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Corrigendum

Corrigendum to “Saturated simplicial complexes” [J. Combin. Theory Ser. A 109 (2005) 149–179]

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1. Introduction

One of the main results in the paper of the title is potentially incorrect as stated. To correct the statement the notion of the *link of a subcomplex* in a simplicial complex is required, and this is more general than the usual definition of link. This new definition is given here and it is shown that it indeed gives the required theorem.

The aim of the paper in the title is to provide a local condition for a complex $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ to be saturated over a subcomplex Γ. If $\Re$ and τ denote the restriction and inner face of the gluing respectively then a *null-continuation* of $\Re$ is an element f in the face module M^Γ such that $(\Re + f)' = 0$. The Theorem 4.4 stated that Δ is saturated over Γ if and only if $\Re$ has a null-continuation $f \in M^\Lambda$ where Λ denotes the link of τ in Γ. After the publication of the paper Hugh Thomas (personal communication, April 2005) pointed out an error in the statement of Theorem 4.15 to us. As a consequence we can no longer assert that saturation implies that the faces of the null-continuation lie in Λ, but only that they lie in a somewhat larger neighbourhood complex of τ.

A suitable notion of neighbourhood which addresses this difficulty is the *link of a subcomplex,* rather than just of a single face. This is defined in Section 2. In Section 3 we state and prove the Null-Link Theorem based on the link of a certain subcomplex. All but two of the results in the paper work with our new version of the Null-Link Theorem which is Theorem 3.1 below. We also make some comments on the remaining cases. We are grateful to Hugh Thomas for making us aware of this problem.

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2. Division by octahedra

Let Δ be a simplicial complex and let M^{Δ} be the associated face module over some field F. In Section 2.4 we defined *generalized octahedra* in M^{Δ} and a process of dividing by an octahedron was described in Section 4.4. The main statement on division was Theorem 4.15. This theorem and its proof are correct except for the last sentence: if $\sigma := \operatorname{supp} c$ is a face of Δ then the faces of f/c may not lie in the link of σ in Δ. There are simple examples where this fails and therefore some larger subcomplex is required.

Definition. Let Υ be a subcomplex of Δ. Then the *link* of Υ in Δ, denoted $\operatorname{link}_{\Delta} \Upsilon$, consists of the faces z in Δ such that (i) z is disjoint from all faces of Υ and (ii) z belongs to $\operatorname{link}_{\Delta} y$ for some facet y of Υ (a maximal face of Υ under inclusion). Furthermore, for $c \in M^{\Delta}$ let Υ be the complex generated by all faces which appear with non-zero coefficient in c and put $\operatorname{link}_{\Delta} c := \operatorname{link}_{\Delta} \Upsilon$.

Now suppose that c is a generalized octahedron in M^{Δ}. Then $c = \bar{c} \cup \gamma$, where $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_\ell\}$ is a set of vertices and

$$\bar{c} = (\alpha_1 - \beta_1) \cup (\alpha_2 - \beta_2) \cup \cdots \cup (\alpha_t - \beta_t)$$

is an octahedron in the proper sense. We say that $\bar{c}$ and γ are the *proper part* and the *cone part* of c, respectively. Note that $c^{(j)} = 0$ when $j > \ell$ and $c^{(j)} = \bar{c} \cup \gamma^{(j)}$ when $j \leqslant \ell$. From the definition on page 168 it is now evident that the faces of $[f, c^{(j)}]$ belong to $\operatorname{link}_{\Delta} c^{(j)}$. The correct form of Theorem 4.15 is therefore

Theorem 2.1. *Let F be a field of characteristic $p > 2$ and let $0 < i < p$. Let Δ be a complex and suppose that $c \in M^{\Delta}$ is an octahedron with $c^{(i)} = 0$ but $c^{(i-1)} \neq 0$. Let $f, h \in M^{\Delta}$ be such that h is disjoint from c and $f^{(i)} = (c \cup h)^{(i)}$. Then $h' = (f/c)'$ and moreover, f/c belongs to M^{Λ} where Λ is the link of the proper part of c.*

3. The Null-Link Theorem

We now return to the proof of the Null-Link Theorem and the issues affected by the adjustment to Theorem 4.15. So let $\Delta = \Gamma \overset{k}{\cup} [\sigma]$ be a k-gluing with inner face τ and restriction $\Re$. In the necessity part of the proof it is shown in Lemma 4.11 that the saturation of $\Delta = \Gamma \overset{k}{\cup} [\sigma]$ over Γ implies the existence of a null-continuation $f \in M^{\Delta}$ with $(\Re + f)' = 0$. From Theorem 2.1 above we know that $f = g/c$, in the notation of the original paper, belongs to M^{Λ} where Λ is the link of the proper part of a suitable octahedron c contained in τ. Furthermore, from the definition of division and from Lemma 4.12 we know that f is disjoint from τ. With this adjustment we can now state

Theorem 3.1 *(Null-Link Theorem). Let Γ be a complex and let $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ be a gluing with restriction $\Re$ and inner face τ. Suppose that $p > 2$. Then Δ is saturated over Γ if and only if $\Re$ has a null continuation $f \in M^{\Lambda}$ with f disjoint from τ, where Λ is the link in Δ of the proper part of a suitable octahedron c with* $\operatorname{supp} c \subseteq \tau$.

Proof. It remains to look at the sufficiency part of the proof in the original paper. The only argument where the shape of the null continuation is of importance is in Section 4.3.1 of the proof, and more precisely in the fifth line after Lemma 4.10. For the argument there to work it is sufficient to assume that f is disjoint from τ. □

Other Comments: (1) We have checked the proofs in Sections 4 and 5 which made use of the Null-Link Theorem in the critical direction: in all cases Theorem 3.1 as above is sufficient. The same is true for Section 6 but with the exception of Theorems 6.11 and 6.12 on the saturation of rank selected complexes. From what we know it appears that both statements are indeed correct as stated. However, the given proofs can not be based on Theorem 3.1. Instead some new arguments need to be established and this question remains open.

(2) We take this opportunity to point out other typographical errors. In Section 2.1 on page 153 it should say that f^{τ} is disjoint from σ. Also, the correct labelling in Fig. 2 is as below.

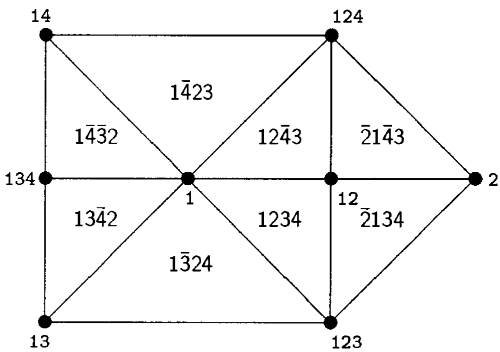

Fig. 2. An initial part of the lexicographic shelling of A_3.