JOURNAL OF PURE AND APPLIED ALGEBRA

ELSEVIER Journal of Pure and Applied Algebra 151 (2000) 51–65

On modular homology in projective space

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Received 2 March 1998; received in revised form 24 February 1999

Communicated by A. Blass

Abstract

For a vector space V over $GF(q)$ let L_k be the collection of subspaces of dimension k. When R is a field let M_k be the vector space over it with basis L_k. The *inclusion map* $\partial : M_k \to M_{k-1}$ then is the linear map defined on this basis via $\partial(X) := \sum Y$ where the sum runs over all subspaces of co-dimension 1 in X. This gives rise to a sequence

$$\mathscr{M} : 0 \leftarrow M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_{k-1} \leftarrow M_k \leftarrow \cdots$$

which has interesting homological properties if R has characteristic $p > 0$ not dividing q. Following on from earlier papers we introduce the notion of π-*homological*, π-*exact* and *almost* π-*exact* sequences where $\pi = \pi(p, q)$ is some elementary function of the two characteristics. We show that $\mathscr{M}$ and many other sequences derived from it are almost π-exact. From this one also obtains an explicit formula for the Brauer character on the homology modules derived from $\mathscr{M}$. For infinite-dimensional spaces we give a general construction which yields π-exact sequences for finitary ideals in the group ring $RP\Gamma L(V)$.

MSC: 51D30; 20C20; 05E25; 06A08

1. Introduction

Let R be an associative ring with identity. Suppose that M_k for $k=0,\ldots$ are R-modules and that $\partial : \oplus_k M_k \to \oplus_k M_k$ is an R-homomorphism with $\partial(M_k) \subseteq M_{k-1}$. From this

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[1] This work is partly supported by a grant from the London Mathematical Society.

0022-4049/00/$ - see front matter
PII: S0022-4049(99)00053-5

we obtain the sequence

$$\mathscr{M} : 0 \xleftarrow{\partial} M_0 \xleftarrow{\partial} M_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} M_{k-1} \xleftarrow{\partial} M_k \xleftarrow{\partial} \cdots .$$

It is convenient to put $M_k = 0$ for $k < 0$ and to define $\partial : M_k \to M_{k-1}$ by putting $\partial(M_k) = 0$ for such values of k. We refer to R also as the *coefficient domain*.

Here we are concerned with the situation when ∂ is nilpotent so that $\partial^\pi = 0$ for some $\pi > 1$. When $\pi = 2$ then $\mathscr{M}$ is a homological sequence in the usual sense. However, for $\pi > 2$ the methods of classical homology theory do not apply immediately. Nevertheless, in a suitable setting, one can talk about homological properties of a sequence such as $\mathscr{M}$ for any nilpotent map ∂.

For this purpose it is necessary to consider certain sequences obtained from $\mathscr{M}$. Thus let $\mathscr{M}$ be as above and suppose that $\pi > 1$ is an arbitrary fixed integer. Now select positive integers $i^* < \pi$ and k^* with $k^* + i^* < \pi$ and consider the sequence

$$\mathscr{M}_{k^*,i^*} : 0 \leftarrow M_{k^*} \leftarrow M_{k^*+i^*} \leftarrow M_{k^*+\pi} \leftarrow M_{k^*+i^*+\pi} \leftarrow M_{k^*+2\pi} \leftarrow \cdots$$

in which each arrow is the appropriate power of ∂. Observe that $\mathscr{M}_{k^*,i^*}$ is homological if $\partial^\pi = 0$. Conversely, if $\mathscr{M}_{k^*,i^*}$ is homological for every choice of k^* and i^* then $\partial^\pi = 0$ and $\mathscr{M}$ is said to be *π-homological.*

One further general notion for homological sequences is important. If

$$\mathscr{A} : 0 \leftarrow A_0 \leftarrow \cdots \leftarrow A_{k-1} \leftarrow A_k \leftarrow A_{k+1} \leftarrow \cdots \leftarrow 0$$

is homological then $\mathscr{A}$ is said to be *almost exact* if at most one of the homology modules in $\mathscr{A}$ is non-zero. We extend this to π-homological sequences and call a π-homological sequence $\mathscr{M}$ *almost π-exact* if each $\mathscr{M}_{k^*,i^*}$ is almost exact.

The purpose of this paper is to investigate sequences such as $\mathscr{M}$ which are naturally associated to the projective geometry over a finite field. More precisely, let V be a vector space of arbitrary dimension over the field $GF(q)$ where q is some prime power and let L be the lattice of all finite-dimensional subspaces of V. If L_k denotes the collection of subspaces of dimension k let M_k be the free R-module with L_k as basis. The containment relation provides a natural *inclusion map* $\partial : \oplus_k M_k \to \oplus_k M_k$ defined on a basis by

$$\partial(X) = \sum Y,$$

where X is a finite-dimensional subspace of V and where the sum runs over all subspaces Y of co-dimension 1 in X.

