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Normal Subgroups of Triply Transitive Permutation Groups of Degree Divisible by 3

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§ 1. Introduction

Jordan [5] proved in 1873 the following well-known theorem on normal subgroups of multiply transitive permutation groups:

Let G be a t-fold transitive permutation group on a finite set Ω, $|\Omega|=n$, $t\geqq 2$ and G not the symmetric group on Ω. If $H\neq 1$ is a non-regular normal subgroup of G then H is $(t-1)$-fold transitive on Ω.

This result has been refined considerably by a number of authors. Wielandt and Huppert [14] introduced the notion of multiple primitivity and $(t+\frac{1}{2})$-fold transitivity. Wagner [13] showed that H is in fact t-fold transitive if $n-t$ is an even number and $t\geqq 3$, and Saxl [8] proved that this is also true if $t\geqq 4$ and $n\leqq 10^6$. The only known examples of triply transitive groups with only doubly transitive normal subgroup are the projective linear groups $PGL(2,q)$ in their usual representation on the projective line. For q odd $PSL(2,q)$ is only doubly transitive on Ω and has exactly two orbits on $\Omega^{(3)}$ where

$$\Omega^{(3)}=\{(\alpha,\beta,\gamma)\,|\,\alpha,\beta,\gamma \text{ distinct in } \Omega\}.$$

In this paper we shall prove the following:

Theorem A. *Let G be a triply transitive permutation group on Ω where $|\Omega|$ is divisible by 3. Then every normal subgroup $H\neq 1$ of G has at most two orbits on $\Omega^{(3)}$. If in addition $|\Omega|\not\equiv 2 \bmod 4$, then H is either triply transitive or $PSL(2,q)\subseteq H\subseteq G\subseteq P\Gamma L(2,q)$ where $q=n-1$.*

Theorem B. *Let G be a triply transitive permutation group on Ω where $|\Omega|$ is divisible by 3. Suppose G_α contains a normal subgroup $M\neq 1$ such that $M_{\beta,\gamma}$ has order prime to 3 for three distinct points α, β and γ in Ω. Then G is isomorphic to a subgroup of $P\Gamma L(2,q)$ containing $PSL(2,q)$ where $q=n-1$.*

This paper is organized in the following way: In Sect. 2 we state a classification theorem of Hering, Kantor and Seitz, and Thompson's result on simple 3′-groups which lead to a straightforward proof of Theorem B. The main work is done in Sect. 3 where we develop an inductive construction. This section

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contains a theorem on the generosity of normal subgroups (3.3) and extensions of Wagner's result (3.6, 3.8).

1.1. *Notation and Definitions.* They are the standard definitions in Wielandt's book [15]. Ω is a finite set of n points $\alpha, \beta, \gamma, \ldots$, and $\Gamma \subseteq \Delta \subseteq \Omega$ denote subsets. G is a permutation group on Ω; $G_{\alpha,\beta}, \ldots$ and G_Γ denote the pointwise stabilizers of $\alpha, \beta, \ldots$ and Γ in G. $G_{\{\Gamma\}}$ is the setwise stabilizer and $G^\Gamma := G_{\{\Gamma\}}/G_\Gamma$ is always considered as a permutation group on Γ. We shall use $\mathrm{Sym}\,\Omega$ and $\mathrm{Alt}\,\Omega$ to denote the symmetric and alternating groups on Ω. For any $k \leqq n$ let $\Omega^{\{k\}}$ be the set of all subsets of Ω of size k and $\Omega^{(k)}$ the set of all ordered sequences of k distinct points. Note that G induces permutation groups on $\Omega^{\{k\}}$ and $\Omega^{(k)}$ in the obvious way. G is t-fold transitive and t-fold homogeneous on Ω if G acts transitively on $\Omega^{(t)}$ and $\Omega^{\{t\}}$ respectively. G is said to be $(t-1)$-fold generously transitive on Ω if $G^\Gamma = \mathrm{Sym}\,\Gamma$ for any Γ in $\Omega^{\{t\}}$. Let B be a subset of $\Omega^{\{k\}}$ for some $k \leqq n$, and $B_\Gamma = \{\Delta \mid \Delta \in B,\ \Gamma \subseteq \Delta\}$ for Γ in $\Omega^{\{t\}}$, $t \leqq k$. Then (Ω, B) is a design with parameters $t-(|\Omega|, |\Delta|, |B_\Gamma|)$ if $|B_\Gamma| \geqq 1$ is independent of Γ in $\Omega^{\{t\}}$.

Let U, H be subgroups of G. Then U^H is the set of all H-conjugates U^h of U and $\mathrm{Syl}_p(H)$ the set of all Sylow p-subgroups of H. The set of all points of Ω fixed by U is denoted by $\mathrm{Fix}\,U$.

§ 2. Preliminaries

We shall frequently use Jordan's result and other basic facts about permutation groups for which Wielandt's book [15] is a good reference.

Lemma 2.1. *Let G be triply transitive on Ω and suppose G has a solvable normal subgroup $H \neq 1$ which is not regular on Ω. Then $|\Omega| = 3$ or 4 and $G = \mathrm{Sym}\,\Omega$.*

Proof. Let $M \neq 1$ be a minimal normal subgroup of G contained in H. Since H is solvable, M is elementary abelian for order p^m and regular on Ω.

