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Efficient reconstruction of partitions

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Abstract

We consider the problem of reconstructing a partition x of the integer n from the set of its t-*subpartitions*. These are the partitions of the integer $n-t$ obtained by deleting a total of t from the parts of x in all possible ways. It was shown (in a forthcoming paper) that all partitions of n can be reconstructed from t-subpartitions if n is sufficiently large in relation to t. In this paper we deal with *efficient reconstruction*, in the following sense: if all partitions of n are t^--reconstructible, what is the minimum number $N=N^-(n,t)$ such that every partition of n can be identified from any $N+1$ distinct subpartitions? We determine the function $N^-(n,t)$ and describe the corresponding algorithm for reconstruction. Superpartitions may be defined in a similar fashion and we determine also the maximum number $N^+(n,t)$ of t-superpartitions common to two distinct partitions of n.

Keywords: Reconstruction; Partitions

1. Introduction

Let $\mathscr{P}(n)$ denote the set of all partitions of the integer n. Thus, if $x \in \mathscr{P}(n)$ then $x = [x_1, x_2, \ldots, x_l]$ is a multiset of integers x_i with $0 \leqslant x_i$ and $\sum_{i=1}^{l} x_i = n$. To avoid cumbersome distinctions we identify two partitions if they differ by parts of size 0 only. If $x \in \mathscr{P}(n)$ and if t is an integer with $0 \leqslant t \leqslant n$ then we say that $x' := [x_1 - e_1, x_2 - e_2, \ldots, x_l - e_l]$ is a t-*subpartition* if $0 \leqslant e_i \leqslant x_i$ and $\sum_{i=1}^{l} e_i = t$.

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The set of all t-subpartitions of x is denoted by $D_t(x)$ and we say that x is *reconstructible* from its t-subpartitions, or that x is t^-*-reconstructible*, if and only if the following is true: Whenever $D_t(x) = D_t(y)$ for some $y \in \mathscr{P}(n)$, then $x = y$. For instance, $x = [3, 2, 2, 1]$ has $D_2(x) = \{[3, 2, 1], [3, 1, 1, 1], [2, 2, 2], [2, 2, 1, 1]\}$ and by elementary arguments one can show that x is 2^--reconstructible. On the other hand, $D_5(x) = \{[3], [2, 1], [1, 1, 1]\} = D_5([5, 2, 1])$ and so x is not 5^--reconstructible. Note also that in contrast to other reconstruction problems here we have no information about the multiplicity with which a partition occurs in $D_t(x)$.

The problem of reconstructing partitions was proposed in [4,1]. Intuitively, if t is small in relation to n then the partitions of n should be t^--reconstructible, and this is proved in [5]. In this paper we deal with an additional issue: if all partitions of n are t^--reconstructible, is there an $N = N^-(n, t)$ such that an arbitrary $x \in \mathscr{P}(n)$ can be identified from any $N + 1$ distinct members of $D_t(x)$? This is the problem of *efficient reconstruction* first considered in Levenshtein's seminal paper [2] on the reconstruction of sequences. Here we also answer the question of efficient reconstruction of partitions. To state these results we need some additional definitions.

If $x = [x_1, x_2, \ldots, x_l] \in \mathscr{P}(n)$ and if $0 \leqslant t$ and $0 \leqslant e_1, e_2, \ldots, e_{l'}$ are integers with $l' \geqslant l$ and $\sum_{i=i}^{l'} e_i = t$ then $x' = [x_1 + e_1, x_2 + e_2, \ldots, x_l + e_l, e_{l+1}, \ldots, e_{l'}]$ is a t*-superpartition* of x. The set of all t-superpartitions of x is denoted by $U_t(x)$. The partition x is said to be *reconstructible* from its t-superpartitions, or t^+*-reconstructible*, if and only if the following is true: whenever $U_t(x) = U_t(y)$ for some $y \in \mathscr{P}(n)$ then $x = y$. For $n \geqslant t \geqslant 0$ we define

$$N^-(n, t) := \max_{x, y \in \mathscr{P}(n);\ x \neq y} |D_t(x) \cap D_t(y)| \quad \text{and}$$

$$D(n, t) := \max_{x \in \mathscr{P}(n)} |D_t(x)|,$$

and for arbitrary $t \geqslant 0$ we define

$$N^+(n, t) := \max_{x, y \in \mathscr{P}(n);\ x \neq y} |U_t(x) \cap U_t(y)|,$$

$$U(n, t) := \max_{x \in \mathscr{P}(n)} |U_t(x)|.$$

Clearly, $N^-(n, t) \leqslant D(n, t)$ and $N^+(n, t) \leqslant U(n, t)$, and if $x \in \mathscr{P}(n)$ satisfies $N^-(n, t) < |D_t(x)|$ then x is t^--reconstructible. In fact, the function $N^-(n, t)$ measures *efficient reconstructibility* in the sense that for an arbitrary partition $x \in \mathscr{P}(n)$ any $N^-(n, t) + 1$ or more distinct members of $D_t(x)$ determine x uniquely, as intended above. A similar statement holds for $N^+(n, t)$ and t^+-reconstructibility.

