2}$ then ${}_k[\mathcal{Q}]_{k+1}$ has maximal linear rank for all choice of k. This follows from the fact that for all facets we have that ${}_k[\mathcal{P}(V)]_{k+1}$ represents an injective map when $k \geqslant \frac{m}{2}$ and we restrict to a single facet. Therefore in the q–poset as a whole this is still true as we have no intersection between the facets in the levels concerned. The result for $k < \frac{m}{2}$ follows from the theorem.
3. The linear rank of incidence matrices for $\mathcal{P}(V)$ over a non-zero characteristic field is known under certain conditions - namely when the characteristic of the field is co-prime to q. For $q = 1$ see Wilson's 1990 paper ([65]) and in the $q > 1$ setting see Frumkin and Yakir's 1990 paper ([15]).
4. Since an incidence matrix represents an inclusion map, if the q–poset is of the form $\mathcal{Q}_X \trianglelefteq \mathcal{A}_q(G)$ for X a union of conjugacy classes of subgroups we can think of G acting on the associated kernel. Through this we obtain a representation of G. This is explored with examples in Appendix A.

2.4 Möbius Inversion for q–Posets

We should not finish the discussion of incidence structures for q–posets without a mention of Möbius inversion. This is a fundamental technique applied to the combinatorics of posets (for more on this see Chapter 25 in [63] or Chapter 3 in [57]).

Let $\mathcal{P}$ be a poset. First consider the set $\mathcal{M}$ of matrices over $\mathbb{Q}$, whose row and columns are indexed by the elements of $\mathcal{P}$. Then if we have a map $f : \mathcal{P} \times \mathcal{P} \to \mathbb{Q}$ we can represent it by $M_f \in \mathcal{M}$ such that $(M_f)_{xy} = f(x, y)$, where $(M_f)_{xy}$ indicates the entry in the row labelled x and column labelled y. Then the *incidence algebra* $I(\mathcal{P})$ consists of all matrices $M \in \mathcal{M}$ such that $M_{xy} = 0$ unless $x \leqslant y$ in $\mathcal{P}$. This is indeed an algebra under matrix multiplication and under suitable arrangement of the elements of $\mathcal{P}$ the elements are upper triangular matrices. We further note that in $I(\mathcal{P})$ matrix multiplication is equivalent to the convolution product for maps $\mathcal{P} \times \mathcal{P} \to \mathcal{Q}$.

We define the *zeta function* ζ of $\mathcal{P}$ as $\zeta(x, y) = 1$ if $x \leqslant y$ and 0 otherwise. This gives rise to an element M_ζ of $I(\mathcal{P})$ which is of interest. After suitable arrangement of the elements of $\mathcal{P}$ we have that M_ζ is an is upper triangular matrix with 1's on the diagonal and therefore is an invertible matrix. Its inverse

M_μ represents the *Möbius function* μ of $\mathcal{P}$. The function μ can also be defined inductively as follows:

$$\begin{aligned} \mu(x,x) &:= 1 && \text{for all } x \in \mathcal{P}, \\ \mu(x,y) &:= -\sum_{z\,:\,x\leqslant z<y} \mu(x,z) && \text{for all } x < y \text{ in } \mathcal{P}. \end{aligned}$$

In the case of q–posets it is possible to give an explicit expression for the Möbius function:

Theorem 2.4.1. *Let $\mathcal{Q}$ be a q–poset ($q \geqslant 1$). Then for all $x, y \in \mathcal{Q}$:*

$$\mu(x,y) = \begin{cases} (-1)^r q^{\binom{r}{2}} & \textit{if } x \leqslant y, \textit{ and } r = |y| - |x|, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If we consider $\mathcal{Q}$ with a single facet then $\mathcal{Q} = \mathcal{P}(V)$. Then we can invoke the standard results (see, for example, [63] p300-301) for the Möbius function for $\mathcal{P}(V)$. For $q = 1$ we have that $\mathcal{P}(V)$ is the power set for a finite set of size n and :

$$\mu(x,y) = \begin{cases} (-1)^{|y|-|x|} & \text{if } x \leqslant y, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Then in the $q > 1$ case for $\mathcal{P}(V)$ we have:

$$\mu(x,y) = \begin{cases} (-1)^{|y|-|x|} q^{\binom{|y|-|x|}{2}} & \text{if } x \leqslant y, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

We note that putting $q = 1$ into (2.5) recovers the expression (2.4) for $q = 1$. Then recall that for a q–poset $\mathcal{Q}$ that any interval $\mathcal{Q}$ is isomorphic to a power set or a projective space (depending on value of q). Thus the inductive definition of μ tells us that the value of $\mu(x,y)$ depends only on the interval $[x,y]$ and so from these facts we conclude that the value of μ for $\mathcal{Q}$ is as quoted. $\square$

Chapter 3

Characterizations of q–Complexes

In Chapter 1 we mentioned briefly the problem of determining when a q–poset forms a q–complex. This problem is fully solved here in the $q = 1$ case, and along the way we pick up some new characterizations of simplicial complexes. For $q > 1$ the situation is non-trivial and we explore the difficulties and show progress in the setting of $\mathcal{A}_q(G)$ for certain finite groups G.

3.1 Joins, Meets, Embeddings and Meet-Semilattice q–Posets

In Chapter 1 we alluded to the result that being a meet-semilattice was a necessary and sufficient condition for a simplicial poset to be a simplicial complex and we will present later a proof of this result. In this section we collect some results about meet-semilattice q–posets which we use in the proof and subsequent characterizations. We start by recalling that in Chapter 1 we defined that the join (respectively meet) of two elements in a poset was their least upper bound (greatest lower bound), where such an element exists. We can extend these definitions in the obvious way to any finite set of elements of a poset. We then have a couple of useful results for the meets and joins of certain sets of elements in meet-semilattice q–posets:

Lemma 3.1.1. *Let $\mathcal{Q}$ be a meet-semilattice q–poset ($q \geqslant 1$) and let $x \in \mathcal{Q}$. Then the join of the set $[0, x]_i$ exists in $\mathcal{Q}$, and is equal to x for all $i \in \mathbb{N}$ with $0 < i \leqslant |x|$.*

Proof. In $[0, x] \trianglelefteq \mathcal{Q}$ we have that x is the join of the set $[0, x]_i$ since $[0, x]$ is isomorphic to a projective space or power set ($q > 1$ and $q = 1$ respectively). Take any $y \in \mathcal{Q}$ such that $y \geqslant_{\mathcal{Q}} z$ for all $z \in [0,x]_i$. Note that this excludes the possibility $y <_{\mathcal{Q}} x$. We need to show that $y \geqslant_{\mathcal{Q}} x$. As $\mathcal{Q}$ is a meet-semilattice $y \cap x := u$ is defined in $\mathcal{Q}$. Then for all $z \in [0,x]_i$ we have $y \geqslant_{\mathcal{Q}} z$ and $x \geqslant_{\mathcal{Q}} z$ and thus $u \geqslant_{\mathcal{Q}} z$. Since u is in $[0,x]$ we conclude that $u = x$ and thus $y \geqslant_{\mathcal{Q}} x$. The argument then holds equally for all $i \leqslant |x|$. □

Remarks. In a ranked poset $\mathcal{P}$ the elements x such that $0 \prec x$ are called *atoms*. If every element of $\mathcal{P}$ is the join of atoms then $\mathcal{P}$ is said to be *atomic*. Therefore if we put $i = 1$ in the statement of the result we see that a meet-semilattice q–poset is atomic.

In the $q = 1$ setting we actually have the following reverse result:

Lemma 3.1.2. *Let $\mathcal{Q}$ be a simplicial poset ($q = 1$). Then $\mathcal{Q}$ is a meet-semilattice if and only if for all $x \in \mathcal{Q}$ the join of the set $[0,x]_1$ exists in $\mathcal{Q}$, and is equal to x.*

Proof. We already have one direction given by Lemma 3.1.1. Conversely let $\mathcal{Q}$ be a simplicial poset for which the join hypothesis holds. Then assume for a contradiction that we have $x, y \in \mathcal{Q}$ such that they have lower bounds $z_1, z_2, \ldots, z_n$ which are pairwise incomparable (and thus $z_j \neq 0$ for any j), $n > 1$ and if $u \leqslant_{\mathcal{Q}} x, y$ then $u \leqslant_{\mathcal{Q}} z_j$ for some $j : 1 \leqslant j \leqslant n$. So we have in terms of sets $[0,x] \cap [0,y] = \bigcup_{j=1}^{n} [0, z_j] \trianglelefteq \mathcal{Q}$ and we also must have $|x|, |y| > 1$. As $[0,x]$ and $[0,y]$ are lattices, there exists e_x the join of the z_i in $[0,x]$ and similarly e_y in $[0,y]$. Since $\mathcal{Q}$ is a simplicial poset and $[0,x], [0,y] \trianglelefteq \mathcal{Q}$ it follows that $[0,e_x]_1 = [0,e_y]_1 = (\bigcup_{j=1}^{n} [0,z_j])_1$. Since e_x and e_y are the joins of their elements of rank 1 in $[0,x]$ and $[0,y]$ respectively; and as $[0,x], [0,y] \trianglelefteq \mathcal{Q}$, by the join hypothesis we must have $e_x = e_y$. Let $e = e_x = e_y$. So $e \geqslant_{\mathcal{Q}} z_i$ for all $i \leqslant n$ and since $e \leqslant_{\mathcal{Q}} x, y$ we have our contradiction. Thus $\mathcal{Q}$ is a meet-semilattice. □

For q–posets the property of being a meet-semilattice is a direct consequence of the existence of the meets of facets:

Lemma 3.1.3. *Let $\mathcal{Q}$ be a q–poset. Then $\mathcal{Q}$ is a meet-semilattice if and only if the meet exists in $\mathcal{Q}$ for all pairs of facets.*

Proof. If $\mathcal{Q}$ is a meet-semilattice then the forward implication holds by definition. Conversely suppose that the meet exists for all pairs of facets and let $x, y \in \mathcal{Q}$. We need to show that $x \cap y$ is uniquely defined in $\mathcal{Q}$. There are facets u, v of $\mathcal{Q}$ (we include the possibility that $u = v$) such that $x \in [0,u]$ and $y \in [0,v]$. By assumption $w := u \cap v$ exists in $\mathcal{Q}$, with $[0,w] \trianglelefteq [0,u]$, $[0,w] \trianglelefteq [0,v]$ and $[0,w] \trianglelefteq \mathcal{Q}$. In $[0,u]$ we have uniquely defined the element $(x \cap w)$. Similarly in $[0,v]$ we have uniquely defined the element $(y \cap w)$. Clearly $(x \cap w), (y \cap w) \in [0,w]$. We claim that $m := (x \cap w) \cap_{[0,w]} (y \cap w)$ is the meet of x and y in $\mathcal{Q}$ (Note: m is uniquely defined element of $[0,w]$, but since $[0,w] \trianglelefteq \mathcal{Q}$ we have $m \in \mathcal{Q}$). As $m \leqslant_{[0,w]} (x \cap w)$ and $[0,w] \trianglelefteq [0,u] \trianglelefteq \mathcal{Q}$ it follows that $m \leqslant_{[0,u]} (x \cap w) \leqslant_{[0,u]} x$ and thus $m \leqslant_{\mathcal{Q}} x$. Similarly $m \leqslant_{[0,v]} (y \cap w) \leqslant_{[0,u]} x$ and thus $m \leqslant_{\mathcal{Q}} y$. Let $z \leqslant_{\mathcal{Q}} x$ and $z \leqslant_{\mathcal{Q}} y$. This implies $z \leqslant_{\mathcal{Q}} u$ and $z \leqslant_{\mathcal{Q}} v$ and so $z \leqslant_{\mathcal{Q}} w$. As $[0,u] \trianglelefteq \mathcal{Q}$ it follows that $z \leqslant_{[0,u]} x$ and $z \leqslant_{[0,u]} w$. Thus $z \leqslant_{[0,u]} (x \cap w)$. Similarly as $[0,v] \trianglelefteq \mathcal{Q}$ then $z \leqslant_{[0,v]} (y \cap w)$. As $[0,w] \trianglelefteq [0,u]$ and $[0,w] \trianglelefteq [0,v]$ this implies $z \leqslant_{[0,w]} (x \cap w)$ and $z \leqslant_{[0,w]} (y \cap w)$. So $z \leqslant_{[0,w]} m$. But as $[0,w] \trianglelefteq \mathcal{Q}$ it follows that $z \leqslant_{\mathcal{Q}} m$. Thus m is the meet of x and y in $\mathcal{Q}$. □

The proof of our theorem on simplicial complexes will be obtained by forming an embedding of a simplicial poset into $\mathcal{P}(V)$ for some finite set V. So we finish this section by collecting some further results on embeddings of q–posets.

Proposition 3.1.4. *Let $\phi : \mathcal{P} \hookrightarrow \mathcal{Q}$ be an embedding where $\mathcal{P}$ is a p-poset with $2 \leqslant |\mathcal{P}|$ with $p \geqslant 1$ and where $\mathcal{Q}$ is a q–poset with $q \geqslant 1$. Assume $\phi(0_{\mathcal{P}}) = 0_{\mathcal{Q}}$ then:*

1. *$p \leqslant q$.*
2. *If $p = q$ it follows that $\phi(\mathcal{P}) \trianglelefteq \mathcal{Q}$. Furthermore if $\mathcal{Q}$ is a q–complex then $\mathcal{P}$ is also q–complex.*
3. *Suppose that $p = q$ and that $\mathcal{Q}$ is a q–complex. If $\theta : \mathcal{P} \to \mathcal{Q}$ is an embedding such that $\phi(y) = \theta(y)$ for all $y \in \mathcal{P}_1$, then $\theta = \phi$.*

Proof. (1.) Let $x \in \mathcal{P}$ such that $|x| = 2$. Since ϕ is an embedding we have $\phi([0,x]) \sqsubseteq [0, \phi(x)]$. Then as ϕ is injective $2 + p + 1 = |\{y \mid y \in [0,x]\}| = |\{y' \mid y' \in \phi([0,x])\}| \leqslant |\{z \mid z \in [0, \phi(x)]\}| = 2 + q + 1$. Therefore $p \leqslant q$.
(2.) Let $z = \phi(x)$ such that $|z| = k$. Then $|[0,z]_{k-1}| = \begin{bmatrix} k \\ k-1 \end{bmatrix}_q$ and since ϕ is cover and rank preserving we have that $\phi([0,x]_{k-1}) \subseteq [0,z]_{k-1}$. But since we also have $|\phi([0,x]_{k-1})| = \begin{bmatrix} k \\ k-1 \end{bmatrix}_p$, with $p = q$, it follows that $\phi([0,x]_{k-1}) = [0,z]_{k-1}$. Therefore, if z covers z' in $\mathcal{Q}$ (so $z' \in [0,z]_{k-1}$) then $z' = \phi(x')$ for some $x' \in [0,x]_{k-1}$ covered by x in $\mathcal{P}$. By extension we see that if $z \in \phi(\mathcal{P})$ then for all $z' < z$ in $\mathcal{Q}$ we have that $z' \in \phi(\mathcal{P})$, thus $\phi(\mathcal{P}) \trianglelefteq \mathcal{Q}$. If $\mathcal{Q}$ is a q–complex then $\mathcal{Q} \cong \mathcal{R}$ with $\mathcal{R} \trianglelefteq \mathcal{P}(V)$ (where V is a finite set if $q = 1$ and V is a finite vector space if $q > 1$). We then recall that an isomorphism preserves the property of being an order ideal by Proposition 1.1.6. Therefore there must exist $\mathcal{S} \trianglelefteq \mathcal{R} \trianglelefteq \mathcal{P}(V)$ such that $\phi(\mathcal{P}) \cong \mathcal{S}$. Since $\mathcal{P} \cong \phi(\mathcal{P})$ it follows that

$\mathcal{P} \cong \mathcal{S} \trianglelefteq \mathcal{P}(V)$ and thus $\mathcal{P}$ is a q–complex.
(3.) Let $x \in \mathcal{P}$ and set $X := [0, x]_1$. Since by Proposition 1.1.6 we have that $\mathcal{P} \cong \phi(\mathcal{P})$ and by (2) above we have $\phi(\mathcal{P}) \trianglelefteq \mathcal{Q}$, it follows that $\phi(X) = [0, \phi(x)]_1$. As $\mathcal{Q}$ is a q–complex we have by Lemma 3.1.1 that $\phi(x) \in \mathcal{Q}$ is the join of all elements in $\phi(X)$. Similarly $\theta(X) = [0, \theta(x)]_1$ and $\theta(x)$ is the join of all elements in $\theta(X)$. Then since $\theta(X) = \phi(X)$ it follows by Lemma 3.1.1 that $\theta(x) = \phi(x)$.

□

Remark. Let $\mathcal{P}$ be a q–poset. We note that if $\mathcal{P}$ is a q–complex then $\mathcal{P} \cong \mathcal{Q} \trianglelefteq \mathcal{P}(V)$, for V a finite set if $q = 1$ or a vector space over $\mathbb{F}_q$ if $q > 1$. Thus combining the isomorphism $\mathcal{P} \cong \mathcal{Q}$ and the inclusion of $\mathcal{Q}$ in $\mathcal{P}(V)$ we have an embedding $\varphi : \mathcal{P} \hookrightarrow \mathcal{P}(V)$ with $\varphi(0_{\mathcal{P}}) = 0_{\mathcal{P}(V)}$. Thus with Proposition 3.1.4 (2.) we have the following result:

Corollary 3.1.5. *A q–poset $\mathcal{Q}$ is a q–complex if and only if there exist an embedding $\phi : \mathcal{Q} \hookrightarrow \mathcal{P}(V)$, where V is a finite set if $q = 1$ or a vector space over $\mathbb{F}_q$ if $q > 1$ and $\phi(0_{\mathcal{Q}}) = 0_{\mathcal{P}(V)}$.*

If an embedding as described in Corollary 3.1.5 exists we call $|V|$ the *embedding dimension*. Clearly such an embedding dimension is never uniquely determined for any q–complex ($q \geqslant 1$).

3.2 Characterizations of Simplicial Complexes

By definition a q–complex is a q–poset but the converse is not always true and examples illustrating this point have been seen in Section 1.3. For $q = 1$ it turns out that if a simplicial poset is a meet-semilattice then this is a sufficient condition for it to be a simplicial complex (this result has been stated, but not explicitly proved, in [13]). We now provide a straightforward proof of this result using the result of Corollary 3.1.5.

Theorem 3.2.1. *Let $\mathcal{Q}$ be a simplicial poset ($q = 1$). Then $\mathcal{Q}$ is a simplicial complex if and only if $\mathcal{Q}$ is a meet-semilattice.*

Proof. By Proposition 1.2.4 we know a simplicial complex is a meet-semilattice. So conversely assume that $\mathcal{Q}$ is a meet-semilattice. We form a map $\varphi : \mathcal{Q} \to \mathcal{P}(V)$, where $V = \mathcal{Q}_1$. For any $x \in \mathcal{Q}$ we define that $\varphi(x) = [0, x]_1$, that is the formal set of rank 1 elements of $[0, x]$. By definition $[0, x] \cong \mathcal{P}(U)$, where U is a finite set of size $|U| = |[0,x]_1| = rk_{\mathcal{Q}}(x)$. Thus φ is rank-preserving. Then if $x \leqslant_{\mathcal{Q}} y$ it follows that $[0,x]_1 \subseteq [0,y]_1$ and thus $\varphi(x) \leqslant_{\mathcal{P}(V)} \varphi(y)$. So φ is order preserving.

Suppose we have $x, y \in \mathcal{Q}$ with $\varphi(x) = \varphi(y)$, hence $[0,x]_1 = [0,y]_1$. Then setting $i = 1$ in Lemma 3.1.1 we have that $x = y$ and thus we conclude φ is injective. Therefore $\varphi : \mathcal{Q} \to \mathcal{P}(V)$ is an embedding by Proposition 1.2.5 and since $\varphi(0_{\mathcal{Q}}) = \emptyset = 0_{\mathcal{P}(V)}$ it follows by Corollary 3.1.5 that $\mathcal{Q}$ is a simplicial complex. □

Remark. A proof of this form would not be possible in the $q > 1$ case as we cannot define the map φ in such an easy manner. In particular setting $V = \mathcal{Q}_1$ would not work, since for $q > 1$ we do not have $\dim(V) = |\mathcal{P}(V)_1|$.

We can combine the results of the previous section on meet-semilattice q–posets and the above theorem to derive two alternative characterizations of simplicial complexes (these do not seem to have appeared in the literature before):

Corollary 3.2.2. *Let $\mathcal{Q}$ be a simplicial poset ($q = 1$). Then $\mathcal{Q}$ is a simplicial complex if and only if the meet exists for all pairs of facets.*

Proof. The result follows immediately from Theorem 3.2.1 and Lemma 3.1.3. □

Corollary 3.2.3. *Let $\mathcal{Q}$ be a simplicial poset ($q = 1$). Then $\mathcal{Q}$ is a simplicial complex if and only if for each $x \in \mathcal{Q}$ the join of $[0,x]_1$ exists in $\mathcal{Q}$ and is equal to x.*

Proof. The result follows immediately from Theorem 3.2.1 and Lemma 3.1.2. □

3.3 Some Characterizations of q–Complexes

We turn our attention now to the problem of determining when q–posets are q–complexes in the case $q > 1$. We would perhaps initially conjecture that the result for $q = 1$ extends to the $q > 1$ case. In this section we exhibit a counterexample to this initial conjecture and show that being a meet-semilattice is not sufficient when $q > 1$. As will become apparent solving the problem in the $q > 1$ case is extremely non-trivial. However, we do discuss a solution in the simplest setting and go onto to show progress can be achieved when the q–poset is an order ideal in $\mathcal{A}_q(G)$, the poset of elementary abelian q-subgroups of some finite group G.

Note: In the rest of this chapter we consider only q–posets where $q > 1$. In this section if $\mathcal{Q}$ is a q–complex for which we have an embedding $\varphi : \mathcal{Q} \hookrightarrow \mathcal{P}(V)$, we mean that it is the embedding as in Corollary 3.1.5. Thus $\varphi(0_{\mathcal{Q}}) = 0_{\mathcal{P}(V)}$ and φ is order and rank preserving.

3.3.1 Simple Beginnings and a Counterexample

Recall that by Corollary 3.1.5 a q–poset is a q–complex if we can find an embedding into $\mathcal{P}(V)$ of our q–poset for some finite dimensional vector space V over $\mathbb{F}_q$. Let us take a q–complex $\mathcal{Q}$ with elements x, y and assume we have such an embedding $\varphi : \mathcal{Q} \hookrightarrow \mathcal{P}(V)$. Then as $\mathcal{P}(V)$ is a projective space for any $x, y \in \mathcal{Q}$ we have the sum space $\alpha = \varphi(x) + \varphi(y) \in \mathcal{P}(V)$, where $|\alpha| = |x| + |y| - |x \cap y|$ (since we have the modular law inside $\mathcal{P}(V)$ and an embedding is order and rank preserving). This must be true of any pair of elements in $\mathcal{Q}$. Using this simple fact we have the following elementary result.

Proposition 3.3.1. *Let $\mathcal{Q}$ be a meet-semilattice q–poset ($q > 1$) with 3 or fewer facets. Then $\mathcal{Q}$ is a q–complex.*

Proof. By definition a single facet q–poset is a q–complex. Now take a meet-semilattice q–poset $\mathcal{Q}$ with two facets x and y. Since $q > 1$ it is quite natural to think of x and y as vector spaces and $[0, x]$, $[0, y]$ as projective spaces. Let c_x and c_y be complements of $x \cap y$ in x and y respectively. Then as vector spaces $x = c_x \oplus (x \cap y)$ and $y = c_y \oplus (x \cap y)$. Now form the vector space $x + y := c_x \oplus (x \cap y) \oplus c_y$ with associated projective space $\mathcal{P}(x + y)$. It follows $\mathcal{P}(x + y)$ forms a projective space where x and y are identified with (and isomorphic to) vector subspaces and where $x + y$ is the sumspace of these subspaces. In particular $\mathcal{Q} = [0, x] \cup [0, y]$ is isomorphic to an order ideal in $\mathcal{P}(x + y)$. Thus $\mathcal{Q}$ is a q–complex.

Now let us consider the case when we extend to a meet-semilattice q–poset $\mathcal{Q}$ which has 3 facets x, y, z. By the above we have an embedding of $[0, x] \cup [0, y]$ into a projective space $\mathcal{P}(x + y)$. We already see under this embedding the image of $(x \cap z)$ and $(y \cap z)$, whose intersection in $\mathcal{Q}$ is $(x \cap y \cap z) \subseteq x \cap y$. So in $\mathcal{P}(x + y)$ we have $u_{\mathcal{P}(x+y)} := (x \cap z) \vee_{\mathcal{P}(x+y)} (y \cap z)$, similarly in $[0, z] \trianglelefteq \mathcal{Q}$ we have $u_{\mathcal{Q}} := (x \cap z) \vee_{\mathcal{Q}} (y \cap z)$. As the original embedding is order and rank preserving these have the same rank in each poset. Thus let us set these equal and so set

$z \cap \mathcal{P}(x+y) = u_{\mathcal{P}(x+y)} = u_{\mathcal{Q}}$ (strictly we have extended the embedding so the image of $u_{\mathcal{Q}}$ is $u_{\mathcal{P}(x+y)}$).

Since $x + y := c_x \oplus (x \cap y) \oplus c_y$ we can write $u_{\mathcal{Q}}$ as $x^* \oplus (x \cap y)^* \oplus y^*$, where $(x \cap y)^* \subseteq (x \cap y)$; $x^* \subseteq c_x$ with $x^* \oplus (x \cap y)^* = (x \cap z)$; and $y^* \subseteq c_y$ with $y^* \oplus (x \cap y)^* = (y \cap z)$. Thus $(u_{\mathcal{Q}}) \cap x = x \cap z$ and $(u_{\mathcal{Q}}) \cap y = y \cap z$. Now we can write z as $z = (u_{\mathcal{Q}}) \oplus c_z$, where c_z is a complement of $(u_{\mathcal{Q}})$ in z. Thus if we set $V := (x + y) \oplus c_z$ we have a natural embedding $\mathcal{Q} \hookrightarrow \mathcal{P}(V)$ extending the embedding $[0, x] \cup [0, y] \hookrightarrow \mathcal{P}(x + y)$ and where $0_{\mathcal{Q}}$ is mapped to $0_{\mathcal{P}(V)}$. It follows that $\mathcal{Q}$ is a q–complex. $\square$

Remarks. This describes a minimal embedding dimension for a q–complex with 2 facets, but in the 3 facet case this is not necessarily minimal. Now if we try to apply a similar analysis to a 4 facet meet-semilattice q–poset this can breakdown and such a case is illustrated by the following counterexample.

Example 3.3.2. Let $\mathcal{Q}$ be a meet-semilattice 2–poset with four facets x_1, x_2, x_3, x_4 all of rank 2 as in Figure 3.1. The claim then is that this q–poset is not a q–complex.

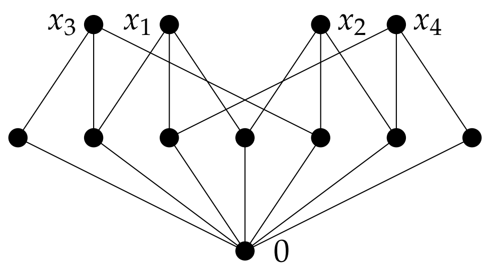

Figure 3.1: Counterexample for $q = 2$.

Proof of Claim: We prove the equivalent statement that we cannot embed this q–poset into any projective space. So assume for a contradiction that we have an embedding $\phi : \mathcal{Q} \hookrightarrow \mathcal{P}(V)$ with $\phi(0_{\mathcal{Q}}) = 0_{\mathcal{P}(V)}$ and V is a vector space over $\mathbb{F}_q$. Then in $\mathcal{P}(V)$ we see the sumspace $z := \phi(x_1) + \phi(x_2)$, with $\dim(z) = 3$ and where $[0, z]$ contains the distinct elements $\phi((x_1 \cap x_3))$, $\phi((x_2 \cap x_3))$. We note that $(x_1 \cap x_3) \vee (x_2 \cap x_3) = x_3$ and since ϕ is rank and order preserving with $\mathcal{Q} \cong \phi(\mathcal{Q}) \trianglelefteq \mathcal{P}(V)$, it follows that $\phi((x_1 \cap x_3)) + \phi((x_2 \cap x_3)) = \phi(x_3)$. Therefore $\phi(x_3) \cap z = \phi(x_3)$ and by a similar argument $\phi(x_4) \cap z = \phi(x_4)$. As ϕ is rank preserving $\dim(\phi(x_3)) = \dim(\phi(x_4)) = 2$. But any pair of 2-dimensional subspaces of a 3-dimensional space intersect in a 1-dimensional subspace, thus

$\dim(\phi(x_3) \cap \phi(x_4)) = 1$ and so we have a contradiction since in $\mathcal{Q}$ we have $|x_3 \cap x_4| = 0$. Therefore we conclude that $\mathcal{Q}$ is not a q–complex. $\square$

Remarks. 1. An analogous example exists for any choice of $q > 1$, with the intersection profile remaining unchanged for the choice of q. As the proof above does not rely on the fact $q = 2$ it follows that for any $q > 1$ the analogous example is not a q–complex.

2. We conclude from Example 3.3.2 that being a meet-semilattice is not a sufficient condition for a q–poset to form a q–complex, when $q > 1$. While we would conjecture that not containing such an order ideal evidenced by Example 3.3.2 is a sufficiency condition for being a q–complex, solving definitively the problem for when a q–poset is a q–complex ($q > 1$) in general remains an open and non-trivial problem.
3. Let $\mathcal{Q}$ be the q–poset of Example 3.3.2 (as we say above we can take any $q > 1$). Then we claim that $\mathcal{Q} \ntrianglelefteq \mathcal{A}_q(G)$ for any finite group G.

   Proof of Claim: So assume for a contradiction that $\mathcal{Q} \trianglelefteq \mathcal{A}_q(G)$ with facets x_1, x_2, x_3, x_4 being elementary abelian q-subgroups of G. Without loss of generality let $x_3 = \langle a, b \rangle$ and $x_4 = \langle c, d \rangle$ where a, b, c, d all have order q and $x_1 \cap x_3 = \langle a \rangle$, $x_2 \cap x_3 = \langle b \rangle$, $x_1 \cap x_4 = \langle c \rangle$, $x_2 \cap x_4 = \langle d \rangle$. Here a and b commute as do c and d since x_3 and x_4 are abelian. Furthermore by assumption we have that that $a^{\alpha_1} b^{\alpha_2} = c^{\beta_1} d^{\beta_2}$ only has the solution $\alpha_1 \equiv 0 \mod q$, $\alpha_2 \equiv 0 \mod q$, $\beta_1 \equiv 0 \mod q$ and $\beta_2 \equiv 0 \mod q$ $(*)$. So we can write $x_1 = \langle a, c \rangle$ and $x_2 = \langle b, d \rangle$. Since $|x_1 \cap x_2| = 1$ it follows that $\exists\, \lambda_1, \lambda_2, \mu_1, \mu_2$ all not equivalent to $0 \mod q$ with $a^{\lambda_1} c^{\lambda_2} = b^{\mu_1} d^{\mu_2}$. But this implies that in G we have:

$$\begin{aligned} & b^{q-\mu_1} a^{\lambda_1} && = d^{\mu_2} c^{q-\lambda_2} \\ \Rightarrow\quad & \underbrace{a^{\lambda_1} b^{q-\mu_1}}_{\in x_3} && = \underbrace{c^{q-\lambda_2} d^{\mu_2}}_{\in x_4} \qquad \text{since } a, b \text{ commute as do } c, d. \end{aligned}$$

   As $\lambda_1, \lambda_2, \mu_1, \mu_2$ all not equivalent to $0 \mod q$ it follows that $\lambda_1, q\lambda_2, q - \mu_1, \mu_2$ are also all not equivalent to $0 \mod q$ and so neither side of this final expression are equal to $\mathbb{1}$. Thus $x_3 \cap x_4$ is non-trivial and we have a contradiction to the assumption $(*)$. $\square$

4. The above result suggests that in the important case of the q–poset $\mathcal{A}_q(G)$ the underlying group structure may allow some progress in solving the problem of characterizations of q–complexes ($q > 1$). This is explored in the next section.