Therefore $|\Omega| = p^m = 3$ or 2^m. Suppose $|\Omega| = 2^m$ and let $F \neq 1$ be a minimal normal subgroup of G_α contained in H_α. Then also F is elementary abelian of order q^r and regular on $\Omega \backslash \{\alpha\}$. Hence $|F| = q^r = 2^m - 1$. Since r must be odd, the second factor in $2^m = (q+1)(q^{r-1} - q^{r-2} + \ldots + 1)$ is 1 and therefore $r = 1$. So $|F| = q$ and since $G_{\alpha,\beta}$ is a complement of F in $G_\alpha = F \cdot G_{\alpha,\beta}$, $G_{\alpha,\beta}$ is cyclic as a group of automorphism of F and this implies that $G_{\alpha,\beta}$ is regular on $\Omega \backslash \{\alpha, \beta\}$. Hence G is sharply triply transitive on Ω of order $2^m \cdot (2^m - 1) \cdot (2^m - 2)$ where $2^m - 1 = q$ is a prime. Since $|G|$ is divisible by 3, we either have $2^m - 1 = 3$, $|\Omega| = 4$ or otherwise 3 divides $2^m - 2$. The latter cannot happen. For let $S \neq 1$ be a maximal subgroup of $G_{\alpha,\beta}$ with order prime to 3. Note that $|S|$ is divisible by 2 since $|G_{\alpha,\beta}| = 2^m - 2$ is even. The group $(M \cdot F) \cdot S$ is normal in $G = G_{\alpha,\beta} \cdot (M \cdot F)$ and has order prime to 3. Let $i = (\alpha)(\beta)(\gamma, \delta)\ldots$ be an involution in S and conjugate i by an appropriate g in G such that $i' = i^g = (\delta)(\beta, \gamma)\ldots$ is contained in $(M \cdot F) \cdot S$. Then also $i \cdot i' = (\beta\gamma\delta)\ldots$ is contained in $(M \cdot F) \cdot S$, a contradiction. Hence $|\Omega| = 3$ or 4. ■

Lemma 2.2 (Bender [2]). *Let G be a doubly transitive permutation group on Ω. Suppose the stabilizer of one point in Ω has odd order. Then G is either solvable or else G contains a normal subgroup isomorphic to $PSL(2, q)$.*

Lemma 2.3 (Suzuki [11]. *Let G be doubly transitive on Ω with no regular normal subgroup such that G_α contains a regular characteristic p-subgroup where α is a point in Ω and p some prime. Then G has no transitive extension unless $|\Omega|=5$ and $G \geqq \mathrm{Alt}(5)$ or $|\Omega|=10$ and G is sharply triply transitive with extension M_{11}, the Mathieu group on 11 points.*

Lemma 2.4 (Hering, Kantor and Seitz [3]). *Let G be a finite doubly transitive permutation group on Ω. Suppose that for a point α in Ω G_α has a normal subgroup regular on $\Omega \backslash \{\alpha\}$. Then G contains a normal subgroup M such that $M \subseteq G \subseteq \mathrm{Aut}\, M$ and M acts on Ω as one of the following groups in its usual doubly transitive representation: a sharply doubly transitive group, $PSL(2,q)$, $Sz(2^{2r+1})$, $PSU(3,q)$ or a group of Ree type.*

Lemma 2.5 (Martineau [6], Thompson [12]). *Let G be a non-abelian finite simple group of order prime to 3. Then G is isomorphic to a Suzuki group $Sz(2^{2r+1})$.*

(This result now is most easily accessible in Glauberman: Factorizations in local subgroups of finite groups. Regional Conference Series in Mathematics No. 33 1977. It is contained in Corollary 7.3 on p. 48, which in turn is based on Goldschmidt: 2-Fusion in finite groups. Ann. of Math. (2) **99**, 70–117 (1974).)

§ 3. Induction for Normal Subgroups

3.1. Let G be a permutation group on Ω of degree n, G not the symmetric group $\mathrm{Sym}\,\Omega$. We will suppose G is t-fold homogeneous or t-fold transitive on Ω for some fixed integer $t \geqq 1$. So G acts transitively on $\Omega^{\{t\}}$ or $\Omega^{(t)}$. Let $H \neq 1$ be a subgroup normal in G. A convenient measure for the drop in transitivity from G to H are the numbers of H-orbits on $\Omega^{\{t\}}$ and $\Omega^{(t)}$. Let therefore $\{U_i | U_i \subseteq \Omega^{\{t\}}\}$ be the set of H-orbits on $\Omega^{\{t\}}$ and $\{O_i | O_i \subseteq \Omega^{(t)}\}$ the set of H-orbits on $\Omega^{(t)}$. Define $y(H) := |\{U_i\}|$ and $x(H) := |\{O_i\}|$.

