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ON SOME CONTINUED FRACTION IDENTITIES OF SRINIVASA RAMANUJAN[1]

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ABSTRACT. The main purpose of this note is to state and prove, in a simple, unified manner, several q-continued fraction expansions found in Ramanujan's "lost" notebook. This is related to some recent works of G. E. Andrews and M. D. Hirschhorn.

0. Introduction. The following continued fraction identities $(1)_R$–$(3)_R$ and $(\mathrm{I})_R$ found in the "lost" notebook of Ramanujan (terminology due to G. E. Andrews [**1**]) contain as special cases many of his other identities:

$$(1)_R \qquad \frac{G(0,\lambda q,b,q)}{G(0,\lambda,b,q)} = \frac{1}{1+}\,\frac{\lambda q}{1+}\,\frac{bq+\lambda q^2}{1} + \cdots \frac{\lambda q^{2n+1}}{1} + \frac{bq^{n+1}+\lambda q^{2n+2}}{1} + \cdots$$

$$(2)_R \qquad = \frac{1}{1+}\,\frac{\lambda q}{1+bq+}\,\frac{\lambda q^2}{1+bq^2+}\cdots\frac{\lambda q^n}{1+bq^n+}\cdots$$

$$(3)_R \qquad = \frac{1}{1-b+}\,\frac{b+\lambda q}{1-b+}\cdots\frac{b+\lambda q^n}{1-b}+\cdots$$

and, more generally

$$(\mathrm{I})_R \qquad \frac{G(aq,\lambda q,b,q)}{G(a,\lambda,b,q)} = \frac{1}{1+}\,\frac{aq+\lambda q}{1}+\frac{bq+\lambda q^2}{1} + \cdots \frac{aq^{n+1}+\lambda q^{2n+1}}{1} + \frac{bq^{n+1}+\lambda q^{2n+2}}{1} + \cdots$$

where

$$(4) \qquad G(a,\lambda,b,q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-\lambda/a)_n a^n}{(q)_n(-bq)_n}.$$

Here and in what follows,

$$(c)_k = \begin{cases} 1 & \text{if } k=0, \\ (1-c)(1-cq)\cdots(1-cq^{k-1}) & \text{if } k>0. \end{cases}$$

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The suffix R signifies that the identity is due to Ramanujan. It is easily seen that $(1)_R$–$(3)_R$ are themselves special cases, respectively, of $(\mathrm{I})_R$ above and (II) and $(\mathrm{III})_H$ below:

$$\text{(II)} \qquad \frac{G(aq, \lambda q, b, q)}{G(a, \lambda, b, q)} = \frac{1}{1} + \frac{aq + \lambda q}{1 - aq + bq} + \cdots \frac{aq + \lambda q^n}{1 - aq + bq^n} + \cdots$$

$$(\mathrm{III})_H \qquad = \frac{1}{1 - b + aq} + \frac{b + \lambda q}{1 - b + aq^2} + \cdots \frac{b + \lambda q^n}{1 - b + aq^{n+1}} + \cdots.$$

The suffix H in $(\mathrm{III})_H$ signifies that the identity is due to M. D. Hirschhorn [**3**]. Identity $(\mathrm{I})_R$, and thereby $(1)_R$, has been proved independently by Andrews [**1**] and by Hirschhorn [**4**]. Andrews has employed G and some auxiliary functions and a transformation of E. Heine; and Hirschhorn has proved it by obtaining a closed form for the nth convergent. While Andrews [**2**] has given a separate proof of the "slightly tricky" identity $(2)_R$, he has extracted $(3)_R$ as a particular case of $(\mathrm{III})_H$ which Hirschhorn [**3**] has proved by finding a closed form for the nth convergent. Many other identities of Ramanujan also follow as pointed out by Andrews and Hirschhorn. However, we have not come across (II) nor a proof of the following Ramanujan identities $(\mathrm{IV})_R$ and $(5)_R$ also listed in the "lost" notebook:

$$(\mathrm{IV})_R \qquad \frac{G(aq, \lambda q, b, q)}{G(a, \lambda, b, q)} = \frac{1}{1 + aq} + \frac{\lambda q - abq^2}{1 + q(aq + b)} + \cdots \frac{\lambda q^n - abq^{2n}}{1 + q^n(aq + b)} + \cdots,$$

$$(5)_R \qquad \frac{1}{a + c} - \frac{ab}{a + b + cq} - \cdots \frac{ab}{a + b + cq^n} - \cdots = \frac{1}{c - b + a} + \frac{bc}{c - b + a/q} + \cdots \frac{bc}{c - b + a/q^n} + \cdots.$$