3.3.2 The Poset of Elementary Abelian q–Subgroups $\mathcal{A}_q(G)$

Now let $\mathcal{Q}$ be a q–poset of elementary abelian q-subgroups of some finite group G such that $\mathcal{Q} \trianglelefteq \mathcal{A}_q(G)$. Here we want to describe a construction that provides an embedding of $\mathcal{Q} \hookrightarrow \mathcal{P}(V)$, subject to certain conditions, for V a vector space of $F := \mathbb{F}_q$. We thus may regard this as a partial solution to the problem of finding suitable conditions on such a q–poset to guarantee that it is a q–complex.

Let G be a group and $X = \{X_1, X_2, \ldots, X_m\}$ be a collection of elementary abelian q–subgroups of G, where q is a prime and $\mathcal{Q}_X \trianglelefteq \mathcal{A}_q(G)$. The following shows that in some cases we can determine a vector space V over F such that $\mathcal{Q}_X$ is isomorphic to an order ideal in $\mathcal{P}(V)$ over F.

We begin by defining $\overline{X} = \{g \in G \mid g \in X_i \text{ for some } X_i \in X\}$. Form the subspace $F\overline{X}$ of the group ring FG, considered as a vector space over F. In $F\overline{X}$ we define for $a, b \in G$ a bracket operation $[a, b] = a + b - ab$ (here the multiplication ab is the group operation). Now let $W = W_X$ be the F-subspace of $F\overline{X}$ which is generated by all elements $[a, b]$, where there is some $i \leqslant m$ such that $a, b \in X_i$. We note at this point some basic facts about the bracket operation:

- $[a, b] = [b, a]$ whenever $a, b \in X_i$ for some i, since X_i is abelian.
- We always have $\mathbb{1} \in W$, since $[\mathbb{1}, \mathbb{1}] = \mathbb{1}$.
- When ϕ is an automorphism of G fixing X then ϕ, as a linear map on $F\overline{X}$, fixes W as a subspace of $F\overline{X}$. Furthermore $\phi([a, b]) = [\phi(a), \phi(b)]$, whenever $a, b \in X_i$ for some i.

Next we define a map $\phi : G \to F\overline{X}/W$ by $\pi(g) = g + W$ for all $g \in G$. We note the following:

- If $X_i \in X$ then $\pi : X_i \to F\overline{X}/W$ is a homomorphism. This is clear since we have $(gh) + W = (g + W) + (h + W) - [g, h]$, if $g, h \in X_i$.
- Furthermore if $\pi : X_i \to F\overline{X}/W$ is injective then π induces an embedding $[0, X_i] \to F\overline{X}/W$.

From the above we conclude the following Theorem:

Theorem 3.3.3. *Let X be a collection of elementary abelian q–subgroups of G, where q is a prime. If the map $\pi : \overline{X} \to F\overline{X}/W_X$ is injective then $\pi : \mathcal{Q}_X \hookrightarrow F\overline{X}/W_X$ is an embedding and thus $\mathcal{Q}_X$ is a q–complex.*

So to understand when we can find since a collection of subgroups meeting the condition of the Theorem we need to understand the condition of $\pi : \overline{X} \hookrightarrow F\overline{X}/W_X$ being injective. First note that for g, h in some $X_i \in X$ we have that $g + W = h + W$ if and only if $g - h \in W$ if and only if $g^{-1}h \in W$. This last step follows since $g - h = g - g(g^{-1}h) = [g, g^{-1}h] - g^{-1}h$. Thus if we define $Ker(\pi) = \{k \in \overline{X} \mid \pi(k) = \mathbb{1}\}$ then π is injective if and only if $Ker(\pi) = \{\mathbb{1}\}$. We should note that for each $X_i \in X$ the intersection $Ker(\pi) \cap X_i$ is the usual kernel of the restriction $\pi : \overline{X_i} \to F\overline{X}/W_X$.

In general however $Ker(\pi)$ is not a subgroup of G. On the other hand, if X is a union of conjugacy classes of subgroups of G, then $\overline{X}$ and $Ker(\pi)$ are the union of conjugacy classes. This is clear since if $a \in G$ normalizes $\overline{X}$ then $\pi(a^{-1}ga) = a^{-1}\pi(g)a$ for all $g \in \overline{X}$. This follows from earlier comments about automorphisms fixing X, which imply that $a^{-1}W_X a = W_X$.

We will therefore concentrate our search on such collections X where it is a union of conjugacy classes of subgroups of the group G. Aiding the search is the following elementary result:

Proposition 3.3.4. *Let X, Y be two collections of q-elementary abelian subgroups of a group G such that $(\mathcal{Q}_X)_2 = (\mathcal{Q}_y)_2$ (equivalently an elementary abelian subgroup of order q^2 in G is contained in some $X_i \in X$ if and only if the same is true for some $Y_j \in Y$). Then $\pi_X : \overline{X} \to F\overline{X}/W_X$ is an embedding if the corresponding map $\pi_Y : \overline{Y} \to F\overline{Y}/W_Y$ is an embedding.*

Proof. The condition $(\mathcal{Q}_X)_2 = (\mathcal{Q}_y)_2$ implies that $\overline{X} = \overline{Y}$. Furthermore the group elements a, b lie in one of the facets X_i of $\mathcal{Q}_X$ if and only if they lie in the elementary abelian subgroup $\langle a, b\rangle \subseteq X_i$ of order q^2. Since the same is true for Y it follows that $W_X = W_Y$. □

Remarks. Consider the situation where X is the set of all maximal elementary abelian subgroups of G and Y the set of all elementary abelian subgroups of order q^2 in the statement of the Proposition. Then $(\mathcal{Q}_X)_2 = (\mathcal{Q}_Y)_2$ and so by the result we need only consider Y. Thus we see that we only ever have to concern ourselves with elementary abelian subgroups of order q^2 in our search for a collection of subgroups meeting the conditions of Theorem 3.3.3.

We now turn to applying our result of Theorem 3.3.3 to some examples. We make use of the computer system GAP to construct FG, $F\overline{X}$ and W_X and determine if the map $\pi : \overline{X} \to F\overline{X}/W_X$ is injective. In particular we check that

$Ker(\pi) \cap \overline{X} = \mathbb{1}$. (The GAP script used can be seen in Appendix D.) We begin with an illustrative example:

Example 3.3.5. Let $G = M_{11}$ the smallest of the Mathieu Groups. In particular we look at the elementary abelian q-subgroups when $q = 2$ and $q = 3$. In both cases $\mathcal{A}_q(G)$ is of rank 2 with one conjugacy class of subgroups of size q^1 and one of size q^2. The corresponding f-vectors are $f(\mathcal{A}_2(G)) = (1, 165, 330)$ and $f(\mathcal{A}_3(G)) = (1, 220, 55)$. So take X to be the set of all subgroups of size q^2. The similarities end with the fact that in the $q = 3$ case the map π is injective and we obtain an embedding of $\mathcal{Q}_X = \mathcal{A}_3(G)$, whereas in the $q = 2$ case the map is not injective and we fail to find an embedding of $\mathcal{Q}_X = \mathcal{A}_2(G)$. As expected we find all elements of $\overline{X}$ appear in W_X for $q = 2$. In the $q = 3$ setting we have that $\dim(F\overline{X}) = 441$, $\dim(W_X) = 331$ and thus the embedding dimension here is $\dim(F\overline{X}/W_X) = 110$. So[1] $\mathcal{A}_3(G)$ is a complex, but we are unable to make this assertion for $\mathcal{A}_2(G)$.

Remarks. The unexpected result is that we have for the same group an injective map when $q = 3$ but this ceases to hold when $q = 2$. Computation suggests that in the $q = 2$ setting the q–poset is some 'way off' from meeting the condition of π being injective. It is possible of course to compute how many groups have to be removed from the collection X which will leave the map π injective. Here $\mathcal{Q} = \mathcal{A}_2(G)$ has 330 facets and computation suggests you would have to remove 192 groups to obtain an injective map. This obviously represents a significant proportion of the facets of $\mathcal{Q}$.

By choice the set $\overline{X}$ is a union of conjugacy classes so it follows that G acts on the basis elements of $F\overline{X}$. Now via the map π we obtain an action of G over $F\overline{X}/W_X$ and therefore we obtain a representation of G over F. So in Example 3.3.5 we have described a representation of $G = M_{11}$ over $F = \mathbb{F}_3$ of degree 110. Then via the Meataxe tool in GAP we can obtain the list of the irreducible composition factors with their multiplicities. For this case we have that the breakdown[2] is $1a + 5ab + 10aab + 24a + 45a$.

[1]In Chapter 4 we discuss an alternative method of determining that this forms a q–complex.
[2]The decomposition is set out in a manner analogous to that used in the $\mathbb{ATLAS}$. So, for example, $1a$ is the trivial character.

3.3.3 Examples of q–Complexes From $\mathcal{A}_q(G)$

We now go onto to set out some examples where we have confirmed that the associated map π, as constructed in the previous section, is injective and thus the q–poset is a q–complex in these cases. Those examples of course only scratch the surface of such possible examples. We separate these results into a number of tables relating different classes of groups. In each table the column $|X|$ gives the number of facets of the complex.

Symmetric Group and Alternating Group

The results for q–posets associated to Symmetric ($\mathfrak{S}_n$) and Alternating ($Alt(n)$) groups are shown in Table 3.1. We note the following:

- 1. $X_1 = \{\langle(5\,6),(3\,4),(1\,2)\rangle^g \mid g \in \mathfrak{S}_6\}$.
  2. $X_2 = \{\langle(5\,6),(1\,2)(3\,4),(1\,3)(2\,4)\rangle^g \mid g \in \mathfrak{S}_6\}$.
  3. $X_3 = \{\langle(4\,5)(6\,7),(2\,3)(6\,7)\rangle^g \mid g \in Alt(7)\}$.
  4. $X_4 = \{\langle(6\,7),(4\,5),(2\,3)\rangle^g \mid g \in \mathfrak{S}_7\}$.
  5. X_5 is a class of groups of size 2^3.
  6. $X_6 = \{\langle(5\,6)(7\,8),(5\,7)(6\,8),(1\,2)(3\,4),(1\,3)(2\,4)\rangle^g \mid g \in \mathfrak{S}_8\}$.
- For $\mathcal{Q} = \mathcal{A}_2(Alt(5))$ we recall that we have previously determined that this is a q–complex - see Example 1.3.6 where we showed the free embedding to be of dimension 10. We note the agreement of embedding dimension between these two methods. The same agreement is also seen for $\mathcal{Q} = \mathcal{A}_3(Alt(6))$, which has a free embedding of dimension 20.
- In the next chapter we take an alternative path to determining when certain q–posets are q–complexes. Some of the complexes identified above will also meet the conditions that arise in this alternative analysis. The examples in Table 3.1 with a (†) represent examples of the *p-cycle complex*, which will receive particular attention in the next chapter.
- It is noticeable that for the alternating group and symmetric group examples that none of the embeddings give a modular representation that is irreducible. This is not a conscious choice of examples, but reflects that we have found none with this property. Compare this with the other classes of groups where we do see irreducible characters turning up.

Table 3.1: Symmetric Group and Alternating Group q–Complexes.

$\mathcal{Q}$	$\|X\|$	$\dim(FX)$	$\dim(W)$	$\dim(FX/W)$	Meataxe Decomposition
$\mathcal{A}_2(Alt(5))$ (†)	5	16	6	10	1aa + 4aa
$\mathcal{A}_2(\mathfrak{S}_4)$ (†)	4	10	5	5	1a + 2aa
$\mathcal{A}_3(\mathfrak{S}_4)$ (†)	4	9	5	4	1a + 3a
$\mathcal{A}_2(\mathfrak{S}_5)$	20	26	21	5	1a + 4a
$\mathcal{A}_3(\mathfrak{S}_5)$ (†)	10	21	11	10	4a + 6a
$\mathcal{A}_3(\mathfrak{S}_6)$ (†)	10	81	61	20	4ab + 6aa
$\mathcal{Q}_{X_1} \trianglelefteq \mathcal{A}_2(\mathfrak{S}_6)$ (†)	15	76	61	15	1aaa+4aab
$\mathcal{Q}_{X_2} \trianglelefteq \mathcal{A}_2(\mathfrak{S}_6)$ (†)	15	76	61	15	1aaa+4aab
$\mathcal{Q}_{X_3} \trianglelefteq \mathcal{A}_2(Alt(7))$	105	106	86	20	6a + 14a
$\mathcal{Q}_{X_4} \trianglelefteq \mathcal{A}_2(\mathfrak{S}_7)$ (†)	105	232	211	21	1a+6a+14a
$\mathcal{Q}_{X_5} \trianglelefteq \mathcal{A}_2(\mathfrak{S}_8)$	30	106	91	15	1a+14a
$\mathcal{Q}_{X_6} \trianglelefteq \mathcal{A}_2(\mathfrak{S}_8)$	35	316	301	15	1a+14a

Projective General Linear Groups

The results for q–posets associated to $PGL_n(d)$, the Projective Linear Group of $n \times n$ matrices with entries in $\mathbb{F}_d$, are shown in Table 3.2.

Table 3.2: Projective General Linear Group q–Complexes.

$\mathcal{Q}$	$\|X\|$	$\dim(FX)$	$\dim(W)$	$\dim(FX/W)$	Meataxe Decomposition
$\mathcal{A}_2(PGL_2(2))$	3	4	1	3	1a + 2a
$\mathcal{A}_2(PGL_3(2))$	14	22	14	8	8a
$\mathcal{A}_3(PGL_3(2))$	28	57	29	28	1a + 6a + 7aaa
$\mathcal{A}_2(PGL_2(3))$	4	10	5	5	1a + 2aa
$\mathcal{A}_3(PGL_2(3))$	4	9	5	4	1a + 3a

Finite Simple Groups

The results for q–posets associated to certain finite simple groups are shown in Table 3.3. The notation for groups is the same as that given in [23].

- X_1 is either of two conjugacy classes of groups of size $2^2 = 4$.

- X_2 is a conjugacy classes of groups of size $2^3 = 8$.

Table 3.3: Finite Simple Group q–Complexes.

$\mathcal{Q}$	$\lvert X\rvert$	$\dim(FX)$	$\dim(W)$	$\dim(FX/W)$	Meataxe Decomposition
$\mathcal{A}_2(L_3(2))$	8	22	14	8	8a
$\mathcal{A}_3(L_3(2))$	28	57	29	28	1a+6a+7aaa
$\mathcal{A}_2(L_2(8))$	9	64	37	27	1aaa+6aa+12a
$\mathcal{A}_3(L_2(8))$	28	57	29	28	7aaaa
$\mathcal{A}_2(L_2(11))$	55	56	46	10	10a
$\mathcal{A}_3(L_2(11))$	55	111	56	55	1a+10aaa+24a
$\mathcal{A}_2(L_2(13))$	91	92	78	14	14a
$\mathcal{A}_7(L_2(13))$	78	469	391	78	12aaa+14abb
$\mathcal{A}_2(L_2(17))$	204	154	138	16	16a
$\mathcal{A}_2(U_3(3))$	63	64	50	14	14a
$\mathcal{Q}_{X_1} \trianglelefteq \mathcal{A}_2(L_2(23))$	253	254	232	22	11ab
$\mathcal{A}_2(L_3(4))$	32	316	300	16	16a
$\mathcal{Q}_{X_2} \trianglelefteq \mathcal{A}_2(U_4(2))$	316	316	296	20	6a+14a

Remarks.

1. We note that in the case of $PGL_3(2)$ and a number of the finite simple groups that the embedding construction nicely gives an irreducible representation of the associated group over $\mathbb{F}_q$, though this unfortunately is not a general property.
2. All the q–posets of the form $\mathcal{Q}_X$ described in the tables above are by Theorem 3.3.3 q–complexes since the associated map π is injective in these cases.
3. As we noted in Example 3.3.5 the map π is not always injective and this is not the only such case. For example if $\mathcal{Q} = \mathcal{A}_2(\mathfrak{S}_6)$ then π is not injective and W contains the conjugacy class of double transpositions e.g.$(1\,2)(3\,4)$. Indeed we get a similar result for $\mathcal{Q} = \mathcal{A}_2(\mathfrak{S}_n)$ when $n = 7, 8$, but we saw above that by suitable choice of conjugacy classes of groups (equivalently group elements) we do obtain an injective map and an embedding of the corresponding q–subposet.
4. The theorem gives only a sufficiency criteria and so it should be noted that if an order ideal $\mathcal{Q} \trianglelefteq \mathcal{A}_q(G)$ fails the conditions of Theorem 3.3.3 this does not necessarily mean that $\mathcal{Q}$ is not a q–complex. It remains at this time an open question as to how the conditions of Theorem 3.3.3 can be described in purely group properties.

5. We would obviously like to have a definitive result which gives a necessary and sufficient condition for when such order ideals are q–complexes. The construction of the embedding in Theorem 3.3.3 seems to be the most elementary and thus there is some support to the conjecture that if $\mathcal{Q}_X \trianglelefteq \mathcal{A}_q(G)$ does not admit such an injective map then $\mathcal{Q}_X$ is not a q–complex. Indeed for $q = 2$, if this method works it would appear to be a universal construction with the relation $a + b - ab = 0$ having to hold in any embedding. Thus in this case any embedding of $\mathcal{Q}_X$ is a quotient of the one constructed in the manner described above (for some background on this see the discussion of nonabelian representation groups and their abelian quotients in Part 1 of [21]).

3.3.4 Some Results in the 1–Facet Case

Let $\mathcal{Q} \trianglelefteq \mathcal{A}_q(G)$ be a q–poset of rank n. If Q has only one facet then it is immediate that $\mathcal{Q}$ is isomorphic to a projective space and indeed is isomorphic to the lattice of subgroups of the elementary abelian group C_q^n. So an obvious question of interest is whether in this case the poset $\mathcal{Q}$ meets the conditions of Theorem 3.3.3. There is no immediate general solution but we do have a full proof in the case $q = 2$ and $q = 3$.

Let G_n be an elementary abelian group of order q^n with basis $g_1, g_2, \ldots, g_n$. We thus denote an element of G as $g_1^{i_1} \cdot g_2^{i_2} \cdots g_n^{i_n}$ and $G_n := C_q^n$. Also let $W_n := \langle [a, b] \mid a, b \in G_n \rangle \subseteq FG_n$.

Consider FG_n/W_n. We note for any element $a \in G_n$ that $a + a \equiv a^2 \mod W_n$ and more generally we have that $za \equiv a^z \mod W$ for $z \in \mathbb{N}$. This implies that any coset in FG_n/W_n can be written as

$$\begin{aligned} z_1 a_1' + z_2 a_2' + \cdots z_r a_r' + W &= (a_1 + a_2 + \cdots a_r) + W \\ &= (a_1 \cdot a_2 \cdots a_r) + W \\ &= a + W \qquad \text{where } a = a_1 \cdot a_2 \cdots a_r \in G_n \,. \end{aligned}$$

Thus we conclude that $\pi : G_n \to FG_n/W_n$ is a surjection. Since $\dim(G_n) = n$ as an F-vector space, if $\dim(W_n) = q^n - n$ this then implies that $G_n \cong FG_n/W$. This is the method used in the proof for $q = 2$ and $q = 3$ below:

$q = 2:$

We define inductively a subset $B_n \subseteq W_n$ of elements of the form $[a, b]$ with $a, b \in G_n$, which is a basis for W_n. As a base step note that if $n = 1$ that $[a, b] = \mathbb{1}$ for all choices of $a, b \in G_1$. Thus a basis for W_1 is $B_1 = \{[\mathbb{1}, \mathbb{1}]\}$. This is then of size $2^1 - 1 = 1$.

Now assume for an inductive hypothesis that we have a basis B_n for W_n of size $2^n - n$, where the elements of B_n are all of the shape $[a, b]$ for some $a, b \in G_n$. We pick a single representative $[a, b]$ for each such element of B_n and further we assume that $\mathbb{1} = [\mathbb{1}, \mathbb{1}] \in B_n$. Next form two sets of elements of W_{n+1}:

$$\begin{aligned} S_1 &= \{[\mathbb{1} \cdot g_{n+1}, c_1 \cdot g_{n+1}] \quad : \quad \forall\, c_1 \in G_n\}\,, \\ S_2 &= \{[c_2 \cdot \mathbb{1}, c_3 \cdot \mathbb{1}] \quad : \quad \forall\, [c_2, c_3] \in B_n\} \quad . \end{aligned}$$

We claim that $B_{n+1} := S_1 \cup S_2$ forms a basis for W_{n+1} of size $2^{n+1} - (n + 1)$.

Proof of Claim:
- Since $|G_n| = 2^n$ it follows that $|S_1| = 2^n$ as no two elements of S_1 are equal. Then by assumption $|S_2| = |B_n| = 2^n - n$. If $x \in S_1 \cap S_2$ then we have: $x = \mathbb{1} \cdot g_{n+1} + c_1 \cdot g_{n+1} + c_1 \cdot \mathbb{1} = c_2 \cdot \mathbb{1} + c_3 \cdot \mathbb{1} + c_2 c_3 \cdot \mathbb{1}$. Equality is only possible if both sides simplify to a single element of FG_{n+1}. This only occurs if we set $c_1 := \mathbb{1}$ and $[c_2, c_3] = \mathbb{1}$. Thus by assumptions about B_n we see that $x = [\mathbb{1} \cdot g_{n+1}, \mathbb{1} \cdot g_{n+1}] = [\mathbb{1}, \mathbb{1}] = \mathbb{1}$. Therefore we conclude that $|S_1 \cap S_2| = 1$ and thus $|B_{n+1}| = 2^n + 2^n - n - 1 = 2^{n+1} - (n + 1)$.
- Next we show that the elements of B_{n+1} are linearly independent. By assumption B_n is a basis for W_n so it follows that S_2 is a linearly independent set. Then for S_1 consider that each element has the form $\mathbb{1} \cdot g_{n+1} + c_1 \cdot g_{n+1} + c_1 \cdot \mathbb{1}$. Then if $c_1 \neq \mathbb{1}$ we note that the summands $c_1 \cdot g_{n+1}$ + $c_1 \cdot \mathbb{1}$ only appear once in the elements of S_1 and therefore we conclude that S_1 is a linearly independent set. Furthermore the summand $c_1 \cdot g_{n+1}$ does not appear in any elements of S_2 and only in one element of S_1 if $c_1 \neq \mathbb{1}_n$. Since the case $c_1 = \mathbb{1}$ gives $\mathbb{1}$, which is the only element of $S_1 \cap S_2$, it follows that $B_{n+1} = S_1 \cup S_2$ is linearly independent.
- Lastly these elements generate all $[a, b]$ for $a, b \in G_{n+1}$ and thus span W_{n+1}. The elements of the form $[a, b]$ in W_{n+1} fall into 3 different types:
  (A) $[d_1^n \cdot \mathbb{1}_1, d_2^n \cdot \mathbb{1}_1]$ where $d_i^n \in G_n$ for $i = 1, 2$.
  (B) $[d_1^n \cdot g_{n+1}, d_2^n \cdot \mathbb{1}_1]$ where $d_i^n \in G_n$ for $i = 1, 2$.

(C) $[d_1^n \cdot g_{n+1}, d_2^n \cdot g_{n+1}]$ where $d_i^n \in G_n$ for $i = 1, 2$.

By inductive assumption those of type (A) can be generated by elements of S_2. Now for types (B) and (C) we note the following relations. These show that types (B) and (C) are linear combinations of elements of B_{n+1}:

$$\begin{aligned}
\text{(B)}\ [d_1^n \cdot g_{n+1}, d_2^n \cdot \mathbb{1}_1] &= [\mathbb{1} \cdot a_{n+1}, d_1^n \cdot g_{n+1}] + [\mathbb{1} \cdot g_{n+1}, (d_1^n d_2^n) \cdot g_{n+1}] \\
&\quad + [d_1^n \cdot \mathbb{1}, d_2^n \cdot \mathbb{1}_1]\,. \\
\text{(C)}\ [d_1^n \cdot g_{n+1}, d_2^n \cdot a_{n+1}] &= [\mathbb{1} \cdot g_{n+1}, d_1^n \cdot a_{n+1}] + [\mathbb{1} \cdot g_{n+1}, d_2^n \cdot g_{n+1}] \\
&\quad + [d_1^n \cdot \mathbb{1}, d_2^n \cdot \mathbb{1}_1]\,.
\end{aligned}$$

Therefore as claimed B_{n+1} spans W_{n+1} and since $[\mathbb{1}, \mathbb{1}] \in B_{n+1}$ we are done by induction. □

$q = 3$:

We repeat the above argument with $q = 3$. Now $G_n = C_3^n$, otherwise notation is unchanged. As a base step note that if $n = 1$ then it is straightforward to show that a basis for W_1 is $B_1 = \{[\mathbb{1}, \mathbb{1}], [g_1, g_1]\}$. This is then of size $3^1 - 1 = 2$.

As for $q = 2$, we make the same inductive hypothesis about B_n (excluding size of B_n which is now $3^n - n$). Now form the following three sets of elements of W_{n+1}:

$$\begin{aligned}
S_{1a} &= \{[\mathbb{1} \cdot g_{n+1}, c_1 \cdot g_{n+1}^2] \quad : \quad \forall c_1 \in G_n\}\,, \\
S_{1b} &= \{[\mathbb{1} \cdot g_{n+1}^2, c_1 \cdot g_{n+1}] \quad : \quad \forall c_1 \in G_n\}\,, \\
S_2 &= \{[c_2 \cdot \mathbb{1}, c_3 \cdot \mathbb{1}] \quad : \quad \forall [c_2, c_3] \in B_n\} \quad .
\end{aligned}$$

We claim that $B_{n+1} := S_{1a} \cup S_{1b} \cup S_2$ forms a basis for W_{n+1} of size $3^{n+1} - (n+1)$.

Proof of Claim:

- By assumption $|S_2| = |B_n| = 3^n - n$. Then since $|G_n| = 3^n$ it follows that $|S_{1a}| = 3^n$ as no two elements of S_{1a} are equal. Similarly $|S_{1b}| = 3^n$. Then we note that $S_{1a} \cap S_2 = S_{1b} \cap S_2 = \emptyset$. However $S_{1a} \cap S_{1b} = \{[\mathbb{1} \cdot g_{n+1}, \mathbb{1} \cdot g_{n+1}^2] = [\mathbb{1} \cdot g_{n+1}^2, \mathbb{1} \cdot g_{n+1}]\}$. Thus $|B_{n+1}| = 2 \cdot 3^n - 1 + 3^n - n = 3^{n+1} - (n+1)$.
- Next we show that the elements of B_{n+1} are linearly independent. By assumption that B_n is a basis for W_n it follows that S_2 is a linearly independent set. Then for S_{1a} and S_{1b} consider that each element has the form $\mathbb{1} \cdot g_{n+1} + c_1 \cdot g_{n+1}^2 - c_1 \cdot \mathbb{1}$ and $\mathbb{1} \cdot g_{n+1}^2 + c_1 \cdot g_{n+1} - c_1 \cdot \mathbb{1}$ respectively. Then

if $c_1 \neq \mathbb{1}$ we note that the summands $c_1 \cdot g_{n+1}^2$ and $c_1 \cdot g_{n+1}$ only appear once in the elements of S_{1a} and S_{1b} respectively, therefore we conclude that S_{1a} and S_{1b} are both linearly independent set. Furthermore the summands $c_1 \cdot g_{n+1}^2$ and $c_1 \cdot g_{n+1}$ do not appear in any elements of S_2 and each only in one element of $S_{1a} \cup S_{1b}$ if $c_1 \neq \mathbb{1}$. Since the case $c_1 = \mathbb{1}$ gives $S_{1a} \cap S_{1b}$ it follows that $B_{n+1} = S_{1a} \cup S_{1b} \cup S_2$ is linearly independent.

- Lastly these elements generate all $[a, b]$ for $a, b \in G_{n+1}$ and thus span W_{n+1}. The elements of the form $[a, b]$ in W_{n+1} fall into 6 different types:

  (A) $[d_1^n \cdot \mathbb{1}_1, d_2^n \cdot \mathbb{1}]$ where $d_i^n \in G_n$ for $i = 1, 2$.
  (B) $[d_1^n \cdot g_{n+1}, d_2^n \cdot \mathbb{1}]$ where $d_i^n \in G_n$ for $i = 1, 2$.
  (C) $[d_1^n \cdot g_{n+1}^2, d_2^n \cdot \mathbb{1}]$ where $d_i^n \in G_n$ for $i = 1, 2$.
  (D) $[d_1^n \cdot g_{n+1}, d_2^n \cdot g_{n+1}]$ where $d_i^n \in G_n$ for $i = 1, 2$.
  (E) $[d_1^n \cdot g_{n+1}^2, d_2^n \cdot g_{n+1}]$ where $d_i^n \in G_n$ for $i = 1, 2$.
  (F) $[d_1^n \cdot g_{n+1}^2, d_2^n \cdot g_{n+1}^2]$ where $d_i^n \in G_n$ for $i = 1, 2$.

  By inductive assumption those of type (A) can be generated by elements of S_2. Now for the other types (B), (C), (D), (E) and (F) we note the following linear relations. These show that these types can be written as linear combinations of elements of B_{n+1}:

$$\begin{aligned}
\text{(B)}\quad [d_1^n \cdot g_{n+1}, d_2^n \cdot \mathbb{1}] &= [\mathbb{1} \cdot g_{n+1}^2, d_1^n \cdot a_{n+1}] - [\mathbb{1} \cdot g_{n+1}^2, (d_1^n d_2^n) \cdot g_{n+1}] \\
&\quad + [d_1^n \cdot \mathbb{1}, d_2^n \cdot \mathbb{1}]. \\
\text{(C)}\quad [d_1^n \cdot g_{n+1}^2, d_2^n \cdot \mathbb{1}] &= [\mathbb{1} \cdot g_{n+1}, d_1^n \cdot g_{n+1}^2] - [\mathbb{1} \cdot g_{n+1}, (d_1^n d_2^n) \cdot g_{n+1}^2] \\
&\quad + [d_1^n \cdot \mathbb{1}, d_2^n \cdot \mathbb{1}]. \\
\text{(D)}\quad [d_1^n \cdot g_{n+1}, d_2^n \cdot g_{n+1}] &= [\mathbb{1} \cdot g_{n+1}^2, d_1^n \cdot g_{n+1}] + [\mathbb{1} \cdot g_{n+1}^2, d_2^n \cdot g_{n+1}] \\
&\quad + [d_1^n \cdot \mathbb{1}, d_2^n \cdot \mathbb{1}] - [\mathbb{1} \cdot g_{n+1}, (d_1^n d_2^n) \cdot g_{n+1}^2] \\
&\quad - 2[\mathbb{1} \cdot g_{n+1}, \mathbb{1} \cdot g_{n+1}^2] - 2[\mathbb{1}, \mathbb{1}]. \\
\text{(E)}\quad [d_1^n \cdot g_{n+1}^2, d_2^n \cdot g_{n+1}] &= [\mathbb{1} \cdot g_{n+1}, d_1^n \cdot g_{n+1}^2] + [\mathbb{1} \cdot g_{n+1}^2, d_2^n \cdot g_{n+1}] \\
&\quad + [d_1^n \cdot \mathbb{1}, d_2^n \cdot \mathbb{1}] - [\mathbb{1} \cdot g_{n+1}, \mathbb{1} \cdot g_{n+1}^2] - [\mathbb{1}, \mathbb{1}]. \\
\text{(F)}\quad [d_1^n \cdot g_{n+1}^2, d_2^n \cdot g_{n+1}^2] &= [\mathbb{1} \cdot g_{n+1}, d_1^n \cdot g_{n+1}^2] + [\mathbb{1} \cdot g_{n+1}, d_2^n \cdot g_{n+1}^2] \\
&\quad + [d_1^n \cdot \mathbb{1}, d_2^n \cdot \mathbb{1}] - [\mathbb{1} \cdot g_{n+1}^2, (d_1^n d_2^n) \cdot g_{n+1}] \\
&\quad - 2[\mathbb{1} \cdot g_{n+1}, \mathbb{1} \cdot g_{n+1}^2] - 2[\mathbb{1}, \mathbb{1}].
\end{aligned}$$

  Therefore as claimed B_{n+1} spans W_{n+1} and since $[\mathbb{1}, \mathbb{1}] \in B_{n+1}$ we are done by induction. □

Thus we have proved the following elementary result:

Proposition 3.3.6. *Let G be an elementary abelian group of order $|G| = q^n$ and F be the field of q elements. Also Let $W \subseteq FG$ be the space spanned by all $[a, b]$ with $a, b \in G$. Then $G \cong FG/W$ if $q = 2$ or $q = 3$.*

Remarks.