Let Δ be a subset of Ω of size at least t and let H^Δ, G^Δ be the groups induced on Δ. In order to relate the transitivity on Δ to the transitivity on Ω, define similar parameters $y(H^\Delta)$ and $x(H^\Delta)$: Let $\{(U^\Delta)_i | (U^\Delta)_i \subseteq \Delta^{\{t\}}\}$ be the orbits of $H_{\{\Delta\}}$ (or H^Δ) on $\Delta^{\{t\}}$, $y(H^\Delta) = |\{(U^\Delta)_i\}|$, and $\{(O^\Delta)_i | (O^\Delta)_i \subseteq \Delta^{(t)}\}$ the orbits of $H_{\{\Delta\}}$ (or H^Δ) on $\Delta^{(t)}$, $x(H^\Delta) = |\{(O^\Delta)_i\}|$. Let $B := \Delta^G$ be the set of all G-images of Δ and $\{B_i | B_i \subseteq B\}$ the set of H-orbits on B. Put $z(H, \Delta) := |\{B_i\}|$ and $B_\Gamma = \{\Delta | \Delta \in B, \Gamma \subseteq \Delta\}$ for any set Γ of size t. As an easy consequence of these definitions we have:

Lemma 3.1. *Let G be t-fold homogeneous on Ω, let $\Gamma' \subset \Gamma$ be subsets of size $t-1$ and t respectively and let $\Delta^* \supseteq \Gamma$ with $B = \Delta^{*G}$. Then $x(H^\Delta)$, $y(H^\Delta)$ and $|B_{\Gamma^*}|$ are independent of the particular choice of Δ in B and Γ^* in $\Omega^{\{t\}}$. Therefore (Ω, B) is a design with parameters $t-(|\Omega|, |\Delta^*|, |B_\Gamma|)$. If G is t-fold transitive on Ω, $x(H) = y(H) \cdot x(H^\Gamma)$. If H is at least $(t-1)$-fold transitive on Ω, $x(H)$ is the number of $H_{\Gamma'}$-orbits on $\Omega \backslash \Gamma'$ and so $x(H)$ divides $(n-t+1)$. If in addition $H_{\{\Gamma\}} \neq H_\Gamma$, then $y(H)$ is the number of the $H_{\{\Gamma'\}}$-orbits on $\Omega \backslash \Gamma'$.*

Proof. Since $\Delta^* = \Delta^g$ for some g in G, $H_{\{\Delta\}}$ and $H_{\{\Delta^*\}}$ are conjugate in G and this implies $y(H^\Delta) = y(H^{\Delta^*})$, $x(H^\Delta) = x(H^{\Delta^*})$. Since G is t-fold homogeneous, $\Gamma^* = \Gamma^g$ for some g in G and so $|B_\Gamma| = |B_{\Gamma^*}|$. Suppose G is t-fold transitive on Ω. Then $x(H) = [G:G_\Gamma]: [H:H_\Gamma] = ([G:G_{\{\Gamma\}}]: [H:H_{\{\Gamma\}}]) \cdot ([G_{\{\Gamma\}}:G_\Gamma]: [H_{\{\Gamma\}}:H_\Gamma]) = y(H) \cdot [G^\Gamma : H^\Gamma]$. Since $G^\Gamma = \operatorname{Sym}\Gamma$ and $x(H^\Gamma) = [\operatorname{Sym}\Delta : H^\Gamma]$, we have $x(H) = y(H) \cdot x(H^\Gamma)$. If H is $(t-1)$-fold transitive we have $G = G_{\Gamma'} \cdot H$ *and so* $x(H) = [G:G_\Gamma): [H:H_\Gamma] = [G_{\Gamma'}:G_\Gamma]: [H_{\Gamma'}:H_\Gamma] = (n-t+1): [H_{\Gamma'}:H_\Gamma]$ where $[H_{\Gamma'}:H_\Gamma]$ is the length of the $H_{\Gamma'}$-orbit containing $\gamma = \Gamma \backslash \Gamma'$. If $H_{\{\Gamma\}} \neq H_\Gamma$, $H_{\{\Gamma\}}$ is transitive on Γ since $H_{\{\Gamma\}} \trianglelefteq G_{\{\Gamma\}}$. Therefore $|H_{\{\Gamma'\},\gamma}| = |H_{\{\Gamma\},\gamma}| = |H_{\{\Gamma\}}| \cdot t^{-1}$. Let $l = [H_{\{\Gamma'\}} : H_{\{\Gamma'\},\gamma}]$ be the length of the $H_{\{\Gamma'\}}$-orbit containing γ. Then $l = (|H^{\Gamma'}| \cdot |H_{\Gamma'}|): (|H^\Gamma| \cdot |H_\Gamma| \cdot t^{-1}) = t! \cdot (n-t+1): (|H^\Gamma| \cdot x(H)) = ((n-t+1): x(H)) \cdot x(H^\Gamma)$. Therefore $y(H) = (n-t+1) \cdot l^{-1}$ is the number of $H_{\{\Gamma'\}}$-orbits on $\Omega \backslash \Gamma'$. ■

The following proposition allows us to discuss the transitivity properties of H in terms of its transitivity on designs (Ω, B):

Proposition 3.2. *Let G be a t-fold homogeneous group on Ω of degree n, $1 \leq t \leq n$ and $H \neq 1$ a normal subgroup of G. Let Γ be a set of size t and $\Gamma \subseteq \Delta \subseteq \Omega$ with $B = \Delta^G$. If $H_{\{\Gamma\}}$ acts transitively on B_Γ, then G^Δ is t-fold homogeneous on Δ and $y(H) = z(H, \Delta) \cdot y(H^\Delta)$. If G is t-fold transitive on Ω and if H_Γ acts transitively on B_Γ, then G^Δ is t-fold transitive on Δ and $x(H) = z(H, \Delta) \cdot x(H^\Delta)$.*

This shows that H is t-fold homogeneous (transitive) on Ω if and only if H is transitive on the set of blocks of (Ω, B) and t-fold homogeneous (transitive) on each block.

