Reconstruction of permutations distorted by single transposition errors

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Abstract— **The reconstruction problem for permutations on n elements from their erroneous patterns which are distorted by transpositions is presented in this paper. It is shown that for any $n \geq 3$ an unknown permutation is uniquely reconstructible from 4 distinct permutations at transposition distance at most one from the unknown permutation. The *transposition distance* between two permutations is defined as the least number of transpositions needed to transform one into the other. The proposed approach is based on the investigation of structural properties of a corresponding Cayley graph. In the case of at most two transposition errors it is shown that $\frac{3}{2}(n-2)(n+1)$ distinct erroneous patterns are required in order to reconstruct an unknown permutation. Similar results are obtained for two particular cases when permutations are distorted by given transpositions. These results confirm some bounds for regular graphs which are also presented in this paper.**

I. Introduction

Efficient reconstruction of arbitrary sequences was introduced and investigated by Levenshtein for combinatorial channels with errors of interest in coding theory such as substitutions, transpositions, deletions and insertions of symbols [1], [2]. Sequences are considered as elements of a vertex set V of a graph $\Gamma = (V, E)$ where an edge $\{x, y\} \in E$ is viewed as the single error transforming x into $y \in V$. One of the metric problems which arises here is the problem of reconstructing an unknown vertex $x \in V$ from a minimum number of vertices in the metric ball $B_r(x)$ of radius r centered at the vertex $x \in V$. It is reduced to finding the value

$$N(\Gamma, r) = \max_{x,y \in V(\Gamma),\ x \neq y} |B_r(x) \cap B_r(y)|, \qquad (1)$$

since $N(\Gamma, r) + 1$ is the least number of distinct vertices in the ball $B_r(x)$ around the unknown vertex x which are sufficient to reconstruct x subject to the condition that at most r single errors have happened. As one can see, this problem is based on considering metric balls in a graph but it differs from traditional packing and covering problems in various ways. It is motivated by a transmission model where information is realized in the presence of noise without encoding or redundancy, and where the ability to reconstruct a message (vertex) uniquely depends on having a sufficiently large number of erroneous patterns of this message.

The value (1) was studied for the Hamming and Johnson graphs [2]. Both graphs are distance–regular and the first is a Cayley graph. The problem of finding the value (1) is much more complicated for graphs which are not distance–regular. Cayley graphs of this kind arise for instance on the symmetric group and the signed permutation group, when the reconstruction of permutations and signed permutations is considered for distortions by single reversal errors [3], [4].

In this paper we continue these investigations and consider the reconstruction problem for permutations on a set $\{1..n\}$ which are distorted by single transposition errors consisting of swapping 1) any two elements of $\{1..n\}$; 2) any two neighboring elements of $\{1..n\}$; and 3) the symbol 1 and j for any $1 < j \leq n$. The corresponding graphs are the *transposition Cayley graph*, the *bubble–sort Cayley graph* and the *star Cayley graph*. They are regular but not distance–regular. We investigate the combinatorial properties of these graphs and present the values (1) when $r = 1, 2$ in each case. Some bounds on $N(\Gamma, 1)$ and $N(\Gamma, 2)$ for regular graphs are also considered. It is shown that the bubble–sort and star Cayley graphs are examples for which these bounds are attained.

II. Definitions, notation, general results

Let G be a finite group and let S be a set of generators of G such that the identity element e of G does not belong to S and such that $S = S^{-1}$, where $S^{-1} = \{s^{-1} : s \in S\}$. In the *Cayley graph* $\Gamma = Cay(G, S) = (V, E)$ vertices correspond to the elements of the group, i.e. $V = G$, and edges correspond to multiplication on the right by generators, i.e. $E = \{\{g, gs\} : g \in G, s \in S\}$. Denote by $d(x, y)$ the *path distance* between the vertices x and y in Γ, and by $d(\Gamma) = \max\{d(x, y) : x, y \in V\}$ the *diameter* of Γ. In other words, in a Cayley graph the diameter is the maximum, over $g \in G$, of the length of a shortest expression for g as a product of generators. For the vertex x let $S_r(x) = \{y \in V : d(x, y) = r\}$ and $B_r(x) = \{y \in V : d(x, y) \leq r\}$ be the *sphere* and the *ball* of radius r centered at x, respectively. The vertices $y \in B_r(x)$ are r-*neighbors* of the vertex x.