1. The above method of proof does not appear to exhibit a pattern that follows through to a general proof for all $q > 1$. Even when moving to the $q = 5$ setting it is not apparent what the shape of sets that span W_{n+1} look like that have the size we seek. However, all small cases done by hand and those done by computer suggest the result maybe true for all $q > 1$ and for all n. By computer (using GAP) we have a positive result for $G = C_5^n, C_7^n$ with $n \leqslant 3$ and for $G = C_q^n$ with $n \leqslant 2$ when $q = 7, 11, 13$ or 19. No counterexample has been found for any $q > 1$ and $n \geqslant 1$.
2. The fact that a proof for the simplest case of C_q^n in generality is almost certainly non-trivial, adds some weight to previous comments on the difficulty of obtaining a more definitive result in the general setting of $\mathcal{Q}_X \trianglelefteq \mathcal{A}_q(G)$ case.

Chapter 4

Vertex Sets and the p-Cycle Complex

We saw in the previous chapter that the problem of characterizing q–complexes is not straightforward when $q > 1$. In this chapter we consider a class of q–posets for which a 'nice' condition tells us that they are a q–complex. To describe this condition we introduce the concept of a *vertex set* for a q–poset, which reflects a generalization of a vertex set of a simplicial complex and the basis of a vector space. Meet-semilattice q–posets which display a particular type of vertex set will be shown to be q–complexes with some interesting properties. We finish this chapter by describing in some detail an important example of such a complex associated with the Symmetric Group.

4.1 Vertex Sets : Definition and Properties

To motivate the definition of a *vertex set* for q–posets we first note that geometrically in a simplicial complex its vertex set is the elements of rank 1. Recall the simplicial complex formed from finite words in Chapter 1. This can be visualized as an octahedron (Figure 1.3). Here its rank 1 elements are the vertices of the octahedron. In the $q > 1$ setting the elements of a basis each generate a 1-dimensional subspace of V and it is these linearly independent 1-spaces that can be thought of as a 'vertex set' for the projective space of V, namely $\mathcal{P}(V)$. The following is then a generalization of this idea to q–posets ($q \geqslant 1$):

Definition 4.1.1. Let $\mathcal{Q}$ be a q–poset ($q \geqslant 1$) and let $y \in \mathcal{Q}$. Then a *vertex set* for y is a set $\{x_1, x_2, \ldots, x_k\} \subseteq [0, y]_1$ such that $k = rk(y)$ and $\bigvee_{i=1}^{k} x_i = y$. Furthermore if $\mathcal{B}$ is a subset of $\mathcal{Q}_1$ such that for all facets z of $\mathcal{Q}$ we have that

$\mathcal{B}(z) := \mathcal{B} \cap ([0,z]_1))$ forms a vertex set for the facet z, then $\mathcal{B}$ is a *vertex set for* $\mathcal{Q}$. In this case we say that $\mathcal{Q}$ *admits* a vertex set.

Remarks.

1. In the $q = 1$ setting the unique vertex set of a facet is the set of elements of rank 1. If $\mathcal{Q}$ is a simplicial complex then $\mathcal{B}$ is the set of vertices of $\mathcal{Q}$, that is $\mathcal{B} = \mathcal{Q}_1$.
2. If a q–poset $\mathcal{Q}$ admits a vertex set $\mathcal{B}$ and $\mathcal{B} \cap ([0,y]_1))$ is a vertex set for y then as for facets we denote this set by $\mathcal{B}(y)$.
3. For the minimal element 0 of a q–poset we set $\mathcal{B}(0) := \varnothing$.
4. If $q > 1$ and we consider an element $y \in \mathcal{Q}$ then by definition $[0,y]$ is a projective space over $\mathbb{F}_q$. Thus a vertex set $\mathcal{B}(y)$ consists of the linearly independent 1-dimensional subspaces $\langle a_i \rangle, \ldots, \langle a_k \rangle$ where $a_i, \ldots, a_k$ is a basis of y. This is then an alternative definition of a vertex set when $q > 1$. Indeed for $q > 1$ this is the most practical definition for us to use in the analysis below.
5. A consequence of the above analysis is that if $\{x_1, x_2, \ldots, x_k\} \subseteq \mathcal{B}(y)$, then their join exists in $[0,y]$. Furthermore $rk(\bigvee_{i=1}^{k} x_i) = k$.
6. For a q–poset there is not a unique vertex set when $q > 1$. In the one facet setting this is a consequence of the fact that the basis of a vector space is not unique.
7. Not all q–posets will admit a vertex set. We can illustrate this in the following simple example for a known q–complex:

Example 4.1.2. Consider $V = \mathbb{F}_2^3 = \langle a,b,c \rangle_{\mathbb{F}_2}$. Then take the 2-sphere $[S_3]_2$ formed by removing V from the projective space $\mathcal{P}(V)$. This is a q–complex, but we claim that it does not admit a vertex set. Let us list its 7 facets:

$$\begin{array}{llllll} x_1 = & \langle a,b \rangle & x_2 = & \langle a,c \rangle & x_3 = & \langle b,c \rangle \\ x_4 = & \langle a+b, a+c \rangle & x_5 = & \langle a+b+c, b+c \rangle & x_6 = & \langle a+b+c, a+b \rangle \\ x_7 = & \langle a+b+c, a+c \rangle & . & & & \end{array}$$

Without loss of generality start with x_1. As $q = 2$ it contains 3 one-dimensional subspaces and therefore we have 3 choices for the vertex set for x_1. We consider these cases in turn and show that for all choices we cannot find a vertex set for the 2-sphere. (Here we drop the angled brackets for ease of notation):

1. Choose $\mathcal{B}(x_1) = \{a, a+b\}$. Then we can have $\mathcal{B}(x_2) = \{a,c\}$ or $\mathcal{B}(x_2) = \{a, a+c\}$. In the first case this forces $\mathcal{B}(x_6) = \{a+b, c\}$. Combined with $\mathcal{B}(x_1) = \{a, a+b\}$ and $\mathcal{B}(x_2) = \{a,c\}$, this means that we are unable to find a vertex set for x_7 since none of the set $\{b, a+c, a+b+c\}$ can lie in

our vertex set. In the second case this forces $\mathcal{B}(x_4) = \{a+b, a+c\}$ and then by the same argument as before we cannot find a vertex set for x_3.

2. Choose $\mathcal{B}(x_1) = \{a, b\}$. Then we can have $\mathcal{B}(x_2) = \{a, c\}$ or $\mathcal{B}(x_2) = \{a, a+c\}$. By similar argument as above we cannot find a vertex set for x_4 in the first case, and in the second case for x_6.
3. Choose $\mathcal{B}(x_1) = \{b, a+b\}$. Then we can have $\mathcal{B}(x_3) = \{b, c\}$ or $\mathcal{B}(x_3) = \{b, b+c\}$. By similar argument as above we cannot find a vertex set for x_5 in the first case, and in the second case for x_2.

We have exhausted all the possibilities for $\mathcal{B}(x_1)$ and in all cases we cannot find a vertex set for our 2-sphere, therefore proving our claim.

In addition we should also note that admitting a vertex set is not necessarily a guarantee of being a q–complex. For example Figure 4.1 exhibits a 2–poset which is not a meet-semilattice so therefore is not a 2–complex, but which does admit a vertex set $\mathcal{B} = \{v_1, v_2, v_3\}$.

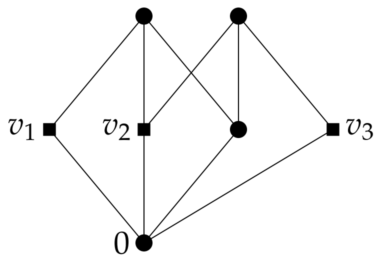

Figure 4.1: A 2–poset which admits the vertex set $\{v_1, v_2, v_3\}$, but which is not a 2–complex.

However, the aim of defining a vertex set is to provide a sufficiency criterion for when certain q–posets are q–complexes and as alluded to at the start of this chapter we need a 'nice' condition on a vertex set to achieve the aim. The existence of a vertex set meeting the following condition will be sufficient for us to show that the q–poset is a q–complex.

Definition 4.1.3. Let $\mathcal{Q}$ be a meet-semilattice q–poset ($q \geqslant 1$) which admits a vertex set $\mathcal{B}$. Then $\mathcal{B}$ is said to be *consistent* if for any pair of facets u, v of $\mathcal{Q}$ we have that $u \cap v$ has a vertex set $\mathcal{B}(u \cap v) = \mathcal{B}(u) \cap \mathcal{B}(v)$.

Remarks. 1. Equivalently the definition says that a consistent vertex set agrees on the intersection between any two facets, so that this intersection has a vertex set which is a subset of $\mathcal{B}$ and equal to the intersection of the vertex sets for the two facets.

2. Though the definition is for the intersection of any pair of facets, we now show that this is sufficient for the condition to hold for the intersection of any number of facets in $\mathcal{Q}$:

Lemma 4.1.4. *Let $\mathcal{Q}$ be a meet-semilattice q–poset ($q \geqslant 1$) which admits a consistent vertex set $\mathcal{B}$. Let the facets of $\mathcal{Q}$ be $x_1, x_2 \ldots, x_n$. Then for any subset $\{j_1, \ldots, j_k\} \subseteq \{1, 2, \ldots, n\}$ we have that $\bigcap_{i=1}^{k} x_{j_i}$ has a vertex set given by $\bigcap_{i=1}^{k} \mathcal{B}(x_{j_i})$.*

Remarks.
- If $q = 1$ the proof of the claim is immediate. The vertex set of $\mathcal{Q}$ would be all elements of rank 1 since $\mathcal{Q}$ is a simplicial complex by Theorem 3.2.1. Thus in this case it is clear $\bigcap_{i=1}^{k} x_{j_i}$ has a vertex set given by $[0, \bigcap_{i=1}^{k} x_{j_i}]_1 = \bigcap_{i=1}^{k} \mathcal{B}(x_{j_i})$.
- The result is not immediate for $q > 1$ since the vertex set is not the complete set of elements of rank 1.
- If $q > 1$ we may think of each facet as a vector space and the elements of the facets as subspaces of the vector space. As we noted before a vertex set is then equivalent to a set of linearly independent and spanning 1-dimensional subspaces. That is the generators of the subspaces form a basis of the facet as a vector space. For each facet of $\mathcal{Q}$ the basis derived from the vertex set is the basis we use throughout the proof below. Let us call this the *vertex basis*. Then the definition of a consistent vertex set says that the vertex basis of each facet (as a vector space) agrees on the subspace which is their intersection (which exists as $\mathcal{Q}$ is a meet-semilattice).
- For ease of notation below let $x_i \cap x_j := x_{ij}$ and $x_i \cap x_j \cap x_k := x_{ijk}$.
- Consider first the case that $\mathcal{Q}$ has 3 facets x_1, x_2, x_3 with $q > 1$. By definition the result holds for the intersection of any two facets. So in particular x_{12} has a vertex set which is a subset of $\mathcal{B}(x_1)$ and $\mathcal{B}(x_2)$. Similarly x_{13} has a vertex set which is a subset of $\mathcal{B}(x_1)$ and $\mathcal{B}(x_3)$ and x_{23} has a vertex set which is a subset of $\mathcal{B}(x_2)$ and $\mathcal{B}(x_3)$. In the language of vector spaces the vertex bases of x_1 and x_2 agree on x_{12}, the vertex bases of x_1 and x_3 agree on x_{13} and the vertex bases of x_2 and x_3 agree on x_{23}. Now $x_{123} = x_{12} \cap x_{13} \cap x_{23}$, since $\mathcal{Q}$ is a meet-semilattice. Consider any 1-dimensional subspace $\langle \alpha \rangle \subseteq x_{123}$. If no such element exists $x_{123} = 0$ and this implies $\bigcap_{j=1}^{3} \mathcal{B}(x_j) = \emptyset = \mathcal{B}(0)$ and so the statement holds. So assume $\langle \alpha \rangle$ exists and as $\langle \alpha \rangle \subseteq x_1$ it follows α can be written in terms of the vertex basis of x_1 and similarly it can be written in terms of the vertex basis of x_2. But $\langle \alpha \rangle \subseteq x_{12}$ so the consistent nature says that this is the same expression. Also $\langle \alpha \rangle \subseteq x_{13}$ and $\langle \alpha \rangle \subseteq x_{23}$ so it follows by extension of this argument that α has the same expression in the vertex bases of x_1,x_2 and x_3. Thus

$\alpha = \sum_{\langle a_i \rangle \in \mathcal{B}} \lambda_i a_i$ ($\lambda_i \in \mathbb{F}_q$), where $\lambda_i \neq 0$ if and only if $\langle a_i \rangle \subseteq \mathcal{B}(x_1)$, $\langle a_i \rangle \subseteq \mathcal{B}(x_2)$ and $\langle a_i \rangle \subseteq \mathcal{B}(x_3)$ if and only if $\langle a_i \rangle \subseteq \bigcap_{j=1}^{3} \mathcal{B}(x_j)$. This is true for any $\langle \alpha \rangle \in [0, x_{123}]_1$ and since all $\langle a_i \rangle \in \bigcap_{j=1}^{3} \mathcal{B}(x_j)$ are elements of x_{123} so it implies they span x_{123}. As $\bigcap_{j=1}^{3} \mathcal{B}(x_j) \subseteq \mathcal{B}(x_1)$, by the definition of vertex set they are linear independent and thus $\bigcap_{j=1}^{3} \mathcal{B}(x_j)$ forms a vertex set of x_{123}.

- This can then be extend to any number of facets:

Proof of Lemma 4.1.4. Let $\{j_1, \ldots, j_k\} \subseteq \{1, 2, \ldots, n\}$ and $q > 1$. Assume we have a non-trivial element $\langle \alpha \rangle \in [0, \bigcap_{i=1}^{k} x_{j_i}]_1$. If no such element exists $\bigcap_{i=1}^{k} x_{j_i} = 0$ and this implies $\bigcap_{i=1}^{k} \mathcal{B}(x_{j_i}) = \emptyset = \mathcal{B}(0)$ and so the statement holds. So assume $\langle \alpha \rangle$ exists and since $\langle \alpha \rangle \in (x_{j_1 j_2})_1$ by the consistent nature of the vertex set of $\mathcal{Q}$ then the expression for α in terms of the vertex basis of x_{j_1} and vertex basis of x_{j_2} is the same. Similarly $\langle \alpha \rangle \in (x_{j_2 j_3})_1$ it follows that the expression for α in terms of the vertex basis of x_{j_2} and vertex basis of x_{j_3} is the same. Combining these says the expression for α in terms of the vertex basis of x_{j_1}, x_{j_2} and x_{j_3} is the same. Continuing in this manner we see that the expression for α in terms of the vertex basis of x_{j_i} for any $i \in \{j_1, \ldots, j_k\}$ is the same. Thus $\alpha = \sum_{\langle a_i \rangle \in \mathcal{B}} \lambda_i a_i$ ($\lambda_i \in \mathbb{F}_q$), where $\lambda_i \neq 0$ if and only if $\langle a_i \rangle \in \bigcap_{i=1}^{k} \mathcal{B}(x_{j_i})$. Since this is true for any $\langle \alpha \rangle \in (\bigcap_{i=1}^{k} x_{j_i})_1$ and all $\langle a_i \rangle \in \bigcap_{i=1}^{k} \mathcal{B}(x_{j_i})$ are elements of $(\bigcap_{i=1}^{k} x_{j_i})_1$ this implies they span $\bigcap_{i=1}^{k} x_{j_i}$. Since $\bigcap_{j=1}^{k} \mathcal{B}(x_{j_i}) \subseteq \mathcal{B}(x_{j_1})$, by the definition of vertex set they are linear independent and so it follows that $\bigcap_{j=1}^{k} \mathcal{B}(x_{j_i})$ forms a vertex set of $\bigcap_{i=1}^{k} x_{j_i}$. □

Now using Definition 4.1.1 and Definition 4.1.3 we can obtain the result alluded to at the start of this chapter and which gives a characterization of certain q–complexes:

Theorem 4.1.5. *Let $\mathcal{Q}$ be a meet-semilattice q–poset ($q \geqslant 1$) with a consistent vertex set $\mathcal{B}$. Then $\mathcal{Q}$ is a q–complex.*

Proof. The result for $q = 1$ is an application of Theorem 3.2.1. So now let us assume that $q > 1$. We proceed by induction on the number of facets in $\mathcal{Q}$. The base step is a q–poset with a single facet, which by definition is always a q–complex. Let the facets of $\mathcal{Q}$ be $x_1, x_2, \ldots, x_n$. Now let us assume that inductively we have an embedding of the first $k-1$ facets with the following properties:

1. We have an embedding $\phi : \left(\mathcal{Q}^{k-1} := \bigcup_{i=1}^{k-1}[0, x_i]\right) \hookrightarrow \mathcal{P}(V^{k-1})$, where V^{k-1} is the $\mathbb{F}_q$ vector space with a basis $\mathcal{B}^{k-1} := \mathcal{B} \cap (\mathcal{Q}^{k-1})_1$. So strictly

there exists a bijection $\theta : \mathcal{B}^{k-1} \to \mathcal{B}(V^{k-1})$, where $\mathcal{B}(V^{k-1})$ is a basis of V^{k-1} with elements labelled by the elements of $\mathcal{B}^{k-1}$ and where θ maps an element of $\mathcal{B}^{k-1}$ to the element of $\mathcal{B}(V^{k-1})$ of the same label.

2. If $z \in \mathcal{B}^{k-1}$ then $\phi(z) = \langle \theta(z) \rangle$.
3. If $y \in \mathcal{Q}^{k-1}$ with a vertex set $\mathcal{B}(y) \subseteq \mathcal{B}$, then $\phi(y)$ is the subspace of V^{k-1} spanned by $\theta(\mathcal{B}(y))$ [1] or equivalently $\phi(y)$ has basis $\theta(\mathcal{B}(y))$ (recall here the remarks after Definition 4.1.1).

Therefore if we also set $\phi(0_{\mathcal{Q}}) = 0_{\mathcal{P}(V^{k-1})}$ the above implies by Corollary 3.1.5 that $\mathcal{Q}^{k-1}$ is a q–complex.

Now consider the next facet x_k. As $\mathcal{Q}$ is a meet-semilattice for all $i \leqslant k-1$ and $\mathcal{Q}^{k-1} \trianglelefteq \mathcal{Q}$ we have that $x_i \cap x_k$ exists in $\mathcal{Q}$ and by the consistent property has a vertex set given by $\mathcal{B}(x_i) \cap \mathcal{B}(x_k)$. Then by property (3.) it follows that as $x_i \cap x_k \in \mathcal{Q}^{k-1}$ we have that $\phi(x_i \cap x_k)$ is defined and has a basis given by $\theta(\mathcal{B}(x_i) \cap \mathcal{B}(x_k))$. Furthermore ϕ induces the isomorphism $[0, x_i \cap x_k] \cong [0, \phi(x_i \cap x_k)]$. From this we see that ϕ is already defined on the subcomplex $\mathcal{P} := \bigcup_{i=1}^{k-1}[0, x_i \cap x_k] \trianglelefteq [0, x_k]$ and as ϕ is an embedding it induces the isomorphism $\mathcal{P} \cong \phi(\mathcal{P})$. Our next step is to extend ϕ to the whole of $[0, x_k]$.

In $[0, x_k]$ consider the element $z = \bigvee_{i=1}^{k-1}(x_i \cap x_k)$ such that $\mathcal{P} \trianglelefteq [0, z] \trianglelefteq [0, x_k]$. We note that $x_i \cap z = x_i \cap x_k$ for all $i \leqslant k-1$. Given the consistent vertex set and the properties of a vertex set it follows that z has a vertex set given by $\bigcup_{i=1}^{k-1} \mathcal{B}(x_i \cap x_k)$. Correspondingly we have an element $z' \in \mathcal{P}(V^{k-1})$ which has a basis given by $\bigcup_{i=1}^{k-1} \theta(\mathcal{B}(x_i \cap x_k))$ and $\phi(\mathcal{P}) \trianglelefteq [0, z']$. By the definition of a vertex set we have that $[0, z] \cong [0, z']$. It is clear as $\mathcal{Q}$ is a meet-semilattice no elements of $[0, x_i]$, for all $i \leqslant k-1$, lie in $[0, z] \setminus \mathcal{P}$. Thus we may extend ϕ to the whole of $[0, z] \trianglelefteq [0, x_k]$ such that it respects the existing definition on $\mathcal{P}$, induces the isomorphism $[0, z] \cong [0, z']$, has $\phi(z) = z'$ and meets properties (2.) and (3.) as above.

But we need to be sure that this is well defined and in particular it preserves intersections of facets, that is for all $i \leqslant k-1$ we have that $\phi(x_i) \cap z' = \phi(x_i) \cap \phi(z) = \phi(x_i \cap z) = \phi(x_i \cap x_k)$. We know by the construction that $x_i \cap z = x_i \cap x_k$ and $z' = \phi(z)$. Also that as sets $[0, \phi(x_i \cap z)] = [0, \phi(x_i \cap x_k)] \subseteq [0, \phi(x_i) \cap \phi(z)]$. We of course want this to be equality i.e $\phi(x_i \cap x_k) = \phi(x_i) \cap \phi(z)$. By property 3. in $\mathcal{P}(V^{k-1})$ it follows that $\phi(x_i)$ is spanned by $\theta(\mathcal{B}(x_i))$ and by construction z' is spanned by $\theta(\mathcal{B}(z)) = \bigcup_{i=1}^{k-1} \theta(\mathcal{B}(x_i \cap x_k))$. This implies that $\phi(x_i) \cap z'$ is spanned by $\theta(\mathcal{B}(x_i) \cap \theta(\mathcal{B}(z))$. But $\mathcal{B}(x_i) \cap \mathcal{B}(z) = \mathcal{B}(x_i \cap x_k)$ and so by property

[1] Where θ is extended to operate on a set of vertex set elements in the obvious way.

(3.) we have that $\phi(x_i \cap x_k)$ is spanned by $\theta(\mathcal{B}(x_i)) \cap \theta(\mathcal{B}(z))$. It follows that $\phi(x_i) \cap z' = \phi(x_i \cap x_k)$ as we require. Thus the extension of ϕ preserves the intersections of facets x_i (for $x \leqslant k-1$) with z.

Thus ϕ is now defined on $\mathcal{Q}^{k-1} \cup [0,z]$ and we are therefore left to define ϕ on $[0,x_k] \setminus [0,z]$, which contains no elements of $\mathcal{Q}^{k-1}$. Considered as a vector space (remember here $q > 1$) we have $x_k = z \oplus c_z$, where c_z is a complement subspace. In particular we have a complement c_z (which can be identified as an element of $[0,x_k] \setminus [0,z]$) which has a vertex set $\mathcal{B}(x_k) \setminus \mathcal{B}(z)$. So let $V^k = V^{k-1} \oplus c_{z'}$, where we extend θ in the obvious way so that $c_{z'}$ has basis given by $\theta(\mathcal{B}(x_k)) \setminus \mathcal{B}(z)$ and where $[0,c_z] \cong [0,c_z']$.

It is clear that we then have a subspace $w \subseteq V^k$ such that $w = z' \oplus c_{z'}$, $w \cap V^{k-1} = z'$, $[0,w] \cong [0,x_k]$ and w is spanned by $\theta(\mathcal{B}(x_k))$. Given theses facts and the properties of vertex sets we may extend ϕ to the whole of $[0,x_k]$ and so become a map $\phi : \left(\mathcal{Q}^k := \bigcup_{i=1}^k [0,x_i]\right) \hookrightarrow \mathcal{P}(V^k)$ which respects the existing definition of ϕ on $\mathcal{Q}^{k-1} \cup [0,z]$, has $\phi(x_k) = w$, induces the isomorphism $[0,x_k] \cong [0,w]$ and meets properties (2.) and (3.) Furthermore as $w \cap V^{k-1} = z'$ we know that ϕ continues to preserve the intersections of facets in the image of $\mathcal{Q}^k$. Thus by induction we are done. $\square$

Remarks.
1. The obvious question to ask is if such a proof can work in the setting of a general meet-semilattice q–poset. In this regard we note that without the properties of a consistent vertex set we cannot guarantee the step of showing $\phi(x_i) \cap \phi(z) = \phi(x_i \cap z)$ (in the notation of the proof), namely that ϕ as constructed respects intersections of facets. Thus the proof does not hold in the general setting. Indeed if you try to apply this construction to Example 3.3.2, this is exactly where the proof breaks done.
2. Of course having a consistent vertex set is a sufficient but not a necessary condition for a q–poset to be a q–complex. This is evidenced by Example 4.1.2.

4.1.1 Examples of Vertex Sets

To illustrate the ideas of a vertex set and a consistent vertex set we include below some examples.

Example 4.1.6. Consider the semihedral group G of order 16 with presentation $\langle t, \beta \mid t^8 = \beta^2 = 1,\ t\beta = \beta t^3 \rangle$. The Hasse diagram of the 2–poset $\mathcal{A}_2(G)$ can be seen in Figure 4.2. Then $\mathcal{A}_2(G)$ admits the following vertex sets (angled brackets dropped for ease of notation): $\mathcal{B}_1 = \{\beta, t^4\beta, \beta t^2, t^2\beta\}$, $\mathcal{B}_2 = \{\beta, t^4, \beta t^2\}$, $\mathcal{B}_3 = \{t^4\beta, t^4, \beta t^2\}$, $\mathcal{B}_4 = \{\beta, t^4, t^2\beta\}$ and $\mathcal{B}_5 = \{t^4\beta, t^4, t^2\beta\}$. For this example all except $\mathcal{B}_1$ are consistent vertex sets. It is already clear (by inspection) that this is a 2–complex which can be embedded in $\mathcal{P}(V)$, where $V = \mathbb{F}_2^3$. We can see this demonstrated pictorially in Figure 4.3 (a minimal embedding). This then concurs with the result of Theorem 4.1.5.

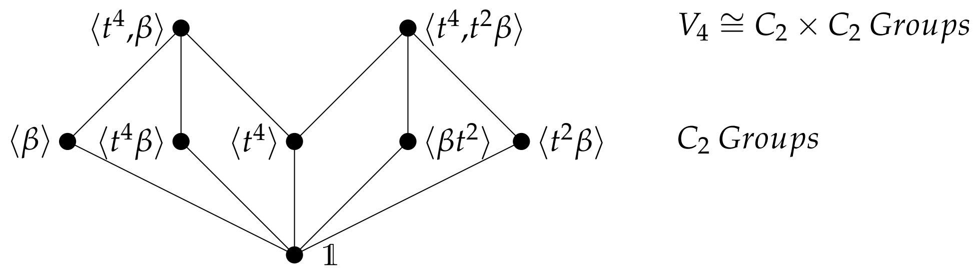

Figure 4.2: Elementary Abelian Groups of G, the Semihedral Group of Order 16.

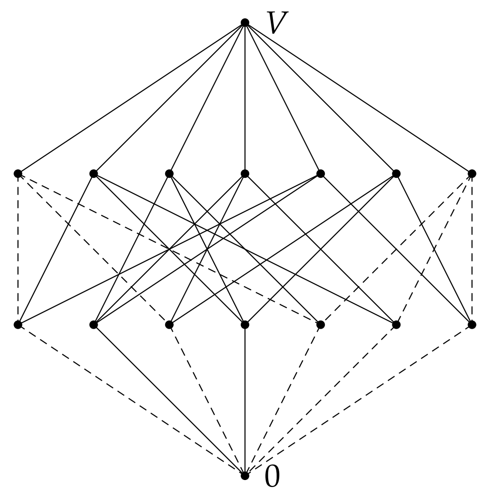

Figure 4.3: The Lattice of Subspaces of $V = \mathbb{F}_2^3$ - With Embedded 2–Complex $\mathcal{A}_2(G)$.

Example 4.1.7. Recall the q–complexes $\mathcal{A}_2(G)$ and $\mathcal{A}_3(G^*)$ of Chapter 1, where G is the alternating group $Alt(5)$ on 5 points and G^* is the alternating group $Alt(6)$ on 6 points. These q–complexes both admit a consistent vertex set. In these q–complexes the facets have mutual trivial intersection and therefore we can take any possible vertex set for each facet (which always exists) and their union forms a consistent vertex set for the whole q–complex.

Example 4.1.8. Following on from the previous example, order ideals of $\mathcal{A}_q(G)$ can give us many examples of a q–poset with a consistent vertex set. We consider

$\mathcal{Q}_X \trianglelefteq \mathcal{A}_q(G)$ for X a conjugacy class of elementary abelian q-subgroups. Indeed in $\mathcal{A}_2(G)$, where $G = \mathfrak{S}_5$ the symmetric group on 5 elements, we have a conjugacy class $X = \{\langle (2\,3)(4\,5), (2\,4)(3\,5)\rangle^g \,|\, g \in G\}$ of size 5 which is such an example. Similarly we have in $\mathfrak{S}_6$ the example $X = \{\langle (5\,6), (1\,2)(3\,4), (1\,3)(2\,4)\rangle^g \,|\, g \in G\}$, which is of size 15. Some further examples are evident among the Mathieu groups of the form $\mathcal{Q}_X \trianglelefteq \mathcal{A}_q(G)$ where X is a conjugacy class of subgroups - see Table 4.1.

Table 4.1: Consistent Vertex Set q-Posets of the Form $\mathcal{Q}_X$ for Mathieu Groups.

Group	q	Order Of Groups	Size of Conjugacy Class X	Size of Vertex Set
M_{11}	3	$3^2 = 9$	55	110
M_{12}	3	$3^2 = 9$	220	440
M_{12}	2	$2^4 = 16$	77	308
M_{23}	3	$3^2 = 9$	253	1012
M_{24}	2	$2^4 = 16$	759	3036

Remarks.

1. Some of the embeddings of the q–complexes (under Theorem 4.1.5) that follow from these vertex sets in Table 4.1 have been known previously. As an example, the one for M_{24} can be found in the discussion of Petersen and tilde geometries in [20].
2. An important general example of this type is found within the subgroup lattice of any Symmetric Group. This example is described and explored in Section 4.3.

4.2 The Skeleton of a q–Poset

A direct consequence of the definition of a vertex set (and the motivation for the name) is that a vertex set of a q–poset forms the vertex set of a certain simplicial complex. We explore here the interaction between the two.

Note: Throughout this section we only consider q–posets $\mathcal{Q}$ which admit a vertex set $\mathcal{B}$.

We form a simplicial complex $\mathcal{S}_{\mathcal{B}}(\mathcal{Q})$ which has the elements of $\mathcal{B}$ as its vertices by setting $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \subseteq \mathcal{B}$ as an element if and only if $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ are all contained in some facet of $\mathcal{Q}$. We call this simplicial complex the *skeleton of $\mathcal{Q}$ for vertex set $\mathcal{B}$*. Where context is clear we just use the notation $\mathcal{S}(\mathcal{Q})$ for the skeleton. We would remark that it is not necessarily true that $\mathcal{S}(\mathcal{Q})$ embeds into $\mathcal{Q}$, however we can give a sufficiency criterion for this occurring:

Proposition 4.2.1. *Let $\mathcal{Q}$ be a q–poset ($q \geqslant 1$) which admits a vertex set $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$. If the join $\bigvee_{j=1}^{k} v_{i_j}$ exists in $\mathcal{Q}$ for all subsets $\mathcal{V} = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \subseteq \mathcal{B}$, with the elements $\mathcal{V}$ contained in some facet of $\mathcal{Q}$, then the skeleton $\mathcal{S}(\mathcal{Q})$ embeds into $\mathcal{Q}$.*

Note: In the $q = 1$ setting this result follows immediately since any vertex set of $\mathcal{Q}$ is equal to $\mathcal{Q}_1$ and in fact the skeleton is isomorphic to $\mathcal{Q}$. Thus we concentrate on the $q > 1$ setting.