Proof. To prove the first part of 3.2 we choose a set Δ_i in each H-orbit B_i on B, $i = 1, \ldots, z = z(H, \Delta)$. Choose in each Δ_i $y' = y(H^{\Delta_i})$ sets Γ_{ij}, $j = 1, \ldots, y'$, such that Γ_{ij} belongs to $(U^{\Delta_i})_j$. By Lemma 3.1 $y(H^{\Delta_i}) = y(H^\Delta)$ and so we obtain exactly $z \cdot y(H^\Delta)$ sets Γ_{ij} and we only need to show that each H-orbit on $\Omega^{\{t\}}$ contains exactly one of these sets.

Suppose $\Gamma^h = \Gamma'$ for some h in H where $\Gamma := \Gamma_{ij} \subseteq \Delta_i =: \Delta$ and $\Gamma' := \Gamma_{i'j'} \subseteq \Delta_{i'} =: \Delta'$. Then both Δ^h and Δ' contain Γ' and since our assumption implies that $H_{\{\Gamma'\}}$ is transitive on $B_{\Gamma'}$, we conclude that there is some k in $H_{\{\Gamma'\}}$ such that $\Delta^{hk} = \Delta'$. This implies that Δ' and Δ belong to the same H-orbit on B and so $i = i'$ and $\Delta = \Delta'$. Therefore $h \cdot k$ is contained in $H_{\{\Delta\}}$ and since $\Gamma^{h \cdot k} = \Gamma'^k = \Gamma'$, Γ and Γ' belong to the same $H_{\{\Delta\}}$-orbit on $\Delta^{\{t\}}$. Thus $j = j'$, $\Gamma' = \Gamma$ and each H-orbit on $\Omega^{\{t\}}$ contains at most one of the Γ_{ij}'s. On the other hand, any Γ^* in $\Omega^{\{t\}}$ is contained in some $\Delta^* \in B$ since G is t-fold homogeneous and this implies that Γ^* is an H-image of one of the Γ_{ij}'s. Hence $y(H) = z(H, \Delta) \cdot y(H^\Delta)$. We evaluate this equation for $H = G$ and $1 = y(G) = z(G, \Delta) \cdot y(G^\Delta)$ implies $y(G^\Delta) = 1$, i.e. G^Δ is t-fold homogeneous on Δ and the same is true for any other Δ' in B. The second part of 3.2 may be proved in very much the same way by considering ordered t-sequences (Γ_{ij}) in $\Delta_i^{(t)}$. ■

3.2. Suppose G is t-fold transitive on Ω and let Δ be a subset of Ω as in proposition 3.2. Then G^Δ is t-fold transitive on Δ but H^Δ does not necessarily inherit the transitivity properties of H on Ω.

Definition. Let G be t-fold transitive on Ω, $1 \leqq t \leqq |\Omega|$ and let $H \neq 1$ be normal in G. Let Γ be a subset of Ω of size t and $\Delta \supseteq \Gamma$. Then Δ is *inductive with respect to* H if the following conditions hold:

i) H_Γ permutes $B_\Gamma = \{\Delta^g | g \in G, \Gamma \subseteq \Delta^g\}$ transitively, and

ii) If H is t'-fold transitive on Ω, $1 \leqq t' \leqq t$, then H^Δ is t'-fold transitive on Δ.

We shall show that certain types of subsets of Ω are inductive. First suppose $G \neq \operatorname{Sym}\Omega$ and $\Delta = \Gamma$ is any set of size t. Then clearly condition i) is satisfied. If $t = 3$ and H is regular, $|\Omega| = |H|$ is a power of 2 and so $H^\Delta = 1$, illustrating the definition. Apart from this case H is at least $(t-1)$-fold transitive and therefore Δ is inductive if and only if $H^\Delta = \operatorname{Sym}\Delta$, i.e. H is generously $(t-1)$-fold transitive on Ω. We prove the following generosity theorem originally proposed by Ito [4] for quadruply transitive groups, see also Neumann (Theorem 9.1 in [7]) and Saxl [8].