As mentioned in the Introduction, the value (1) was investigated initially for distance–regular graphs such as the Hamming and Johnson graphs. Let us recall that a simple connected graph Γ is *distance–regular* if there are integers b_i, c_i for $i \geq 0$ such that for any two vertices x and y at distance $d(x,y) = i$ there are precisely c_i neighbors of y in $S_{i-1}(x)$ and b_i neighbors of y in $S_{i+1}(x)$. Evidently Γ is *regular* of valency $k = b_0$, or *k-regular*. A k-regular simple graph Γ is *strongly regular* if there exist integers λ and μ such that every adjacent pair of vertices has λ common neighbors, and every nonadjacent pair of vertices has μ common neighbors.

The Hamming space F_q^n consists of the q^n vectors of length n over the alphabet $\{0,1,...,q-1\},\ q \geq 2$. It is endowed with the Hamming distance d where $d(x,y)$ is the number of coordinate positions in which x and y differ. It can be viewed as a graph $L_n(q)$ with vertex set given by the vector space F_q^n (where F_q is the field of q elements) where $\{x,y\}$ is an edge of $L_n(q)$ if and only if $d(x,y)=1$. This Hamming graph is the Cayley graph on the additive group F_q^n when we take the generator set $S = \{xe_i : x \in (F_q)^\times, 1 \leq i \leq n\}$ where the $e_i = (0,...,0,1,0,...0)$ are the standard basis vectors of F_q^n. It was shown in [1], [2] that for any $n \geq 2$, $q \geq 2$ and $r \geq 1$,

$$N(L_n(q), r) = q \sum_{i=0}^{r-1} \binom{n-1}{i} (q-1)^i. \tag{2}$$

For the particular case $n=2$ the Hamming graph $L_2(q)$ is the *lattice graph* over F_q. This graph is strongly regular with parameters $v = q^2$, $k = 2(q-1)$, $\lambda = q-2$, $\mu = 2$, and from (2) we get $N(L_2(q),1) = q$ and $N(L_2(q),2) = q^2$.

For the integer parameters $n > e \geq 1$ the Johnson graph J_e^n is defined on the subset $V = J_e^n \subseteq F_2^n$ consisting of all vectors of Hamming weight e. On J_e^n the Johnson distance is defined as half the (even) Hamming distance, and two vertices x, y are joined by an edge if and only if they are at Johnson distance 1 from each other. In general J_e^n is not a Cayley graph although the notion of errors being represented by edges makes sense all the same. In particular, two vertices are at distance 1 from each other if and only if one is obtained from the other by the interchange of two coordinate positions. In [1], [2] it was shown that

$$N(J_e^n, r) = n \sum_{i=0}^{r-1} \binom{e-1}{i} \binom{n-e-1}{i} \frac{1}{i+1} \tag{3}$$

for any $n \geq 2$, $e \geq 1$ and $r \geq 1$. In the particular case $e = 2$ and $n \geq 4$ the Johnson graph J_2^n is the *triangular graph* $T(n)$. As vertices it has the 2-element subsets of an n-set and two vertices are adjacent if and only if they are not disjoint. This graph is strongly regular with parameters $v = \frac{n(n-1)}{2}$, $k = 2(n-2)$, $\lambda = n-2$, $\mu = 4$, and from (3) we obtain $N(T(n),1) = n$ and $N(T(n),2) = \frac{n(n-1)}{2}$.

These two results were the first analytic formulas for the reconstruction problem we are interested in. Their uniformity depends on the fact that these graphs are distance–regular. What then are the general results for simple graphs, regular graphs and Cayley graphs? We start with a few observations from [5] for any connected simple graphs $\Gamma = (V,E)$. In the spirit of distance regularity we put $k_i(x) = |S_i(x)|$ and define numbers $c_i(x,y)$, $b_i(x,y)$ and $a_i(x,y)$ for any two vertices $x \in V$ and $y \in S_i(x)$ such that

$$c_i(x,y) = |\{z \in S_{i-1}(x) \ : \ d(z,y) = 1\}|,$$
$$b_i(x,y) = |\{z \in S_{i+1}(x) \ : \ d(z,y) = 1\}|,$$
$$a_i(x,y) = |\{z \in S_i(x) \ : \ d(z,y) = 1\}|.$$