Theorem 1.1. *Let $n \geqslant 2$ and $t \geqslant 1$. Then $N^-(n, t) = D(n-1, t-1)$ for $n \geqslant t$ and $N^+(n, t) = U(n + 1, t - 1)$ for all t.*

This theorem is the exact analogue of the main result of Levenshtein in [2] on the efficient reconstruction of a sequence from its sub- and supersequences. In particular, every partition $x \in \mathscr{P}(n)$ is reconstructible from any two of its 1-subpartitions, as $D(n - 1, t - 1) = 1$ for $t = 1$. In Section 2 it is shown that $D(n, t) = |\mathscr{P}(n - t)|$ for some values of n and t.

In the same sections we obtain the proof of the theorem above and we give algorithms to reconstruct the partitions in each case.

We may regard partitions as Young diagrams and hence as elements of the Young lattice. In this lattice the set of elements of rank n is $\mathscr{P}(n)$, see [6, Chapter 7] for instance, and for $x \in \mathscr{P}(n)$ the set $D_1(x)$ are the partitions which appear in the Murnaghan–Nakayama rule. Therefore, partition reconstruction can be placed into the larger context of reconstruction problems in lattices. In a recent paper Stanley [7] determines the number of standard Young diagrams of $n+t$ cells which contain a given partition of n. It may well be possible to use these techniques to give formulae for the values of $D(n,t)$ and $U(n,t)$ in general.

One further general comment should be made. As we have seen, partitions and sequences both belong to a class of reconstruction problems where reconstruction is guaranteed, and where even the problem of efficient reconstruction has a satisfactory answer. One may therefore ask which other kinds of reconstruction problems belong to that class. For orbit reconstruction of permutation groups we know that also cyclic, and possibly solvable groups belong to this class. In [3] we consider the reconstruction problems associated with finite primitive groups. In this generality efficient reconstruction cannot be expected any longer.

2. Subpartition reconstruction

If $x \in \mathscr{P}(n)$ is a partition, we shall always assume that x is in *standard form* $x = [x_1, x_2, \ldots, x_l]$ with $x_i \geqslant x_{i+1}$ for $i = 1, 2, \ldots, l-1$. Also, we let $\|x\| := \sum_{1 \leqslant i \leqslant l} x_i$ be the number partitioned by x. If also $y = [y_1, y_2, \ldots, y_m]$ is a partition, possibly of a different integer, the *intersection* $x \cap y$ is the partition

$$x \cap y = [\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_v, y_v\}],$$

where $v = \min\{l, m\}$. Similarly, the *union* $x \cup y$ is the partition

$$x \cup y = [\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots, \max\{x_w, y_w\}],$$

where $w = \max\{l, m\}$ and where we assume that $x_{l'} = 0$ for $l' > l$ and $y_{m'} = 0$ for $m' > m$. These definitions make sense since we are assuming that our partitions are in standard form. Note that then also $x \cup y$ and $x \cap y$ are in standard form and further that $x \cap y = x$ if and only if $x \cup y = y$. We can now define a partial order on partitions by

$$x \subseteq y \quad \text{if and only if } x \cap y = x.$$

For any integer $m > 0$ we define the partition

$$S(m) := [m, \lfloor m/2 \rfloor, \lfloor m/3 \rfloor, \ldots \lfloor m/(m-2) \rfloor, \lfloor m/(m-1) \rfloor, 1]$$

of $s(m) := \sum_{i=1}^{m} \lfloor m/i \rfloor$. It is easy to see that for any $u \in \mathscr{P}(m)$ there exist subpartitions of $S(m)$ that are equal to u and $S(m)$ is the smallest partition with this property. That is to say, if x is a partition of some integer $m' < s(m)$ then there are partitions $u \in \mathscr{P}(m)$ which are not subpartitions of x. We can now prove