Theorem 3.3. *Let G be a t-fold transitive permutation group on Ω of degree n, $G \neq \operatorname{Sym}\Omega$ and $t \geqq 2$. Suppose $H \neq 1$ is a normal subgroup of G.*

i) *If $t = 2$, H is generously transitive if and only if H has even order. In particular H is generously transitive if G contains no regular normal subgroup.*

ii) *If $t = 3$, H is generously 2-fold transitive except if H is regular or if $PSL(2, q) \subseteq H \subseteq G \subseteq P\Gamma L(2, q)$ in their usual representation on the projective line, $q + 1 = n \equiv 0 \bmod 4$.*

iii) *If $t \geqq 4$, H is generously $(t-1)$-fold transitive.*

Proof. Let Γ be a set of size t. Since G is t-fold transitive, $G_{\{\Gamma\}}$ acts on Γ like $\operatorname{Sym}\Gamma$. To show that H is $(t-1)$-fold generously transitive, we have to prove the same is true for $H_{\{\Gamma\}}$. Since $H_{\{\Gamma\}}$ is normal in $G_{\{\Gamma\}}$, it suffices to show that $H_{\{\Gamma\}}$ contains an element h which acts on Γ like a transposition. Since G is also t-fold homogeneous on Ω one may choose a particular Γ to show the required property. Let therefore Δ be a subset of Ω with $|\Delta| = t - 2$. Then H is $(t-1)$-fold generously transitive on Ω if and only if H_Δ has even order. For assume that $|H_\Delta|$ is even. Then there is some element h in H_Δ interchanging two points α and β in $\Omega \setminus \Delta$. Take $\Gamma = \Delta \cup \{\alpha, \beta\}$ and h acts on Γ like a transposition. The converse implication is trivial.

(a) If $t = 2$, $\Delta = \varnothing$ and (i) is proved if we can show, that H has even order if G contains no regular normal subgroup. Let M be a minimal normal subgroup of G contained in H. By a result of Burnside, M is simple and by the Feit-Thompsom Theorem H has even order.

(b) Now let $t = 3$. Then H is either regular or doubly transitive. If H is regular, $H_\alpha = 1$ and by the above remark H is certainly not generously doubly transitive. So suppose H is doubly transitive. Put $\{\alpha\} = \Delta$. Then we have to show that H_α has even order if G is not contained in $P\Gamma L(2, n-1)$. Hence assume H_α has odd order. Then a theorem of Bender, Lemma 2.2, applies and H is either solvable or otherwise contains $PSL(2, q)$ as a normal subgroup for some prime power q. Lemma 2.1 implies that H cannot be solvable unless $G = \operatorname{Sym}(3)$ or $\operatorname{Sym}(4)$ which contradicts our assumption. Hence $PSL(2, q) \subseteq H$ and using results of Burnside one concludes that $PSL(2, q)$ is characteristic in H and hence normal in G. Therefore $PSL(2, q)$ is doubly transitive on Ω and checking through Dickson's list of subgroups of $PSL(2, q)$ we find that $PSL(2, q)$ acts on the projective line $PG(1, q)$ in its usual representation. Hence $q+1=n$,

$$PSL(2, n-1) \subseteq H \subseteq G \subseteq P\Gamma L(2, n-1)$$

and $n \equiv 0 \bmod 4$ since $|H_\alpha|$ is odd.

(c) Now suppose $t \geqq 4$. Since $G \neq \mathrm{Sym}\,\Omega$, H is at least $(t-1)$-fold transitive on Ω. Let Γ' be a subset of Ω with $|\Gamma'|=t-3$. Then $G_{\Gamma'}$ is triply transitive on $\Omega \setminus \Gamma'$ and at the beginning of the proof we saw that H is $(t-1)$-fold generously transitive on Ω if and only if $H_{\Gamma'}$ is doubly transitive on $\Omega \setminus \Gamma'$. This is the case if $H_{\Gamma'}$ is not one of the exceptions in (ii). But $H_{\Gamma'}$ cannot be regular on $\Omega \setminus \Gamma'$, because then H could only be $(t-2)$-fold transitive on Ω. Similarly if

$$PSL(2, n-|\Gamma'|-1) \subseteq H_{\Gamma'} \subseteq G_{\Gamma'} \subseteq P\Gamma L(2, n-|\Gamma'|-1),$$

as permutation groups on $\Omega \setminus \Gamma'$, let Γ^* be a subset of Γ' with $|\Gamma^*|=|\Gamma'|-1$. Then G_{Γ^*} and H_{Γ^*} are transitive extensions of $G_{\Gamma'}$ and $H_{\Gamma'}$. Since $n>t+2$, $n-|\Gamma'|>5$ and therefore by Lemma 2.3, $G_{\Gamma^*}=M_{11}$, the Mathieu group on 11 points. Since M_{11} is simple, $H_{\Gamma^*}=M_{11}$ is quadruply transitive on $\Omega \setminus \Gamma^*$. In particular H_{Γ^*} is 3-fold generously transitive on $\Omega \setminus \Gamma^*$ and hence $H_{\Gamma'}$ is doubly generously transitive on $\Omega \setminus \Gamma'$. ■

A second type of inductive sets arises from subgroups of H_Γ where Γ is in $\Omega^{\{t\}}$. Let p be some prime and U a p-subgroup of H_Γ with $\Delta := \mathrm{Fix}\,U$. If U is a Sylow p-subgroup of $H_{\Gamma'}$ then H_Γ acts transitively on $B_\Gamma=\{\Delta^g \mid \Delta^g \supseteq \Gamma,\ g \in G\}$ by Sylow's theorem and the same is true if U is a G-strongly closed subgroup of a Sylow p-subgroup of H_Γ.