If R has finite characteristic p not dividing q then ∂ is nilpotent and we define the function $\pi := \pi(p, q)$ as the least value π for which $\partial^\pi = 0$. Our main results show that $\mathscr{M}$ and various sequences constructed from it are almost π-exact.

Almost exact sequences play an important role in representation theory and combinatorics. For many important classes of partially ordered sets the order complex is known to be almost exact, these include for example Cohen–Macauley posets, see for instance [3]. Here the differential operator satisfies $\partial^2 = 0$ independently of the characteristic. This is not the case for our construction which leads to nilpotent maps only if the characteristic of R is non-zero. This may explain why we use the term *modular homology*. While modular homology is quite different from order homology it appears, as we shall see, that quite general classes of partially ordered sets give rise to almost exact sequences in both homology theories.

From the view point of group representation theory almost exact sequences are important because the Hopf–Lefschetz trace formula relates representations on the modules of the sequence directly to the representation on its only non-trivial homology module. This is very well studied for order homology and the Steinberg representations which are fundamental in many respects. While the modular homology discussed here has not yet been studied as extensively, nevertheless, several important families of representations have already been described in these terms.

The construction above for projective space can obviously be set up for quite general classes of partially ordered sets. It makes sense therefore to study modular homology in greater generality. This has been carried out for the Boolean algebra and certain of its rank-selected sublattices in [10,11]. Here typically the sequence attached to the poset, for coefficient domain of characteristic $p > 0$, is almost p-exact and the non-trivial homologies are interesting, often irreducible representations of the symmetric group. The results of this paper extend some of these observations to projective spaces and may indicate that modular homology is an interesting general tool to study both the poset and the modular representations of its automorphism group.

The Boolean algebra and projective spaces are examples of *differential posets* and in Section 3.2 we investigate modular homologies of such posets. We show that almost exactness can be derived from two facts: One are suitable conditions on the structure constants of the poset, the other is a statement to the extend that the kernel of ∂ can be generated by elements of suitably small 'support'. This leads to the general problem of determining the support size for kernels of inclusion maps in such posets. In [10,2,12] we have done this by elementary means for the Boolean algebra and products of chains but unfortunately we were not quite able to extend these techniques to projective spaces. In the same section we shall also briefly indicate how homotopy arguments can be used to establish almost π-exactness.

Modular homology appears to be mentioned first in two papers [9] by Mayer in 1947, further historical remarks and references can be found also in [2]. More recent papers[2] on the subject include Dubois-Violette [4] and Kapranov [8].

2. Notation

Let V be a vector space of arbitrary dimension over $F = GF(q)$ with q a prime power and let L be the lattice of all finite-dimensional subspaces of V. Using affine

[2] We wish to thank the referee for pointing these out to us and for other suggestions which led to improvements in an earlier draft of this paper.

dimensions, the *dimension* of L is $dim\, L := dim\, V$ and for $k < \infty$ the collection of all k-dimensional subspaces of V is denoted by L_k. The following standard analogues of combinatorial functions will be useful. We put

$$[i]_q := 1 + q + q^2 + \cdots + q^{i-1}$$

and define the *q-factorial function* by

$$(i!)_q := [1]_q \cdot [2]_q \cdot \cdots \cdot [i]_q.$$

The *q-binomial function* $\binom{n}{k}_q$ is the number of k-dimensional subspaces in V:

$$\binom{n}{k}_q := |L_k| = \frac{[n]_q \cdot [n-1]_q \cdots [n-k+1]_q}{[k]_q \cdot [k-1]_q \cdots [1]_q}$$

and it is convenient to put $\binom{n}{k}_q = 0$ if $k < 0$ or $k > n$. As a reference to Gaussian polynomials see Chapter 3 of [7] or the book of Andrews [1].

For the remainder we will assume that R is a finite field of characteristic $p > 0$ and in order to avoid confusion with the elements of L we will refer to vector spaces over R as, evidently free, R-modules. This will also be useful later when we are concerned with group actions over R. Thus the R-modules with L and L_k as basis will be denoted by RL and RL_k respectively. In particular, $RL = \bigoplus_{0 \le k} RL_k$. For RL_k and RL we will often write M_k and M respectively as the context is usually clear. In particular, M_0 is the one-dimensional module with the null space as basis and for convenience we put $M_k = 0$ for $k < 0$ or $k > dim\, V$. For $f = \sum r_X X \in M$ the *support* is the subspace *supp*(f) generated by all X with $r_X \neq 0$ and its dimension is the *support dimension* of f.

The *inclusion map* $\partial : M \to M$ is the R-homomorphism defined by $\partial(X) := \sum Y$ for $X \in L$ where the sum runs over all subspaces Y of co-dimension 1 in X. Instead of $\partial(f)$ we also write f' and $f^{(s)}$ denotes $[f^{(s-1)}]'$. It is clear that ∂ restricts to maps $\partial : M_k \to M_{k-1}$ for all k.