Proof of Proposition 4.2.1. We show that a map $\varphi : \mathcal{S}(\mathcal{Q}) \to \mathcal{Q}$ defined by setting $\varphi(\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}) = \bigvee_{j=1}^{k} v_{i_j}$ with $\varphi(\emptyset) = 0$, is an embedding. This is a well defined and injective map since $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ is an element of $\mathcal{S}(\mathcal{Q})$ if and only if the elements $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ are all contained in some facet of $\mathcal{Q}$ and if so, by assumption, their join is uniquely defined in $\mathcal{Q}$. We are left to show that if $x, y \in \mathcal{S}(\mathcal{Q})$ we have $x \prec_{\mathcal{S}(\mathcal{Q})} y$ if and only if $\varphi(x) \prec_{\mathcal{Q}} \varphi(y)$. Without loss of generality let $y = \{v_1, v_2, \ldots, v_m\}$ and $x = \{v_1, v_2, \ldots, v_{m-1}\}$ so that $x \prec_{\mathcal{S}(\mathcal{Q})} y$. Then $\varphi(x) = \bigvee_{j=1}^{m-1} v_j$ and $\varphi(y) = \bigvee_{j=1}^{m} v_j$ and so clearly $\varphi(x) \leqslant_{\mathcal{Q}} \varphi(y)$. By remarks (6) after Definition 4.1.1 we have that $rk(\varphi(x)) = rk(\varphi(y)) - 1$ and so it follows immediately that $\varphi(x) \prec_{\mathcal{Q}} y$. Conversely if for some x', y' in $\mathcal{S}(\mathcal{Q})$ we have $\phi(x') \prec_{\mathcal{Q}} \phi(y')$ then $\phi(x')$ and $\phi(y')$ are elements of some facet W of $\mathcal{Q}$, which has vertex set $\mathcal{B}(W) \subseteq \mathcal{B}$. Furthermore they are (uniquely defined) joins of subsets of elements of $\mathcal{B}(W)$ and $rk_{\mathcal{Q}}(\phi(y')) = rk_{\mathcal{Q}}(\phi(x')) + 1 = k$, say. As a consequence of the definition of a vertex set and the map ϕ we must have, without loss of generality, $\phi(y') = \bigvee(\{v_1, v_2, \ldots, v_{k-1}, v_k\})$, where $\{v_1, v_2, \ldots, v_{k-1}, v_k\} \subseteq \mathcal{B}(W)$ and $\phi(x') = \bigvee(\{v_1, v_2, \ldots, v_{k-1}\})$. Thus $y' = \{v_1, v_2, \ldots, v_k\}, x' = \{v_1, v_2, \ldots, v_{k-1}\}$ and therefore $x \prec_{\mathcal{S}(\mathcal{Q})} y$. $\square$

Remarks.
1. By Proposition 1.1.6 (i) the above result tells us that $\mathcal{Q}$ contains a subposet isomorphic to $\mathcal{S}(\mathcal{Q})$.
2. The result says that a 1–complex (namely $\mathcal{S}(\mathcal{Q})$) embeds into a q–poset and so this provides an illustration of Proposition 3.1.4 (i) for the setting $p \lneqq q$.
3. Any meet-semilattice q–poset $\mathcal{Q}$ with a vertex set is an example of a q–poset meeting the conditions of Proposition 4.2.1. To see this consider a subset $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ of $\mathcal{B}$. If these elements appear only in one facet of $\mathcal{Q}$ then their join is uniquely defined inside that facet. If they are contained in two facets x_i, x_j then as $\mathcal{Q}$ is a meet-semilattice it follows that $\exists z \in \mathcal{Q}$ such that $x_i \cap x_j = z$. So inside $[0, z]$ we have $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ as elements

and $\bigvee_{j=1}^{k} v_{i_j}$. Furthermore $\bigvee_{j=1}^{k} v_{i_j}$ is the same element in $[0, x_i]$ and $[0, x_j]$. Since this argument holds equally for any number of facets we conclude that the conditions of the proposition are met.

4. We can find q–posets which satisfy the conditions of the proposition but are not meet-semilattices. For an example see Figure 4.4. This shows a 2–poset $\mathcal{Q}$ which is not a meet-semilattice, but where the dashed lines and black squares indicate the embedded skeleton for vertex set $\mathcal{B} = \{v_1, v_2, v_3\}$.

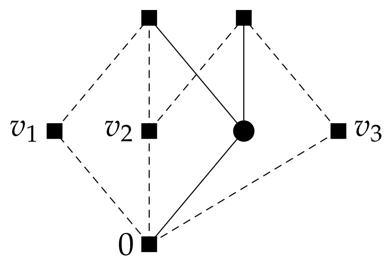

Figure 4.4: A 2–poset which is a not a meet-semilattice but has a skeleton.

Now if $\mathcal{Q}$ is a meet-semilattice q–poset with a consistent vertex set $\mathcal{B}$ then the link between the skeleton $\mathcal{S}(\mathcal{Q})$ and $\mathcal{Q}$ preserves some properties of the posets. Let $\varphi : \mathcal{S}(\mathcal{Q}) \hookrightarrow \mathcal{Q}$ be an embedding as per Proposition 4.2.1. By definition for any $x \in \mathcal{S}(\mathcal{Q})$ we have that x is a facet if and only if $\varphi(x)$ is a facet. Additionally φ preserves the intersection profile in the sense that if x, y are facets of $\mathcal{S}(\mathcal{Q})$ then (†):

- $\varphi(x \cap_{\mathcal{S}(\mathcal{Q})} y) = \varphi(x) \cap_{\mathcal{Q}} \varphi(y)$.
- $rk_{\mathcal{S}(\mathcal{Q})}(x \cap_{\mathcal{S}(\mathcal{Q})} y) = rk_{\mathcal{Q}}(\varphi(x) \cap_{\mathcal{Q}} \varphi(y))$. This is an application of Proposition 1.1.6 (iii), as by definition $\varphi(0_{\mathcal{S}(\mathcal{Q})}) = 0_{\mathcal{Q}}$.

Since φ is injective Proposition 1.1.6 (i) tells us $\mathcal{S}(\mathcal{Q}) \cong \varphi(\mathcal{S}(\mathcal{Q}))$ so φ^{-1} is an embedding defined on the image $\varphi(\mathcal{S}(\mathcal{Q}))$. Therefore as $\mathcal{S}(\mathcal{Q})$ and $\mathcal{Q}$ have the same number of facets, if $x', y' \in \mathcal{Q}$ are facets we also have the dual to the facts above (††):

- $\varphi^{-1}(x') \cap_{\mathcal{S}(\mathcal{Q})} \varphi^{-1}(y') = \varphi^{-1}(x' \cap_{\mathcal{Q}} y')$.
- $rk_{\mathcal{S}(\mathcal{Q})}(\varphi^{-1}(x') \cap_{\mathcal{S}(\mathcal{Q})} \varphi^{-1}(y')) = rk_{\mathcal{Q}}(x' \cap_{\mathcal{Q}} y')$.

By Lemma 4.1.4 it follows that (†) and (††) will hold equally for any number of facets. As a consequence of these facts we have the following result on the shellability of $\mathcal{S}(\mathcal{Q})$ and $\mathcal{Q}$ when both are pure complexes.

Proposition 4.2.2. *Let $\mathcal{Q}$ be a pure q–complex ($q \geqslant 1$) of rank d with a consistent vertex set $\mathcal{B}(\mathcal{Q})$. Then $\mathcal{Q}$ is shellable if and only if its skeleton $\mathcal{S}(\mathcal{Q})$ for vertex set $\mathcal{B}$ is shellable.*

Proof. First note that as $\mathcal{Q}$ is a meet-semilattice by Proposition 4.2.1 we have an embedding $\varphi : \mathcal{S}(\mathcal{Q}) \hookrightarrow \mathcal{Q}$. Let $x_1, x_2, \ldots, x_n$ be a shelling order of facets for $\mathcal{Q}$. So $(\bigcup_{j=1}^{i-1}[0, x_j]) \cap [0, x_i])$ is pure of rank $d-1$ for $2 \leqslant i \leqslant n$. We have $\mathcal{S}(\mathcal{Q})$ for vertex set $\mathcal{B}$ has facets $\varphi^{-1}(x_1), \varphi^{-1}(x_2), \ldots, \varphi^{-1}(x_n)$. Then (††) tells us that φ^{-1} maps facets to facets and preserves the rank of their mutual intersections. Thus we have $(\bigcup_{j=1}^{i-1}[0, \varphi^{-1}(x_j)]) \cap \varphi^{-1}(x_i)$ is pure of rank $d-1$ for $2 \leqslant i \leqslant n$ and it follows that $\varphi^{-1}(x_1), \varphi^{-1}(x_2), \ldots, \varphi^{-1}(x_n)$ forms a shelling order for $\mathcal{S}(\mathcal{Q})$. Equally this argument works in the opposite direction. By (†) and the fact φ maps facets to facets if we start with a shelling order for $\mathcal{S}(\mathcal{Q})$ the reverse implication follows by the same argument. Thus $\mathcal{Q}$ is shellable if and only if $\mathcal{S}(\mathcal{Q})$ is shellable. □

Remarks. We should note that we may reverse the process of forming the skeleton. That is starting from a simplicial complex $\mathcal{P}$ we may always find a q–poset ($q > 1$), for which $\mathcal{P}$ forms a skeleton. This will not be uniquely determined. If we took the simplicial complex $\mathcal{P}$ of Figure 4.5, then we can see that in Figure 4.6 we have two different 2–posets both containing a subposet isomorphic to $\mathcal{P}$ (given by dashed lines and black squares) as a skeleton and which both admit the vertex set $\{v_1, v_2, v_3, v_4\}$. Only in the second q–poset is the vertex set consistent.

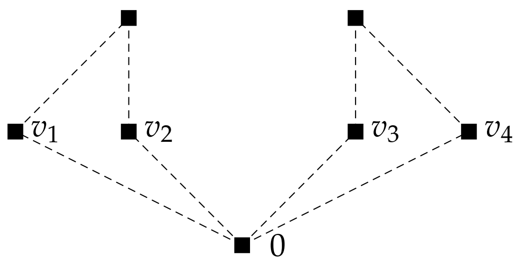

Figure 4.5: Simplicial Complex $\mathcal{P}$.

4.3 The p-Cycle Complex of $\mathfrak{S}_n$

We set out in detail a further example of a q–poset ($q > 1$) which admits a consistent vertex set. This complex appears as a subposet of the subgroup lattice

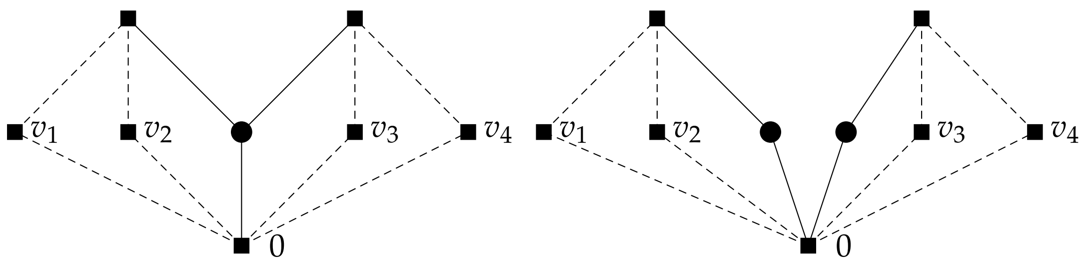

Figure 4.6: Two q–Posets Containing Isomorphic Skeletons.

of the symmetric group. The complex was discussed in papers by Ksontini [28], [29]; Shareshian [48] and Shareshian & Wachs [49]. Their interest was in using this to study the order complex[2] $\Delta(\mathcal{A}_q(G))$ for G the symmetric group.

4.3.1 Introduction to the Complex

Let $\mathfrak{S}_n$ be the *symmetric group* on $\{1, 2, \ldots, n\}$ and fix a prime $p \leqslant n$. Then we call a subgroup $C \subseteq \mathfrak{S}_n$ a *cycle subgroup* of $\mathfrak{S}_n$ for p or just a *cycle group*, if C is abelian and it can be generated by single cycles of length p. As C is abelian such cycles are necessarily disjoint. It follows that a maximal cycle subgroup is an elementary abelian group of order p^λ, where λ is the largest integer below $\frac{n}{p}$ and thus $\lambda = \lfloor \frac{n}{p} \rfloor$.

Let $\mathcal{Q}^{n,p} \trianglelefteq \mathcal{A}_p(\mathfrak{S}_n)$ be the p-poset of all subgroups contained in some cycle group for the given prime p. We call $\mathcal{Q}^{n,p}$ the *p-cycle complex of* $\mathfrak{S}_n$.

Remarks. 1. In the papers mentioned in the introduction to the section above, the term *p-cycle complex* was actually used to describe a simplicial complex embedded inside $\mathcal{Q}^{n,p}$, namely its skeleton. We use the term here to describe the whole p–poset.

2. The facets of $\mathcal{Q}^{n,p}$ form a single conjugacy class in $\mathfrak{S}_n$ and thus $\mathcal{Q}^{n,p}$ is pure. Take two such groups:

$$\begin{aligned} A = &\quad \langle (a_{11}\, a_{12}\, \ldots\, a_{1p}), (a_{21}\, a_{22}\, \ldots\, a_{2p}), \ldots, (a_{\lambda 1}\, a_{\lambda 2}\, \ldots\, a_{\lambda p}) \rangle \quad , \\ B = &\quad \langle (b_{11}\, b_{12}\, \ldots\, b_{1p}), (b_{21}\, b_{22}\, \ldots\, b_{2p}), \ldots, (b_{\lambda 1}\, b_{\lambda 2}\, \ldots\, b_{\lambda p}) \rangle \quad . \end{aligned}$$

Let $a = (a_{11}\, a_{12}\, \ldots\, a_{1p})(a_{21}\, a_{22}\, \ldots\, a_{2p}) \cdots (a_{\lambda 1}\, a_{\lambda 2}\, \ldots\, a_{\lambda p})$ and similarly let $b = (b_{11}\, b_{12}\, \ldots\, b_{1p})(b_{21}\, b_{22}\, \ldots\, b_{2p}) \cdots (b_{\lambda 1}\, b_{\lambda 2}\, \ldots\, b_{\lambda p})$. Since the set

[2]See Section 1.4.

of generators of A, B are disjoint cycles, an element $g \in \mathfrak{S}_n$ such that $gag^{-1} = b$ is exactly the element g that conjugates A to B.

3. The facets are maximal elementary abelian p-subgroups in $\mathfrak{S}_n$. This result follows from Lemma 4.1 in [48] and Lemma 3.1 in [49].
4. Where clarity is required we write $\lambda(n, p) := \lfloor \frac{n}{p} \rfloor$ in place of λ.

The first pertinent fact about this poset is that it admits a consistent vertex set and is thus a q–complex.

Proposition 4.3.1. *$\mathcal{Q}^{n,p}$ admits a consistent vertex set and is thus a q–complex.*

Proof. Let $\mathcal{B} \subset \mathcal{Q}_1^{n,p}$ be the set of all subgroups of the form $\langle (a_{11}\, a_{12}\, \ldots\, a_{1p}) \rangle \cong C_p$ in $\mathfrak{S}_n$. Each facet z of $\mathcal{Q}^{n,p}$ is generated as a group by exactly λ disjoint p cycles and so $[0, z]_1$ contains exactly λ elements of $\mathcal{B}$ and they form a vertex set $\mathcal{B}(z)$ for z. Thus $\mathcal{B}$ as defined is a vertex set for $\mathcal{Q}$.

Then take two facets x, y of $\mathcal{Q}^{n,p}$ with vertex sets $\mathcal{B}(x)$ and $\mathcal{B}(y)$ respectively. As groups, the generators of the facets are disjoint cycles. So $x \cap y$ is a group whose generators are given by the intersection of the set of generators of x and y. Thus $x \cap y$ has a vertex set given by $\mathcal{B}(x) \cap \mathcal{B}(y) = [0, x \cap y]_1 \cap \mathcal{B}$. Therefore $\mathcal{B}$ is a consistent vertex set and so by Theorem 4.1.5 we conclude that $\mathcal{Q}^{n,p}$ is a q–complex. □

Remarks.

1. It should be noted that we may be able to construct other vertex sets for $\mathcal{Q}^{n,p}$. For example consider $\mathcal{Q}^{4,2}$. Here $\mathcal{Q}^{n,p}$ has 3 facets which have mutual trivial intersection. Therefore we find ourselves in the same situation as Example 4.1.7 and thus we do not have a unique vertex set.
2. The consistent vertex set $\mathcal{B}$ described above is the *standard vertex set for $\mathcal{Q}^{n,p}$*, which we assume whenever we discuss this complex. This set then has size $(p-2)!\binom{n}{p}$ (we use throughout the convention $0! = 1$). To see this note that there are $\binom{n}{p} \cdot (p-1)!$ distinct p-cycles making up $\binom{n}{p} \cdot (p-2)!$ such groups, since each C_p contains $(p-1)$ p-cycles of length p.
3. In Section 4.2 we described how a q–poset with a consistent vertex set contains a subposet isomorphic to a simplicial complex called its skeleton. In this case the skeleton has received some attention in the papers by Ksontini, Shareshian and Wachs referred to at the start of this section, in which the skeleton has been used to obtain results on the homology of the Quillen complex for the symmetric group. The skeleton $\mathcal{S}(\mathcal{Q}^{n,p})$ is the subposet of $\mathcal{Q}^{n,p}$ of all the groups whose generators can be written as disjoint single

p cycles. That is if we let (p_i) represent a single p cycle then it is the set of all groups of the form $\langle (p_1), (p_2), \ldots, (p_k) \rangle \subseteq \mathfrak{S}_n$ for all $0 \leqslant k \leqslant \lambda$ and where all the (p_i) are disjoint.

4. Some further combinatorial results on the p-cycle complex including the derivation of its f-vector and the intersection profile of the facets can be found in Appendix B.
5. The p-cycle complex is pure and therefore it is a natural question to ask if we can determine when it is shellable. In this regard we recall Proposition 4.2.2 which tells us that $\mathcal{Q}^{n,p}$ is shellable if and only if its skeleton $\mathcal{S}(\mathcal{Q}^{n,p})$ (as described above) is shellable. While we have no general result, we do note that if $p = 2$ or $p = 3$ then $\mathcal{Q}^{n,p}$ (and of course its skeleton) is not shellable when $n \equiv 0 \mod p$. This follows, since if $p = 2$ or $p = 3$ then the intersection of facets can never be of co-rank 1 as $(p-2)! = 1$ (see remark 2. above). Clearly it then follows that result does not hold if $p \geqslant 5$.

4.3.2 Free Embedding of $\mathcal{Q}^{n,p}$

We know that $\mathcal{Q}^{n,p}$ is a q–complex and thus the obvious question to ask is if we can say anything about an embedding of the complex into a projective space.

An immediate, and most obvious embedding, is that described in the proof of Theorem 4.1.5. That is $\varphi : \mathcal{Q}^{n,p} \hookrightarrow \mathcal{P}(V)$, with $V = F\mathcal{B}$, where $F = \mathbb{F}_q$ and $\mathcal{B}$ is the standard vertex set. Thus $\dim(V) = (p-2)!\binom{n}{p}$. We refer to this embedding as the *free embedding* of $\mathcal{Q}^{n,p}$.

Recall the work in Section 3.3.2 where we described a sufficiency criterion for order ideals in $\mathcal{A}_q(G)$ (for a finite group G) to form a q–complex. Theorem 3.3.3 described an embedding for the order ideals meeting certain properties. Amongst the examples exhibiting this property were the p-cycle complexes $\mathcal{Q}^{6,2}$, $\mathcal{Q}^{7,2}$ and $\mathcal{Q}^{8,2}$ and it is interesting to note that in these cases the embedding dimension obtained by that method is equal to the dimension of the free embedding. Although not proved, we suggest that all p-cycle complexes meet the criterion of Theorem 3.3.3 and that this agreement of dimension of the two methods also holds for all p-cycle complexes.

The free embedding gives us a representation $\rho : \mathfrak{S}_n \to GL(V)$, where $V = F\mathcal{B}$ and so the degree of the representation is $|\mathcal{B}|$. This derives from the fact that $\mathfrak{S}_n$ acts (transitively) by conjugation on the elements of the standard vertex set of $\mathcal{Q}^{n,p}$. Thus via the embedding φ this transfers into a transitive action on the basis elements of the vector space V. This representation is faithful and furthermore is a good representation as per Definition 1.3.7. We can once again use the Meataxe tool of GAP to decompose this representation into its irreducible components. Some example calculations are show in Table 4.2.

Table 4.2: Meataxe Decomposition Table for Modular Representations of $\mathcal{Q}^{n,p}$.

$\mathcal{Q}^{n,p}$	Degree of Representation	Meataxe Decomposition
$\mathcal{Q}^{3,2}$	3	1a+2a
$\mathcal{Q}^{4,2}$	6	1aa + 2aa
$\mathcal{Q}^{4,3}$	4	1a+3a
$\mathcal{Q}^{5,2}$	10	1aa+4ab
$\mathcal{Q}^{6,2}$	15	1aaa+4aab
$\mathcal{Q}^{7,2}$	21	1a+6a+14a
$\mathcal{Q}^{8,2}$	28	1aa+6aa+14a
$\mathcal{Q}^{8,3}$	56	1a +7aa+13a+28a

4.3.3 Refining the Embedding of $\mathcal{Q}^{n,p}$

It is evident that the free embedding of the previous section results in an embedding into a projective space of potentially high dimension. In this section we describe the construction of a refined embedding which gives a dimension significantly less than the free embedding.

Consider $\mathcal{Q}^{n,p}$ for some fixed prime p with $p \leqslant n$ and the standard vertex set $\mathcal{B}(\mathcal{Q}^{n,p})$. For all $n \leqslant 2p-1$ we have that $rk(\mathcal{Q}^{n,p}) = 1$. In such cases a minimal embedding (in terms of dimension) can be seen trivially - namely for a vector space V of dimension m over $\mathbb{F}_p$, such that m is the minimal value with $\left[{m \atop 1}\right]_p \geqslant |(\mathcal{Q}^{n,p})_1|$. Any arbitrary injective map $\varphi : (\mathcal{Q}^{n,p})_1 \hookrightarrow \mathcal{P}(V)_1$ gives such an embedding when combined with setting $\varphi(\mathbb{1}) = 0_{\mathcal{P}(V)}$.

Now we set out a construction for a refined embedding of $\mathcal{Q}^{n,p}$ with $n \geqslant 2p$ in a inductive manner. We will assume that we have an embedding $\phi_n : \mathcal{Q}^{n,p} \hookrightarrow V^n$, for V^n a vector space over $\mathbb{F}_p$, and extend this to an embedding ϕ_{n+1} of $\mathcal{Q}^{n+1,p}$. Let $\dim(d_p(n)$:=*embedding dimension of* $\mathcal{Q}^{n,p}$ under this construction.

To describe ϕ_n we concentrate on certain single p cycles - namely the generators of the elements of the standard vertex set $\mathcal{B}(\mathcal{Q}^{n,p})$ of $\mathcal{Q}^{n,p}$. Let $\mathscr{G}$ be the set of such cycles. Then assume inductively we have an injective map $\theta_n : \mathscr{G} \to V^n$ which is linearly defined. So for $\alpha, \beta \in \mathscr{G}$ and $\gamma \in \mathbb{F}_p$ we have that:

$$\theta_n(\alpha\beta) = \theta_n(\alpha) + \theta_n(\beta) \quad ,$$
$$\theta_n(\alpha^\gamma) = \gamma\theta_n(\alpha) \quad .$$

Let $\mathcal{G}_n := \{g \in \mathfrak{S}_n \mid g \text{ is a non-trivial group element of some facet of } \mathcal{Q}^{n,p}\}$. For any $x \in (\mathcal{Q}^{n,p})_l$ we have $x = \langle g_1, g_2, \ldots, g_l\rangle$, where each $g_i = \prod_j r_j^{b_j} \in \mathcal{G}_n$ for $r_j \in \mathscr{G}_n$ and $b_j \in \mathbb{F}_p$. We then note that θ_n extends to a map $\mathcal{G}_n \to V^n$ by defining $\theta_n(g_i) = \sum_j \theta_n(r_j^{b_j})$. Then the embedding $\phi_n : \mathcal{Q}^{n,p} \hookrightarrow \mathcal{P}(V^n)$ induced by θ_n is given by $\phi_n(x) = \bigvee_{i=1}^{l}\langle\theta_n(g_i)\rangle$ or equivalently $\phi_n(x)$ is the vector space with basis $\{\theta_n(g_1), \theta_n(g_2), \ldots, \theta_n(g_l)\}$. Also assume that if $x = \mathbb{1}$ we have $\phi_n(\mathbb{1}) = 0_{\mathcal{P}(V^n)}$.

Our base step for the induction is given by the map previously described for $n = 2p-1$, which clearly meets all the conditions above. We note here that if $n \leqslant 2p-1$ that the only non-trivial elements of $\mathcal{Q}^{n,p}$ are the groups $\langle(p)\rangle$, where (p) is a cycle of length p.

We now proceed to describe an embedding for $\mathcal{Q}^{n+1,p}$ extending the embedding ϕ_n of $\mathcal{Q}^{n,p}$. Without loss of generality consider the point stabilizer $(\mathfrak{S}_{n+1})_1$. Then $\mathcal{Q}^{n+1,p}$ contains the subposet formed by the cycle subgroups of $(\mathfrak{S}_{n+1})_1$, which is isomorphic to $\mathcal{Q}^{n,p}$. That is all subgroups in $\mathcal{Q}^{n+1,p}$ which are subgroups of $(\mathfrak{S}_{n+1})_1$. Let us call this subposet $\mathcal{R}$, with its standard vertex set $\mathcal{B}(\mathcal{R}) \subseteq \mathcal{B}(\mathcal{Q}^{n+1,p})$. By assumption this embeds into a projective space $\mathcal{P}(V)$ of dimension $d_p(n)$ over $\mathbb{F}_p$, under the map ϕ_n. We are left to consider the elements of $\mathcal{Q}^{n+1,p}$ which have a generator which contains 1.

We concentrate on the single cycles which contain the element 1. Let $\mathscr{C}$ be this set of single cycles which generate the elements of $\mathcal{B}(\mathcal{Q}^{n+1,}) \setminus \mathcal{B}(\mathcal{R})$. We note that no element of $\mathcal{Q}^{n+1,p}$ can be generated using more than one element of $\mathscr{C}$. Now we take a vector space U over $\mathbb{F}_p$ of minimal dimension ℓ such that $\left[{\ell \atop 1}\right]_p \geqslant |\mathscr{C}|$. We then take an arbitrary injective map $\varphi : \mathscr{C} \to U$ such that for $c, d \in \mathscr{C}$ we have $\langle\phi(c)\rangle \neq \langle\phi(d)\rangle$. Then form the vector space $V^{n+1} := V^n \oplus U$.

The embedding $\phi_{n+1} : \mathcal{Q}^{n+1,p} \to \mathcal{P}(V^{n+1})$ is determined by the action on the vertex set elements as we now describe. A general element of $\mathcal{Q}^{n+1,p}$ is generated by a combination of the generators of the elements of $\mathcal{B}(\mathcal{R})$ and the single cycles of $\mathscr{C}$. Let $\mathscr{R}$ be the single cycles which are generators of the elements in $\mathcal{B}(\mathcal{R})$. Then any element $y \in (\mathcal{Q}^{n+1,p})_l$ is of form $y = \langle g_1, g_2, \ldots, g_l \rangle$, where[3] $g_i = c^a \prod_j r_j^{b_j}$ with $c \in \mathscr{C}$ (or possibly identity $\mathbb{1}$), $r_j \in \mathscr{R}$ and $a, b_j \in \mathbb{F}_p$. Then if $\mathcal{G}_{n+1} := \{g \in \mathfrak{S}_{n+1} \mid g \text{ is a non-trivial group element of some facet of } \mathcal{Q}^{n+1,p}\}$ we define the following map $\theta_{n+1} : \mathcal{G}_{n+1} \to V^{n+1}$ which we use to induce the embedding $\phi_{n+1} : \mathcal{Q}^{n+1,p} \to \mathcal{P}(V^{n+1})$:

$$\theta_{n+1}(g_i) = \varphi(c^a) + \sum_j \theta_n(r_j^{b_j}) \qquad \text{for any } g_i \in \mathcal{G} \text{ and from this we obtain}$$

$$\phi_{n+1}(y) = \bigvee_{i=1}^{l} \phi_{n+1}(g_i) = \langle \theta_{n+1}(g_1), \theta_{n+1}(g_2), \ldots, \theta_{n+1}(g_l) \rangle \quad .$$

If $c^a = \mathbb{1}$ we set $\varphi(c) = 0_{\mathcal{P}(V^{n+1})}$, thus $\phi_{n+1}|_{\mathcal{R}} = \phi_n$ and $\phi_{n+1}(\mathbb{1}) = 0_{\mathcal{P}(V^{n+1})}$. This map is injective since ϕ_n is injective by assumption and any element of $\mathcal{Q}^{n+1,p}$ can only be generated by one element of $\mathscr{C}$. Furthermore ϕ_{n+1} is rank and order preserving and so is an embedding by Proposition 1.2.5. We also note that ϕ_{n+1} has the same properties as we assumed for ϕ_n and that $d_p(n+1) = \dim(V^n \oplus \mathcal{U}) = d_p(n) + l$.

If we take the case $n \leqslant 2p-1$ as a base step we have described above an inductive construction of an embedding $\phi : \mathcal{Q}^{n,p} \hookrightarrow \mathcal{P}(V^n)$, where V^n has dimension given by $d_p(n)$ and $\phi(\mathbb{1}) = 0_{\mathcal{P}(V)}$. Tables B.2, B.3 and B.4 in Appendix B give calculated values of $d_p(n)$ for $p = 2, 3, 5$ respectively.

Remarks. 1. At each step (as described above) we have that:

$$|\mathscr{C}| = (p-2)! \left[\binom{n+1}{p} - \binom{n}{p} \right] \quad ,$$
$$= \frac{n!}{(p-1)((n+1-p)!)} \quad .$$

and since we require $\left[\begin{smallmatrix}\ell\\1\end{smallmatrix}\right]_p \geqslant |\mathscr{C}|$ from this we get that:

$$\ell = \left\lceil \frac{\log\left(\frac{n!}{((n+1-p)!)} + 1\right)}{\log(p)} \right\rceil \quad .$$

[3]Since the generators of a facet are disjoint cycles, any group element of a facet can be written in this form.

2. An example of this method at work is exhibited in Appendix B.
3. As there are choices at each step of the construction this is not a uniquely defined embedding, however the embedding dimension that results is always the same for the pair (n, p).
4. Unlike with the free embedding this refined embedding does not necessarily translate directly an action of $\mathfrak{S}_n$ on $\mathcal{Q}^{n,p}$ into an action on the basis of V^n. As a result this embedding may not generate a representation $\rho : \mathfrak{S}_n \to GL(V^n)$. However, for example, in the case $\mathcal{Q}^{3,2}$ it is straightforward to show that we can obtain a 2 dimensional irreducible representation over $\mathbb{F}_2$ via the embedding.
5. By construction the dimension of this refined embedding will always be less than or equal to that given by the free embedding. In the tables B.2, B.3, B.4 of Appendix B we see that the dimension is considerably less than the free embedding dimension, with this disparity more apparent as p or n increases. In fact based on our calculations we would go further and suggest the following:

   Conjecture 4.3.2. *The refined embedding $\phi : \mathcal{Q}^{n,p} \hookrightarrow \mathcal{P}(V)$ described above gives a minimal embedding dimension.*

Chapter 5

The Modular Homology of q–Posets

The concept of modular homology seems to have been first mentioned in a paper by Tucker ([62]) in 1932. Then independently (it would also appear) Mayer & Campbell ([32]) proposed a similar concept in 1940 with a follow up paper in 1942 ([33]) and a further one by Mayer ([31]) also in 1942. This is an area which has seen some interest in recent years. We will begin with an overview of the subject, as applied to q–posets, before going on to consider a result for a class of q–complexes.