Proposition 3.4. *Let G be a t-fold transitive group on Ω, $2 \leqq t$, $G \neq \mathrm{Sym}\,\Omega$ and let $H \neq 1$ be a normal subgroup of G. Suppose $\Gamma \subset \Omega$ is a set of size t and p some prime. Let U be a p-subgroup of H_Γ with $\Delta := \mathrm{Fix}\,U$ such that H_Γ acts transitively on B_Γ. Then Δ is inductive with respect to G and H.*

Proof. G^Δ is t-fold transitive on Δ by Proposition 3.2 and so Δ is inductive with respect to G. The same argument applies if H is t-fold transitive on Ω. If H is regular, $\Delta=\Omega$ is inductive with respect to H. Hence assume H is exactly $(t-1)$-fold transitive on Ω and we show that then H^Δ is $(t-1)$-fold transitive on Δ. Since $H^\Delta=H_{\{\Delta\}}/H_\Delta \cong H_{\{\Delta\}} \cdot G_\Delta/G_\Delta \trianglelefteq G^\Delta$, we consider H^Δ as a normal subgroup of G^Δ. Therefore H^Δ is either a) $(t-1)$-fold transitive on Δ, b) regular on Δ and $t=3$, c) $G^\Delta=\mathrm{Sym}\,\Delta$, $H^\Delta=\mathrm{Alt}\,\Delta$ and $|\Delta|=t$ or d) $H^\Delta=1$. Let $\Gamma' \subset \Gamma$ be a set of size $t-1$. If $H_{\Gamma'}$ should be transitive on $B_{\Gamma'}=\{\Delta^g \mid \Gamma' \subset \Delta^g,\ g \in G\}$, we use Proposition 3.2 (replacing G by H and t by $t-1$) to show that H^Δ is $(t-1)$-fold transitive on Δ. Hence assume that $H_{\Gamma'}$ acts not transitively on $B_{\Gamma'}$. Let S be in $\mathrm{Syl}_p(H_{\Gamma'})$ containing U. Then $U \neq S$ and the same is true for any subgroup $T \subseteq S$ with $\mathrm{Fix}\,T=\Delta$. Let therefore T be maximal in S with $\mathrm{Fix}\,T=\Delta$ and let h be in $N_S(T) \setminus T \neq \varnothing$. Then h is in $H_{\{\Delta\}} \cap H_{\Gamma'}$ but not in H_Δ. This implies $|\Delta| \geqq t-1+p>t$ and $(H^\Delta)_{\Gamma'} \neq 1$. Hence b)–d) do not occur and the proposition is proved. ■

Let G be t-fold transitive on Ω and H $(t-1)$-fold transitive. Suppose U and Δ satisfy the hypotheses of 3.4 such that U is maximal with $\Delta=\mathrm{Fix}\,U$, i.e. $U \in \mathrm{Syl}_p(H_\Delta)$. Then $G^\Delta \trianglerighteq H^\Delta$ are t- and $(t-1)$-fold transitive groups on Δ respectively. Note that $|\Delta| \equiv |\Omega| \bmod p$. Let $N=N_G(U) \subseteq G_{\{\Delta\}}$ and $M=N \cap H$. Then

$G_{\{\Delta\}} = G_\Delta \cdot N$ and $H_{\{\Delta\}} = H_\Delta \cdot M$ by the Frattini argument. Hence $G^\Delta = N \cdot G_\Delta / G_\Delta \cong N / N \cap G_\Delta = N^\Delta$ and $H^\Delta = M \cdot H_\Delta / H_\Delta \cong M^\Delta$. Our assumption is in particular true if U is a Sylow p-subgroup of H_Γ. In this case the last remark implies that $(H^\Delta)_{\Gamma^*}$ is a p'-group for every $\Gamma^* \subseteq \Delta$ of size t. By 3.2 we obtain $x(H) = z(H, \Delta) \cdot x(H^\Delta)$ and we calculate $z = z(H, \Delta)$. By definition

$$z = [G : G_{\{\Delta\}}] : [H : H_{\{\Delta\}}] = [G : (H \cdot G_{\{\Delta\}})] = [G : H \cdot N] = |U^G| : |U^H|.$$

Suppose $U \in \mathrm{Syl}_p(H_\Gamma)$ and p does not divide $(n-t+1)/x(H)$. Then, if $\Gamma' \subset \Gamma$ has size $t-1$, U is contained in $\mathrm{Syl}_p(H_{\Gamma'})$ since $[H_{\Gamma'} : H_\Gamma] = (n-t+1)/x(H)$ by 3.1. By the Frattini argument $G_{\Gamma'} = N_{\Gamma'} \cdot H_{\Gamma'}$. Since $G = G_{\Gamma'} \cdot H$, we obtain $G = N \cdot H$ and therefore $z = 1$. Hence we have proved our main induction theorem:

Theorem 3.5. *Let G be a t-fold transitive group on Ω of degree n, $t \geqq 2$, $G \neq \mathrm{Sym}\,\Omega$ and let $H \neq 1$ be a non-regular normal subgroup of G. Let Γ be in $\Omega^{\{t\}}$, p some prime and U a Sylow p-subgroup of H_Γ with $\mathrm{Fix}\, U = \Delta$.*

Then G^Δ is a t-fold transitive group on Δ of degree $|\Delta| \equiv n \bmod p$. H^Δ is at least $(t-1)$-fold transitive on Δ, $H^\Delta \trianglelefteq G^\Delta$ and $(H^\Delta)_\Gamma$ is a p'-group. Furthermore $x(H) = (|U^G| : |U^H|) \cdot x(H^\Delta)$ and $|U^G| : |U^H| = 1$ if p does not divide $(n-t+1)/x(H)$. In this case H is t-fold transitive on Ω if and only if H^Δ is t-fold transitive on Δ.