Lemma 2.1. *Let x be a partition of n. Then $x \cap S(n-t)$ determines $D_t(x)$ uniquely. That is, if x and $y \in \mathscr{P}(n)$ then $D_t(x) = D_t(y)$ if and only if $x \cap S(n-t) = y \cap S(n-t)$.*

Proof. First we prove that if $x \cap S(n-t) = y \cap S(n-t)$ then $D_t(x) = D_t(y)$. It is sufficient to show that if $x \cap S(n-t) = u$ then $D_t(x) = D_r(u)$, where $r = \|u\| + t - n$. Certainly, $D_r(u) \subseteq D_t(x)$. Thus, assume that there is some $z \in D_t(x)$ with $z \notin D_r(u)$. However, then we would have $z \cap S(n-t) \neq z$ for some $z \in \mathscr{P}(n-t)$, a contradiction.

Next assume that $x \cap S(n-t) \neq y \cap S(n-t)$. Let $x \cap S(t) = S = [s_1, s_2, \ldots, s_l]$ and $y \cap S(t) = T = [t_1, t_2, \ldots, t_{l'}]$ for some $l, l' \in \mathbb{N}$. Let the first instance of inequality of these sequences be, without loss, $s_i > t_i$ for some $i \in \{1, 2, \ldots, \min\{l, l'\}\}$. Consider the partition $u = [s_i, s_i, \ldots, s_i]$ of the integer is_i. Now $s_i \leqslant \lfloor (n-t)/i \rfloor$ and so $\|u\| \leqslant i\lfloor (n-t)/i \rfloor \leqslant n-t$. Therefore there is some $v \in D_t(x)$ such that $u \cap v = u$. It is clear that $v \notin D_t(y)$. □

We can use Lemma 2.1 to manufacture non-reconstructible partitions: take $x, y \in \mathscr{P}(n)$ with $x \neq y$ and $x \cap S(m) = y \cap S(m)$ for some $m \in \mathbb{N}$. Then $D_{n-m}(x) = D_{n-m}(y)$ and so x and y are not reconstructible from their $(n-m)$-subpartitions. For example, take the partition $S(m)$ for any $m \in \mathbb{N}$ and consider $x, y \in U_i(S(m))$, different partitions of the integer $s(m)+i$ for any $i \in \mathbb{N}$. By Lemma 2.1 we have $D_v(x) = D_v(y) = D_u(S(m)) = \mathscr{P}(m)$ where $u = s(m) - m$ and $v = u + i$.

Next we need the following simple lemma.

Lemma 2.2. *For any $n, t, i \in \mathbb{N}$ with $n \geqslant t$ we have $D(n,t) \leqslant D(n+i, t+i)$. Furthermore, for any $n, t, i \in \mathbb{N}$ with $1 \leqslant i < n$ we have $U(n,t) < U(n-i, t+i)$.*

Proof. Assume that $y \in \mathscr{P}(n)$. For the first inequality observe that $D_t(y) \subseteq D_{t+i}(x)$ for any $x \in U_i(y)$ for any $i \in \mathbb{N}$. This implies that $D(n,t) \leqslant D(n+i, t+i)$. Secondly, for $1 \leqslant i < n$ we have $U_t(y) \subset U_{t+i}(x)$ for any $x \in D_i(y)$ and so $U(n,t) < U(n-i, t+i)$. □

We note that in general we cannot have strict inequality in the first inequality. Indeed, it follows from Lemma 2.1 that

$$D(n,t) = D(n+i, t+i) = |\mathscr{P}(n-t)|$$

for any $i \in \mathbb{N}$, if $n \geqslant s(n-t)$.

Theorem 2.3. *If $n \geqslant 2$ and $n \geqslant t \geqslant 1$, then $N^-(n,t) = D(n-1, t-1)$. Furthermore, there exist partitions x and y of n such that $D_1(x) \cap D_1(y)$ consists of a single partition z of $n-1$ for which $|D_{t-1}(z)| = N^-(n, t)$.*

Proof. First we show that $N^-(n,t) \geqslant D(n-1, t-1)$. Let z be a partition of $n-1$ corresponding to $D(n-1, t-1)$, i.e., $|D_{t-1}(z)|$ is maximal among the partitions of $n-1$. We note that for any $1 \leqslant m \in \mathbb{N}$ and $p \in \mathscr{P}(m)$ we have $|U_1(p)| \geqslant 2$. Thus let $x, y \in U_1(z)$ with $x \neq y$. Certainly $D_{t-1}(z) \subseteq D_t(x), D_t(y)$ and hence it follows that

$$N^-(n,t) \geqslant |D_t(x) \cap D_t(y)| \geqslant |D_{t-1}(z)| = D(n-1, t-1). \tag{1}$$