From this $a_1(x,y) = a_1(y,x)$ is the number of triangles over the edge $\{x,y\}$ and $c_2(x,y)$ is the number of common neighbors of $x \in V$ and $y \in S_2(x)$. Let

$$\lambda = \lambda(\Gamma) = \max_{x \in V,\ y \in S_1(x)} a_1(x,y) \tag{4}$$

$$\mu = \mu(\Gamma) = \max_{x \in V,\ y \in S_2(x)} c_2(x,y). \tag{5}$$

Since $|B_r(x) \cap B_r(y)| > 0$ for $x \neq y,\ x,y \in V(\Gamma)$, if and only if $1 \leq d(x,y) \leq 2r$, we have

$$N(\Gamma, r) = \max_{1 \leq s \leq 2r} N_s(\Gamma, r) \tag{6}$$

where $N_s(\Gamma,r) = \max\{|B_r(x) \cap B_r(y)| : d(x,y) = s\}$. In particular, $N_1(\Gamma,1) = \lambda + 2$ and $N_2(\Gamma,1) = \mu$ so that

$$N(\Gamma,1) = \max(\lambda+2, \mu). \tag{7}$$

One can easily check that using this formula for the lattice graph $L_2(q)$ and the triangular graph $T(n)$ we obtain again the earlier formulas (2) and (3). Indeed, since $\lambda = n-2$ and $\mu = 4$ for $T(n), n \geq 4$, we have $N(T(n),1) = n$ from (7). By the same reason we have $N(L_2(q),1) = q$ since $\lambda = q-2$ and $\mu = 2$ for the lattice graph $L_2(q)$.

We have no general results for $N(\Gamma,r)$ when Γ is a regular graph. The numbers $c_i(x,y)$ and $b_i(x,y)$ usually depend on $y \in S_i(x)$ and this causes difficulties when searching for general estimates of $N(\Gamma,r)$. However, some bounds on $N(\Gamma,1)$ and $N(\Gamma,2)$ were obtained in [5]. Here it is assumed that Γ is connected, k-regular of diameter $d(\Gamma) \geq 2$ with $v \geq 4$ vertices and parameters $0 \leq \lambda \leq k-2,\ 1 \leq \mu \leq k$, where $2 \leq k \leq v-2$.

Theorem 1: For any k-regular graph Γ we have

$$N(\Gamma,1) \leq \frac{1}{2}(v+\lambda). \tag{8}$$

This theorem is proved by checking that $\lambda + 2 \leq \frac{1}{2}(v+\lambda)$ and $\mu \leq \frac{1}{2}(v+\lambda)$. The first inequality takes place since $k \leq v-2$ and $\lambda \leq k-2$. Moreover, there is equality only if $\lambda = v-4$ and $k = v-2$. The second inequality is true since counting edges between $S_1(x)$ and $S_2(x)$ for any $x \in V$ we have

$$\sum_{y \in S_1(x)} (k-1-a_1(x,y)) = \sum_{z \in S_2(x)} c_2(x,z).$$

From (4), (5) and the fact that $k_2(x) \leq v-k-1$ we get $k(k-1-\lambda) \leq \mu k_2(x) \leq \mu(v-k-1)$ with equality if and only if Γ is strongly regular. Let us note here that the equality

$k(k-1-\lambda) = \mu(v-k-1)$ is well-known for strongly regular graphs. From this and the fact that $1 \leq \mu \leq k$ we have $k-1-\lambda \leq v-k-1$ and hence $\mu \leq k \leq \frac{1}{2}(v+\lambda)$ is valid for any regular graph Γ. By taking into account these two inequalities for λ and μ we get (8) from (7). Moreover, (8) is attained on the strongly regular t-partite graph $K^{(t)}_{k-\lambda}$ with $t(k-\lambda)$ vertices partitioned into $t \geq 2$ parts, where $t = \frac{2k-\lambda}{k-\lambda}$ is an integer, and with edges connecting any two vertices of different parts.