3. Homology

We will now examine the homological properties of the inclusion maps just defined. In the first part this is done for projective spaces. The results we obtain are very similar to those holding in the Boolean lattice [10,11,2]. This is not accidental: In the second part we shall explore this connection and explain how results of this kind can be derived for *join-meet regular* and *differential posets*.

3.1. Projective spaces

Let L be the projective geometry associated to the vector space V of dimension $dim\, V \le \infty$. Then $\partial : M_k \to M_{k-1}$ gives rise to the sequence

$$\mathcal{M} : 0 \xleftarrow{\partial} M_0 \xleftarrow{\partial} M_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} M_{k-1} \xleftarrow{\partial} M_k \xleftarrow{\partial} \cdots .$$

Table 1
The function $\pi(p,q)$

p	Values of q														
	2	3	4	5	7	8	9	11	13	16	17	19	23	25	27
2	–	2	–	2	2	–	2	2	2	–	2	2	2	2	2
3	2	–	3	2	3	2	–	2	3	3	2	3	2	3	–
5	4	4	2	–	4	4	2	5	4	5	4	2	4	–	4
7	3	6	3	6	–	7	3	3	2	3	6	6	3	3	2
11	10	5	5	5	10	10	5	–	10	5	10	10	11	5	5
13	12	3	6	4	12	4	3	12	–	3	6	12	6	2	13
17	8	16	4	16	16	8	8	16	4	2	–	8	16	8	16
19	18	18	9	9	3	6	9	3	18	9	9	–	9	9	6

We are interested in the homological properties of $\mathcal{M}$. So fix some $i \leq k$ and let $X \in L_k$. Then $\partial^i(X) = c \sum Y$ where the sum runs over all $Y \in L_{k-i}$ with $Y \subset X$ and where c is the number of saturated chains $Y = Y_0 \subset Y_1 \subset \cdots \subset Y_i = X$. It is easy to see that $c = 1 \cdot (1+q) \cdot \cdots \cdot (1+q+q^2+\cdots+q^{i-1})$ and so $\partial^i(X) = (i!)_q \sum Y$. If p divides q then $\partial^i(X) = \sum Y \neq 0$ for any $i \leq j$ and this case will be of no importance to us. On the other hand, if p does not divide q then there will be values of i with $(i!)_q \equiv 0 \bmod p$.

Definition 3.1. For co-prime integers p and q let $\pi(p,q) > 0$ be the least integer π for which $[\pi]_q \equiv 0 \bmod p$.

We have tabulated some values for $\pi(p,q)$ in Table 1.

Instead of $\pi(p,q)$ we often write just π. Note that $\pi \leq p$ and $\pi = p$ if $q \equiv 1 \bmod p$. If p does not divide $q-1$ then π is the order of q modulo p since $(1+q+q^2+\cdots+q^{\pi-1}) = (q^\pi - 1)/(q-1)$. In either case $q^\pi \equiv 1 \bmod p$ and if $\pi \geq 2$ then $((\pi-1)!)_q \not\equiv 0 \bmod p$ while $(\pi!)_q \equiv 0 \bmod p$. Therefore ∂^π is the least power of ∂ that vanishes on M.

For general parameters $0 < i < \pi$ and k we put $K_{k,i} := ker\, \partial^i \cap M_k$ and $I_{k,i} := \partial^{\pi-i}(M_{k+\pi-i})$. As $\partial^\pi = 0$ we have $I_{k,i} \subseteq K_{k,i}$ and we denote by

$$H_{k,i} := K_{k,i}/I_{k,i},$$

the homology module of the sequence

$$M_{k-i} \xleftarrow{\partial^i} M_k \xleftarrow{\partial^{\pi-i}} M_{k+\pi-i}.$$

Our next result shows that this sequence is exact except for particular choices of k and i:

Theorem 3.1. *Let R be a field of characteristic $p > 0$ and let L be the projective geometry over the finite field $GF(q)$ where p does not divide q. Let $i < \pi := \pi(p,q)$ and k be positive integers. Then $H_{k,i} = 0$ unless $dim\, L < 2k + \pi - i < dim\ Ł + \pi$.*

Proof. To show that $I_{k,i} \supseteq K_{k,i}$ let $f \in K_{k,i}$. As $supp(f)$ has finite-dimension it will be sufficient to assume that $V \supseteq supp(f)$ has finite dimension, say $dim\, V = n$.

The Case $2k+\pi-i \le n$: Using the rank formula of Theorem 3.1 in [6] one can show that the rank of $\partial^{\pi-i} : M_{k+\pi-i} \to M_k$ is

$$\binom{n}{k}_q - \binom{n}{k-i}_q + \binom{n}{k-\pi}_q - \binom{n}{k-i-\pi}_q + \cdots .$$

The same expression gives the dimension of $K_{k,i}$ and hence $I_{k,i} = K_{k,i}$.