5.1 Modular Homology : Background and Overview

We start by setting the scene for describing the modular homology of a poset. Recall the description in Chapter 2 of the inclusion map associated to a ranked poset $\mathcal{P}$. Where before we considered $\mathbb{R}$ modules, we now consider F modules where $F := \mathbb{F}_p$ and $p > 0$ is a prime. As before we have F-modules with $\mathcal{P}$ and $\mathcal{P}_k$ (for $0 \leqslant k \leqslant rk(\mathcal{P})$) as basis, that is $F\mathcal{P}$ and $F\mathcal{P}_k$ respectively. For ease we now write M and M_k for $F\mathcal{P}$ and $F\mathcal{P}_k$ and note $M_k = 0$ if $k < 0$ or $k > rk(\mathcal{P})$. We retain the inclusion map $\partial : M \to M$ with the restriction to maps $\partial : M_k \to M_{k-1}$ with $\partial(M_k) = 0$ if $k < 0$ or $k > rk(\mathcal{P})$. So for $\mathcal{P}$ we have a chain of inclusion maps, which form the following sequence if $rk(\mathcal{P}) = n$:

$$\mathcal{M} : 0 \leftarrow M_0 \overset{\partial}{\leftarrow} M_1 \overset{\partial}{\leftarrow} M_1 \cdots \overset{\partial}{\leftarrow} M_n \overset{\partial}{\leftarrow} 0 \qquad .$$

In the obvious way we define powers of ∂ of the form $\partial^i : M_k \to M_{k-i}$. Then if $i \geqslant 1$ we have for $x \in \mathcal{P}$ that:

$$\partial^i(x) = \sum c(x,z) \cdot z \qquad . \tag{5.1}$$

where the sum runs over all elements $z < x$ of co-rank i in $[0,x]$, and where $c(x,z)$ counts the number of saturated chains $z = x_0 < x_1 < \cdots < x_i = x$ for a fixed z. We once again note at this point the importance of order ideals in our studies by showing that being an order ideal respects the inclusion map.

Proposition 5.1.1. *Let $\mathcal{P} \sqsubseteq \mathcal{Q}$ be posets, with inclusion maps $\partial_{\mathcal{P}}$ and $\partial_{\mathcal{Q}}$ respectively. Then $\partial_{\mathcal{P}}$ is the restriction of $\partial_{\mathcal{Q}}$ to $F\mathcal{P}$ if and only if $\mathcal{P}$ is an order ideal in $\mathcal{Q}$.*

Proof. This is clear from the expression (5.1). If $\mathcal{P}$ is an order ideal of $\mathcal{Q}$ then for any $x \in \mathcal{P}_k$ we have that $[0,x]_{\mathcal{P}} = [0,x]_{\mathcal{Q}}$. Therefore the number of saturated chains $z = x_0 < x_1 < \cdots < x_i = x$ for a fixed z is the same in $\mathcal{P}$ and $\mathcal{Q}$ and thus we conclude that $\partial^i_{\mathcal{P}}(x) = \partial^i_{\mathcal{Q}}(x)$. Conversely if $\partial_{\mathcal{P}}$ is the restriction of $\partial_{\mathcal{Q}}$ to $F\mathcal{P}$ take any $x \in \mathcal{P} \sqsubseteq \mathcal{Q}$ and $y \leqslant_{\mathcal{Q}} x$. Since $\partial^i_{\mathcal{P}}(x) = \partial^i_{\mathcal{Q}}(x)$, for all $i \leqslant rk(x)$, this implies by definition that $[0,x]_i$ is the same set in $\mathcal{P}$ and $\mathcal{Q}$ and so $y \leqslant_{\mathcal{P}} x$ and therefore $\mathcal{P} \trianglelefteq \mathcal{Q}$. $\square$

Now consider the above for $\mathcal{Q}$ a q–poset. By definition for any element $x \in \mathcal{Q}$ we have that $[0,x]$ is isomorphic to an interval in $\mathcal{P}(V)$ for some V with $|V| = |x|$. Thus the number $c(x,z)$ is invariant under the choice of $x \in \mathcal{Q}_k$. So the expression (5.1) simplifies with $c(x,z) := c$ a constant. It is straightforward to see the constant is given for all q by[1]:

$$c = [i]_q \cdot [i-1]_q \cdots [1]_q = (i!)_q \qquad . \tag{5.2}$$

As an aside recall from Chapter 2 that the incidence matrix ${}_k[\mathcal{Q}]_l$ represents the map which sends an element $x \in \mathcal{Q}_l$ to $\sum z$, where this sum runs over all $z \in \mathcal{Q}_k$ such that $z \leqslant x$. Thus by the result above we have that $((l-k)!)_q \cdot_k [\mathcal{Q}]_l$ is the matrix representing the inclusion map $\partial^{l-k} : M_l \to M_k$.

We are interested in finding when ∂ is nilpotent, that is when there exists an m such that $\partial^m = 0$. Let $m > 1$ by the such that $(m!)_q = 0$ in F. This is equivalently the least value for which $[m]_q = 0$ in F. Thus by (5.1) and (5.2) we have that $\partial^m = 0$. The term $m := m(p,q)$ is commonly called the *quantum*

[1] We are using here the natural identification $(i!)_1 := i!$ and $[i]_1 := i$.

characteristic of q in F. It should be noted that if $p|q$ then $(i!)_q = 1$ in F for all i so that $m(p,q)$ is not defined, we must therefore assume throughout that $p \not| q$ and $p > 0$.

We now apply this to the chain of maps $\mathcal{M}$ for the q–poset $\mathcal{Q}$. We note immediately that if $m(p,q) = 2$ we have $\partial \cdot \partial = \partial^2 = 0$ and so $\mathcal{M}$ is a homological sequence in the classical sense. However, for $m(p,q) > 2$ classic homology theory does not immediately apply and we have further work to define a homological sequence in this setting.

If $m(p,q) > 2$ we need to consider certain subsequences obtained from $\mathcal{M}$. Select some i,k with $0 < i < m = m(p,q)$ and $k \leqslant rk(\mathcal{Q})$. Then we may consider the following subsequence of $\mathcal{M}$:

$$\mathcal{M}_{k,i} \,:\, 0 \xleftarrow{\partial^*} \cdots M_{k-m} \xleftarrow{\partial^*} M_{k-i} \xleftarrow{\partial^*} M_k \xleftarrow{\partial^*} M_{k+m-i} \xleftarrow{\partial^*} \cdots \xleftarrow{\partial^*} 0 \quad .$$

Here $\partial^* = \partial^i$ or $\partial^* = \partial^{m-i}$, alternating in the value it takes as appropriate. Then for two adjacent maps in the sequence we have $\partial^*\partial^* = \partial^m = 0$ and so $\mathcal{M}_{k,i}$ is homological. The module M_k is said to be at position k in the subsequence and we have:

$$M_{k-i} \xleftarrow{\partial^i} M_k \xleftarrow{\partial^{m-i}} M_{k+m-i} \qquad .$$

Therefore, when we put

$$K_{k,i,} := ker\,\partial^i \cap M_k \quad \text{and} \quad I_{k,i} := \partial^{m-i}(M_{k+m-i})$$

as respectively the kernel and image of the appropriate ∂^* at position k, then

$$H_{k,i} := \frac{K_{k,i,}}{I_{k,i}}$$

is the *inclusion homology* of $\mathcal{M}_{k,i}$ at the position k. For different choices of the pair k,i we do not necessarily obtain distinct sequences. For example $(k,i), (k-i, m-i)$, and $(k \pm m, i)$ all give the same sequence.

As per standard terminology we say that the sequence $\mathcal{M}_{k,i}$ is *exact* at position k if the homology module $H_{k,i}$ vanishes, that is $K_{k,i} = I_{k,i}$. If this is true for all positions in the sequence it is an *exact sequence*. If the homology modules of the sequence vanish at all but most one position $H_{k,i}$ then the sequence is *almost exact*.

Remarks. The above analysis also holds if $m(p,q) = 2$. In that situation we have $i = 1$ and $m - i = 1$ when the sequence $\mathcal{M}_{k,i}$ is an ordinary homological sequence.

5.1.1 Modular Homology for $\mathcal{P}(V)$

In this section we briefly outline the known results on the homology of sequences $\mathcal{M}_{k,i}$ when $\mathcal{Q} = \mathcal{P}(V)$, for $q \geqslant 1$. Let $|V| = n$ where V is a finite set if $q = 1$ and a finite vector space over $\mathbb{F}_q$ if $q > 1$. As in previous section we have $m = m(p,q)$ and $p \nmid q$. Let us label the inclusion homology at position k (as defined above) as $H^n_{k,i}$.

The homology of sequences associated to $\mathcal{P}(V)$ has been studied in a number of papers - Mnukhin and Siemons ([35], [36]); Bell, Jones and Siemons ([3]); Jones and Siemons ([24]); Siemons and Smith ([52]); with some initial work in the $q > 1$ case by Fisk in his 1997 paper [14]. Among the interesting results on the associated homology modules, the following were obtained for $q \geqslant 1$:

Theorem 5.1.2 ([3], [36]). *The homology module $H^n_{k,i} \neq 0$ if and only if $n < 2k + m - i < n + m$.*

Theorem 5.1.3 ([3], [36], [52]). *If $2k - i = n - 1$ then the homology module $H^n_{k,i}$ is irreducible.*

Remarks.
1. If $q = 1$ then it is straightforward to see that $m(p,q) = p$.
2. There have been a number of papers on modular homology related to simplicial complexes, see the following papers by Mnukhin and Siemons: [37] in 2001, [38] in 2002 and [39] in 2005.
3. If $H^n_{k,i} \neq 0$, as per Theorem 5.1.2, we then say that M_k for (k,i) is a *middle term* of the sequence $\mathcal{M}_{k,i}$. It is fact that any such sequence for $\mathcal{P}(V)$ has at most one middle term and thus $\mathcal{M}_{k,i}$ is almost exact if such a middle term exists.

5.2 Modular Homology and q–Spheres

We now apply the results about modular homology in projective space to q–spheres. Recall that a q-sphere $[S_n]_q$ is the q–complex formed by all the n-dimensional spaces (and their subspaces) in an $n+1$-dimensional vector space over $\mathbb{F}_q$ (for $q > 1$).

In the same way as $\mathcal{M}_{k,i}$ for $\mathcal{P}(V) = \mathcal{P}(\mathbb{F}_q^{n+1})$ we may form the homological sequence $\mathcal{M}_{k,i}^{\mathcal{Q}}$ for the q-sphere $\mathcal{Q} := [S_n]_q \trianglelefteq \mathcal{P}(V)$, where as before we have i, k with $0 < i < m = m(p,q)$ and $k \leqslant rk(\mathcal{Q})$. For the q-sphere we have:

$$M_\ell^{\mathcal{Q}} = \begin{cases} 0 & \text{if } \ell = n+1, \\ M_\ell & \text{otherwise.} \end{cases}$$

Thus we have two sequences (for the same choice of (k,i)), the first for $\mathcal{P}(V)$ the second for $\mathcal{Q}$ (assuming here that $M_s \neq 0$):

$$\begin{array}{lllllllll} \mathcal{M}_{k,i} : 0 & \stackrel{\partial^*}{\longleftarrow} \cdots & \stackrel{\partial^*}{\longleftarrow} M_{k-i} & \stackrel{\partial^*}{\longleftarrow} M_k & \stackrel{\partial^*}{\longleftarrow} M_{k+m-i} & \stackrel{\partial^*}{\longleftarrow} \cdots & \stackrel{\partial^*}{\longleftarrow} M_s & \stackrel{\partial^*}{\longleftarrow} 0 & , \\ \mathcal{M}_{k,i}^{\mathcal{Q}} : 0 & \stackrel{\partial^*}{\longleftarrow} \cdots & \stackrel{\partial^*}{\longleftarrow} M_{k-i}^{\mathcal{Q}} & \stackrel{\partial^*}{\longleftarrow} M_k^{\mathcal{Q}} & \stackrel{\partial^*}{\longleftarrow} M_{k+m-i}^{\mathcal{Q}} & \stackrel{\partial^*}{\longleftarrow} \cdots & \stackrel{\partial^*}{\longleftarrow} M_s^{\mathcal{Q}} & \stackrel{\partial^*}{\longleftarrow} 0 & . \end{array}$$

These sequences agree if $M_s \neq M_{n+1}$. Let H_t^{n+1} be the homology module at position t in $\mathcal{M}_{k,i}$ for the projective space and $H_t^{\mathcal{Q}}$ be the homology module at position t in $\mathcal{M}_{k,i}^{\mathcal{Q}}$. Therefore if $M_s \neq M_{n+1}$ then $H_t^{\mathcal{Q}} = H_t^{n+1}$ for all t. So assume we have $M_s = M_{n+1}$ and thus $M_s^{\mathcal{Q}} = 0$, the trivial module. In this setting we have the following two cases:

(a) M_s does not form a middle term for $\mathcal{M}_{k,i}$. Then we concentrate on the right hand end of the two sequences:

$$\begin{array}{llllll} \mathcal{M}_{k,i} : & \cdots M_u & \stackrel{\partial^*}{\longleftarrow} M_t & \stackrel{\partial^*}{\longleftarrow} M_s & \stackrel{\partial^*}{\longleftarrow} 0 & , \\ \mathcal{M}_{k,i}^{\mathcal{Q}} : & \cdots M_u^{\mathcal{Q}} & \stackrel{\partial^*}{\longleftarrow} M_t^{\mathcal{Q}} & \stackrel{\partial^*}{\longleftarrow} 0 & \stackrel{\partial^*}{\longleftarrow} 0 & . \end{array}$$

Since $\dim(M_s) = \dim(M_{n+1}) = 1$, it is straightforward to see that $\dim(H_t^{\mathcal{Q}}) = \dim(H_t^{n+1}) + 1$.

(b) M_s forms a middle term for $\mathcal{M}_{k,i}$. So take $s = n+1 = k$ and thus by Theorem 5.1.2 we must have $(n+1) < 2(n+1) + m - i < (n+1) + m$ or equivalently we have $0 < (n+1) + m - i < m$. But $(n+1) + m - i < m$ if and only if $(n+1) < i$. The only possibility is that we in fact have the following sequence for $\mathcal{M}_{k,i}$:

$$0 \leftarrow (M_{n+1}) \leftarrow 0 \qquad .$$

However the associated sequence $\mathcal{M}_{k,i}^{\mathcal{Q}}$ for the q-sphere is the degenerate case of a sequence of only trivial modules which is clearly exact.

From these facts we have proved the following:

Theorem 5.2.1. *Let $\mathcal{Q} = [S_n]_q$ be the q-sphere with $\mathcal{Q} \trianglelefteq \mathcal{P}(\mathbb{F}_q^{n+1}) = \mathcal{P}(V)$. Let $\mathcal{M}_{k,i}^{\mathcal{Q}}$ and $\mathcal{M}_{k,i}$ be the homological sequences (as above) for $\mathcal{Q}$ and $\mathcal{P}(V)$ respectively. Let the homology module at $M_k^{\mathcal{Q}}$ be $H_{k,i}^{\mathcal{Q}}$ and the homology module at M_k be $H_{k,i}^{n+1}$. Then if $k = n+1$ we have $H_{k,i}^{\mathcal{Q}} = 0$. For $k < (n+1)$ we fall into one of the following 2 cases:*

- $\dim(H_{k,i}^{\mathcal{Q}}) = \dim(H_{k,i}^{n+1}) + 1$ *if* $k = (n+1) - (m-i)$ *, otherwise*
- $H_{k,i}^{\mathcal{Q}} = H_{k,i}^{n+1}$.

Remarks.

1. This result tells us that such a sequence $\mathcal{M}_{k,i}^{\mathcal{Q}}$ is exact, almost-exact, or has two non-trivial homology modules.
2. We can also derive from this result the dimension of the non-trivial homology modules. We can do this since the homology modules are directly related to the homologies of the sequence $\mathcal{M}_{k,i}$ for $\mathcal{P}(V)$. In particular we have an expression[2] for $\dim(H_{k,i}^{n+1})$ in [52]. From Theorem 5.2.1 the dimension of $H_{k,i}^{\mathcal{Q}}$ will be equal to that of $H_{k,i}^{n+1}$ or equal to $\dim(H_{k,i}^{n+1}) + 1$ depending on the value of k.
3. We would naturally like to extend this result to any q–complex or perhaps q–posets in general. However this is probably unrealistic as it appears too difficult to determine homology for such sequences in general. We can though suggest a result based on some computation of homology modules for sequences derived from q–posets. In particular we have calculated the dimension of modules for sequences associated to certain q–posets that are order ideals of $\mathcal{A}_q(G)$ for a finite group G (here q a prime dividing the order of G) and these are set out in Appendix C. Let $\mathcal{M}_{k,i}^{\mathcal{Q}}$ be a homological sequence for one of these q–posets. Although we are restricted by computing power/memory on the examples that can be tested, the computational results of Appendix C suggest a result akin to Theorem 5.2.1 on the number of non-trivial homology modules of $\mathcal{M}_{k,i}^{\mathcal{Q}}$:

   Conjecture 5.2.2. *Let G be some finite group and q a prime dividing the order of G. Furthermore let $\mathcal{Q} := \mathcal{Q}_X \trianglelefteq \mathcal{A}_q(G)$ where X is a union of conjugacy classes of subgroups in $\mathcal{A}_q(G)$. Then $\mathcal{M}_{k,i}^{\mathcal{Q}}$ has at most 2 non-trivial homology modules.*

[2] This is expanded upon in Appendix C.

Appendix A

Representations and Characters From q–Posets

We consider the case where a q–poset is an order ideal in $\mathcal{A}_q(G)$ of a certain form for some finite group G. A natural question to ask is whether we may derive any representations of G from these q–posets and in particular the characters of these representations. We explore this below first in the form of permutation characters and then in terms of characters derived from associated inclusion maps and incidence matrices. We illustrate the ideas throughout with computed examples.

A.1 Permutation Characters

Let G be some finite group and $\mathcal{Q} \trianglelefteq \mathcal{A}_q(G)$ an order ideal in the poset of elementary abelian q-groups for q a prime dividing the order of G. Furthermore assume that $\mathcal{Q} := \mathcal{Q}_X$, where X is a union of conjugacy classes of subgroups of G. Then G acts on the facets by conjugation, with the stabilizer of a facet x under this action being the normalizer $N_G(x)$. So we note the following facts:

- The group $G \subseteq Aut(\mathcal{Q})$.
- Via the action described above the group G becomes a group of permutations on the facets and therefore we obtain a *permutation character* for G over $\mathbb{C}$. Let π be the permutation character. Then:

$$\pi(g) = |\{x \in X \mid x^g = x\}| = |\{x \in X \mid g \in N_G(x)\}| \quad .$$

Example A.1.1. Let $G := M_{11}$, the smallest of the Mathieu groups. Let $\mathcal{Q} = \mathcal{A}_2(G)$. Then the facets of this q–poset form a single conjugacy class of 330 subgroups of size 2^2. GAP tells us that the complex permutation character associated to these facets is as in Table A.1. (Note: the conjugacy classes are in the order given by those in GAP for the inbuilt presentation of M_{11}). Using standard results for characters we can also decompose the character into its irreducible factors. Using the notation seen in the $\mathbb{ATLAS}$ ([12]) we have that $\pi = 1a + 10aa + 11aa + 44aaa + 45a + 55a$.

Conjugacy Class No.	1	2	3	4	5	6	7	8	9	10
π	330	0	0	0	0	2	18	6	0	0

Table A.1: Permutation Character of M_{11}.

Remarks. In fact if $\mathcal{Q} = \mathcal{Q}_X$, for X a union of conjugacy classes of subgroups of G, then $\mathcal{Q}_i$ $(0 < i \leqslant rk(\mathcal{Q}))$ is also such a union of conjugacy classes of subgroups of G. Therefore in this case we may also construct a permutation character from G's action on the module $\mathbb{C}\mathcal{Q}_i$ in the same way as above.

A.2 Kernels of Inclusion Maps and Irreducible Ordinary Characters

Let $F = \mathbb{C}$ and our q–poset be $\mathcal{Q}$. Then in Chapter 5 we described the inclusion map $F(\mathcal{Q}_i) \xrightarrow{\delta^j} F(\mathcal{Q}_{i-j})$. We are interested here in the situation that the kernel $ker(\delta^j)$ is non-trivial. In particular we note that the elements of the kernel are linear combinations of elements of the poset, namely elementary abelian-q subgroups of the group G. The elements of G therefore act on the elements of $ker(\delta^j)$ by conjugation and so we can then consider the representation over $\mathbb{C}$ associated to $ker(\delta^j)$ and given by this action. In this we think of the module over $\mathbb{C}$, whose basis is the elements of $ker(\delta^j)$. Of particular interest is that the associated characters are often irreducible (though this is not always the case), and we include below some examples demonstrating this outcome.

Recall from Chapter 5 that, subject to multiplication by a constant, the incidence matrix ${}_{i-j}[\mathcal{Q}]_i$ represents the inclusion map δ^j. The kernel of this matrix then describes $ker(\delta^j)$. Therefore via the incidence matrix we can compute easily the corresponding character of the representation described above.

Below we have have calculated (using GAP) a selection of such examples of irreducible characters of groups, derived in the manner set out above.

Remarks. Where appropriate, character labels are as given in $\mathbb{ATLAS}$ [12]. The conjugacy class numbers are those given by the group presentation in GAP and are labelled cc_i in the tables below.

- The following are associated to the Alternating Group ($Alt(n)$) and the Symmetric Group ($\mathfrak{S}_n$):
  - $G = Alt(5)$. Let $\mathcal{Q} = \mathcal{A}_2(G)$ for ${}_0[\mathcal{Q}]_2$ then the corresponding irreducible character is χ_4:

	cc_1	cc_2	cc_3	cc_4	cc_5
χ_4	4	0	1	−1	−1

  - $G = Alt(5)$. Let $\mathcal{Q} = \mathcal{A}_2(G)$ for ${}_0[\mathcal{Q}]_1$ then the corresponding irreducible character is χ_5:

	cc_1	cc_2	cc_3	cc_4	cc_5
χ_5	5	1	−1	0	0

  - $G = Alt(6)$. Let $\mathcal{Q} = \mathcal{A}_3(G)$ for ${}_0[\mathcal{Q}]_2$ then the corresponding irreducible character is χ_6:

	cc_1	cc_2	cc_3	cc_4	cc_5	cc_6	cc_7
χ_6	9	1	0	0	1	−1	1

  - $G = Alt(7)$. Let $\mathcal{Q} = \mathcal{Q}_X \trianglelefteq \mathcal{A}_2(G)$, where $|X| = 105$ and $X = \{\langle (1\,3)(4\,5), (1\,3)(6\,7)\rangle^g \mid g \in G\}$. Here for ${}_1[\mathcal{Q}]_2$ the corresponding irreducible character is χ_6:

	cc_1	cc_2	cc_3	cc_4	cc_5	cc_6	cc_7	cc_8	cc_9
χ_6	14	2	−1	−1	2	0	−1	0	0

  - $G = \mathfrak{S}_7$. Let $\mathcal{Q} = \mathcal{Q}_X \trianglelefteq \mathcal{A}_2(G)$, where $|X| = 105$ and $X = \{\langle (4\,5), (1\,3)(2\,7), (1\,7)(2\,3)\rangle^g \mid g \in G\}$. Here for ${}_1[\mathcal{Q}]_3$ the corresponding irreducible character is χ_7:

	cc_1	cc_2	cc_3	cc_4	cc_5	cc_6	cc_7	cc_8
χ_7	15	5	−1	−3	3	−1	−1	0

	cc_9	cc_{10}	cc_{11}	cc_{12}	cc_{13}	cc_{14}	cc_{15}
χ_7	1	-1	1	0	0	0	1

- $G = Alt(8)$. Let $\mathcal{Q} = \mathcal{Q}_X \trianglelefteq \mathcal{A}_2(G)$, where $X = \{\langle(1\,4)(2\,5)(3\,7)(6\,8), (1\,3)(2\,6)(4\,7)(5\,8), (1\,8)(2\,7)(3\,5)(4\,6)\rangle^g \mid g \in G\}$ and $|X| = 15$. Here for ${}_0[\mathcal{Q}]_3$ the corresponding irreducible character is χ_3 :

	cc_1	cc_2	cc_3	cc_4	cc_5	cc_6	cc_7
χ_3	14	2	6	-1	-1	2	0

	cc_8	cc_9	cc_{10}	cc_{11}	cc_{12}	cc_{13}	cc_{14}
χ_3	2	-1	-1	-1	0	0	0

- The following are associated to finite simple groups, with group labels as in [12] and [23]:
  - $G = L_2(8)$. Let $\mathcal{Q} = \mathcal{A}_2(G)$ for ${}_0[\mathcal{Q}]_3$ then the corresponding irreducible character is χ_6:

	cc_1	cc_2	cc_3	cc_4	cc_5	cc_6	cc_7	cc_8	cc_9
χ_6	8	0	-1	-1	1	-1	1	1	-1

  - $G = L_2(11)$. Let $\mathcal{Q} = \mathcal{A}_2(G)$ for ${}_1[\mathcal{Q}]_2$ then the corresponding irreducible character is χ_4:

	cc_1	cc_2	cc_3	cc_4	cc_5	cc_6	cc_7	cc_8
χ_4	10	0	0	-1	-1	-2	1	1

  - $G = L_2(16)$. Let $\mathcal{Q} = \mathcal{A}_2(G)$ for ${}_0[\mathcal{Q}]_4$ then the corresponding irreducible character is χ_{10}:

	cc_1	cc_2	cc_3	cc_4	cc_5	cc_6	cc_7	cc_8	cc_9
χ_{10}	16	-1	-1	-1	-1	-1	-1	-1	-1

	cc_{10}	cc_{11}	cc_{12}	cc_{13}	cc_{14}	cc_{15}	cc_{16}	cc_{17}
χ_{10}	1	1	1	1	1	1	1	0

  - $G = U_4(2)$. Let $\mathcal{Q} = \mathcal{Q}_X \trianglelefteq \mathcal{A}_2(G)$ with X being either of the two conjugacy classes of size 125 containing groups of size 2^3. Then for

${}_1[\mathcal{Q}]_3$ the corresponding irreducible character is χ_8:

	cc_1	cc_2	cc_3	cc_4	cc_5	cc_6	cc_7	cc_8	cc_9	cc_{10}	cc_{11}
χ_8	15	0	0	−3	−3	1	1	7	1	3	−1

	cc_{12}	cc_{13}	cc_{14}	cc_{15}	cc_{16}	cc_{17}	cc_{18}	cc_{19}	cc_{20}
χ_8	−1	−1	0	0	3	−2	−2	1	0

– $G = L_3(4)$. Let $\mathcal{Q} = \mathcal{Q}_X \trianglelefteq \mathcal{A}_2(G)$ with X being either of the two conjugacy classes of size 21 containing groups of size 2^4. Then for ${}_0[\mathcal{Q}]_4$ the corresponding irreducible character is χ_2:

	cc_1	cc_2	cc_3	cc_4	cc_5	cc_6	cc_7	cc_8	cc_9	cc_{10}
χ_2	20	−1	−1	0	0	2	0	4	0	0

– $G = Sz(8)$. Let $\mathcal{Q} = \mathcal{A}_2(G)$ for ${}_0[\mathcal{Q}]_3$ then the corresponding irreducible character is χ_7:

	cc_1	cc_2	cc_3	cc_4	cc_5	cc_6	cc_7	cc_8	cc_9	cc_{10}	cc_{11}
χ_7	64	−1	1	1	1	−1	−1	−1	0	0	0

- The following are associated to $PGL_n(d)$, the projective linear group of $n \times n$ matrices with entries in $\mathbb{F}_d$:
  – $PGL_3(2)$. Let $\mathcal{Q} = \mathcal{Q}_X \trianglelefteq \mathcal{A}_2(G)$ with X being either of the two conjugacy classes of size 4 containing groups of size 2^2. Then for ${}_0[\mathcal{Q}]_2$ the associated corresponding character is χ:

	cc_1	cc_2	cc_3	cc_4	cc_5	cc_6
χ	6	2	0	0	−1	−1

  – $PGL_4(2)$. Let $\mathcal{Q} = \mathcal{Q}_X \trianglelefteq \mathcal{A}_2(G)$ with X being either of the two conjugacy classes of size 15 containing groups of size 2^3. Then for ${}_0[\mathcal{Q}]_3$ the corresponding irreducible character is χ:

	cc_1	cc_2	cc_3	cc_4	cc_5	cc_6	cc_7
χ	14	0	0	−1	−1	2	2

	cc_8	cc_9	cc_{10}	cc_{11}	cc_{12}	cc_{13}	cc_{14}
χ	0	6	0	2	−1	−1	−1

Remarks. 1. The examples above then are evidence that a group's action on the kernel of associated inclusion maps can often provide a way to construct irreducible ordinary representation of the group.

2. Also associated to the inclusion maps we have described is of course the image of the map. The elements of the image are linear combinations of elements of the poset as with the kernel described before. So similar to above we have a representation (over $\mathbb{C}$) given by G's action on the elements of the image. As with the kernel the image is described by the image of the associated incidence matrix. Thus the characters of the representations associated to the image of inclusion maps can be calculated using a similar GAP code to that used for the kernel representations.
3. We finish with an example where we do not find an irreducible character but where we do find a character corresponding to the group's action on one of its maximal subgroup:

Example A.2.1. Let $G = M_{22}$ be the third of the sporadic Mathieu groups and let $\mathcal{Q} := \mathcal{A}_2(G)$. Then for ${}_1[\mathcal{Q}]_4$ the kernel has dimension 22 and the associated character χ is:

	cc_1	cc_2	cc_3	cc_4	cc_5	cc_6	cc_7	cc_8	cc_9	cc_{10}	cc_{11}	cc_{12}
χ	22	0	2	6	0	4	2	1	1	0	0	2

Using the $\mathbb{ATLAS}$ notation χ's decomposition into irreducible parts is given by $\chi = 1a + 21a$. Of interest to note is that M_{22} has a maximal subgroup of index 22, which is isomorphic to the linear group $L_3(4)$. The character of the representation given by the action on the cosets of this group is equivalent to χ. Thus χ is a permutation character.

Appendix B

Combinatorics of the p-Cycle Complex

In this appendix we include for reference some results on the combinatorics of $\mathcal{Q}^{n,p}$, in particular the derivation of its f–vector for varying parameters n, p (here p is a prime such that $p \leqslant n$). We also describe the intersection profile for the facets of $\mathcal{Q}^{n,p}$ and give examples of some results from Chapter 4.

B.1 The f-vector of $\mathcal{Q}^{n,p}$

By comments in Chapter 4 we have that $\mathcal{Q}^{n,p}$ is a pure p-complex of rank $\lambda(n,p) := \lfloor \frac{n}{p} \rfloor$ (we use just λ, where meaning is clear). Therefore its f-vector is $f(\mathcal{Q}^{n,p}) = (1, f_1, \ldots, f_\lambda)$ where:

$$\begin{aligned} f_\lambda &= \frac{1}{\lambda!} \prod_{i=0}^{\lambda-1} \left((p-2)! \binom{n-ip}{p} \right) \\ &= \frac{n!((p-2)!)^\lambda}{(p!)^\lambda((n-\lambda p)!)(\lambda!)} \qquad = \frac{n!}{(p(p-1))^\lambda((n-\lambda p)!)(\lambda!)} \qquad . \end{aligned} \tag{B.1}$$

To justify this think of the different ways to pick the λ generators. The number of possibilities for the first generator is equal to the size of the standard vertex set for $\mathcal{Q}^{n,p}$, which in Chapter 4 was shown to be $(p-2)!\binom{n}{p}$. Now out of n elements we have $n-p$ left to choose for the second generator. So by the same argument as for the first generator there are $(p-2)!\binom{n-p}{p}$ possibilities for the second generator. Combining these we have $\frac{(p-2)!\binom{n}{p}(p-2)!\binom{n-p}{p}}{2!}$ possibilities for two generators. Continuing then in the same manner gives the required result.