Theorem 2: For any k-regular graph Γ we have

$$N_2(\Gamma,2) \geq \mu\left(k-1-\frac{3}{4}(\mu-1)(N(\Gamma,1)-2)\right)+2. \quad (9)$$

In proving (9) the linear programming problem arises for the vertex subset $U = \bigcup_{i=1}^{\mu} B_1(z_i)\setminus\{x,y\}$, where $x,y \in V$ with $d(x,y)=2$ and $z_i,\ i=1,...,\mu$, are the vertices at distance 1 from both x and y. The task is to minimize $|U| = \sum_{h=1}^{\mu} u_h$ for nonnegative numbers u_h satisfying the following conditions

$$\sum_{h=1}^{\mu} u_h h^2 \geq \mu(k-1),$$

$$\sum_{h=1}^{\mu} u_h h\binom{h}{2} \leq \binom{\mu}{2}(N(G,1)-2),$$

where $u_h = |U(h)|/h$, and $U(h)$ is the set of vertices in U belonging to h sets $B_1(z_i)$, $i=1,...,\mu$.

The details of the proof of this theorem as well as the proofs of most other results in this article can be found in [5]. From the last theorem one can immediately get the following corollaries.

Corollary 1: For a k-regular graph Γ,
(i) if $\mu = 1$, then $N_2(\Gamma,2) \geq k+1$;
(ii) if $\mu = 2$ and $N(\Gamma,1)=2$, then $N_2(\Gamma,2) \geq 2k$;
(iii) if $\mu = 3$ and $N(\Gamma,1)=3$, then $N_2(\Gamma,2) \geq 3k-5$.

Corollary 2: Let Γ be a k-regular graph without triangles or pentagons, with $\mu \geq 2$ and $k \geq 1+\frac{3}{4}(\mu-1)\mu$. Then

$$N_2(\Gamma,2) \geq N_1(\Gamma,2). \quad (10)$$

Actually, since Γ does not contain triangles or pentagons we have $N_1(\Gamma,2)=2k$ and $N(\Gamma,1)=\mu$ by (7) since $\lambda=0$ and $\mu \geq 2$. Using (9) we get

$$N_2(\Gamma,2)-2k \geq (\mu-2)(k-1-\frac{3}{4}(\mu-1)\mu) \geq 0,$$

and finally we obtain (10).

In the remainder of this section it is assumed that $\Gamma = Cay(G,S)$ is a Cayley graph on the group G for the generator set S. Let us put $S^0 = \{e\}$ and set $S^i = SS^{i-1}$. Moreover, by vertex–transitivity it is sufficient to consider only the spheres and balls with center e so that $S_i = S_i(e)$.

Lemma 1: For any Cayley graph Γ on the group G and for $i>0$ we have $S_i = S^i\setminus(S^{i-1}\cup S^{i-2}\cup...\cup S^0)$. In particular, μ is the maximum number of representations of an element in $S^2\setminus(S\cup S^0)$ as a product of two elements of S and λ is the maximum number of representations of an element in S as a product of two elements of S, i,e.

$$\lambda(\Gamma) = \max_{s\in S} \mid \{(s_i s_j)\in S^2 \,:\, s = s_i s_j\} \mid,$$

$$\mu(\Gamma) = \max_{s\in S^2\setminus(S\cup S^0)} \mid \{(s_i s_j)\in S^2 \,:\, s = s_i s_j\} \mid.$$

This lemma allows us to find $N(\Gamma,1)$ from (7) for a general Cayley graph. The results for estimating the values $N(\Gamma,r)$ for small r in Cayley graphs on the symmetric group Sym_n will be presented in the next section when the generator set S consists of transpositions.

III. The Reconstruction of Permutations in Cayley Graphs Generated by Transpositions

Let Sym_n be the symmetric group on n symbols. We write a permutation π in one–line notation as $\pi = [\pi_1,\pi_2,\ldots,\pi_n]$ where $\pi_i = \pi(i)$ for every $i \in \{1,\ldots,n\}$.