The Case $n+\pi \le 2k+\pi-i$: Corresponding to $\partial : M_k \to M_{k-1}$ we define a new map $\varepsilon : M_k \to M_{k+1}$ by $\varepsilon(X) := \sum Z$ for $X \in L_k$ where the summation runs over all $Z \supset X$ with $dim\, Z = k+1$. The matrices of $\partial^i : M_{k+i} \to M_k$ and $\varepsilon^i : M_k \to M_{k+i}$ are transposed to each other so that in particular $dim\, \partial^i(M_k) = dim\, \varepsilon^i(M_{k-i})$ and $dim\, \partial^{\pi-i}(M_{k+\pi-i}) = dim\, \varepsilon^{\pi-i}(M_k)$. Hence,

$$\begin{aligned} dim\, H_{k,i} &= dim\, K_{k,i} - dim\, I_{k,i} = dim\, M_k - dim\, \partial^i(M_k) - dim\, \partial^{\pi-i}(M_{k+\pi-i}) \\ &= dim\, M_k - dim\, \varepsilon^i(M_{k-i}) - dim\, \varepsilon^{\pi-i}(M_k) \\ &= dim\, M_{n-k} - dim\, \partial^i(M_{n-k+i}) - dim\, \partial^{\pi-i}(M_{n-k}) \\ &= dim\, K_{n-k,\pi-i} - dim\, \partial^i(M_{n-k+i}) = dim\, H_{n-k,\pi-i}. \end{aligned}$$

As $n+\pi \le 2k+\pi-i$ we have $2(n-k)+\pi-(\pi-i) \le n$ and the result follows from the first part. $\square$

It would be desirable to have a direct proof of this result as in [10,2] and in Section 3.2 we will discuss how this might be done.

In view of Theorem 3.1 we make the following definition. If k and $i < \pi$ are integers with

$$dim\, V < 2k + \pi - i < dim\, V + \pi,$$

then the three consecutive terms $M_{k-i} \leftarrow M_k \leftarrow M_{k+\pi-i}$ in $\mathcal{M}_{k^*,i^*}$ — or simply the parameters (k,i) — are called a *middle term*. So note that either $k-1 \equiv k^* \bmod \pi$ or $k \equiv k^* \bmod \pi$. Clearly, middle terms occur only if L is finite and there may be no middle term for a particular $\mathcal{M}_{k^*,i^*}$. However, if $\mathcal{M}_{k^*,i^*}$ has a middle term then it is also unique and we speak of *the* middle term of $\mathcal{M}_{k^*,i^*}$. A restatement of Theorem 3.1 is therefore

Theorem 3.1′. *For $i^* < \pi$ and k^* with $k^* + i^* < \pi$ the sequence $\mathcal{M}_{k^*,i^*}$ is almost exact. If $\mathcal{M}_{k^*,i^*}$ contains no middle term then it is exact. If $\mathcal{M}_{k^*,i^*}$ has middle term (k,i) then its only non-trivial homology is the homology of $M_{k-i} \leftarrow M_k \leftarrow M_{k+\pi-i}$.*

Remark. Analogous results for the Boolean lattice have been proved in [10,11]. Here certain *inductive systems* of modular respresentations of the symmetric group occur, such as for instance the representations of Ryba [14]. These do not generalize in any obvious way to projective spaces. However, some case-by-case computations for low-dimensional projective spaces show that also here homologies often are irreducible $P\Gamma L(n,q)$-modules. While the inductive structure of these modules is probably a little more complicated, we shall see in Section 4 that the Brauer character can be computed explicitly from Theorem 3.1 and so these homology modules are in some sense described rather well. Their full classification has not yet been completed and remains an open problem.

A further application of Theorem 3.1 concerns generators of the kernel of ∂^i. If (k,i) is not a middle term then the set $\{\partial^{\pi-i}(X): X \in L_{k+\pi-i}\}$ generates the kernel of $\partial^i : M_k \to M_{k-i}$. Hence

Corollary 3.2. *Unless (k,i) is a middle term the kernel of $\partial^i : M_k \to M_{k-i}$ is generated by elements of support dimension at most $k+\pi-i$.*

3.2. Join-meet regular posets

Instead of projective space we consider more generally a partially ordered set $(L,\leq)$ with rank function. We adopt the same notation as before: the collection of elements of rank k is denoted L_k and RL_k or M_k is the free R-module with basis L_k.