3.3. *Wagner's Theorem and Related Results.* In 1966 Wagner [13] proved that normal subgroups of triply transitive groups of odd degree $n > 3$ are also triply transitive. A similar result is due to Ito [4] who has shown that normal subgroups of quadruply transitive groups of degree $n > 5$, $n \not\equiv 0 \bmod 3$, are quadruply transitive. As an application of 3.5 we prove the following theorem containing Wagner's result and a major part of Ito's theorem.

Theorem 3.6. *Let G be t-fold transitive on Ω of degree $n, t \geqq 3$, $G \neq \mathrm{Sym}\,\Omega$ and let $H \neq 1$ be a normal subgroup of G. Suppose p is a prime, $p < t$, not dividing $(n-t+1)$ and let x be the number of H-orbits on $\Omega^{(t)}$. Let r be the smallest positive number with $r \equiv (n-t+1) : x \bmod p$. Then $0 < x \cdot r < p < t$.*

Proof. The assumptions imply that H is at least $(t-1)$-fold transitive since, if $t = 3$, $p = 2$ and H is regular then $|\Omega|$ is a power of 2, i.e. 2 divides $n-2$. Let Γ be in $\Omega^{\{t\}}$ and U in $\mathrm{Syl}_p(H_\Gamma)$ with $\Delta = \mathrm{Fix}\, U$. By 3.5 $H^\Delta \trianglelefteq G^\Delta$, $x = x(H^\Delta)$, $r \equiv (|\Delta| - t + 1) : x(H^\Delta) \not\equiv 0 \bmod p$ and so H^Δ, G^Δ satisfy the hypotheses of 3.6 except if $G^\Delta = \mathrm{Sym}\,\Delta$. In the latter case we can assume $H^\Delta = \mathrm{Alt}\,\Delta$ and $|\Delta| = t+1$ since otherwise H^Δ is t-fold transitive on Δ and so $x(H^\Delta) = x = 1$. Since $H^\Delta = \mathrm{Alt}\,\Delta$ implies $x(H^\Delta) = 2$, we have $2 = |\Delta| - t + 1 \equiv n - t + 1 \not\equiv 0 \bmod p$, i.e. $p > 2$, and $r \equiv (n-t+1) : x \equiv 1$. Hence $0 < r \cdot x = 2 < p$. Assume therefore by induction that H_Γ is a p'-group. Since $[H_{\Gamma'} : H_\Gamma] = (n-t+1) : x$ for any $\Gamma' \subset \Gamma$ of size $t-1$, also $H_{\Gamma'}$ is a p'-group and so any element in H of p-order fixes at most $t-2$ points. Since H is $(t-1)$-fold transitive on Ω, $H_{\{\Gamma'\}}$ acts on Γ' like $\mathrm{Sym}\,\Gamma'$.

Choose some element h in $H_{\{\Gamma'\}}$ consisting of a single p-cycle and $t-1-p$ fixed points inside Γ'. We can assume that h has order p. Let $T_1, \ldots, T_x$ be the $H_{\Gamma'}$-orbits on $\Omega \backslash \Gamma'$ (Lemma 3.1). Since h normalizes $H_{\Gamma'}$, h induces a permutation on the set $\{T_1, \ldots, T_x\}$. We show that h acts trivially on this set. For, since $x \not\equiv 0 \bmod p, h$ fixes at least one of the T_i's, say $T_1^h = T_1$. Suppose $T_2^h = T_3$ and choose some g_1 in $G_{\{\Gamma'\}}$ such that $h^{g_1}|_{\Gamma'} = h^{-1}|_{\Gamma'}$. Since $G_{\Gamma'}$ acts transitively on $\{T_1, \ldots, T_x\}$ there is some g_2 in $G_{\Gamma'}$ such that $T_2^{g_1 \cdot g_2} = T_1$. Put $g = g_1 \cdot g_2$ and note that also $h^g|_{\Gamma'} = h^{-1}|_{\Gamma'}$, i.e. $h \cdot h^g$ is contained in $H_{\Gamma'}$ and therefore fixes all the T_i's. Hence $T_1 = T_1^{h \cdot h^g} = T_1^{h^g} = T_2^{h \cdot g} = T_3^g$ and so $T_2 = T_3$. Thus h fixes all the T_i's as sets and has in each of them at least $r \equiv (n-t+1):x = |T_i|$ fixed points. Therefore we count in all at least $t-1-p+x \cdot r$ points fixed by h. Since this number is at most $t-2$, the required property follows. ■

Remark. In Theorem 3.6 H is in fact t-fold transitive if $t \leqq 6$. A straightforward but rather length proof eliminates the various possibilities $1 < x < p < t$. For details the interested reader is referred to my Ph.D. thesis [9].