Next we show that $N^-(n,t) \leqslant D(n-1, t-1)$. Thus assume that $x, y \in \mathscr{P}(n)$, $x \neq y$ are such that $|D_t(x) \cap D_t(y)|$ is maximal. Set $w := x \cap y$. Then $D_t(x) \cap D_t(y) \subseteq D_v(w)$, where $v = \|x \cap y\| + t - n$. In particular,

$$|D_t(x) \cap D_t(y)| \leqslant |D_v(w)| \leqslant D(\|x \cap y\|, v) \leqslant D(n-1, t-1).$$

The last inequality following from Lemma 2.2 upon taking $i = n - 1 - \|x \cap y\|$.

To prove the final part we consider again the partitions x, y constructed at the beginning of the proof. It now follows that the inequalities of (1) are actually equalities and $|D_t(x) \cap D_t(y)| = |D_{t-1}(z)| = D(n-1, t-1)$. □

In particular, for any $x \in \mathscr{P}(n)$ it follows that if $|D_t(x)| \geqslant D(n-1, t-1) + 1$ then x is reconstructible from its t-subpartitions.

Corollary 2.4. *Any two different* 1*-subpartitions of a partition allow its reconstruction. All partitions of* $n \geqslant 3$ *are reconstructible from their* 1*-subpartitions.*

Proof. Taking $t = 1$ in Theorem 2.3 we get $N^-(n, 1) = D(n-1, 0) = 1$. This proves the first statement. For the second, note that the only partitions of n that have only one different type of 1-subpartition are $x_l = [l, l, l, \ldots, l] \in \mathscr{P}(n)$ for any $l \mid n$. For $n \geqslant 3$ it is not hard to see that $D_1(x_i) \neq D_1(x_j)$ for any $i, j \mid n$ with $i \neq j$. For $n = 2$ the two partitions of 2 given by [1, 1] and [2] clearly have the same 1-subpartitions, hence the restriction. □

Algorithm for recovering a partition from its subpartitions. The following simple procedure reconstructs a partition $x \in \mathscr{P}(n)$ if one knows n and at least $N^-(n, t) + 1$ members from the set $D_t(x)$.

(i) Take *any* $y_1, y_2, \ldots, y_l \in D_t(x)$, where $l = N^-(n, t) + 1$ and put them in standard form.
(ii) Form the union $\overline{x} = \bigcup_{i=1,\ldots,l} y_i$.

We claim that $x = \overline{x}$, as follows. Since all partitions are in standard form, we have that $y_i \subseteq x$ and so $\bigcup_{i=1,\ldots,l} y_i \subseteq x$. Assume that $\bigcup_{i=1,\ldots,l} y_i = x' \subset x$ with, say, $\|x'\| = n' < n$. By Theorem 2.3 and Lemma 2.2 it follows that $l > D(n-1, t-1) \geqslant D(n', n'+t-n) \geqslant |D_v(x')|$, where $v = n'+t-n$. Conversely, since $y_i \subseteq x'$ for $i = 1, 2, \ldots, l$ it follows that $|D_v(x')| \geqslant l$, a contradiction.

No explicit formula for $D(n, t)$ seems to be known for general n and t. For $t = 1$ and $x = [x_1, x_2, \ldots, x_l]$ it is easy to see that $|D_1(x)|$ is the number of distinct x_i, that is $|D_1(x)| = |\{x_1, x_2, \ldots, x_l\}|$. Thus $N^-(n, 1)$ is the greatest integer j for which $j(j+1)/2 \leqslant n$. Already for $t = 2$ it is much more difficult to give an explicit value for $D(n, t)$.

3. Superpartition reconstruction

In contrast to reconstruction from subpartitions, partitions are always reconstructible from superpartitions.