For the *transposition Cayley graph* $Sym_n(T)$ on Sym_n the generator set consists of all transpositions $T = \{t_{i,j} \in Sym_n,\ 1 \leq i < j \leq n\}$, $|T| = \binom{n}{2}$, where $t_{i,j}$ interchanges positions i and j when multiplied on the right, i.e., $[\ldots,\pi_i,\ldots,\pi_j,\ldots]\cdot t_{i,j} = [\ldots,\pi_j,\ldots,\pi_i,\ldots]$. For $x,\ y \in Sym_n$ the distance $d(x,y)$ is the least number of transpositions $t_1,\ ...,t_r$ such that $x\cdot t_1\cdot...\cdot t_r = y$, or $t_1\cdot...\cdot t_r = x^{-1}\cdot y$. As any k-cycle can be written as a product of $k-1$ transpositions (but no fewer), the diameter of $Sym_n(T)$ is $(n-1)$. The graph is bipartite since any edge joins an even permutation to an odd permutation. The symmetry properties of $Sym_n(T)$ have been discussed in [6]. The graph is edge–transitive but not distance–regular and hence not distance–transitive. Let us recall, that a simple connected graph Γ is *distance–transitive* if, for any two arbitrary–chosen pairs of vertices (x,y) and (x',y') at the same distance $d(x,y)=d(x',y')$, there is an automorphism σ of Γ satisfying $\sigma(x)=x'$ and $\sigma(y)=y'$, where an automorphism σ is a permutation of the vertex–set of a graph Γ provided that $\{x,y\}$ is an edge of Γ if and only if $\{\sigma(x),\sigma(y)\}$ is an edge of Γ. All these properties and other basic facts are collected in the following statements.

Lemma 2: The transposition graph $Sym_n(T),\ n\geq 3$,
(i) is a connected bipartite $\binom{n}{2}$-regular graph of order $n!$ and diameter $(n-1)$;
(ii) is not distance–regular and hence not distance–transitive;
(iii) it does not contain subgraphs isomorphic to $K_{2,4}$, and each of its vertices belongs to $\binom{n}{3}$ subgraphs isomorphic to $K_{3,3}$.

Here $K_{p,q}$ is the complete bipartite graph with p and q vertices in the two parts, respectively.

Theorem 3: For any $n \geq 3$ we have $N(Sym_n(T),1)=3$.

This means that any unknown permutation is uniquely reconstructible from 4 distinct permutations at transposition distance at most one from the unknown permutation. The proof of these statements is based on considering a permutation $\pi \in Sym_n$ in cycle notation, with *cycle type* $\mathrm{ct}(\pi) = 1^{h_1}2^{h_2}..\, n^{h_n}$, where h_i is the number of cycles of length i. In particular $\sum_i^n ih_i = n$. The permutation π can be also presented as a product of a least number of transpositions. Each such product represents a shortest path in $Sym_n(T)$ from e to π. The number of such paths was obtained in [7]. This result is based on Ore's theorem on the number of trees with n labeled vertices and presented by the following theorem.

Theorem 4: [7] Let $\pi \in Sym_n$ have cycle type $\mathrm{ct}(\pi) = 1^{h_1}2^{h_2}...n^{h_n}$, consisting of $\sum_{j=1}^n h_j = n - i$ cycles where $1 \le i \le n-1$. Then the number of distinct ways to express π as a product of i transpositions is equal to

$$i! \prod_{j=1}^{n} \left(\frac{j^{j-2}}{(j-1)!} \right)^{h_j}.$$

The following simple fact about multiplication by a transposition t_{ij} is essential: if a single cycle containing i and j is multiplied by $t_{i,j}$ then the resulting product consists of two disjoint cycles, each containing one of i and j. And *vice versa*, when two cycles each containing one of i and j are multiplied by $t_{i,j}$ then the product consists of a single cycle. It follows from this that $S_i := S_i(e)$, where e is the identity permutation, consists of all permutations having exactly $(n-i)$ disjoint cycles when the 1-cycles are included. Furthermore, the number of edges from a permutation $\pi \in S_i$ leading to a vertex in S_{i-1} corresponds to the distinct ways of splitting one of the cycles in π into two. In addition, as the elements in S_i have determinant $(-1)^i$ we must have that $a_i(\pi) := a_i(\pi, e) = 0$. We collect these facts in the following lemma where we abbreviate $c_i(\pi) := c_i(\pi, e)$ and $b_i(\pi) := b_i(\pi, e)$.