Corollary 3.7. *Let G be t-fold transitive of degree n, $3 \leqq t \leqq 6$, $G \neq \operatorname{Sym} \Omega$ and let $H \neq 1$ be a normal subgroup of G. Suppose $(n-t+1)$ is not divisible by some prime p less than t. Then H is t-fold transitive.*

As a second application of Theorem 3.5 we deal with the case $(n-t+1) \equiv 0 \bmod p$ for some prime $p < t$ but $(n-t+1)$: $x(H) \not\equiv 0 \bmod p$. In this situation we show $x(H) \neq y(H)$. As in the proof of Theorem 3.6 we can assume that elements of p-order in H fix at most $t-2$ points. Let Γ' be a set of size $t-1$ and let h be an element in $H_{\{\Gamma'\}}$ consisting of a p-cycle and $t-1-p$ fixed points. If $x(H) = y(H)$, h fixes the T_i's setwise by Lemma 3.1 and so h has at least $t-1-p+r \cdot x$ fixed points where $0 < r \equiv |T_i| \bmod p$. This is a contradiction since in particular p divides x. Hence $x(H) \neq y(H)$ and by 3.1 $x(H^\Gamma) \neq 1$, i.e. H is not generously $(t-1)$-fold transitive. Together with Theorem 3.3 we therefore obtain:

Theorem 3.8. *Let G be t-fold transitive on Ω of degree n, $3 \leqq t$ and $G \neq \operatorname{Sym} \Omega$. Let $H \neq 1$ be a non-regular normal subgroup of G such that H has $x > 1$ orbits on $\Omega^{(t)}$. Suppose there is some prime $p < t$ dividing $n-t+1$. Then p also divides $(n-t+1):x$ except if $t = 3$ and $PSL(2,q) \subseteq H \subseteq G \subseteq P\Gamma L(2,q)$ with $q = n-1 \equiv 3 \bmod 4$.*

The last result shows that in general $x(H)$ is odd for triply transitive groups of degree $n \not\equiv 2 \bmod 4$. A similar result is due to Bannai [1] who showed that x is odd for $t \geqq 6$ and arbitrary degree.

§ 4. Proof of the Theorems

The projective linear groups $PGL(2,q)$ are triply transitive on the projective line $\Omega = PG_1(q)$. For q even, $PGL(2,q) = PSL(2,q)$ is triply transitive on $PG_1(q)$ (illustrating Wagner's theorem) while for q odd, $PSL(2,q)$ has exactly two orbits on $\Omega^{(3)}$. We show that a triply transitive group G of degree $n \equiv 0 \bmod 3$ with $|G_{\alpha,\beta,\gamma}| \not\equiv 0 \bmod 3$ is one of these groups.

Proof of Theorem B. Let M^* be a minimal normal subgroup of G_α contained in M. By a result of Burnside M^* is either simple and primitive on $\Omega \setminus \{\alpha\}$ or else an elementary abelian p-group, regular on $\Omega \setminus \{\alpha\}$. In the first case M^* has order prime to 3 since M^* is 3/2 transitive on $\Omega \setminus \{\alpha\}$ and $|M^*_{\beta,\gamma}| \not\equiv 0 \bmod 3$. Hence by Lemma 2.5, $M^* \cong Sz(2^{2r+1})$ and the primitivity of M^* on $\Omega \setminus \{\alpha\}$ implies that G_α / M^* is contained in the outer automorphism group of M^*. From this information we conclude that M^* acts on $\Omega \setminus \{\alpha\}$ in the usual representation of a Suzuki group on q^2+1 points where $q = 2^{2r+1}$. (See Theorems 9 and 11 in

Suzuki's paper [10].) The Sylow 2-subgroup of M^*_β is characteristic in $G_{\alpha,\beta}$ and regular on $\Omega\setminus\{\alpha,\beta\}$ and so G_α is not extendable by Lemma 2.3, a contradiction. Hence M^* is elementary abelian and G must be one of the groups in Lemma 2.4 with minimal normal subgroup H^*. Since H^* is not solvable by Lemma 2.1, H^* cannot be sharply doubly transitive. $H^* \cong Sz$ is impossible since 3 divides n and similarly H^* cannot be a group of Ree type since their representation degree is $\equiv 1 \bmod 3$. Finally $H^* \cong PSU(3,q)$ leads to a contradiction since these groups are not normal in a triply transitive group. Hence $H^* = PSL(2,q) \subseteq H \subseteq G \subseteq P\Gamma L(2,q)$, $q=n-1$ and the theorem is proved. ■

Proof of Theorem A. Let S be a Sylow 3-subgroup of $H_{\alpha,\beta,\gamma}$ for 3 distinct points and $\Delta = \mathrm{Fix}\, S$. By Theorem 3.5, G^Δ is one of the groups in theorem B and therefore $x(H) = x(H^\Delta) \leqq 2$. If in addition $n \not\equiv 2 \bmod 4$, by Theorem 3.8 either $x(H) = 1$ or $PSL(2,q) \leqq H \leqq G \leqq P\Gamma L(2,q)$ where $q=n-1$ and the proof is completed. ■

As a final note we remark that for the proof of Theorem A very little information from Theorem B is required. If therefore one could show that a group as in Theorem B with $n \equiv 1 \bmod 3$ is a "known" group, then Theorem A would also hold for triply transitive groups of degree $n \equiv 1 \bmod 3$.

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