If $p = 2$ then f_λ is equal to the product of the first $\lceil \frac{n}{2} \rceil$ odd numbers, as we prove below (for ease in the following we write $\lambda(n)$ in place of $\lambda(n,2)$):

Proposition B.1.1. *Let $p = 2$ and $f(\mathcal{Q}^{n,2}) = (1, f_1^n, \ldots, f_{\lambda(n)}^n)$. Then:*

$$f_{\lambda(n)}^n = \begin{cases} 1 \cdot 3 \cdot 5 \cdots (n-1) & \text{if } n \text{ is even,} \\ 1 \cdot 3 \cdot 5 \cdot \cdots \cdot n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The proof is by induction on n, even and odd. First note that $f_{\lambda(2)}^2 = \frac{\binom{2}{2}}{1} = 1$ (base step for n even) and $f_{\lambda(3)}^3 = \frac{\binom{3}{2}}{1} = 1 \times 3$ (base step for n odd). Then for inductive step assume that:

$$f_{\lambda(k)}^k = \begin{cases} 1 \cdot 3 \cdot \cdots \cdot (k-1) & \text{if } k \text{ is even,} \\ 1 \cdot 3 \cdot \cdots \cdot k & \text{if } k \text{ is odd.} \end{cases}$$

By the final expression in (B.1) we see that $f_{\lambda(k)}^k = \frac{k!}{2^{\lambda(k)}(\lambda(k))!}$. Further we note that if k is even we have $\lambda(k+2) = \frac{k+2}{2} = \lambda(k) + 1$ and if k is odd we have $\lambda(k+2) = \frac{k+1}{2} = \lambda(k) + 1$. Therefore:

$$\begin{aligned} f_{\lambda(k+2)}^{k+2} &= \frac{(k+2)!}{2^{\lambda(k)+1}((\lambda(k)+1)!)} \\ &= \begin{cases} \frac{(k+2)(k+1)}{2(\frac{k+2}{2})} \cdot f_{\lambda(k)}^k & \text{if } k \text{ is even,} \\ \frac{(k+2)(k+1)}{2(\frac{k+1}{2})} \cdot f_{\lambda(k)}^k & \text{if } k \text{ is odd,} \end{cases} \\ &= \begin{cases} (k+1) \cdot f_{\lambda(k)}^k & \text{if } k \text{ is even,} \\ (k+2) \cdot f_{\lambda(k)}^k & \text{if } k \text{ is odd,} \end{cases} \\ &= \begin{cases} 1 \cdot 3 \cdot \cdots \cdot (k-1) \cdot (k+1) & \text{if } k \text{ is even,} \\ 1 \cdot 3 \cdot \cdots \cdot k \cdot (k+2) & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

Thus by induction the proof is complete. □

Notation. Let (p) and (p_i) (for $i \in \mathbb{N}$) both indicate a cycle of length p.

Next we claim that f_1 is given by the following expression:

$$f_1 = (p-2)!\binom{n}{p} + \sum_{i=2}^{\lambda}\left[\frac{(p-2)!}{i!}\binom{n}{p}\left(\prod_{j=1}^{i-1}(p-1)!\binom{n-jp}{p}\right)\right] \quad .$$

Proof of Claim: We count the number of ways we can obtain groups of the form $\langle(p_1)\rangle, \langle(p_1)(p_2)\rangle, \ldots, \langle(p_1)(p_2)\cdots(p_\lambda)\rangle$, where the p-cycles in a generator are disjoint. Then the first term in the formula is easily seen to be the number of choices for groups of the form $\langle(p_1)\rangle$, the argument follows that seen previously for f_λ. The sum in the formula then gives the number of groups with more than 1 cycle in the generator, that is of the form $\langle(p_1)(p_2)\cdots(p_k)\rangle$ with the cycles disjoint. To see this first consider groups of the form $\langle(p_1)(p_2)\rangle$. Again we have $(p-2)!\binom{n}{p}$ choices for the first generator. As the cycles are disjoint we have $\binom{n-p}{p}$ choices for the elements of the next cycle, and this cycle can be written in $(p-1)!$ different ways. Therefore we have $\frac{1}{2!}(p-2)!\binom{n}{p}(p-1)!\binom{n-p}{p}$ choices for groups of the form $\langle(p_1)(p_2)\rangle$. By extension of this argument for $k \leqslant \lambda$ we have:

$$\frac{1}{k!}\left((p-2)!\binom{n}{p}\right)\left((p-1)!\binom{n-p}{p}\right)\cdots\left((p-1)!\binom{n-(k-1)p}{p}\right) \quad .$$

groups of the form $\langle(p_1)(p_2)\cdots(p_k)\rangle$ with the (p_i) disjoint. The result follows by summing from $k=2$ to $k=\lambda$. $\square$

As we see above, even the formula for f_1 is complicated and this complexity increases when we start to consider f_2. So for the rest of the f-vector we do not show an explicit formula but work inductively and apply a result from Chapter 2 on f-vectors. We exhibit this by a worked example.

Example B.1.2. We consider $\mathcal{Q}^{9,2}$ and by the results above we have that the f-vector is of the form $f(\mathcal{Q}^{9,2}) = (1, 2619, f_2, f_3, 945)$. We assume here inductively that we know the f-vector for $\mathcal{Q}^{n,2}$ in the case $n < 9$ (see Table B.1 below for these calculated values).

We now apply Proposition 2.2.5 to what we know about the elements of $\mathcal{Q}^{9,2}_1$. To do this we need the conjugacy classes that make up $\mathcal{Q}^{9,2}_1$ (note in the following all cycles are disjoint):

- **Class 1:** of form $\langle(a\,b)\rangle$ of which there are $\binom{9}{2} = 36$.
- **Class 2:** of form $\langle(a\,b)(c\,d)\rangle$ of which there are $\frac{\binom{9}{2}\binom{7}{2}}{2!} = 378$.
- **Class 3:** of form $\langle(a\,b)(c\,d)(e\,f)\rangle$ of which there are $\frac{\binom{9}{2}\binom{7}{2}\binom{5}{2}}{3!} = 1260$.

- **Class 4:** of form $\langle (a\,b)(c\,d)(e\,f)(g\,h)\rangle$ of which there are $\frac{\binom{9}{2}\binom{7}{2}\binom{5}{2}\binom{3}{2}}{4!} = 945$.

Then we need to work out the f-vector $f^{\mathcal{Q}^{9,2}_{x\leqslant}}$ for an x in each class above to obtain f_2 and f_3:

- **Class 1:** If $x = \langle (a\,b)\rangle$ then it is contained in the facets of $\mathcal{Q}^{9,2}$ of the form $\langle (a\,b), (c\,d), (e\,f), (g\,h)\rangle$ for all possible choices of $(c\,d), (e\,f), (g\,h)$ with no appearance of a or b. But then $\mathcal{Q}^{9,2}_{x\leqslant} \cong \mathcal{Q}^{7,2}$. So by Table B.1 we have $f(\mathcal{Q}^{9,2}_{x\leqslant}) = (1, 231, 525, 105)$.
- **Class 2:** If $x = \langle (a\,b)(c\,d)\rangle$ then it is contained in $\frac{\binom{5}{5}\binom{3}{3}}{2!} = 15$ facets of $\mathcal{Q}^{9,2}$ and thus $f_3^{\mathcal{Q}^{9,2}_{x\leqslant}} = 15$. In $(\mathcal{Q}^{9,2}_{x\leqslant})_1$ we see that x is a subgroup of the following classes of groups:
  - $\langle (a\,b)(c,\,d), (e,\,f)\rangle$ of which there are $\binom{5}{2} = 10$.
  - $\langle (a,\,b)(c\,d), (e\,f)(g\,h)\rangle$ of which there are $\frac{\binom{5}{2}\binom{3}{2}}{2!} = 15$.
  - $\langle (a\,b), (c\,d)\rangle$ of which there are 1.
  - $\langle (a,\,b)(c\,d), (e\,f)(g\,h)(a\,b)\rangle$ of which there are $\frac{\binom{5}{2}\binom{3}{2}}{2!} = 15$.
  - $\langle (a,\,b)(c\,d), (e\,f)(a\,b)\rangle$ of which there are $\frac{\binom{5}{2}\binom{3}{2}}{2!} = 15$.

  So in total we have $f_1^{\mathcal{Q}^{9,2}_{x\leqslant}} = 51$. Then in $(\mathcal{Q}^{9,2}_{x\leqslant})_3$ we have that x is a subgroup of the following classes of groups:
  - $\langle (a\,b), (c,\,d), (e,\,f)\rangle$ of which there are $\binom{5}{2} = 10$.
  - $\langle (a,\,b)(c\,d), (e\,f), (g\,h)\rangle$ of which there are $\frac{\binom{5}{2}\binom{3}{2}}{2!} = 15$.
  - $\langle (a\,b), (c\,d), (e\,f)(g\,h)\rangle$ of which there are $\frac{\binom{5}{2}\binom{3}{2}}{2!} = 15$.
  - $\langle (a,\,b)(c\,d), (e\,f)(a\,b), (g\,h)(a\,b)\rangle$ of which there are $\frac{\binom{5}{2}\binom{3}{2}}{2!} = 15$.
  - $\langle (a,\,b)(e\,f), (c\,d)(e\,f), (e\,f)(g\,h)\rangle$ of which there are $\frac{\binom{5}{2}\binom{3}{2}}{2!} = 15$.
  - $\langle (a,\,b)(e\,f), (a\,b)(g\,h), (c\,d)\rangle$ of which there are $\frac{\binom{5}{2}\binom{3}{2}}{2!} = 15$.

  So in total we have $f_2^{\mathcal{Q}^{9,2}_{x\leqslant}} = 85$ and thus $f^{\mathcal{Q}^{9,2}_{x\leqslant}} = (1, 51, 85, 15)$.
- **Class 3:** If $x = \langle (a\,b)(c\,d)(e\,f)\rangle$ then it is contained in $\binom{3}{2}$ facets of $\mathcal{Q}^{9,2}$ with the form $\langle (a\,b), (c\,d), (e\,f), (g\,h)\rangle$. Furthermore all these facets intersect in the same group of size 2^8, namely $\langle (a\,b), (c\,d), (e\,f)\rangle$. So $f_2^{\mathcal{Q}^{9,2}_{x\leqslant}} = 3 \times \begin{bmatrix}3\\2\end{bmatrix}_2 - 2 = 19$ and $f_1^{\mathcal{Q}^{9,2}_{x\leqslant}} = 3 \times \begin{bmatrix}3\\1\end{bmatrix}_2 - 2 \times \begin{bmatrix}2\\1\end{bmatrix}_2 = 15$. Thus $f^{\mathcal{Q}^{9,2}_{x\leqslant}} = (1, 15, 19, 3)$.
- **Class 4:** If $x = \langle (a\,b)(c\,d)(e\,f)(g\,h)\rangle$ then if is contained in one facet of $\mathcal{Q}^{9,2}$, that is of $x = \langle (a\,b), (c\,d), (e\,f), (g\,h)\rangle$. Thus $\mathcal{Q}^{9,2}_{x\leqslant} \cong \mathcal{P}(\mathbb{F}_2^3)$ and so $f(\mathcal{Q}^{9,2}_{x\leqslant}) = (1, 7, 7, 1)$.

Now applying Proposition 2.2.5 we see that:

$$f_2 = \frac{1}{\left[{2 \atop 1}\right]_2}(36 \times 231 + 378 \times 51 + 1260 \times 15 + 945 \times 7) \quad = 17703 \quad ,$$
$$f_3 = \frac{1}{\left[{3 \atop 1}\right]_2}(36 \times 525 + 378 \times 85 + 1260 \times 19 + 945 \times 7) \quad = 11655 \quad .$$

Compiling these results we conclude that $f(\mathcal{Q}^{9,2}) = (1, 2619, 17703, 11655, 945)$.

Remark. Table B.1 below sets out the computed f-vectors of $\mathcal{Q}^{n,p}$ for $4 \leqslant n \leqslant 11$ when $n \geqslant 2p$. When $n < 2p$ we have that $\mathcal{Q}^{n,p}$ has rank 1 and these are excluded from the table.

Table B.1: f–Vectors for $\mathcal{Q}^{n,p}$.

n	p	$f(\mathcal{Q}^{n,p})$
4	2	(1,9,3)
5	2	(1,25,15)
6	2	(1,75,105,15)
6	3	(1,40,10)
7	2	(1,231,525,105)
7	3	(1,175,70)
8	2	(1,763,3255,1575,105)
8	3	(1,616,280)
9	2	(1,2619,17703,11655,945)
9	3	(1,2884,3640,280)
10	2	(1,9495,112455,140175,29295,945)
10	3	(1,15520,30100,2800)
10	5	(1,19656,4536)
11	2	(1,35695,669735,1133055,287595,10395)
11	3	(1,71005,158620,15400)
11	5	(1,202356,49896)

B.2 The f-vector of the Skeleton of $\mathcal{Q}^{n,p}$

In Chapter 4 we introduced the skeleton $\mathcal{S}(\mathcal{Q}^{n,p})$ of $\mathcal{Q}^{n,p}$, which is derived from the standard vertex set. Here the f-vector can be given in an explicit form. Let the f-vector of $\mathcal{S}(\mathcal{Q}^{n,p})$ be $(1, f_1^{\mathcal{S}}, \ldots f_\lambda^{\mathcal{S}})$. Then for $0 < i \leqslant \lambda$ we have that $f_i^{\mathcal{S}}$ is equal to the number of subgroups of $\mathfrak{S}_n$ of the form $\langle (p_1), (p_2), \ldots, (p_i) \rangle$, where all the cycles are disjoint. Thus by the same argument used in the calculation of

f_λ in the f-vector of $\mathcal{Q}^{n,p}$, we have that:

$$f_i^{\mathcal{S}} = \frac{1}{i!}\prod_{j=0}^{i-1}\left((p-2)!\binom{n-jp}{p}\right) \qquad .$$

B.3 Intersection Profile of the Facets of $\mathcal{Q}^{n,p}$

We consider here the facets of $\mathcal{Q}^{n,p}$ and calculate their intersection profile. We use throughout the standard vertex set $\mathcal{B}$ for $\mathcal{Q}^{n,p}$. Since $\mathcal{B}$ is a consistent vertex set we know that $x_i \cap x_j$ has a vertex set that is equal to $\mathcal{B}(x_i) \cap \mathcal{B}(x_j)$. Equivalently if we consider the facets as groups generated by λ disjoint p-cycles, then the intersection of any two facets is a group generated by the intersection of the generator sets of the two facets.

Clearly the size (as groups) of any intersection of two distinct facets is p^s (for $0 \leqslant s \leqslant \lambda - 1$). Then for any group $g \in (\mathcal{Q}^{n,p})_\lambda$ we set $I(n,p,s)$ as the number of other facets it intersects exactly in a subgroup of order p^s, that is:

$$I(n,p,s)_\lambda := |\{h \in (\mathcal{Q}^{n,p})_\lambda \setminus g \mid |h \cap g| = p^s\}| \qquad .$$

Remarks. 1. $I(n,p,s)_\lambda$ is independent of the choice of g.

2. If $s = \lambda$ we have $I(n,p,s)_\lambda = 0$.

We can calculate $I(n,p,s)_\lambda$ in a recursive manner, as set out in the following propositions:

Proposition B.3.1. *If $\lambda > 1$ and $1 \leqslant s \leqslant \lambda - 1$ then we have the following relation:*

$$I(n,p,s)_\lambda = \binom{\lambda}{s} \cdot \frac{I(n-(s-1)p,p,1)_{[\lambda-(s-1)]}}{\lambda-(s-1)} \qquad .$$

Proof. Consider our group $g \in (\mathcal{Q}^{n,p})_\lambda$ with λ generators. Then intersecting in s such generators is equivalent to removing $(s-1)$ of the intersecting generators and then intersecting in one generator. Now there are $I(n-(s-1)p,p,1)_{[\lambda-(s-1)]}$ intersections of one generator for the reduced set. Since the number intersecting in a particular generator is the same for each generator we see $\frac{I(n-(s-1)p,p,1)_{[\lambda-(s-1)]}}{\lambda-(s-1)}$ intersections of a particular generator. There are then $\binom{\lambda}{s}$ ways of picking the original s generators we want to intersect with. So we see that:

$$I(n,p,s)_\lambda = \binom{\lambda}{s} \cdot \frac{I(n-(s-1)p,p,1)_{(\lambda-(s-1))}}{\lambda-(s-1)} \quad .$$

□

Remarks. If $s = 1$ we note the right hand side of the expression is equal to $I(n,p,1)_\lambda$ as we would expect.

Using this result we can express $I(n,p,1)_\lambda$ for $\lambda > 1$ in the following recursive form:

Proposition B.3.2. *Let* $\lambda > 1$ *then:*

$$I(n,p,1)_\lambda = \lambda\left[\frac{1}{(\lambda-1)!}\prod_{i=0}^{\lambda-2}\left((p-2)!\binom{n-(i+1)p}{p}\right) - 1 - \sum_{j=1}^{\lambda-2}\binom{\lambda-1}{j}\frac{I(n-jp,p,1)_{\lambda-j}}{\lambda-j}\right] \quad .$$

Proof. Start with one facet g of $\mathcal{Q}^{n,p}$ (the result is independent of choice). Then pick a subgroup $f \subseteq g$ such that $f \in \mathcal{B}(g)$ and so $f = \langle (p) \rangle$ for (p) a p-cycle. Let $\mathcal{I}$ be the set of facets in $\mathcal{Q}^{n,p}$ which contain f. By the same argument as in the expression for f_λ we see that $|\mathcal{I}| = \frac{1}{(\lambda-1)!}\prod_{i=0}^{\lambda-2}\left((p-2)!\binom{n-(i+1)p}{p}\right)$. Now to find the number of $h \in \mathcal{I}$ such that $h \cap g = f$, we need to deduct from $|\mathcal{I}|$ all those $h \in \mathcal{I}$ with $|h \cap g| > p$. First we take 1 from the total, representing the group g itself. Then we need to deduct from $(|\mathcal{I}| - 1)$ the number of groups whose intersection with g contains the subgroup f, but for which the intersection is of size greater than p, that is of size $p^2, p^3, \ldots, p^{\lambda-1}$. As all the groups in $\mathcal{I}$ contain the subgroup f the number of groups in $\mathcal{I}$ such that their intersection with g is of size p^i ($2 \leqslant i \leqslant \lambda - 1$) is equal to $I(n-p,p,i-1)_{\lambda-1}$. Finally as we had λ choices for f we must multiply this sum λ. Thus we arrive at the following expression:

$$I(n,p,1)_\lambda = \lambda\left[\frac{1}{(\lambda-1)!}\prod_{i=0}^{\lambda-2}\left((p-2)!\binom{n-(i+1)p}{p}\right) - 1 - \sum_{j=1}^{\lambda-2} I(n-p,p,j)_{\lambda-1}\right] \quad .$$

By application of Proposition B.3.1 this is equal to the required expression. □

Remarks. 1. We already know $I(n,p,1)_1 = 0$. So by repeated application of Proposition B.3.2, in a recursive manner we can calculate $I(n,p,1)_\lambda$ for any pair n, p.

2. By Proposition B.3.1 for general n, p, s $(1 \leqslant s \leqslant (\lambda - 1))$ we can write $I(n,p,s)_\lambda$ in terms of $I(n',p,1)_{\lambda'}$ for some $n' < n$ and $\lambda' < \lambda$, where $\lambda' := \lfloor \frac{n'}{p} \rfloor$. Thus via Proposition B.3.2 we have an algorithm to calculate $I(n,p,s)_\lambda$ for any $s : 1 \leqslant s \leqslant (\lambda - 1)$.
3. Finally the case $I(n,p,s)_\lambda$ when $s = 0$ is given by:

$$I(n,p,0)_\lambda = f_\lambda - 1 - \sum_{i=1}^{\lambda-1} I(n,p,i)_\lambda \qquad ,$$

where $f_\lambda = |(\mathcal{Q}^{n,p})_\lambda|$.
4. By comments and results in Section 4.2 the above also gives the intersection profile for the skeleton $\mathcal{S}(\mathcal{Q}^{n,p})$ derived from the standard vertex set $\mathcal{B}$.

B.4 Refined Embedding for $\mathcal{Q}^{n,p}$

We exhibit here a small worked example of the refined embedding described in Chapter 4 for $\mathcal{Q}^{4,2}$, in particular we exhibit the inductive construction of this embedding. We use throughout the notation of Section 4.3.3.

Example B.4.1. Start with $\mathcal{Q}^{3,2}$. This is a poset whose f-vector is $f(\mathcal{Q}^{3,2}) = (1,3)$ and whose vertex set is $\{\langle(1\,2)\rangle, \langle(1\,3)\rangle, \langle(2\,3)\rangle\}$. Then it is clear that the minimal dimension projective space into which we can embed this complex is $\mathbb{F}_2$. Let $V^3 = \langle a, b\rangle_{\mathbb{F}_2}$, we then have an embedding $\phi_3 : \mathcal{Q}^{3,2} \hookrightarrow \mathcal{P}(V^3)$ given by:

$$\phi_3(\langle(1\,2)\rangle) = \langle a\rangle \qquad \phi_3(\langle(1\,3)\rangle) = \langle a+b\rangle \qquad \phi_3(\langle(2\,3)\rangle) = \langle b\rangle \quad .$$

Associated to this we have the injective map θ_3 between the generators of the elements of the vertex set and the elements of the vector space V^3. This is given by:

$$\theta_3((1\,2)) = a \qquad \theta_3((1\,3)) = a+b \qquad \theta_3((2\,3)) = b \quad .$$

We now use this to construct an embedding of $\mathcal{Q}^{4,2}$. This has f-vector $f(\mathcal{Q}^{4,2}) = (1,9,3)$ and vertex set $\{\langle(1\,2)\rangle, \langle(1\,3)\rangle, \langle(1\,4)\rangle\ \langle(2\,3)\rangle, \langle(2\,4)\rangle, \langle(3\,4)\rangle\}$. We see an order ideal $\mathcal{R} \trianglelefteq \mathcal{Q}^{4,2}$ with facets $\{\langle(2\,3)\rangle, \langle(2\,4)\rangle, \langle(3\,4)\rangle\}$, which is isomorphic to $\mathcal{Q}^{3,2}$. Let $\mathscr{R} = \{(2\,3), (2\,4), (3\,4)\}$. Then we have as described above, the map θ_3 between $\mathscr{R}$ and the elements of V^3 via the isomorphism $\mathcal{R} \cong \mathcal{Q}^{3,2}$. This gives[1]:

$$\theta_3((2\,3)) = a \qquad \theta_3((2\,4)) = a + b \qquad \theta_3((3\,4)) = b \quad .$$

Then we are left with the remaining single cycles $\mathscr{C} = \{(1\,2), (1\,3), (1\,4)\}$ of the vertex set of $\mathcal{Q}^{2.4}$. So let $U = \langle c, d \rangle_{\mathbb{F}_2}$ (a minimal choice with respect to dimension) and define the map $\varphi : \mathscr{C} \to U$ given by:

$$\varphi((1\,2)) = c \qquad \varphi((1\,3)) = c + d \qquad \varphi((1\,4)) = d \quad .$$

Let $V^4 := V^3 \oplus U$. Then using the maps above we define as in Section 4.3.3 an embedding $\phi_4 : \mathcal{Q}^{4,2} \hookrightarrow V^4$. Under this map the image of the facets of $\mathcal{Q}^{4,2}$ is as follows:

$$\begin{aligned}
\phi_4(\langle (1\,2), (3\,4) \rangle) &= \langle c, b \rangle \quad , \\
\phi_4(\langle (1\,3), (2\,4) \rangle) &= \langle c + d, a + b \rangle \quad , \\
\phi_4(\langle (1\,4), (2\,3) \rangle) &= \langle d, a \rangle \quad .
\end{aligned}$$

In conclusion we have that $d_2(3) = 2$ and $d_2(4) = 4$.

Remark. As there are choices at each step of the construction this is not a uniquely defined embedding, however the embedding dimension that results is always the same for $\mathcal{Q}^{3,2}$ and $\mathcal{Q}^{4,2}$.

B.4.1 Embedding Dimension Tables for $\mathcal{Q}^{n,p}$

These tables give a comparison of the embedding dimension for $\mathcal{Q}^{n,p}$ under varying values of n and p, using the free embedding and refined embedding methods. The column headed *'free'* gives the dimension of the free embedding of Section 4.3.2. The column headed '$d_p(n)$' gives the dimension of the refined embedding of Section 4.3.3. The column headed '$|\mathscr{C}|$ ' is the size of set $\mathscr{C}$ as described in Section 4.3.3.

[1] Of course this varies with our choice in the definition of the isomorphism.

Table B.2: Embedding Dimension Table for $\mathcal{Q}^{n,p}$ with $p = 2$.

n	*Free*	$d_2(n)$	$\lvert\mathscr{C}\rvert$
2	1	1	n/a
3	3	2	n/a
4	6	4	3
5	10	7	4
6	15	10	5
7	21	13	6
8	28	16	7
9	36	20	8
10	45	24	9

Table B.3: Embedding Dimension Table for $\mathcal{Q}^{n,p}$ with $p = 3$.

n	*Free*	$d_3(n)$	$\lvert\mathscr{C}\rvert$
3	1	1	n/a
4	4	2	n/a
5	10	3	n/a
6	20	6	10
7	35	10	15
8	56	14	21
9	84	18	28
10	120	22	36
11	165	27	45
12	220	32	55

Table B.4: Embedding Dimension Table for $\mathcal{Q}^{n,p}$ with $p = 5$.

n	*Free*	$d_5(n)$	$\lvert\mathscr{C}\rvert$
5	6	2	n/a
6	36	4	n/a
7	126	4	n/a
8	336	5	n/a
9	756	5	n/a
10	1512	10	756
11	2772	16	1260
12	4752	22	1980
13	7722	28	2970
14	12012	35	4290
15	18018	42	6006

Appendix C

Homology Calculations

In Chapter 5 we discussed the modular homology derived from homological sequences associated to q–complexes. In that Chapter we made the conjecture that sequences associated to particular order ideals in $\mathcal{A}_q(G)$ (for some finite group G and q a prime), have at most two non-zero homology modules. Here we provide evidence for this by calculating the dimension of such modules for varying choices of G and q. We also look at the decomposition of the associated modular representations of G.

C.1 Method of Calculations

We will initially not directly construct the homology modules, rather we calculate the dimension of the modules. First recall the setup from Chapter 5.

Let $\mathcal{Q}_X \trianglelefteq \mathcal{A}_q(G)$, where X is a union of conjugacy classes of subgroups for a finite group G and q a prime dividing the order of g. Then let $F := \mathbb{F}_p$ such that $p \not| q$. We write M and M_k for the modules $F\mathcal{Q}$ and $F\mathcal{Q}_k$ and note $M_k = 0$ if $k < 0$ or $k > rk(\mathcal{Q})$. We have the inclusion map $\partial : M \to M$ with the restriction to maps $\partial : M_k \to M_{k-1}$ with $\partial(M_k) = 0$ if $k < 0$ or $k > rk(\mathcal{Q})$. So for $\mathcal{Q}$, if $rk(\mathcal{Q}) = n$, we have a chain of inclusion maps:

$$\mathcal{M} : 0 \leftarrow M_0 \overset{\partial}{\leftarrow} M_1 \overset{\partial}{\leftarrow} M_1 \cdots \overset{\partial}{\leftarrow} M_n \overset{\partial}{\leftarrow} 0 \qquad .$$

In the obvious way we define powers of ∂ of the form $\partial^i : M_k \to M_{k-i}$. Now let $m := m(p,q)$ be the quantum characteristic, that is value such that $(m!)_q = 0$ in

F. Select some i, k with $0 < i < m = m(p,q)$ and $k \leqslant rk(\mathcal{Q})$. We include Table[1] C.1 which gives some calculated values for $m(p,q)$. Then we may consider the following subsequence of $\mathcal{M}$:

$$\mathcal{M}_{k,i} : 0 \overset{\partial^*}{\longleftarrow} \cdots M_{k-m} \overset{\partial^*}{\longleftarrow} M_{k-i} \overset{\partial^*}{\longleftarrow} M_k \overset{\partial^*}{\longleftarrow} M_{k+m-i} \overset{\partial^*}{\longleftarrow} \cdots \overset{\partial^*}{\longleftarrow} 0 \qquad .$$

Here $\partial^* = \partial^i$ or $\partial^* = \partial^{m-i}$, alternating in the value it takes as appropriate. Then for two adjacent maps in the sequence we have $\partial^*\partial^* = \partial^m = 0$ and so $\mathcal{M}_{k,i}$ is homological. Therefore, when we put $K_{k,i,} := ker\, \partial^i \cap M_k$ and $I_{k,i} := \partial^{m-i}(M_{k+m-i})$ as respectively the kernel and image of ∂^* at position k, then $H_{k,i} := \frac{K_{k,i,}}{I_{k,i}}$ is the *inclusion homology* of $\mathcal{M}_{k,i}$ at the position k.

Table C.1: Quantum Characteristic Table - $m(p,q)$.

	Values for q							
p	2	3	5	7	11	13	17	19
2	-	2	2	2	2	2	2	2
3	2	-	2	3	2	3	2	3
5	4	4	-	4	5	4	4	2
7	3	6	6	-	3	2	6	6
11	10	5	5	10	-	10	10	10
13	12	6	4	12	12	-	6	12
17	8	16	16	16	16	4	-	8
19	18	18	9	3	3	18	9	-

Now we use the fact discussed in Chapter 5 that $((l-k)!)_q \cdot_k [\mathcal{Q}]_l$ is the matrix representing the inclusion map $\partial^{l-k} : M_l \to M_k$. Thus the linear rank of ${}_k[\mathcal{Q}]_l$ over F gives us the dimension[2] of the image of ∂^{l-k} and so the dimension of the kernel of ∂^{l-k}. Therefore finding the dimension of the homology modules comes down to finding the linear rank of incidence matrices associated to $\mathcal{Q}$. We will do this by calculating the *Smith normal form factors* of the matrix. Recall the well known result of Smith (original paper [53]):

Theorem C.1.1. *([30] p.44) Let A be an $m \times n$ integer matrix and let $s = min\{m, n\}$. Then there exists invertible matrices $P \in GL_m(\mathbb{Z})$ and $Q \in GL_n(\mathbb{Z})$ such that $S = PAQ$ is diagonal. Here P and Q have determinant ± 1. Additionally the entries on the leading diagonal of S, that is $S(ii) = d_i$ $(1 \leqslant i \leqslant s)$, hold the following properties:*

- *$d_i \in \mathbb{Z}_{\geqslant 0}$. The entries are such that any zero entries appear at the end of the list $d_1, \ldots, d_s$.*

[1] Reproduced from a table in [52].
[2] As a module over F.

- $d_i \big| d_{i+1}$ *for* $1 \leqslant i \leqslant (s-1)$. *(We allow that* $0 \mid 0$ *and* $a \mid 0$ *for any integer a).*
- *If r is the linear rank of A (so* $r \leqslant s$*) then there are exactly r non-zero entries amongst the* d_i*, that is* $d_i = 0, \forall\, i > r$.
- *The* d_i *are uniquely determined by A thus S is uniquely determined.*

The matrix S of the theorem is called the Smith normal form *of A and the* d_i *are called the* Smith normal form factors *of A or* elementary divisors *of A.*

So calculating the Smith normal form factors for the incidence matrices under consideration will allow us to read off the linear rank of the matrices in any characteristic. The calculations themselves are done using GAP ([16]) and we present them in table form in the following section. For ease of notation the entries will be displayed in the form $a_1^{r_1} a_2^{r_2} \dots a_n^{r_n}$ (we only list the non-zero entries) to represent the factors:

$$\underbrace{a_1, \dots, a_1}_{r_1 \textit{ times}}, \underbrace{a_2, \dots, a_2}_{r_2 \textit{ times}}, \dots, \underbrace{a_n, \dots, a_n}_{r_n \textit{ times}} \qquad .$$

Then over some F the linear rank is given by $\sum_{i=1}^{n} f(a_i) r_i$, where $f(a_i) = 1$ if $a_i \neq 0 \mod p$ or $f(a_i) = 0$ otherwise.

C.2 Tables of Smith Normal Form Factors

Throughout the following as $\mathcal{Q}$ is always a q–poset we have that $|\mathcal{Q}_0| = 1$ and for ${}_0[\mathcal{Q}]_l$ the linear rank (over any field) is always equal to 1 for any $l : 0 \leqslant l \leqslant rk(\mathcal{Q})$. In each case the q–poset is of the form $\mathcal{Q}_X \trianglelefteq \mathcal{A}_q(G)$, for a finite group G and X a union of conjugacy classes of elementary abelian q-subgroups of G.

Remarks. 1. In the following tables the entry in row $\mathcal{Q}_i$ and column $\mathcal{Q}_j$ gives the non-zero Smith normal form factors of ${}_i[\mathcal{Q}]_j$.