Lemma 3: In the transposition graph $Sym_n(T)$ the sets $S_i,\ 1 \le i \le n-1$, are the permutations consisting of $(n-i)$ disjoint cycles, counting also 1-cycles. For any $\pi \in S_i$ with cycle type $\mathrm{ct}(\pi) = 1^{h_1}2^{h_2}...\, n^{h_n}$, we have $a_i(\pi) = 0$ and

$$c_i(\pi) = \frac{1}{2}\left(\sum_{j=1}^{n} j^2 h_j - n \right),\ b_i(\pi) = \frac{1}{2}\left(n^2 - \sum_{j=1}^{n} j^2 h_j \right).$$

In particular, since $a_i(\pi) = 0$ for all $1 \le i \le n-1$, we have from this lemma and (4) that $\lambda(Sym_n(T)) = 0$. Moreover, it is well-known that two permutations are conjugate by an element of $G := Sym_n$ if and only if they have the same cycle type. If $(1^{h_1}2^{h_2}...\, n^{h_n})^G$ denotes the conjugacy class of an element of cycle type $1^{h_1}2^{h_2}...\, n^{h_n}$ then it is shown in [5] that $S_i,\ 1 \le i \le n-1$, is the disjoint union

$$S_i = \bigcup_{h_1+h_2+\cdots+h_n = n-i} (1^{h_1}\, 2^{h_2}\, ...\, n^{h_n})^G, \tag{11}$$

where

$$|(1^{h_1}2^{h_2}...\, n^{h_n})^G| = \frac{n!}{1^{h_1}h_1!2^{h_2}h_2! \cdots n^{h_n}h_n!}. \tag{12}$$

Hence, from (11) we have $S_2 = (1^{n-3}\, 3^1)^G \cup (1^{n-4}\, 2^2)^G$ and then by Lemma 3 we get $c_2(\pi) = 3$ if $ct(\pi) = 1^{n-3}\, 3^1$, and $c_2(\pi) = 2$ if $ct(\pi) = 1^{n-4}\, 2^2$. From these and (5) we have $\mu(Sym_n(T)) = 3$, and therefore, by (7) we get Theorem 3. Moreover, there are no subgraphs isomorphic to $K_{2,4}$ in $Sym_n(T)$ since $\mu(Sym_n(T)) = 3$. The number $\binom{n}{3}$ of subgraphs isomorphic to $K_{3,3}$ and having e as one of its vertices is obtained from (12) for any $\pi \in (1^{n-3}\, 3^1)^G$. By vertex–transitivity the same holds for any vertex in $Sym_n(T)$ (see condition (iii) in Lemma 2).

So, any unknown permutation is uniquely reconstructible from 4 distinct permutations at transposition distance at most 1 from the unknown permutation. As the following shows, in the case of at most two transposition errors the reconstruction of the permutation π requires many more its distinct 2–neighbors.

Theorem 5: For $n \ge 3$ we have

$$N(Sym_n(T), 2) = \frac{3}{2}(n-2)(n+1). \tag{13}$$

The details of the proof can be found in [5]. One important ingredient in the proof is the following observation which relies on the fact that conjugation of $G = Sym_n$ on itself is an automorphism of the Cayley graph $Sym_n(T)$:

Lemma 4: For any $\pi \in S_i,\ 1 \le i \le n-1$ the number of vertices in $(1^{h_1}2^{h_2}...\, n^{h_n})^G$ at a given distance from π depends only on the conjugacy class to which π belongs.

To prove Theorem 5 it is therefore sufficient to consider the numbers of vertices in all subsets of $B_2(e)$ at minimal distance at most 2 from a given vertex $\pi \in S_i,\ 1 \le i \le 4$. By (11) we have $S_1 = (1^{n-2}\, 2^1)^G$, $S_2 = (1^{n-3}\, 3^1)^G \cup (1^{n-4}\, 2^2)^G$, $S_3 = (1^{n-4}\, 4^1)^G \cup (1^{n-5}\, 2^1\, 3^1)^G \cup (1^{n-6}\, 2^3)^G$, $S_4 = (1^{n-5}\, 5^1)^G \cup (1^{n-6}\, 2^1\, 4^1)^G \cup (1^{n-6}\, 3^2)^G \cup (1^{n-7}\, 2^2\, 3^1)^G \cup (1^{n-8}\, 2^4)^G$. By direct analysis and counting it can be shown easily that $N_4(Sym_n(T), 2) = 20$ for $n \ge 5$, $N_3(Sym_n(T), 2) = 12$ for $n \ge 4$, $N_2(Sym_n(T), 2) = \frac{3}{2}(n-2)(n+1)$ and $N_1(Sym_n(T), 2) = n(n-1)$ for all $n \ge 3$. From these values we conclude (13) by using (6).

The statements of Theorem 5 and Corollary 2 are generalized in the following conjecture.