In what follows we employ, as auxiliary function instead of $G(a, \lambda, b, q)$ a multiple of it, namely,

$$(4^*) \qquad g(a, \lambda, b, q) = (-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-\lambda/a)_n a^n}{(q)_n(-bq)_n}$$

and will thereby be able to give a simple, unified and self-contained approach to proving $(\mathrm{I})_R$, (II), $(\mathrm{III})_H$, $(\mathrm{IV})_R$ and $(5)_R$. We may observe that in all the identities (I)–(IV) and in $(1)_R$–$(3)_R$ we may replace G by g. We deduce (I)–(IV) directly (§§2–5) from three easily proved canonical functional relations (6)–(8) for g (§1) and extract $(5)_R$ (§6) as a particular case of the identity (II) = $(\mathrm{IV})_R$ with $\lambda = 0$.

1. Three canonical functional relations satisfied by g.

LEMMA 1 (KEY LEMMA). *If* $|q| < 1$, *then g satisfies the following functional relations*:

$$(6) \qquad g(a, \lambda, b, q) - g(aq, \lambda, b, q) = aqg(aq, \lambda q, bq, q),$$

$$(7) \qquad g(a, \lambda, b, q) - g(a, \lambda q, b, q) = \lambda qg(aq, \lambda q^2, bq, q),$$

$$(8) \qquad g(a, \lambda, b, q) - g(a, \lambda, bq, q) = bqg(aq, \lambda q, bq, q).$$

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PROOF. Since,

$$(-\lambda/a)_n - q^n(-\lambda/aq)_n = \begin{cases} 0 & \text{if } n = 0, \\ (-\lambda/a)_{n-1}(1-q^n) & \text{if } n > 0, \end{cases}$$

as easily verified, we have

$$\begin{aligned} \text{left side of (6)} &= (-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}a^n}{(q)_n(-bq)_n}\left\{\left(-\frac{\lambda}{a}\right)_n - q^n\left(-\frac{\lambda}{aq}\right)_n\right\} \\ &= aq(-bq^2)_\infty \sum_{n=1}^{\infty} \frac{q^{n(n-1)/2}(aq)^{n-1}(-\lambda/a)_{n-1}}{(q)_{n-1}(-bq^2)_{n-1}} \\ &= aqg(aq, \lambda q, bq, q), \quad \text{proving (6).} \end{aligned}$$

In the penultimate step we have used the obvious identity

$$\frac{(-bq)_\infty}{(-bq)_n} = \frac{(-bq^2)_\infty}{(-bq^2)_{n-1}}.$$

Relations (7) and (8) follow in exactly the same way using

$$(-\lambda/a)_n - (-\lambda q/a)_n = \begin{cases} 0 & \text{if } n = 0, \\ \dfrac{\lambda}{a}(-\lambda q/a)_{n-1}(1-q^n) & \text{if } n > 0, \end{cases}$$

and

$$\frac{(-bq)_\infty}{(-bq)_n} - \frac{(-bq^2)_\infty}{(-bq^2)_n} = \frac{(-bq^2)_\infty}{(-bq^2)_n} bq^{n+1}.$$

Lemmas 2–5 proved below are simple combinations of relations (6)–(8). Also, Theorems 1–4 follow directly from Lemmas 2–5, respectively, in a simple manner.

2. Proof of the Ramanujan identity $(\mathrm{I})_{\mathrm{R}}$.

LEMMA 2. *g satisfies*

(9) $$g(a, \lambda, b, q) = g(aq, \lambda q, b, q) + (aq + \lambda q)g(aq, \lambda q^2, bq, q),$$

(10) $$g(a, \lambda, b, q) = g(a, \lambda q, bq, q) + (bq + \lambda q)g(aq, \lambda q^2, bq, q).$$

PROOF. Changing λ to λq in (6) and adding to (7) gives (9), while changing λ to λq in (8) and adding to (7) gives (10).

THEOREM 1. *If* $|q| < 1$, *then*

$(\mathrm{I})_{\mathrm{R}}$ $$\rho = \frac{1}{1}\,+\,\frac{aq + \lambda q}{1}\,+\,\frac{bq + \lambda q^2}{1} + \cdots \frac{aq^{n+1} + \lambda q^{2n+1}}{1}\,+\,\frac{bq^{n+1} + \lambda q^{2n+2}}{1} + \cdots$$

where

$$\rho = \frac{G(aq, \lambda q, b, q)}{G(a, \lambda, b, q)} = \frac{g(aq, \lambda q, b, q)}{g(a, \lambda, b, q)}.$$