2. The entry in row $\mathcal{Q}_i$ and column headed *cc* gives the number of conjugacy classes of elementary abelian subgroups of size q^i in $\mathcal{Q}_i$.

- **Alternating Groups.**
  - $\mathcal{Q} := \mathcal{A}_2(Alt(5))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{15}	1^{5}	1	15
$\mathcal{Q}_2$	0	1^{5}	1	5

– $\mathcal{Q} := \mathcal{A}_3(Alt(6))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	cc	$\|\mathcal{Q}_i\|$
$\mathcal{Q}_1$	1^{40}	1^{10}	2	40
$\mathcal{Q}_2$	0	1^{10}	1	10

– $\mathcal{Q} := \mathcal{A}_2(Alt(7))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	cc	$\|\mathcal{Q}_i\|$
$\mathcal{Q}_1$	1^{105}	$1^{97}5^{7}15^{1}$	1	105
$\mathcal{Q}_2$	0	1^{140}	2	140

– $\mathcal{Q} := \mathcal{Q}_{X_1} \trianglelefteq \mathcal{A}_2(Alt(7))$, where X_1 is one of the conjugacy classes of groups of size 2^2 :

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	cc	$\|\mathcal{Q}_i\|$
$\mathcal{Q}_1$	1^{105}	$1^{85}2^{6}$	1	105
$\mathcal{Q}_2$	0	1^{140}	1	105

– $\mathcal{Q} := \mathcal{A}_3(Alt(7))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	cc	$\|\mathcal{Q}_i\|$
$\mathcal{Q}_1$	1^{175}	1^{70}	2	175
$\mathcal{Q}_2$	0	1^{70}	1	70

– $\mathcal{Q} := \mathcal{A}_2(Alt(8))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	cc	$\|\mathcal{Q}_i\|$
$\mathcal{Q}_1$	1^{315}	$1^{300}2^{14}6^{1}$	$1^{175}2^{125}4^{14}28^{1}$	1^{35}	2	315
$\mathcal{Q}_2$	0	1^{1225}	$1^{385}2^{105}6^{35}$	1^{35}	5	1225
$\mathcal{Q}_3$	0	0	1^{525}	1^{35}	2	525
$\mathcal{Q}_4$	0	0	0	1^{35}	1	35

– $\mathcal{Q} := \mathcal{Q}_{X_2} \trianglelefteq \mathcal{A}_2(Alt(8))$, where X_2 is one of the conjugacy classes of groups of size 2^3:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	cc	$\|\mathcal{Q}_i\|$
$\mathcal{Q}_1$	1^{315}	$1^{288}2^{26}6^{1}$	1^{105}	2	315
$\mathcal{Q}_2$	0	1^{735}	1^{105}	2	735
$\mathcal{Q}_3$	0	0	1^{105}	1	105

– $\mathcal{Q} := \mathcal{Q}_{X_3} \trianglelefteq \mathcal{A}_2(Alt(8))$, where X_3 is one of the conjugacy classes of groups of size 2^3:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	cc	$\|\mathcal{Q}_i\|$
$\mathcal{Q}_1$	1^{105}	$1^{60}2^{30}6^{15}$	1^{15}	1	105
$\mathcal{Q}_2$	0	1^{105}	1^{15}	1	105
$\mathcal{Q}_3$	0	0	1^{15}	1	15

- **Mathieu Groups.**
  – $\mathcal{Q} := \mathcal{A}_2(M_{11})$, where M_{11} is the smallest of the Mathieu Groups :

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	cc	$\|\mathcal{Q}_i\|$
$\mathcal{Q}_1$	1^{165}	$1^{159}3^{6}$	1	165
$\mathcal{Q}_2$	0	1^{330}	1	330

  – $\mathcal{Q} := \mathcal{Q}_{X_1} \trianglelefteq \mathcal{A}_2(M_{12})$, where M_{12} is the second of the Mathieu Groups and X_1 is one of 3 conjugacy classes of groups of size 2^3:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	cc	$\|\mathcal{Q}_i\|$
$\mathcal{Q}_1$	1^{891}	$1^{881}2^{9}6^{1}$	$1^{715}3^{33}6^{2}$	2	891
$\mathcal{Q}_2$	0	1^{4290}	1^{990}	2	4290
$\mathcal{Q}_3$	0	0	1^{990}	1	990

- $\mathcal{Q} := \mathcal{Q}_{X_2} \trianglelefteq \mathcal{A}_2(M_{12})$, where X_2 is one of 3 conjugacy classes of groups of size 2^3:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{891}	$1^{880}2^{10}6^1$	1^{450}	2	891
$\mathcal{Q}_2$	0	1^{3465}	1^{495}	2	3465
$\mathcal{Q}_3$	0	0	1^{495}	1	495

- $\mathcal{Q} := \mathcal{Q}_W \trianglelefteq \mathcal{A}_2(M_{22})$, where M_{22} is one of the Mathieu Groups and W is a conjugacy class of groups of size 2^3:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{1155}	$1^{1064}2^{90}6^1$	$1^{326}2^1$	1	1155
$\mathcal{Q}_2$	0	1^{2310}	1^{330}	1	2310
$\mathcal{Q}_3$	0	0	1^{330}	1	330

- $\mathcal{Q} := \mathcal{Q}_Y \trianglelefteq \mathcal{A}_2(M_{22})$, where Y is a conjugacy class of groups of size 2^4:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{1155}	$1^{847}2^{231}6^{77}$	$1^{385}2^{462}4^{231}28^{77}$	1^{77}	1	1155
$\mathcal{Q}_2$	0	1^{2695}	$1^{847}2^{231}6^{77}$	1^{77}	2	2695
$\mathcal{Q}_3$	0	0	1^{1155}	1^{77}	1	1155
$\mathcal{Q}_4$	0	0	0	1^{77}	1	77

- **Other Finite Simple Groups**. Notation for groups as in [23].
  - $\mathcal{Q} := \mathcal{A}_2(L_3(2))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{21}	1^{13}	1	21
$\mathcal{Q}_2$	0	1^{14}	2	14

  - $\mathcal{Q} := \mathcal{A}(L_2(8))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{63}	$1^{36}2^{18}6^9$	1^9	1	63
$\mathcal{Q}_2$	0	1^{63}	1^9	1	63
$\mathcal{Q}_3$	0	0	1^9	1	9

  - $\mathcal{Q} := \mathcal{A}_2(L_2(11))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{55}	$1^{44}3^1$	1	55
$\mathcal{Q}_2$	0	1^{55}	1	55

  - $\mathcal{Q} := \mathcal{A}_2(L_2(17))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{153}	$1^{136}3^1$	1	153
$\mathcal{Q}_2$	0	1^{204}	2	204

  - $\mathcal{Q} := \mathcal{A}_2(L_2(16))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{255}	$1^{187}2^{51}6^{17}$	$1^{85}2^{102}4^{51}28^{17}$	1^{17}	1	255
$\mathcal{Q}_2$	0	1^{595}	$1^{187}2^{51}6^{17}$	1^{17}	3	595
$\mathcal{Q}_3$	0	0	1^{255}	1^{17}	8	255
$\mathcal{Q}_4$	0	0	0	1^{17}	1	17

  - $\mathcal{Q} := \mathcal{A}_3(L_3(3))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{117}	$1^{109}3^7 9^1$	1	117
$\mathcal{Q}_2$	0	1^{234}	1	234

- $\mathcal{Q} := \mathcal{A}_3(U_3(3))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{364}	1^{112}	2	364
$\mathcal{Q}_2$	0	1^{112}	1	112

- $\mathcal{Q} := \mathcal{A}_2(L_3(4))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{315}	$1^{299}2^{15}6^{1}$	$1^{193}2^{106}4^{15}28^{1}$	1^{41}	1	315
$\mathcal{Q}_2$	0	1^{1365}	$1^{462}2^{126}6^{42}$	1^{42}	7	1365
$\mathcal{Q}_3$	0	0	1^{630}	1^{42}	2	630
$\mathcal{Q}_4$	0	0	0	1^{42}	2	42

- $\mathcal{Q} := \mathcal{Q}_Z \trianglelefteq \mathcal{A}_2(U_4(2))$, where Z is a conjugacy class of groups of size 2^4:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{315}	$1^{291}2^{23}6^{1}$	$1^{135}2^{156}4^{23}28^{1}$	1^{27}	2	315
$\mathcal{Q}_2$	0	1^{945}	$1^{297}2^{81}6^{27}$	1^{27}	3	945
$\mathcal{Q}_3$	0	0	1^{405}	1^{27}	2	405
$\mathcal{Q}_4$	0	0	0	1^{27}	1	27

- $\mathcal{Q} := \mathcal{A}_2(Sz(8))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{455}	$1^{260}2^{130}6^{65}$	1^{65}	1	455
$\mathcal{Q}_2$	0	1^{455}	1^{65}	1	455
$\mathcal{Q}_3$	0	0	1^{65}	1	65

- $PGL_n(d)$**, the projective linear group of** $n \times n$ **matrices with entries in** $\mathbb{F}_d$.
  - $\mathcal{Q} := \mathcal{Q}_X \trianglelefteq \mathcal{A}_2(PGL_4(2))$, where X is the conjugacy class of groups of size 2^4:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{315}	$1^{300}2^{14}6^{1}$	$1^{175}2^{125}4^{14}28^{1}$	1^{35}	2	315
$\mathcal{Q}_2$	0	1^{1225}	$1^{385}2^{105}6^{35}$	1^{35}	5	1225
$\mathcal{Q}_3$	0	0	1^{525}	1^{35}	2	525
$\mathcal{Q}_4$	0	0	0	1^{35}	1	35

  - $\mathcal{Q} := \mathcal{Q}_{Y_1} \trianglelefteq \mathcal{A}_2(PGL_4(2))$, where Y_1 is a conjugacy of groups of size 2^3 (there are two such classes of the same size but both give the same outcomes):

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{105}	$1^{60}2^{30}6^{15}$	1^{15}	1	105
$\mathcal{Q}_2$	0	1^{105}	1^{15}	1	105
$\mathcal{Q}_3$	0	0	1^{15}	1	15

  - $\mathcal{Q} := \mathcal{Q}_{Y_2} \trianglelefteq \mathcal{A}_2(PGL_4(2))$, where Y_2 is a conjugacy class of groups of size 2^3:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{315}	$1^{288}2^{26}6^{1}$	1^{105}	2	315
$\mathcal{Q}_2$	0	1^{735}	1^{105}	2	735
$\mathcal{Q}_3$	0	0	1^{105}	1	105

  - $\mathcal{Q} := \mathcal{A}_2(PGL_3(3))$:

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	cc	$\lvert\mathcal{Q}_i\rvert$
$\mathcal{Q}_1$	1^{117}	$1^{109}3^{7}9^{1}$	1	117
$\mathcal{Q}_2$	0	1^{234}	1	234

Remarks. 1. Calculating the Smith normal form factors in the tables above requires the manipulation of potentially large sparse matrices. As a result the scope of the examples included is restricted by the computer facilities available. Due to these restrictions we have been limited to considering q–posets of maximum rank 4 and mostly where $q = 2$. Thus it is difficult to make many observations, however the following patterns in the case $q = 2$ are clear:

- If $\mathcal{Q}$ is of rank 3 or rank 4 then the Smith normal form factors of ${}_1[\mathcal{Q}]_2$ are $1^a 2^b 6^c$ for $a, b, c \in \mathbb{N}$.
- If $\mathcal{Q}$ is of rank 4 then the Smith normal form factors of ${}_2[\mathcal{Q}]_3$ are $1^d 2^e 6^f$ for $d, e, f \in \mathbb{N}$.
- If $\mathcal{Q}$ is of rank 4 then the Smith normal form factors of ${}_1[\mathcal{Q}]_3$ are $1^g 2^h 4^i 28^j$ for $g, h, i, j \in \mathbb{N}$.

We can however suggest an explanation for this phenomenon. If we consider $\mathcal{Q} = \mathcal{P}(V)$ for $V = \mathbb{F}_2^n$ with $n = 4$ then the Smith normal form factors associated with ${}_1[\mathcal{Q}]_2$, ${}_2[\mathcal{Q}]_3$ and ${}_1[\mathcal{Q}]_3$ have exactly the patterns described above. Similarly for ${}_1[\mathcal{Q}]_2$ when $n = 3$. As an illustrative example, the table below sets out the Smith normal form factors for $\mathcal{Q} = \mathcal{P}(\mathbb{F}_2^4)$ in the same manner as in the preceding tables.

	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\lvert Q_i \rvert$
$\mathcal{Q}_1$	1^{15}	$1^{11}\,2^3\,6^1$	$1^5\,2^6\,4^3\,28^1$	1^1	15
$\mathcal{Q}_2$	0	1^{35}	$1^{11}\,2^3\,6^1$	1^1	35
$\mathcal{Q}_3$	0	0	1^{15}	1^1	15
$\mathcal{Q}_4$	0	0	0	1^1	1

Each facet of our q–posets being considered in this section is an elementary abelian q-group. We have previously noted that such a group is a vector space over $\mathbb{F}_q$, therefore this agreement on the pattern of the Smith normal form factors is not unexpected. It should be noted that the Smith normal form factors for incidence matrices associated with $\mathcal{P}(\mathbb{F}_q^n)$ have been studied. For some background on this work see papers by Kantor, Sin and Xiang - [25], [50], [51], [67]. A formula for their calculation has been derived in a paper by Chandler, Sin and Xiang in 2006 ([9]).

2. This connection to the Smith normal form factors of a projective space can be used to derive and explain some of the entries seen in the tables above. As an example if $\mathcal{Q} \trianglelefteq \mathcal{A}_q(G)$ is pure and has facets which have

only trivial mutual intersection then the values in the tables above are directly computable from the values for $\mathcal{P}(V)$, where V is a vector space over $\mathbb{F}_q$ of dimension equal to $rk(\mathcal{Q})$. This can be seen in the table for $\mathcal{Q} = \mathcal{A}_2(L_2(16))$. Here the facets have mutual trivial intersection and $rk(\mathcal{Q}) = 4$. So the Smith normal form factors are the same as those in the table for $\mathcal{P}(\mathbb{F}_2^4)$ above, but with the exponents multiplied by 17, the number of facets of $\mathcal{Q}$. A similar explanation applies to the values in the table for $\mathcal{Q} = \mathcal{Q}_Y \trianglelefteq \mathcal{A}_2(M_{22})$. Since we know that $\mathcal{P}(\mathbb{F}_2^4)$ is a dual poset it follows that $\left({}_1[\mathcal{P}(\mathbb{F}_2^4)]_2\right) = \left({}_2[\mathcal{P}(\mathbb{F}_2^4)]_3\right)^T$ and so their (non-zero) Smith normal form factors are equal. This fact then explains the equivalence of the factors for ${}_1[\mathcal{Q}]_2$ and ${}_2[\mathcal{Q}]_3$ seen for $\mathcal{Q} = \mathcal{A}_2(L_2(16))$ and $\mathcal{Q} = \mathcal{Q}_Y \trianglelefteq \mathcal{A}_2(M_{22})$.

3. Following on from the previous remarks we should note that if the facets do not have mutual trivial intersection then the corresponding analysis is non-trivial and no such direct result is known. There is however something we can say if $\mathcal{Q}$ is a q–complex ($q > 1$). In this case we have that $\mathcal{Q}$ is isomorphic to an order ideal in some $\mathcal{P}(V)$ for V a vector space over $\mathbb{F}_q$. In this setting let $A =_i [\mathcal{Q}]_j$ and $B =_i [\mathcal{P}(V)]_j$, then via the isomorphism mentioned we have that A forms a submatrix of B. In Thompson's 1979 paper [61] we find results on the Smith normal form factors of submatrices (Theorem 1 and Theorem 2 of this paper). Then using the notation above, we have the following corollary of these results[3] applicable to the situation we describe:

   Corollary C.2.1. *Let the Smith normal form factors of A be $a_1, a_2, \ldots, a_n$ and the Smith normal form factors of B be $b_1, b_2, \ldots, b_m$ (where $m \geqslant n$) then for all $k : 1 \leqslant k \leqslant n$ we have that $a_k | b_k$.*

   As mentioned previously the values of b_i in the statement of the corollary are derived in [9].

4. We mention for completeness that there has been study in the area of Smith normal form factors for the $q = 1$ case. If V is a finite set and $\mathcal{Q} = \mathcal{P}(V)$ then the Smith normal form factors of ${}_i[\mathcal{Q}]_j$ are fully described in Wilson's 1990 paper [65].

[3]Note: here we allow that $0|d$ for $d = 0$ and $0|d$ for all $d \in \mathbb{N}$.

C.3 Homology Calculations

Using the tables of the previous section we can read off the dimension of homology modules associated to a selection of sequences for the q–posets in question. Before completing the calculations we briefly return to the setting of the projective space $\mathcal{P}(V)$ for $V = \mathbb{F}_q^n$. If we have an associated homological sequence:

$$\mathcal{M}_{k,i} : 0 \leftarrow A_1 \leftarrow A_2 \leftarrow A_3 \cdots \leftarrow A_s \leftarrow 0 \qquad .$$

then the dimension of the homology modules for $\mathcal{M}_{k,i}$ are given by the following result (cf. *reduced Euler characteristic* of a simplicial complex [56] p.20):

Theorem C.3.1. *([36] Theorem 4.4) If $\mathcal{M}_{k,i}$ is almost exact with only non-trivial homology H then H has dimension equal to $\left|\sum_{j=1}^{s}(-1)^j \dim(A_j)\right|$.*

We suggest that we have a similar result associated to the homological sequences under consideration in this chapter. So we may extend the conjecture proposed at the end of Chapter 5:

Conjecture C.3.2. *Let $\mathcal{M}$ be a sequence as below associated to $\mathcal{Q} := \mathcal{Q}_X \trianglelefteq \mathcal{A}_q(G)$ for some finite group G and where X is a union of conjugacy classes of subgroups in $\mathcal{A}_q(G)$. Then there are at most 2 non-zero homology modules, H_1 and H_2 and we have that $\left|\sum_{k=1}^{2}(-1)^k |H_i|\right| = \left|\sum_{j=1}^{s}(-1)^j \dim(M_{i_j})\right|$.*

$$\mathcal{M} : 0 \leftarrow M_{i_1} \xleftarrow{\partial^*} M_{i_2} \xleftarrow{\partial^*} M_{i_3} \cdots \xleftarrow{\partial^*} M_{i_s} \xleftarrow{\partial^*} 0 \qquad .$$

We do such a calculation of dimensions for a selection of homological sequences associated to various q-posets $\mathcal{Q} = \mathcal{Q}_X \trianglelefteq \mathcal{A}_q(G)$ to support this conjecture. The results are presented in tables below, in which we label the homology at position[4] i_j as H_{i_j} and detail all the non-zero homology modules for each sequence. There are 5 different sequences $\mathcal{M}_*$ considered, which we label as follows:

- $m(p,q) = 2$ and $rk(\mathcal{Q}) \leqslant 4$.
  - $\mathcal{M}_A : 0 \xleftarrow{\partial} M_0 \xleftarrow{\partial} M_1 \xleftarrow{\partial} M_2 \xleftarrow{\partial} 0$.
  - $\mathcal{M}_B : 0 \xleftarrow{\partial} M_0 \xleftarrow{\partial} M_1 \xleftarrow{\partial} M_2 \xleftarrow{\partial} M_3 \xleftarrow{\partial} 0$.
  - $\mathcal{M}_C : 0 \xleftarrow{\partial} M_0 \xleftarrow{\partial} M_1 \xleftarrow{\partial} M_2 \xleftarrow{\partial} M_3 \xleftarrow{\partial} M_4 \xleftarrow{\partial} 0$.

[4]That is homology of subsequence $M_{i_{j-1}} \leftarrow M_{i_j} \leftarrow M_{i_{j+1}}$

- $m(p,q) = 3$ and $rk(\mathcal{Q}) \leqslant 4$.
  - $\mathcal{M}_D : 0 \overset{\partial}{\longleftarrow} M_0 \overset{\partial^2}{\longleftarrow} M_1 \overset{\partial}{\longleftarrow} M_3 \overset{\partial^2}{\longleftarrow} 0$.
  - $\mathcal{M}_E : 0 \overset{\partial}{\longleftarrow} M_0 \overset{\partial^2}{\longleftarrow} M_2 \overset{\partial}{\longleftarrow} M_3 \overset{\partial^2}{\longleftarrow} 0$.
  - $\mathcal{M}_F : 0 \overset{\partial^2}{\longleftarrow} M_0 \overset{\partial}{\longleftarrow} M_1 \overset{\partial^2}{\longleftarrow} M_3 \overset{\partial}{\longleftarrow} M_4 \overset{\partial^2}{\longleftarrow} 0$.

For each sequence the basis of each module is a set of subgroups of a group G, namely $\mathcal{Q}_i$. In particular as each $\mathcal{Q}_i$ is by choice a union of conjugacy classes of subgroups we see that by conjugation G acts on the modules of the sequences and thus on the homology modules H_{i_j}. Thus we have a modular representation of G over the field F. So in addition to calculating the dimension of the homology modules using the data of the tables of Smith normal form factors of the previous section, we also use the Meataxe tool in GAP to decompose the associated representations into their irreducible components.

As in Chapter 3 these decompositions are set out in a manner analogous to that used in the $\mathbb{ATLAS}$ and for ease we replace, for example, $1aaaaaaaaa$ by $1a^9$ where $1a$ is always the trivial character. A separate table is included below for each class of groups described in the previous section of Smith normal form factors:

- Table C.2 - alternating groups $Alt(n)$.
- Table C.3 - the simple Mathieu groups.
- Table C.4 - finite simple groups (with group labels as in [23]).
- Table C.5 - $PGL_n(d)$, the Projective General Linear Group.

Remarks.
1. As we can see the results given in the tables support Conjecture C.3.2 and we confirm at most two non-zero homology modules for each sequence. Also it is interesting to note that in some nice cases (especially for finite simple groups) we find the associated modular representations are irreducible.
2. These results come with a rather large caveat when trying to extrapolate a general result. The calculations require the manipulation of large (sparse) matrices to obtain the data in the tables above. As a result the size of the group that can be analysed and the length of sequences obtainable is restricted by the computing power to hand. Thus there are possibly longer sequences obtainable with more than two non-zero homology modules, though no such example has been encountered.

Table C.2: Alternating Group Homology Sequences.

$\mathcal{Q}$	q	p	$\mathcal{M}_*$	Module Dimensions	Meataxe Decomposition	Dimension Check
$\mathcal{A}_2(Alt(5))$	2	3	$\mathcal{M}_A$	$\lvert H_1\rvert = 9$	$1a + 4aa$	$\lvert -1+15-5\rvert = 9$
$\mathcal{A}_3(Alt(6))$	3	2	$\mathcal{M}_A$	$\lvert H_1\rvert = 29$	$1a^5 + 4a^3b^3$	$\lvert -1+40-10\rvert = 29$
$\mathcal{A}_2(Alt(7))$	2	3	$\mathcal{M}_A$	$\lvert H_2\rvert = 36$	$1a^4 + 6ab + 13aa$	$\lvert -1+105-140\rvert = 36$
$\mathcal{Q}_{X_1} \trianglelefteq \mathcal{A}_2(Alt(7))$	2	3	$\mathcal{M}_A$	$\lvert H_1\rvert = 13$ $\lvert H_2\rvert = 14$	$13a$ $1a + 13a$	$\lvert -1+105-105\rvert = 1$ =14-13
$\mathcal{A}_3(Alt(7))$	3	2	$\mathcal{M}_A$	$\lvert H_1\rvert = 104$	$1a^2 + 4aabb + 6a^4 + 14a^3 + 20a$	$\lvert -1+175-70\rvert = 104$
$\mathcal{A}_2(Alt(8)$	2	3	$\mathcal{M}_C$	$\lvert H_2\rvert = 421$	$1a^{11} + 7a^7 + 13a^{10} + 28aa + 35a^5$	$\lvert -1+315-1225+525-35\rvert = 421$
$\mathcal{A}_2(Alt(8)$	2	7	$\mathcal{M}_F$	$\lvert H_3\rvert = 176$	$19a + 45a + 56a^2$	$\lvert -1+315-525+35\rvert = 176$
$\mathcal{Q}_{X_2} \trianglelefteq \mathcal{A}_2(Alt(8))$	2	3	$\mathcal{M}_E$	$\lvert H_2\rvert = 316$	$1a^3 + 7a^4 + 13aa + 28a^3 + 35a^5$	$\lvert -1+315-735+105\rvert = 316$
$\mathcal{Q}_{X_3} \trianglelefteq \mathcal{A}_2(Alt(8))$	2	3	$\mathcal{M}_B$	$\lvert H_1\rvert = 14$	$1a + 13a$	$\lvert -1+105-105+15\rvert = 14$
$\mathcal{Q}_{X_3} \trianglelefteq \mathcal{A}_2(Alt(8))$	2	7	$\mathcal{M}_D$	$\lvert H_1\rvert = 89$	$14a + 19a + 56a$	$\lvert -1+105-15\rvert = 89$
$\mathcal{Q}_{X_3} \trianglelefteq \mathcal{A}_2(Alt(8))$	2	7	$\mathcal{M}_E$	$\lvert H_2\rvert = 89$	$14a + 19a + 56a$	$\lvert -1+105-15\rvert = 89$

Table C.3: Mathieu Groups Homology Sequences.

$\mathcal{Q}$	q	p	$\mathcal{M}_*$	Module Dimensions	Meataxe Decomposition	Dimension Check
$\mathcal{A}_2(M_{11})$	2	3	$\mathcal{M}_A$	$\lvert H_1\rvert = 5$ $\lvert H_2\rvert = 171$	$5a$ $1a^3 + 5a^4b^3 + 10aabc + 24aa + 45a$	$\lvert -1 + 165 - 330\rvert = 166 = 171 - 5$
$\mathcal{Q}_{X_1} \trianglelefteq \mathcal{A}_2(M_{12})$	2	7	$\mathcal{M}_D$	$\lvert H_1\rvert = 140$ $\lvert H_3\rvert = 240$	$32a + 54aa$ $120aa$	$\lvert -1 + 891 - 990\rvert = 100$ $= 240 - 140$
$\mathcal{Q}_{X_2} \trianglelefteq \mathcal{A}_2(M_{12})$	2	7	$\mathcal{M}_D$	$\lvert H_1\rvert = 440$ $\lvert H_3\rvert = 45$	$11ab + 45a + 54aa + 55a + 66a + 144a$ $45a$	$\lvert -1 + 891 - 495\rvert = 395$ $= 440 - 45$
$\mathcal{Q}_W \trianglelefteq \mathcal{A}_2(M_{22})$	2	3	$\mathcal{M}_B$	$\lvert H_2\rvert = 826$	$1a + 21a + 49ab + 55a + 210aa + 231a$	$\lvert -1 + 1155 - 2310 + 330\rvert = 826$
$\mathcal{Q}_Y \trianglelefteq \mathcal{A}_2(M_{22})$	2	3	$\mathcal{M}_C$	$\lvert H_1\rvert = 76$ $\lvert H_2\rvert = 539$	$21a + 55a$ $1a^3 + 21a^3 + 49ab + 55a^3 + 210a$	$\lvert -1 + 1155 - 2695 + 1155 - 77\rvert = 463$ $= 539 - 76$
$\mathcal{Q}_Y \trianglelefteq \mathcal{A}_2(M_{22})$	2	7	$\mathcal{M}_F$	$\lvert H_1\rvert = 76$	$1a + 21a + 54a$	$\lvert -1 + 1155 - 1155 + 77\rvert = 76$

Table C.4: Finite Simple Groups Homology Sequences.

$\mathcal{Q}$	q	p	$\mathcal{M}_*$	Module Dimensions	Meataxe Decomposition	Dimension Check
$\mathcal{A}_2(L_3(2))$	2	3	$\mathcal{M}_A$	$\lvert H_1\rvert = 7$ $\lvert H_2\rvert = 1$	$7a$ $1a$	$\lvert -1+21-14\rvert = 6$ $= 7-1$
$\mathcal{A}_2(L_2(8))$	2	3	$\mathcal{M}_B$	$\lvert H_1\rvert = 8$	$1a + 7a$	$\lvert -1+63-63+9\rvert = 8$
$\mathcal{A}_2(L_2(11))$	2	3	$\mathcal{M}_A$	$\lvert H_1\rvert = 10$ $\lvert H_2\rvert = 11$	$5ab$ $1a + 10a$	$\lvert -1+55-55\rvert = 1$ $= 11-10$
$\mathcal{A}_2(L_2(17))$	2	3	$\mathcal{M}_A$	$\lvert H_1\rvert = 16$ $\lvert H_2\rvert = 68$	$16a$ $1aa + 16aaa + 18a$	$\lvert -1+153-204\rvert = 52$ $= 68-16$
$\mathcal{A}_2(L_2(16))$	2	7	$\mathcal{M}_F$	$\lvert H_1\rvert = 16$	$16a$	$\lvert -1+255-255+17\rvert = 16$
$\mathcal{A}_3(L_3(3))$	2	3	$\mathcal{M}_A$	$\lvert H_1\rvert = 7$ $\lvert H_2\rvert = 125$	$7a$ $1a^4 + 3aabb + 6aabb + 7a^4 + 15ab + 27a$	$\lvert -1+117-234\rvert = 118$ $= 125-7$
$\mathcal{A}_3(U_3(3))$	3	2	$\mathcal{M}_A$	$\lvert H_1\rvert = 251$	$1a^9 + 6a^{11} + 14a^8 + 32ab$	$\lvert -1+364-112\rvert = 251$
$\mathcal{A}_2(L_3(4))$	2	7	$\mathcal{M}_F$	$\lvert H_3\rvert = 274$	$19a + 35aabb + 45a$	$\lvert -1+315-630+42\rvert = 274$
$\mathcal{Q}_Z \trianglelefteq \mathcal{A}_2(U_4(2))$	2	7	$\mathcal{M}_f$	$\lvert H_3\rvert = 64a$	$64a$	$\lvert -1+315-405+27\rvert = 64$
$\mathcal{A}_2(Sz(8))$	2	3	$\mathcal{M}_B$	$\lvert H_1\rvert = 64$	$64a$	$\lvert -1+455-455+65\rvert = 64$

Table C.5: Projective General Linear Group Homology Sequences.

$\mathcal{Q}$	q	p	$\mathcal{M}_*$	Module Dimensions	Meataxe Decomposition	Dimension Check
$\mathcal{Q}_X \trianglelefteq \mathcal{A}_2(PGL_4(2))$	2	3	$\mathcal{M}_C$	$\lvert H_2\rvert = 421$	$1a^{11} + 7a^7 + 13a^{10} + 28aa + 35a^5$	$\lvert -1 + 315 - 1225 + 525 - 35\rvert = 421$
$\mathcal{Q}_X \trianglelefteq \mathcal{A}_2(PGL_4(2))$	2	7	$\mathcal{M}_F$	$\lvert H_3\rvert = 176$	$19a + 45a + 56aa$	$\lvert -1 + 315 - 525 + 35\rvert = 176$
$\mathcal{Q}_{Y_1} \trianglelefteq \mathcal{A}_2(PGL_4(2))$	2	3	$\mathcal{M}_B$	$\lvert H_1\rvert = 14$	$1a + 13a$	$\lvert -1 + 15 - 105 + 105\rvert = 14$
$\mathcal{Q}_{Y_2} \trianglelefteq \mathcal{A}_2(PGL_4(2))$	2	3	$\mathcal{M}_B$	$\lvert H_2\rvert = 316$	$1a^3 + 7a^4 + 13aa + 28a^3 + 35a^5$	$\lvert -1 + 315 - 735 + 105\rvert = 316$
$\mathcal{A}_2(PGL_3(3))$	2	3	$\mathcal{M}_A$	$\lvert H_1\rvert = 7$ $\lvert H_2\rvert = 125$	$7a$ $1a^4 + 3aabb + 6aabb + 7a^4 + 15ab + 27a$	$\lvert -1 + 117 - 234\rvert = 118$ $= 125 - 7$

Appendix D

GAP Scripts

We collect in this appendix the GAP ([16]) scripts that have been used to obtain the computed results seen elsewhere in the text.