For $X \in L_k$ the 'up-degree' is $d^+(X) := |\{Z \in L_{k+1}: X < Z\}|$ and $d^-(X) := |\{Y \in L_{k-1}: Y < X\}|$ is the 'down-degree'. The following two regularity conditions are of importance to us:

- R_1: The up- and the down-degree are finite and constant on L_k for all k. Thus $d_k^+ := d^+(X) < \infty$ and $d_k^- := d^-(X) < \infty$ are independent of $X \in L_k$, and
- R_2: For each k there is a constant c_k such that $X, X^* \in L_k$ implies $|\{Z \in L_{k+1}: X < Z > X^*\}| = c_k \cdot |\{Y \in L_{k-1}: X > Y < X^*\}|$.

The Boolean lattice 2^n satisfies these conditions (with $d_k^+ = n-k$, $d_k^- = k$ and $c_k = 1$ for all $k \leq n$) and so do projective spaces (with $d_k^+ = (q^{n-k}-1)/(q-1)$, $d_k^- = (q^k-1)/(q-1)$ and $c_k = 1$ for all $k \leq n$). For an incidence structure with 'point set' P and 'block set' B consider the poset $L = \{o\} \cup P \cup B$ with $o < p < b$ iff $p \in P$ is incident with $b \in B$. Then L satisfies R_1 and R_2 iff (P,B) is a 2-design. (Here c_1 is the parameter λ of the design and so c_k is not always equal to 1.) Similarly, symmetric designs can be characterised by the condition that R_1 and R_2 hold in the poset $\{o\} \cup P \cup B \cup \{1\}$ with $o < p < b < 1$ iff p is incident with b. In [15] we called a poset satisfying R_1 and R_2 *join-meet regular*. The posets for which $c_k = 1$ for all k are the **r**-*differential posets* of [16].

The notion of an inclusion map can, of course, be extended to such more general objects. In fact, for any ranked locally finite poset there are two natural homomorphisms which are dual to each other: $\partial : M_k \to M_{k-1}$ and $\varepsilon : M_k \to M_{k+1}$ defined by $\partial(X) = \sum Y$ where the summation runs over all $Y \in L_{k-1}$ covered by X, and $\varepsilon(X) = \sum Z$ where the summation runs over all $Z \in L_{k+1}$ which cover X.

Lemma 3.3. *If L is join-meet regular and if $c_k \in R$ then $f \in M_k$ implies*

$$\partial\varepsilon(f) - c_k \cdot \varepsilon\partial(f) = (d_k^+ - c_k d_k^-) \cdot f.$$

Proof. This follows directly from the regularity conditions if $f \in L_k$ and hence by linearity in general, alternatively see Lemma 2.1 in [15]. □

Lemma 3.4. *Let L be join-meet regular, suppose that* $c_k = \cdots = c_{k+i} = 1$ *for some k and* $i \geq 1$, *and put* $d_j := (d_j^+ - d_j^-)$. *Then* $f \in Ker\,\partial \cap M_k$ *implies that*

$$\partial^i \varepsilon^i(f) = (d_k)(d_k + d_{k+1}) \cdots (d_k + d_{k+1} + \cdots + d_{k+i-2} + d_{k+i-1}) \cdot f.$$

Proof. For $i = 1$ this is Lemma 3.3. Hence suppose the result holds for i. Then $\psi := \partial^i \partial \varepsilon \varepsilon^i = \partial^i (d_{k+i} + \varepsilon \partial) \varepsilon^i$ by Lemma 3.3. Now continue $\psi = d_{k+i} \partial^i \varepsilon^i + \partial^i \varepsilon (\partial \varepsilon) \varepsilon^{i-1} = (d_{k+i} + d_{k+i-1}) \partial^i \varepsilon^i + \partial^i \varepsilon^2 \partial \varepsilon^{i-1}$, etc. As $\partial f = 0$ we get $\psi(f) = (d_{k+i} + d_{k+i-1} + \cdots + d_k) \partial^i \varepsilon^i(f)$ which completes the proof. □

Thus $Ker\,\partial \cap M_k \subseteq \partial^i(M_{k+i})$ if $(d_k)(d_k + d_{k+1}) \cdots (d_k + d_{k+1} + \cdots + d_{k+i-1}) \neq 0$ in R. That this observation can be made to work is illustrated in the following examples. As before the characteristic of the coefficient domain is denoted by p.

Lemma 3.5 (projective space)**.** *Let L be the projective space associated to* $GF(q)^n$. *For given* $k \leq n$ *and* $0 \leq j \leq n - k$ *we have*

(a) $d(j) := (d_k + \cdots + d_{k+j-1} + d_{k+j}) = (q^k/(q-1))(q^{n-2k-j} - 1)(1 + q + \cdots + q^j)$, *and in particular, when* $\pi := \pi(p, q)$ *then,*

(b) $d(0)d(1) \cdots d(\pi - 2) \not\equiv 0 \bmod p$ *if and only if* $2k - 1 \equiv n \bmod \pi$.