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PROOF. Changing a to aq^n, λ to λq^{2n}, b to bq^n in (9) and changing a to aq^{n+1}, λ to λq^{2n+1} and b to bq^n in (10) we can write (9) and (10), respectively, as

$$Q_n \equiv \frac{g(aq^n, \lambda q^{2n}, bq^n, q)}{g(aq^{n+1}, \lambda q^{2n+1}, bq^n, q)} = 1 + \frac{aq^{n+1} + \lambda q^{2n+1}}{Q'_n},$$

$$Q'_n \equiv \frac{g(aq^{n+1}, \lambda q^{2n+1}, bq^n, q)}{g(aq^{n+1}, \lambda q^{2n+2}, bq^{n+1}, q)} = 1 + \frac{bq^{n+1} + \lambda q^{2n+2}}{Q_{n+1}}.$$

Iterating the last two identities alternately with $n = 0, 1, 2, \ldots,$ we have $(\mathrm{I})_\mathrm{R}$. Convergence of the continued fraction follows since Q_n, $Q'_n \to 1$ as $n \to \infty$ when $|q| < 1$.

3. Proof of identity (II).

LEMMA 3. *g satisfies*

$$g(a, \lambda, b, q) = g(aq, \lambda q, b, q) + (aq + \lambda q) g(aq, \lambda q^2, bq, q), \tag{11}$$

$$g(aq, \lambda, b, q) = (1 - aq + bq) g(aq, \lambda q, bq, q) + (aq + \lambda q) g(aq, \lambda q^2, bq^2, q). \tag{12}$$

PROOF. Changing λ to λq in (6) and adding it to (7), we have (11). Changing λ to λq and b to bq in (6), b to bq in (7), taking the negative of (6) and adding these three equalities to (8), we deduce (12).

THEOREM 2. *If* $|q| < 1$, *then*

$$\rho = \frac{1}{1} + \frac{aq + \lambda q}{1 - aq + bq} + \cdots \frac{aq + \lambda q^n}{1 - aq + bq^n} + \cdots \tag{II}$$

where ρ *is as in Theorem* 1.

PROOF. (11) can be written as

$$\frac{g(aq, \lambda q, b, q)}{g(a, \lambda, b, q)} = \frac{1}{1} + \frac{aq + \lambda q}{\dfrac{g(aq, \lambda q, b, q)}{g(aq, \lambda q^2, bq, q)}}. \tag{13}$$

Changing λ to λq^{n+1} and b to bq^n, (12) can be written as

$$S_n \equiv \frac{g(aq, \lambda q^{n+1}, bq^n, q)}{g(aq, \lambda q^{n+2}, bq^{n+1}, q)} = (1 - aq + bq^{n+1}) + \frac{aq + \lambda q^{n+2}}{S_{n+1}}.$$

Iterating this with $n = 0, 1, 2, \ldots,$ and using (13) we have (II). Convergence of (II) follows since $S_n \to 1$ as $n \to \infty$ when $|q| < 1$.

4. Proof of identity $(\mathrm{III})_\mathrm{H}$ of Hirschhorn.

LEMMA 4. *g satisfies*

$$g(a, \lambda, bq, q) = (1 - bq + aq) g(aq, \lambda q, bq, q) + (bq + \lambda q) g(aq^2, \lambda q^2, bq, q). \tag{14}$$

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PROOF. Changing a to aq in (7), a to aq and λ to λq in (8), taking the negative of (8) and adding these three equalities to (6), we deduce (14).

THEOREM 3. *If* $|q| < 1$, *then*

$$(\mathrm{III})_{\mathrm{H}} \qquad \rho = \frac{1}{1 - b + aq} + \frac{b + \lambda q}{1 - b + aq^2} + \cdots \frac{b + \lambda q^n}{1 - b + aq^{n+1}} + \cdots$$

where ρ *is as in Theorems* 1 *and* 2.

PROOF. Changing a to aq^n, λ to λq^n and b to b/q, (14) can be written as

$$T_n \equiv \frac{g(aq^n, \lambda q^n, b, q)}{g(aq^{n+1}, \lambda q^{n+1}, b, q)} = (1 - b + aq^{n+1}) + \frac{b + \lambda q^{n+1}}{T_{n+1}}.$$

Iterating this with $n = 0, 1, 2, \ldots$, we have $(\mathrm{III})_{\mathrm{H}}$. Convergence of $(\mathrm{III})_{\mathrm{H}}$ follows as in the proof of Theorem 2.