Lemma 3.1. *Every partition is reconstructible from its t-superpartitions, for any* $t \in \mathbb{N}$. *In fact, if t is given, then just one suitably chosen member from* $U_t(x)$ *allows the unique reconstruction of x. If t is not given then just two suitably chosen members from* $U_t(x)$ *suffice to reconstruct x uniquely.*

Proof. Assume that $x \in \mathscr{P}(n)$ and without loss that $n \geqslant 2$ since $|\mathscr{P}(1)| = 1$. If t is known, select $q = [q_1, q_2, \ldots, q_r] \in U_t(x)$ with the property that $q_1 > p_1$ for any $p = [p_1, p_2, \ldots, p_{r'}] \in U_t(x)$ with $p \neq q$. It is clear then that $x = [(q_1 - t), q_2, \ldots, q_r]$. To determine t we select in addition the (unique) partition in $U_t(x)$ which has the largest number of parts. If this number is l then clearly $t = l - r$. This completes the proof. □

The above lemma shows that for any $x \in \mathscr{P}(x)$ some two *suitably chosen* members from $U_t(x)$ allows the reconstruction of x. The next theorem gives the minimum number of *arbitrarily chosen* members from $U_t(x)$ needed to reconstruct x.

Theorem 3.2. *If* $n \geqslant 2$ *and* $t \geqslant 1$ *then* $N^+(n, t) = U(n + 1, t - 1)$. *Furthermore, there exist partitions x and y of n such that* $U_1(x) \cap U_1(y)$ *consists of a single partition z of* $n - 1$ *for which* $|U_{t-1}(z)| = N^+(n, t)$.

Proof. First we show that $N^+(n, t) \geqslant U(n + 1, t - 1)$. Thus let $z \in \mathscr{P}(n + 1)$ be a partition corresponding to $U(n + 1, t - 1)$. Since $n \geqslant 2$ there are at least two different 1-subpartitions of z unless possibly $z = z_v = [v, v, \ldots, v]$ for any $v \mid n$. However, consider the partitions of $n + 1$ defined by $w_v = [v + 1, v, v, \ldots, v, v - 1]$ for $v < n$ and $w_n = [n, 1]$. It is not hard to see that $U_m(w_v) \geqslant U_m(z_v)$ for any $v \mid n$ and $m \in \mathbb{N}$. In particular, we may assume that there exist $x, y \in D_1(z)$ with $x \neq y$. Then we have $U_{t-1}(z) \subseteq U_t(x), U_t(y)$. Hence

$$N^+(n, t) \geqslant |U_t(x) \cap U_t(y)| \geqslant |U_{t-1}(z)| = U(n + 1, t - 1). \tag{2}$$

Next we prove that $N^+(n, t) \leqslant U(n + 1, t - 1)$. So let x and y be distinct partitions of n such that $|U_t(x) \cap U_t(y)|$ is maximal. Clearly, any $z \in U_t(x) \cap U_t(y)$ must contain both x and y, i.e. $p = (x \cup y) \subset z$. It follows that $|U_t(x) \cap U_t(y)| \leqslant |U_v(p)|$, where $v = n + t - l$ and $l = \|x \cup y\|$. In particular,

$$N^+(n, t) = |U_t(x) \cap U_t(y)| \leqslant |U_v(p)| \leqslant U(l, n + t - l).$$

Now if $l > n+1$, then by Lemma 2.2 it follows that $N^+(n, t) < U(n+1, t-1)$, in contradiction to the first part of the proof. It follows that $l = n+1$ and then $N^+(n, t) \leqslant U(n+1, t-1)$.

To prove the final part, we consider again the partitions x, y constructed at the beginning of the proof. It now follows that the inequalities of (2) are equalities and $|U_t(x) \cap U_t(y)| = |U_{t-1}(z)| = U(n + 1, t - 1)$. □

Algorithm for recovering a partition from its superpartitions. The following simple procedure reconstructs a partition $x \in \mathscr{P}(n)$ if one knows n and at least $N^+(n, t) + 1$ members from the set $U_t(x)$.

(i) Take *any* $y_1, y_2, \ldots, y_l \in U_t(x)$ where $l = N^+(n, t) + 1$ and put them in standard form.
(ii) Form the intersection $\overline{x} = \bigcap_{i=1,\ldots,l} y_i$.

We claim that $x=\overline{x}$, as follows. Since all partitions are in standard form we have that $x \subseteq y_i$ and so $x \subseteq \bigcap_{i=1,\dots,l} y_i$. Assume that $\bigcap_{i=1,\dots,l} y_i = x'$ with $x \subset x'$. Say, $\|x'\| = n' > n$. By Theorem 3.2 and Lemma 2.2 it follows that $l > U(n+1, t-1) \geqslant U(n', n=n'+t) \geqslant |U_v(x')|$, where $v = n + t - n'$. On the other hand, since $x' \subseteq y_i$ for $i = 1, 2, \dots, l$ we also have that $|U_v(x')| \geqslant l$, a contradiction. $\square$

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