Proof. From the definition we get $d_{k+j} = [(q^{n-k-j} - 1) - (q^{k+j} - 1)]/(q-1)$ and (a) is checked easily for $j \leq 2$. Thus compute

$$\begin{aligned} d(j) - d(j-1) &= \frac{q^k}{q-1}[(q^{n-2k-j} - 1)(1 + q + \cdots + q^j) \\ &\quad - (q^{n-2k-j+1} - 1)(1 + q + \cdots + q^{j-1})] \\ &= \frac{q^k}{q-1}(q^{n-2k-j} - q^j) = d_{k+j} \end{aligned}$$

which proves (a). For (b) note that $q^{s\pi} \equiv 1 \bmod p$ for any integer s, see the paragraph after Definition 3.1. Now apply (a). □

Precisely the same arguments can be applied to the lattice 2^Ω of all subsets of a set Ω of cardinality n:

Lemma 3.6 (subset lattice)**.** *Let* $L = 2^\Omega$ *with* $|\Omega| = n$ *and suppose that* $k \leq n$ *and* $0 \leq j \leq n - k$ *are given. Then*

(a) $d(j) := (d_k + \cdots + d_{k+j-1} + d_{k+j}) = (n - 2k - j)(1 + j)$, *and in particular,*

(b) $d(0) \cdot d(1) \cdot \cdots \cdot d(p-2) \not\equiv 0 \bmod p$ *if and only if* $2k - 1 \equiv n \bmod p$.

Proof. By definition $d_{k+j} = (n - k - j) - (k + j)$ and applying the same procedure as in Lemma 3.5 yields the result. As is often the case, 2^n appears as the 'projective space with $q = 1$' and Lemma 3.6 is the special case of Lemma 3.5 for $q = 1$. □

In [10] we have proved the analogue of Theorem 3.1 above for the Boolean lattice 2^{Ω}, stating that any interval of $\mathscr{M}$ is p-exact unless it contains a middle term. To show how useful the Lemmas 3.5 and 3.6 are we sketch an independent proof of this result. As starting step use the 'support size theorem' (Theorem 2.2 in [10], see also its generalization in [2]), stating that the kernel of $\partial : M_k \to M_{k-1}$ is generated by elements of support size at most $2k$, irrespective of the characteristic of the coefficient domain. Secondly, if $f \in Ker\,\partial \cap M_k$, we select some set Ω^* of cardinality $2k$ with $supp(f) \subseteq \Omega^* \subseteq \Omega$. Now apply Lemma 3.5 to $n^* = 2k$ to show that $d := d(0)d(1)\cdots d(p-2) \neq 0$ in R. Denote by ε^* the map relative to Ω^* and put $F := d^{-1}(\varepsilon^*)^{p-1}(f)$. Then $f = \partial^{p-1}(F)$ and we have $H_{k,1} = 0$. For $i > 1$ and $2k + p - i \leq n$ similar arguments can be used to determine $H_{k,i}$ and for parameters $2k - i \geq n$ easy symmetry arguments as in the second part of the proof of Theorem 3.1 will suffice. This argument shows also that the modular homology of an infinite but locally finite join-meet regular poset can be decided entirely from the structure coefficients.

This proof thus falls into two parts, one relying on a good bound for the support size or support dimension of generators for the kernel of ∂, and the other based on the structure coefficients of the poset. The essential tool is Lemma 3.3 which in effect gives a homotopy equivalence. This may indicate that some of the standard notions from homotopy theory can be adapted also to modular homology. For further information on this question we refer also to Chapter 1.12 in [13], Section 2 in [4] and Kapranov's paper [8]. As to the first part: A proof of Theorem 3.1 along the same lines would require a theorem about generators of the kernel of ∂. We have the following:

Projective Support Dimension Conjecture. *Let R be an arbitrary associative ring with 1 and let $\partial : M_k \to M_{k-1}$ be the inclusion map for a projective geometry over a finite field. Then $Ker\,\partial \cap M_k$ can be generated by elements of support dimension at most $2k$.*

In [7] (p. 65) it is shown that the Specht module attached to M_k has generators of support dimension $2k$. This does not yet imply the conjecture as the Specht module may be strictly contained in the kernel of ∂. Parts of the conjecture are implied by Corollary 3.2 when $k \geq \pi$ and it may be possible to settle the remainder with the help of Theorem 3.1. However, the real task is to find an elementary and independent proof as in [10,2]. We have been unable to make much progress on this question and leave it as an open problem.

4. The Lefschetz character and group actions

We will now investigate group actions on the modules of a π-homological sequence and describe two general constructions which will lead to a wealth of almost π-exact sequences.