5. Proof of the Ramanujan identity $(\mathrm{IV})_{\mathrm{R}}$.

LEMMA 5. *g satisfies*

$$(15) \qquad g(a, \lambda, bq, q) = (1 + aq)g(aq, \lambda q, bq, q) + (\lambda q - abq^3)g(aq^2, \lambda q^2, bq^2, q),$$

$$(16) \qquad g(aq, \lambda q, b, q) = \{1 + q(aq + b)\}g(aq^2, \lambda q^2, bq, q) + (\lambda q^2 - abq^4)g(aq^3, \lambda q^3, bq^2, q).$$

PROOF. Change a to aq, λ to λq, b to bq in (6) and multiply the result by $-bq$; change a to aq, b to bq in (7); change a to aq in (8); take the negative of (8) and add all these to (6) to obtain (15). Change a to aq, λ to λq in (15); change a to aq and λ to λq in (8) and add to get (16).

THEOREM 4. *If* $|q| < 1$, *then*

$$(\mathrm{IV})_{\mathrm{R}} \qquad \rho = \frac{1}{1 + aq} + \frac{\lambda q - abq^2}{1 + q(aq + b)} + \cdots \frac{\lambda q^n - abq^{2n}}{1 + q^n(aq + b)} + \cdots$$

where ρ *is as in Theorems* 1–3.

PROOF. Changing b to b/q, (15) can be written as

$$(17) \qquad \frac{g(aq, \lambda q, b, q)}{g(a, \lambda, b, q)} = \frac{1}{1 + aq + \dfrac{\lambda q - abq^2}{\dfrac{g(aq, \lambda q, b, q)}{g(aq^2, \lambda q^2, bq, q)}}}.$$

Changing a, λ and b to aq^{n-1}, λq^{n-1} and bq^{n-1}, respectively, (16) can be written as

$$(18) \quad U_n \equiv \frac{g(aq^n, \lambda q^n, bq^{n-1}, q)}{g(aq^{n+1}, \lambda q^{n+1}, bq^n, q)} = 1 + q^n(aq + b) + \frac{\lambda q^{n+1} - abq^{2n+2}}{U_{n+1}}.$$

Iterating (18) with $n = 1, 2, \ldots$, and using (17) we have $(\mathrm{IV})_{\mathrm{R}}$. Convergence of $(\mathrm{IV})_{\mathrm{R}}$ follows since $U_n \to 1$ as $n \to \infty$ when $|q| < 1$.

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6. Proof of the Ramanujan identity $(5)_{\mathrm{R}}$. The following theorem is a corollary to §§3 and 5. In fact, it is a particular case of the identity (II) = $(\mathrm{IV})_{\mathrm{R}}$.

THEOREM 5. *If* $|q| < 1$, *then*

$$(5)_{\mathrm{R}} \quad \frac{1}{a+c}\,\frac{-}{}\,\frac{ab}{a+b+cq}\,\frac{-}{}\cdots\frac{ab}{a+b+cq^n}\,\frac{-}{}\cdots$$

$$= \frac{1}{c-b+a}\,\frac{+}{}\,\frac{bc}{c-b+a/q}\,\frac{+}{}\cdots\frac{bc}{c-b+a/q^n}\,\frac{+}{}\cdots.$$

PROOF. Changing λ to 0, a to $-b/aq$ and b to c/a in (II) = $(\mathrm{IV})_{\mathrm{R}}$ and taking reciprocal we have

$$(19) \quad \frac{g(-b/aq,0,c/a,q)}{g(-b/a,0,c/a,q)}$$

$$= 1 + \frac{-b/a}{1+(b+cq)/a}\,\frac{+}{}\cdots\frac{-b/a}{1+(b+cq^n)/a}\,\frac{+}{}\cdots$$

$$= \left(1-\frac{b}{a}\right) + \frac{bcq/a^2}{1+q(c-b)/a}\,\frac{+}{}\cdots\frac{bcq^{2n-1}/a^2}{1+q^n(c-b)/a}\,\frac{+}{}\cdots.$$

Multiplying (19) by a throughout and adding c throughout we have

$$(20) \quad c + \frac{ag(-b/ag,0,c/a,q)}{g(-b/a,0,c/a,q)}$$

$$= a + c + \frac{-ab}{a+b+cq}\,\frac{+}{}\cdots\frac{-ab}{a+b+cq^n}\,\frac{+}{}\cdots$$

$$= a + c - b + \frac{bcq}{a+(c-b)q}\,\frac{+}{}\cdots\frac{bcq^{2n-1}}{a+(c-b)q^n}\,\frac{+}{}\cdots.$$

We complete the proof by taking the reciprocal of (20) throughout. In addition to proving $(5)_{\mathrm{R}}$ we have thus obtained that each side of $(5)_{\mathrm{R}}$ equals

$$\left\{c + \frac{ag(-b/aq,0,c/a,q)}{g(-b/a,0,c/a,q)}\right\}^{-1}.$$

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