D.1 Chapter 1: Good Representations

The following is the GAP script and output used in checking the good representation for $\mathcal{A}_2(A_5)$:

```
% Generators of representation from
http://brauer.maths.qmul.ac.uk/Atlas/v3/matrep/A5G1-f2r4bB0
gap> generators := [
> [[1,0,0,0],
> [0,0,1,0],
> [0,1,0,0],
> [1,0,0,1]
> ]*Z(2)
> ,
> [[0,1,0,0],
> [1,1,0,0],
> [0,0,0,1],
> [0,0,1,1]
> ]*Z(2)
> ];
gap> Display(generators[1]);
 1 . . .
 . . 1 .
 . 1 . .
 1 . . 1
gap> Display(generators[2]);
 . 1 . .
 1 1 . .
 . . . 1
 . . 1 1

% Define matrix group over GF(2) with these generators, it is isomorphic to A5
gap> G:=Group(generators);
<matrix group with 2 generators>
gap> Size(G);
```

```
60

% Create a variable s2, with all the 2 spaces in a 4-dimensional space over GF(2)
gap> v:=GF(2)^4;
( GF(2)^4 )
gap> d2:=Subspaces(v,2);
Subspaces( ( GF(2)^4 ), 2 )
gap> s2:=[];
[  ]
gap> Read("dimvs");
gap> Display(dimvs);
function ( a, b )
    local  i, iter;
    iter := Iterator( a );
    for i  in [ 1 .. Size( a ) ]  do
        Add( b, NextIterator( iter ) );
    od;
    return;
end
gap> dimvs(d2,s2);
gap> Size(s2);
35

% look at the orbits biven by the action of our matrix group on s2
gap> o:=OrbitsDomain(G,s2);;
% 4 orbits, we pick the one of size 5
gap> Size(o);
4
gap> Size(o[1]);Size(o[2]);Size(o[3]);Size(o[4]);
10
10
5
10

% set o to the set of five 2-dimensional spaces - this is our complex
which we claim is isomorphic to that in A5
gap> o:=o[3];;


% Now check that this 5 spaces intersect trivially like the Klein groups in A5
gap> Read("INT4");
gap> Display(int4);
function ( o, a )
    local  s, t1, i, s1;
    s := Size( o );
    t1 := o[a];
    for i  in [ 1 .. s ]  do
        s1 := Dimension( Intersection( t1, o[i] ) );
        if s1 = 0  then
            Print( "" );
        else
            Print( i, " ", Dimension( Intersection( t1, o[i] ) ), "\n" );
        fi;
    od;
    return;
end
% following confirms that each 2 -space only intersects with itself non-triviailly
gap> int4(o,1);
1 2
gap> int4(o,2);
2 2
gap> int4(o,3);
3 2
gap> int4(o,4);
4 2
gap> int4(o,5);
5 2

% look at the conjugacy classes of elementary abelian groups in our matrix
representation of A5
gap> ccm:=Filtered(ConjugacyClassesSubgroups(G),cl->IsElementaryAbelian(Representative(cl)));;
```

```
gap> Size(ccm);
5
gap> conjc(ccm,G);
Class 1 Class Size 1 Size of Groups 1
Group([],[ [ Z(2)^0, 0*Z(2), 0*Z(2), 0*Z(2) ], [ 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ],
[ 0*Z(2), 0*Z(2), Z(2)^0, 0*Z(2) ], [ 0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0 ]
 ])

Class 2 Class Size 15 Size of Groups 2
Group([ [ [ 0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0 ], [ 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ],
[ 0*Z(2), Z(2)^0, Z(2)^0, 0*Z(2) ], [ Z(2)^0, 0*Z(2), 0*Z(2), 0*Z(2) ]
     ] ])

Class 3 Class Size 10 Size of Groups 3
Group([ [ [ 0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0 ], [ 0*Z(2), Z(2)^0, Z(2)^0, 0*Z(2) ],
[ 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ], [ Z(2)^0, 0*Z(2), 0*Z(2), Z(2)^0 ]
     ] ])

Class 4 Class Size 5 Size of Groups 4
Group([ [ [ 0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0 ], [ 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ],
[ 0*Z(2), Z(2)^0, Z(2)^0, 0*Z(2) ], [ Z(2)^0, 0*Z(2), 0*Z(2), 0*Z(2) ] ]
    , [ [ 0*Z(2), Z(2)^0, 0*Z(2), Z(2)^0 ], [ 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ],
[ Z(2)^0, 0*Z(2), Z(2)^0, Z(2)^0 ], [ Z(2)^0, Z(2)^0, 0*Z(2), 0*Z(2) ] ]
 ])

Class 5 Class Size 6 Size of Groups 5
Group([ [ [ 0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0 ], [ 0*Z(2), Z(2)^0, 0*Z(2), Z(2)^0 ],
[ 0*Z(2), Z(2)^0, Z(2)^0, Z(2)^0 ], [ Z(2)^0, 0*Z(2), Z(2)^0, Z(2)^0 ]
     ] ])

% Class no 4 is the set of 5 Klein groups which intersect trivially
gap> Display(cc[4]);
ConjugacyClassSubgroups(Group( [ a, b ] ),
Group( [ a*b*a*b^-1*a^-1, b*a*b*a*b^-1*a^-1*b^-1 ] ))
gap> Display(ccm[4]);
ConjugacyClassSubgroups(Group([ [ [ Z(2)^0, 0*Z(2), 0*Z(2), 0*Z(2) ],
[ 0*Z(2), 0*Z(2), Z(2)^0, 0*Z(2) ], [ 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ],
      [ Z(2)^0, 0*Z(2), 0*Z(2), Z(2)^0 ] ], [ [ 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ],
[ Z(2)^0, Z(2)^0, 0*Z(2), 0*Z(2) ], [ 0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0 ],
      [ 0*Z(2), 0*Z(2), Z(2)^0, Z(2)^0 ] ] ]),Group(
[ [ [ 0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0 ], [ 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ],
[ 0*Z(2), Z(2)^0, Z(2)^0, 0*Z(2) ], [ Z(2)^0, 0*Z(2), 0*Z(2), 0*Z(2) ] ],
  [ [ 0*Z(2), Z(2)^0, 0*Z(2), Z(2)^0 ], [ 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ],
[ Z(2)^0, 0*Z(2), Z(2)^0, Z(2)^0 ], [ Z(2)^0, Z(2)^0, 0*Z(2), 0*Z(2) ] ] ]))

% variable to contain all the elements of A5
gap> elemG:=Elements(G);;
gap> Size(elemG);
60

%load programs to look at the action of A5 on the klein 4 groups and compare
with action of A5 (as a matrix group) on the set of 2 dimensional spaces held
in our variable s2
gap> Read("batchpermG");
gap> Display(perm);
function ( F, d, o, q )
    local  i, h, t, k, o1, o2, v1, l, j, f, c, g1, g2, O1, O2, L;
    f := elemG[F];
    h := ccm[d];
    L := Size( h );
    for i  in [ 1 .. L ]  do
        c := ConjugateSubgroup( h[i], f );
        t := 0;
        for j  in [ 1 .. L ]  do
            if h[j] = c  then
                t := j;
            else
                t := t;
            fi;
        od;
        Print( "r", i, " moved to r", t );
```

```
        if i = t  then
            Print( "   FIXED", "\n" );
        else
            Print( "\n" );
        fi;
    od;
    for k  in [ 1 .. L ]  do
        O1 := Basis( o[k] )[1];
        O2 := Basis( o[k] )[2];
        g1 := O1 * f;
        g2 := O2 * f;
        v1 := VectorSpace( GF( q ), [ g1, g2 ] );
        t := 0;
        for l  in [ 1 .. L ]  do
            if o[l] = v1  then
                t := l;
            else
                t := t;
            fi;
        od;
        Print( "o", k, " moved to o", t );
        if k = t  then
            Print( "   FIXED", "\n" );
        else
            Print( "\n" );
        fi;
    od;
    return;
end
gap> Display(batchperm);
function ( f, d, o, q )
    local  I;
    for I  in [ 1 .. f ]  do
        Print( "Element in A5 ", "\n" );
        Display( elemG[I] );
        perm( I, d, o, q );
        Print( "\n" );
    od;
    return;
end
%run program to check for all elements of A5. Here r1,r2,r3,r4,r5 are the elementary
abelian subgroups of A5 and o1,o2,o3,o4,o5 are the subspaces of dimension 2. They are
not produced in the same order but if we take the following identification the operation
of A5 on the two sets is identical

r1 <-> o2
r2 <-> o1
r3 <-> o5
r4 <-> o3
r5 <-> o4
gap> batchperm(60,4,o,2);
```

D.2 Chapter 3: Embeddings of $\mathcal{A}_q(G)$

The following is the GAP script associated with the results of Chapter 3 in considering embedding of q–complexes associated to $\mathcal{A}_q(G)$ for certain groups G. This first script is used when we have $\mathcal{Q} := \mathcal{A}_q(G)$:

```
# output is dimension of FX,W and also
# any elements of G that exist in W
# also output (as double check) that no element of
# form a-b for a,b in G
inject:=function(q,G)  # where G is the group in question and q is our standard parameter
local FG,E,gens,c,i,a,b,W,dimW,BW,co,e,t,l,bracket,FX,d,coa,cob;
FG:=GroupRing(GF(q),G);
e:=Elements(Basis(FG));
gens:=[];
E:=Filtered(e,n->n^q=e[1]);
FX:=Submodule(FG,E);
c:=UnorderedTuples(E,2);
for i in [1..Size(c)] do
a:=c[i][1];
b:=c[i][2];
 if a*b=b*a and (a*b)^q=e[1] then
   bracket:=a+b-a*b;
   if Size(gens)=0 then
     Add(gens,bracket);
   else
   t:=0; l:=0;
        repeat
         l:=l+1;
         if gens[l]=bracket then
          t:=1;
          else
          t:=t;
         fi;
        until t<>0 or l=Size(gens);
        if t=0 then
         Add(gens,bracket);
        else
        gens:=gens;
        fi;fi;
 else
 gens:=gens;
fi;
od;
W:=Submodule(FX,gens);
dimW:=Dimension(W);
Print("FG has dimension ",Dimension(FG),"\n");
Print("\n");
Print("FX has dimension ",Dimension(FX),"\n");
Print("\n");
Print("W has dimension ",dimW,"\n");
Print("\n");
BW:=Basis(W);
Print("Elements of group in span of W:","\n");
Print("\n");
for i in [1..Size(E)] do
co:=Coefficients(BW,E[i]);
if co <> fail then
Print(E[i],"\n");
else
Print("");
fi;
od;
Print("Elements of form a-b in span of W:","\n");
c:=Combinations(E,2);
for i in [1..Size(c)] do
a:=c[i][1];
```

```
b:=c[i][2];
d:=a-b;
coa:=Coefficients(BW,a);
cob:=Coefficients(BW,b);
if coa<>fail and cob<>fail then
 Print("");
else
 co:=Coefficients(BW,d);
 if co<>fail then
  Print(a," ",b,"\n");
 else
  Print("");
 fi;
fi;
od;
end;
```

The following script is used when we consider $\mathcal{Q}_X \trianglelefteq \mathcal{A}_q(G)$ where X is a conjugacy class of elementary abelian q–subgroups of G. The input parameters are:

- q - our standard parameter for q–posets.
- G - the group in question.
- cc -variable containing the conjugacy classes of elementary abelian q-subgroups of G.
- k - the class number in cc for the conjugacy class X.

```
# output is dimension of FX,W and also
# any elements of G that exist in W
# also output (as double check) that no element of
# form a-b for a,b in G
injectcca:=function(q,G,cc,k)
local FG,E,gens,c,i,a,b,W,dimW,BW,co,e,emb,t,FX,l,j,el,bracket,d,coa,cob;
FG:=GroupRing(GF(q),G);
e:=Elements(Basis(FG));
emb:=Embedding(G,FG);
E:=[];
for i in [1.. Size(cc[k])] do
el:=Elements(cc[k][i]);
   for j in [1..Size(el)] do
      a:=Image(emb,(el[j]));
      if Size(E)=0 then
       Add(E,a);
       else
        t:=0; l:=0;
        repeat
         l:=l+1;
         if E[l]=a then
          t:=1;
          else
          t:=t;
         fi;
        until t<>0 or l=Size(E);
        if t=0 then
         Add(E,a);
        else
```

```
        E:=E;
        fi;
      fi;
   od;
od;
FX:=Submodule(FG,E);
gens:=[];
for i in [1.. Size(cc[k])] do
el:=Elements(cc[k][i]);
   c:=UnorderedTuples(el,2);
   for j in [1..Size(c)] do
      a:=Image(emb,(c[j][1]));
      b:=Image(emb,(c[j][2]));
      bracket:=a+b-a*b;
      if Size(gens)=0 then
       Add(gens,bracket);
       else
       t:=0; l:=0;
       repeat
        l:=l+1;
        if gens[l]=bracket then
         t:=1;
         else
         t:=t;
        fi;
       until t<>0 or l=Size(gens);
       if t=0 then
       Add(gens,bracket);
       else
       gens:=gens;
       fi;
      fi;
   od;
od;
W:=Submodule(FX,gens);
dimW:=Dimension(W);
Print("FX has dimension ",Dimension(FX),"\n");
Print("\n");
Print("W has dimension ",dimW,"\n");
Print("\n");
BW:=Basis(W);
Print("Elements of group in conjugacy class in span of W:","\n");
Print("\n");
for i in [1..Size(E)] do
co:=Coefficients(BW,E[i]);
if co <> fail then
Print(E[i],"\n");
else
Print("");
fi;
od;
Print("Elements of form a-b in span of W, a.b not in W:","\n");
c:=Combinations(E,2);
for i in [1..Size(c)] do
a:=c[i][1];
b:=c[i][2];
d:=a-b;
coa:=Coefficients(BW,a);
cob:=Coefficients(BW,b);
if coa<>fail and cob<>fail then
 Print("");
else
 co:=Coefficients(BW,d);
 if co<>fail then
  Print(a," ",b,"\n");
 else
  Print("");
 fi;
fi;
od;
end;
```

D.2.1 Modular Representations

The following is the script used to determine the modular representations associated to the embeddings of the previous section.

```
# inputs are standard parameter q, Group G, list of conjugacy classes of elementary abelian q groups cc,
# and k is the conjugacy class number of groups forming facets of the complex.
# output is dimension of the representation and the Meataxe decomposition
# into factors along with their multiplicities
modrepcc:=function(q,G,cc,k)
local FG,E,gens,c,i,a,b,W,dimW,BW,e,emb,t,FX,l,j,el,bracket,d,coa,cob,cod,h,Q,id,r1,image,embgens,m,
p,rep,homimage,hom,basis,gmod,col,mgens,cola,geng;
FG:=GroupRing(GF(q),G);
e:=Elements(Basis(FG));
emb:=Embedding(G,FG);
E:=[];
for i in [1.. Size(cc[k])] do
el:=Elements(cc[k][i]);
   for j in [1..Size(el)] do
      a:=Image(emb,(el[j]));
      if Size(E)=0 then
       Add(E,a);
       else
        t:=0; l:=0;
        repeat
         l:=l+1;
         if E[l]=a then
          t:=1;
          else
          t:=t;
         fi;
        until t<>0 or l=Size(E);
        if t=0 then
         Add(E,a);
        else
        E:=E;
        fi;
      fi;
   od;
od;
FX:=Submodule(FG,E);
gens:=[];
for i in [1.. Size(cc[k])] do
el:=Elements(cc[k][i]);
   c:=UnorderedTuples(el,2);
   for j in [1..Size(c)] do
      a:=Image(emb,(c[j][1]));
      b:=Image(emb,(c[j][2]));
      bracket:=a+b-a*b;
      if Size(gens)=0 then
       Add(gens,bracket);
       else
       t:=0; l:=0;
       repeat
        l:=l+1;
        if gens[l]=bracket then
         t:=1;
         else
         t:=t;
        fi;
       until t<>0 or l=Size(gens);
       if t=0 then
       Add(gens,bracket);
       else
       gens:=gens;
       fi;
      fi;
   od;
```

```
od;
W:=Submodule(FX,gens);
dimW:=Dimension(W);
Print("FX has dimension: ",Dimension(FX),"\n");
Print("\n");
Print("W has dimension: ",dimW,"\n");
Print("\n");
hom:=NaturalHomomorphismBySubspace(FX,W);
Q:=ImagesSource(hom);
basis:=Basis(Q);
Print("Dimension of FX/W and resulting module: ",Dimension(Q),"\n");
Print("\n");
geng:=GeneratorsOfGroup(G);
e:=BasisVectors(basis);
e:=AsList(e);
embgens:=[];
gens:=[];
for i in [1..Size(geng)] do
Add(embgens,Image(emb,geng[i]));
od;
embgens:=AsList(embgens);
for j in [1..Size(embgens)] do
m:=[];
for i in [1..Size(e)] do
p:=PreImages(hom,e[i]);
rep:=Representative(p);
image:=OnPoints(rep,embgens[j]);
homimage:=Image(hom,image);
r1:=Coefficients(basis,homimage);
Add(m,r1);
od;
m:=TransposedMat(m);
Add(gens,m);
od;
gmod:=GModuleByMats(gens,GF(q));
col:=MTX.CollectedFactors(gmod);
Print("Number of Different Composition Factors: ",Length(col),"\n");
Print("\n");
Print("Dimension and Frequencies of Factors:","\n");
Print("\n");
for i in [1..Length(col)] do
cola:=col[i];
Print(i,". Dimension: ",MTX.Dimension(col[i][1])," Frequency: ",col[i][2],"\n");
od;
Print("\n");
Print("Composition Factors in Detail:","\n");
for i in [1..Length(col)] do
Print(i,". ");
Display(col[i]);
Print("\n");
Print(" Generating Matrices:","\n");
mgens:=MTX.Generators(col[i][1]);
for j in [1..Size(mgens)] do
Display(mgens[j]);
Print("\n");
od;
od;
end;
```

D.3 Chapter 4 : p-Cycle Complex Representations

The following is the GAP script used for the calculations of the modular representations of Symmetric Groups through the p-cycle complex.

```
# following script is for use when G is a symmetric group. Then inputs are as for previous
# script but here k is the conjugacy class which forms the vertex set
# of the p-cycle complex.
# output is dimension of the representation and the Meataxe decomposition
# into factors along with their multiplicities
vertexrep:=function(q,G,cc,k)
local geng,e,gens,i,m,j,image,r1,k1,gmod,col,cola,mgens;
geng:=GeneratorsOfGroup(G);
e:=Elements(cc[k]);
e:=AsList(e);
gens:=[];
for i in [1..Size(geng)] do
m:=[];
for j in [1..Size(e)] do
image:=OnPoints(e[j],geng[i]);
r1:=[];
for k1 in [1..Size(e)] do
if e[k1]=image then
Add(r1,Z(q)^0);
else
Add(r1,0*Z(q));
fi;
od;
Add(m,r1);
od;
m:=TransposedMat(m);
Display(m);
Add(gens,m);
od;
gmod:=GModuleByMats(gens,GF(q));
Print("Dimension of Representation ",MTX.Dimension(gmod),"\n");
col:=MTX.CollectedFactors(gmod);
Print("Number of Different Composition Factors: ",Length(col),"\n");
Print("\n");
Print("Dimension and Frequencies of Factors:","\n");
Print("\n");
for i in [1..Length(col)] do
cola:=col[i];
Print(i,". Dimension: ",MTX.Dimension(col[i][1])," Frequency: ",col[i][2],"\n");
od;
Print("\n");
Print("Composition Factors in Detail:","\n");
for i in [1..Length(col)] do
Print(i,". ");
Display(col[i]);
Print("\n");
Print(" Generating Matrices:","\n");
mgens:=MTX.Generators(col[i][1]);
for j in [1..Size(mgens)] do
Display(mgens[j]);
Print("\n");
od;
od;
end;
```

D.4 Appendix A : Kernels of Inclusion Maps

The following is the GAP script used for the calculation of characters associated to kernels of inclusion maps[1].

[1]The code can also be used to obtain characters associated to the image of inclusion maps. To do this replace the line ‘k:=NullspaceMat(r2);’ by ‘k:=BaseMat(r2);’ and call ‘contrace(cc,l,cg,G)’ to run.

```
%Set up variables first:
%G - the group in question
%cc - list of conjugacy classes of elementary abelian $q$-subgroups
%l,l2 the list of classes under consideration for the incidence matrix
%i.e. where those in list l contained in list l2
%r1, r2 empty variables to hold the rows of matrix and the final matrix itself.
%Functions to derive the incidence matrix - call matrix(l,l2,G) to run:

r1:=[];r2:=[];
matrix:=function(l,l2,G)
local s,t;
r1:=[];
r2:=[];
for s in l do
for t in [1..Size(cc[s])] do
r1:=[];
row(s,t,l2,G);
Add(r2,r1);
od;
od;
r2:=TransposedMat(r2);
Print("Classes ",l," contained in Classes ",l2,"\n");
end;

row:=function(a,b,l3,G)
local r,g,k,i,j;
g:=ClassElementLattice(cc[a],b);
r:=[];
k:=0;
for i in l3 do
for j in [1..Size(cc[i])] do
k:=0;
if IsSubgroup(cc[i][j],g) then
k:=1;
else
k:=0;
fi;
Add(r,k);
od;
od;
r1:=r;
end;

%Next derive the kernel of our incidence matrix r2, using the standard GAP function NullspaceMat

k:=NullspaceMat(r2);

%If kernel is non-trivial then run 'contrace' function with the parameters:
%cc - conjugacy class of subgroups as above
%l2 - as above
%cg - conjugacy classes of group under question
%G - group as above
% This outputs the character table of G; the character calculated;
%whether the character is irreducible
%and its decomposition into irreducible parts.
%Call contrace(cc,l2,cg,G) to run


contrace:=function(cc,l2,cg,G)
local p,q,m1,n1,chi,tab,const,k1;
k1:=Size(k);
m1:=[];
n1:=[];
chi:=[];
for p in [1..Size(cg)] do
m1:=[];
for q in [1..k1] do
con(p,q,cc,l2,cg);
n1:=SolutionMat(k,d);
Add(m1,n1);
od;
```

```
Print("Conjugacy Class ",p," Trace of matrix ",Trace(m1),"\n");
Add(chi,Trace(m1));
od;
Print("\n");
tab:=CharacterTable(G);
Display(tab);
Print("\n");
chi:=Character(tab,chi);
Display(chi);
Print("\n");
Print("Is Irreducible? ",IsIrreducibleCharacter(tab,chi),"\n");
Print("\n");
const:=ConstituentsOfCharacter(chi);
Display(const);
Print("\n");
end;

con:=function(x,y,cc,e,cg)
local c,i,j,dummy,l,m,n,t,a,b,i1,i2,a1,rt;
t:=Representative(cg[x]);
dummy:=0;
b:=[];
a:=0;
for i1 in [1..Length(e)] do
 a:=a+Size(cc[e[i1]]);
od;
rt:=0;
for i2 in [1..Length(e)] do
a1:=Size(cc[e[i2]]);
for i in [1..a1] do
if k[y][rt+i]<>0 then
c:=ConjugateGroup(cc[e[i2]][i],t);
for j in [1..a1] do
if IsSubgroup(cc[e[i2]][j],c) and IsSubgroup(c,cc[e[i2]][j]) then
b[j+rt]:=k[y][rt+i];
else
dummy:=dummy;
fi;
od;
else
dummy:=dummy;
fi;
od;
rt:=rt+a1;
od;
d:=[];
for l in [1..Length(b)] do
if IsBound(b[l]) then
Add(d,b[l]);
else
Add(d,0);
fi;
od;
for m in [(Length(d)+1)..a] do
Add(d,0);
od;
end;
```

D.5 Appendix C : Homology Calculations

The following is the GAP script used for the calculation of the Smith Normal Form Factors of incidence matrices associated to q–posets which are order ideals in $\mathcal{A}_q(G)$.

```
# G is the group under investigation
# cc is a list of conjugacy classes of q-subgroups of G
# if we are considering a matrix between Q_k and Q_l
# then a is the list of conjugacy classes numbers of cc making up Q_k
# and b is the list of conjugacy classes numbers of cc making up Q_k
# outputs the smith normal form factors and their multiplicities
sm:=function(a,b,G,cc)
local s,t,r2,r,g,k,i,j,i1,j1,k1,l1,L,l2,i2,q,a1,b1;
r2:=[];
a1:=Size(a);
b1:=Size(b);
for s in [1..a1] do
for t in [1..Size(ConjugacyClassSubgroups(G,Representative(cc[a[s]])))] do
g:=ClassElementLattice(cc[a[s]],t);
r:=[];
k:=0;
for i in [1..b1] do
for j in [1..Size(ConjugacyClassSubgroups(G,Representative(cc[b[i]])))] do
k:=0;
if IsSubgroup(ClassElementLattice(cc[b[i]],j),g) then
k:=1;
else
k:=0;
fi;
Add(r,k);
od;
od;
Add(r2,r);
od;
od;
Print("Classes ",a," contained in Classes ",b,"\n");
L:=ElementaryDivisorsMatDestructive(Integers,r2);
l2:=Size(L);
Print("Rank of Smith Normal Form ",l2,"\n");
Print(L,"\n");
for q in [1..Maximum(L)] do
t:=0;
for i2 in [1..l2] do
if L[i2]=q then
t:=t+1;
else
t:=t;
fi;
od;
if t=0 then
Print("");
else
Print(q," ",t,"\n");
fi;
od;
end;
```

The following script calculates the decomposition of a modular representation associated to a particular homology module.

```
# G is the group under investigation
# cc is a list of conjugacy classes of q-subgroups of G
# p is the parameter p is the characteristic of the Field F
# if homology module is H_i then
# then b is the list of conjugacy classes numbers of cc making up Q_i
# and b is the list of conjugacy classes numbers of cc making up Q_i*
# where M_i* is the module to left of M_i in the sequence
# and c is the list of conjugacy classes numbers of cc making up Q_i'
# where M_i* is the module to right of M_i in the sequence
# output is dimension of the representation and the Meataxe decomposition
# into factors along with their multiplicities
homologyrepb:=function(p,a,b,c,G,cc)
```

```
local l,ge,pos,conj,ker,pk,mk,m1,a1,b1,s,t,g,r,k,i,m2,K,I,hom,Q,basis,e,gens,j,m,pre,rep,geng,image,
r1,k1,gmod,col,cola,mgens,homimage;
pk:=0;
for i in[1..Size(b)] do
 pk:=pk+Size(cc[b[i]]);
od;
ge:=[];
for i in[1..Size(b)] do
  for j in[1..Size(cc[b[i]])] do
    Add(ge,cc[b[i]][j]);
  od;
od;
mk:=GF(p)^pk;
m1:=[];
a1:=Size(a);
b1:=Size(b);
for s in [1..a1] do
  for t in [1..Size(ConjugacyClassSubgroups(G,Representative(cc[a[s]])))] do
    g:=ClassElementLattice(cc[a[s]],t);
    r:=[];
    for i in [1..b1] do
      for j in [1..Size(ConjugacyClassSubgroups(G,Representative(cc[b[i]])))] do
if IsSubgroup(ClassElementLattice(cc[b[i]],j),g) then
  k:=Z(p)^0;
  else
  k:=0*Z(p);
fi;
Add(r,k);
      od;
     od;
     Add(m1,r);
   od;
od;
m1:=TransposedMatDestructive(m1);
ker:=NullspaceMatDestructive(m1);
m2:=[];
a1:=Size(b);
b1:=Size(c);
for s in [1..a1] do
  for t in [1..Size(ConjugacyClassSubgroups(G,Representative(cc[b[s]])))] do
    g:=ClassElementLattice(cc[b[s]],t);
    r:=[];
    for i in [1..b1] do
      for j in [1..Size(ConjugacyClassSubgroups(G,Representative(cc[c[i]])))] do
if IsSubgroup(ClassElementLattice(cc[c[i]],j),g) then
  k:=Z(p)^0;
  else
  k:=0*Z(p);
fi;
Add(r,k);
      od;
    od;
    Add(m2,r);
  od;
od;
m2:=TransposedMatDestructive(m2);
i:=BaseMatDestructive(m2);
K:=Subspace(mk,ker);
I:=Subspace(K,i);
hom:=NaturalHomomorphismBySubspace(K,I);
Q:=ImagesSource(hom);
basis:=CanonicalBasis(Q);
geng:=GeneratorsOfGroup(G);
e:=BasisVectors(basis);
e:=AsList(e);
gens:=[];
for j in [1..Size(geng)] do
  t:=[];
   for i in [1..Size(ge)] do
      r:=[];
      conj:=OnPoints(ge[i],geng[j]);
```

```
      pos:=Position(ge,conj);
      for l in [1..Size(ge)] do
if l= pos then
  Add(r,Z(p)^0);
  else
  Add(r,0*Z(p));
fi;
      od;
  Add(t,r);
  od;
  m:=[];
  for i in [1..Size(e)] do
pre:=PreImages(hom,e[i]);
rep:=Representative(pre);
image:=t*rep;
homimage:=Image(hom,image);
r1:=Coefficients(basis,homimage);
Add(m,r1);
  od;
m:=TransposedMat(m);
Add(gens,m);
od;
gmod:=GModuleByMats(gens,GF(p));
Print("Dimension of Representation ",MTX.Dimension(gmod),"\n");
col:=MTX.CollectedFactors(gmod);
Print("Number of Different Composition Factors: ",Length(col),"\n");
Print("\n");
Print("Dimension and Frequencies of Factors:","\n");
Print("\n");
for i in [1..Length(col)] do
  cola:=col[i];
  Print(i,". Dimension: ",MTX.Dimension(col[i][1])," Frequency: ",col[i][2],"\n");
od;
Print("\n");
end;
```

The following is variant of the above where the module on the right is a zero module.

```
homologyrepz:=function(p,a,b,G,cc)
local l,ge,pos,conj,ker,pk,mk,m1,a1,b1,s,t,g,r,k,i,m2,K,I,hom,Q,basis,e,gens,j,m,pre,rep,geng,image,
r1,k1,gmod,col,cola,mgens,homimage;
ge:=[];
for i in[1..Size(b)] do
  for j in[1..Size(cc[b[i]])] do
    Add(ge,cc[b[i]][j]);
  od;
od;
pk:=Size(ge);
mk:=GF(p)^pk;
m1:=[];
a1:=Size(a);
b1:=Size(b);
for s in [1..a1] do
  for t in [1..Size(cc[a[s]])] do
    g:=cc[a[s]][t];
    r:=[];
    for i in [1..b1] do
      for j in [1..Size(cc[b[i]])] do
if IsSubgroup(cc[b[i]][j],g) then
  k:=Z(p)^0;
  else
  k:=0*Z(p);
fi;
Add(r,k);
      od;
     od;
```

```
     Add(m1,r);
   od;
od;
m1:=TransposedMatDestructive(m1);
ker:=NullspaceMatDestructive(m1);
i:=[Zero(mk)];
K:=Subspace(mk,ker);
I:=Subspace(K,i);
hom:=NaturalHomomorphismBySubspace(K,I);
Q:=ImagesSource(hom);
basis:=CanonicalBasis(Q);
geng:=GeneratorsOfGroup(G);
e:=BasisVectors(basis);
e:=AsList(e);
gens:=[];
for j in [1..Size(geng)] do
  t:=[];
   for i in [1..Size(ge)] do
      r:=[];
      conj:=OnPoints(ge[i],geng[j]);
      pos:=Position(ge,conj);
      for l in [1..Size(ge)] do
if l= pos then
  Add(r,Z(p)^0);
  else
  Add(r,0*Z(p));
fi;
      od;
  Add(t,r);
  od;
  m:=[];
  for i in [1..Size(e)] do
pre:=PreImages(hom,e[i]);
rep:=Representative(pre);
image:=t*rep;
homimage:=Image(hom,image);
r1:=Coefficients(basis,homimage);
Add(m,r1);
  od;
m:=TransposedMat(m);
Add(gens,m);
od;
gmod:=GModuleByMats(gens,GF(p));
Print("Dimension of Representation ",MTX.Dimension(gmod),"\n");
col:=MTX.CollectedFactors(gmod);
Print("Number of Different Composition Factors: ",Length(col),"\n");
Print("\n");
Print("Dimension and Frequencies of Factors:","\n");
Print("\n");
for i in [1..Length(col)] do
  cola:=col[i];
  Print(i,". Dimension: ",MTX.Dimension(col[i][1])," Frequency: ",col[i][2],"\n");
od;
Print("\n");
end;
```

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