Conjecture 1: For any $\pi \in (1^{n-3}\, 3^1)^G$, for any $r \ge 1$ and $n \ge 2r+1$ we have

$$N(Sym_n(T), r) = N_2(Sym_n(T), 2) = |B_r(I) \cap B_r(\pi)|.$$

Now let us consider the *bubble–sort graph* $Sym_n(t)$. This is the Cayley graph on the symmetric group Sym_n for the generator set $t = \{t_{i,i+1} \in Sym_n,\ 1 \le i < n\}$, $|t| = n-1$. These *bubble–sort transpositions* are 2-cycles $t_{i,i+1}$ interchanging i and $i+1$ and determine the graph distance in $Sym_n(t)$ in the usual way. It is known that the diameter of $Sym_n(t)$ is $\binom{n}{2}$.

Lemma 5: The bubble–sort graph $Sym_n(t),\ n \ge 3$,
(i) is a connected bipartite $(n-1)$-regular graph of order $n!$ and diameter $\binom{n}{2}$;
(ii) it does not contain subgraphs isomorphic to $K_{2,3}$;
(iii) each of its vertices belongs to $\binom{n-2}{2}, n \ge 4$, subgraphs isomorphic to $K_{2,2}$.

The symmetry properties of the bubble–sort graph were discussed in [6] where it was shown that this graph is not distance–regular. As it is bipartite there are no triangles and hence $\lambda(Sym_n(t)) = 0$. If an element $\pi \in S_2(e)$ has at least two neighbors $t_{i,i+1} \neq t_{j,j+1}$ in $S_1(e)$ then necessarily $t_{i,i+1}t_{j,j+1} = \pi = t_{j,j+1}t_{i,i+1}$ with $\{j, j+1\}$ and $\{i, i+1\}$ disjoint. It suffices to verify this for permutations on 4 letters. Hence there are at most two such neighbors and so $\mu(Sym_n(t)) = 2$. It can be also verified that we have $N_4(Sym_n(t), 2) = 4$ for $n \geq 5$, $N_3(Sym_n(t), 2) = 2$ for $n \geq 4$, $N_2(Sym_n(t), 2) = N_1(Sym_n(t), 2) = 2(n-1)$ for $n \geq 3$. From all these and by (6) and (7) we get the following theorem.

Theorem 6: For any $n \geq 3$ we have

$$N(Sym_n(t), 1) = 2 \quad \text{and} \quad N(Sym_n(t), 2) = 2(n-1).$$

Almost the same results appear for the *star Cayley graph* $Sym_n(st)$ generated by the *prefix–transpositions* from the set $st = \{(1, i) \in Sym_n,\ 1 < i \leq n\},\ |st| = n-1$. It is one of the most investigated graphs in the theory of interconnection networks since many parallel algorithms are efficiently mapped on the star Cayley graph.

Lemma 6: [8] The star Cayley graph $Sym_n(st), n \geq 3$, is a connected bipartite $(n-1)$-regular graph of order $n!$ with diameter $\lfloor \frac{3(n-1)}{2} \rfloor$.

The star Cayley graph $Sym_n(st)$ is not distance–regular for $n \geq 4$ [6] and has no cycles of lengths of 3, 4, 5 or 7. Hence $\lambda(Sym_n(st)) = 0$ and $\mu(Sym_n(st)) = 1$. Moreover, it is easy to verify that $N_4(Sym_n(st), 2) = 4$ for $n \geq 5$, $N_3(Sym_n(st), 2) = 4$ for $n \geq 4$, $N_2(Sym_n(st), 2) = 2(n-1)$ for $n \geq 5$ and $N_1(Sym_n(st), 2) = 2(n-1)$ for $n \geq 4$. From these properties and by (6) and (7) we get the following theorem.

Theorem 7: For any $n \geq 4$ we have

$$N(Sym_n(st), 1) = 2 \quad \text{and} \quad N(Sym_n(st), 2) = 2(n-1).$$

Thus, in the bubble–sort and star Cayley graphs any unknown permutation π is uniquely reconstructible from 3 distinct 1–neighbors of π. Similarly, for the unique reconstruction of π from neighbors at distance at most 2 we see that any $2n-1$ distinct 2–neighbors of π are sufficient. These two graphs are examples for which the inequality (ii) in Corollary 1 is attained.

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