4.1. The Hopf–Lefschetz trace formula

We begin by recollecting some standard results from algebraic topology. Let R now be a field and let A_k for $k \geq 0$ be finite-dimensional R-modules with an R-homomorphism $\partial\colon \bigoplus_k A_k \to \bigoplus_k A_k$. Suppose that

$$\mathscr{A} : 0 \leftarrow A_0 \leftarrow \cdots \leftarrow A_{k-1} \xleftarrow{\partial} A_k \leftarrow A_{k+1} \leftarrow \cdots \leftarrow 0$$

is a homological sequence of finitely many non-zero terms. We use the natural notation $K_k := A_k \cap Ker\,\partial$ and $I_k := \partial(A_k) \subseteq K_{k-1}$, and for $k = 0, 1, \ldots$ we denote the homology modules by $H_k := K_k/I_{k+1}$. Their dimension are the *Betti numbers* $\beta_k := \dim H_k$.

Suppose that the group G acts on each A_k and commutes with ∂. Then there is a natural G action on each H_k. We denote the character of G on A_k by $\chi(g, A_k)$ and the character of G on H_k by $\chi(g, H_k)$. The Hopf–Lefschetz trace formula then states:

Theorem 4.1. $\sum (-1)^k \chi(g, H_k) = \sum (-1)^k \chi(g, A_k)$.

For the elementary proof see for instance [13]. In an almost exact sequence there will be at most one non-trivial homology module and this is usually called the *Lefschetz module* of the sequence. As a corollary we have therefore

Theorem 4.2. *If $\mathscr{A}$ is almost exact with Lefschetz module H then $\chi(g, H) = \pm \sum (-1)^k \chi(g, A_k)$.*

The $\pm$ sign is not an ambiguity: it depends on the position of H in the sequence and so choose the parity such that $\chi(1, H) \geq 0$. An interesting further observation can be made when each A_k is a permutation module for G. Here let $\mathit{fix}(g, A_k)$ denote the number of fixed points of g on the set underlying A_k. Then clearly $\mathit{fix}(g, A_k) \equiv \chi(g, A_k) \bmod p$.

Theorem 4.3. *If $\mathscr{A}$ is almost exact with Lefschetz module H define $\bar{\chi}(g, H) := \pm \sum (-1)^k \mathit{fix}(g, A_k)$. Then $\bar{\chi}(g, H)$ is the Brauer character of G on H.*

The proof of this result is not very difficult and is left to the reader. A special case which can be proved independently is the following dimension formula:

Theorem 4.4. *If $\mathscr{A}$ is almost exact with only non-trivial homology H then H has dimension $dim(H) = |\sum (-1)^k dim\, A_k|$.*

4.2. Finite-dimensional spaces

We now return to the more specific situation when V is a vector space over $GF(q)$ of dimension $n \leq \infty$ and R a field of characteristic p not dividing q. Throughout π stands for $\pi(p, q)$.

The projective general semi-linear group $P\Gamma L(V)$ of V acts on L by $X \mapsto X^g$ and this action extends linearly to M. Also the group ring $RP\Gamma L(V)$ over R acts naturally on M: for $a = \sum_g a_g g \in RP\Gamma L(V)$ we put $X^a := \sum_g a_g X^g$ and extend this to M by $f = \sum f_X X \mapsto f^a := \sum f_X X^a$. This action commutes of course with the inclusion map and so all homologies are $RP\Gamma L(V)$-modules.

First suppose that $V = (GF(q))^n$ has finite dimension and that G is a subgroup of $P\Gamma L(V)$. Then the *orbit module* of G is

$$M_k^G := \{f \in M_k\colon\ f^g = f,\ \forall g \in G\}.$$

Its natural basis are the 'orbit sums' $\sum_{X^* \in X^G} X^*$ where as usual $X^G := \{X^g\colon\ g \in G\}$ and, in particular,

$$n_k^G := dim\, M_k^G$$

is the number of G-orbits on L_k. Such orbit numbers play an important role in enumerating combinatorial structures and configurations in projective spaces, see also [17,18]. As $\partial(M_k^G) \subseteq (M_{k-1}^G)$ we obtain a sequence of orbit modules

$$\mathscr{M}^G\colon 0 \xleftarrow{\partial} M_0^G \xleftarrow{\partial} M_1^G \xleftarrow{\partial} \cdots \xleftarrow{\partial} M_{k-1}^G \xleftarrow{\partial} M_k^G \xleftarrow{\partial} \cdots$$

which is always π-homological. For $0 < i^* < \pi$ and $0 \le k^*$ with $k^* + i^* < \pi$ we obtain as before a sequence

$$\mathscr{M}_{k^*,i^*}^G : 0 \leftarrow M_{k^*}^G \leftarrow M_{k^*+i^*}^G \leftarrow\ M_{k^*+\pi}^G \leftarrow M_{k^*+i^*+\pi}^G \leftarrow M_{k^*+2\pi}^G \cdots$$

in which arrows are appropriate powers of ∂. For $0 < i < \pi$ let $K_{k,i}^G$ denote $ker\,\partial^i \cap M_k^G$ and let $H_{k,i}^G := K_{k,i}^G / \partial^{\pi-i}(M_{k+\pi-i}^G)$ be the corresponding homology module. The dimension of $H_{k,i}^G$ is the *Betti number* $\beta_{k,i}^G := dim\, H_{k,i}^G$. In particular, if G is the identity group then $H_{k,i}^G = H_{k,i}$ and we put $\beta_{k,i}^n := dim\, H_{k,i}$.

Theorem 4.5. *Let $\mathscr{M}$ be the sequence associated to the subspace lattice of $GF(q)^n$ over a coefficient field R of characteristic $p > 0$ co-prime to q and let G be a subgroup of $P\Gamma L(V)$ whose order is co-prime to p. Then $H_{k,i}^G = 0$ unless (k,i) is a middle term. In particular, $\mathscr{M}^G$ is almost π-exact.*

Denote the Lefschetz modules of $\mathscr{M}_{k^*,i^*}$ and $\mathscr{M}_{k^*,i^*}^G$ by $H_{k,i}$ and $H_{k,i}^G$, respectively. If $G \trianglelefteq K \subseteq P\Gamma L(n,q)$ then $H_{k,i}^G$ is an RK-module with Brauer character

$$\chi(x, H_{k,i}^G) = \pm \sum_{t \in \mathbb{Z}} fix(x, M_{k-\pi t}^G) - fix(x, M_{k-i-\pi t}^G),$$

where $fix(x, M_k^G)$ denotes the number of G-orbits on L_k left invariant by $x \in K$. Further, if C denotes the fixed module of G on $H_{k,i}$ then $C \cong H_{k,i}^G$ and G has a fixed-point free representation on $H_{k,i}/C$ whose Brauer character is

$$\chi(x, H_{k,i}/C) = \chi(x, H_{k,i}) - \chi(x, H_{k,i}^G),$$

where $fix(x, M_k)$ is the number of spaces in L_k left invariant by $x \in G$.

For the dimensions of the various homology modules we have therefore

Corollary 4.6. (1) *$\dim H = \beta^n_{k,i} = \sum_{t\in\mathbb{Z}} \binom{n}{k-\pi t}_q - \binom{n}{k-i-\pi t}_q$,*

(2) *$\beta^n_{k,i} \geq \dim H^G = \beta^G_{k,i} = \sum_{t\in\mathbb{Z}} n^G_{k-\pi t} - n^G_{k-i-\pi t}$, and*

(3) *If p does not divide $|K|$ then $\beta^G_{k,i} \geq \beta^K_{k,i}$.*

Proof. If (k,i) is not a middle term suppose that $f \in M^G_k \cap K_{k,i}$. By Theorem 3.1 there is some $F \in M_{k+\pi-i}$ with $\partial^{\pi-i}(F) = f$. Therefore $F^* := |G|^{-1}\sum_{g\in G} F^g$ belongs to $M^G_{k+\pi-i}$ and $\partial^{\pi-i}(F^*) = f$. Thus $\mathscr{M}^G$ is almost π-exact.

As $G \trianglelefteq K$ elements of K permute the orbits of G so that M^G_k is a permutation module for K and each k. The formula for $\chi(x, H^G_{k,i})$ now follows from Theorems 4.5 and 4.3. In Proposition 4.4 in [2] it is shown more generally that the fixed module of the homology is isomorphic to the homologies of the fixed modules, at least as long as the group order is co-prime to the characteristic. The latter also implies that no element of G fixes all cosets of C in $H_{k,i}$ so that G is fixed point free on $H_{k,i}/C$. The formula for the character on $H_{k,i}/C$ follows from the first part of the Theorem with $G = 1$. The corollary is straightforward. □

Remark. (1) The inequality $\beta^n_{k,i} \geqslant \sum_{t\in\mathbb{Z}} n^G_{k-\pi t} - n^G_{k-i-\pi t}$ is specific to the group order, it may not hold when $|G|$ has order divisible by p.

(2) If the order of G is co-prime to two different prime p_1 and p_2 and if $\pi(p_1,q) = \pi(p_2,q)$ then the Lefschetz modules in these characteristics have the same dimension. In fact, more generally, the Brauer characters for characteristics p_1 and p_2 are the same.

(3) For fixed π the function $\varphi^n_{k,i} := \sum_{t\in\mathbf{Z}} \binom{n}{k-\pi t}_q - \binom{n}{k-i-\pi t}_q$ is periodic in k and i of period at most π. When $q = 1$ the functional relations for binomial coefficients immediately carry over to $\varphi^n_{k,i}$ and hence to the Betti numbers. For instance, we have $\beta^n_{k,i} = \beta^{n-1}_{k,i} + \beta^{n-1}_{k-1,i}$ as this holds for binomial coefficients. More interestingly we have also $\beta^n_{k,i} = \beta^{n-1}_{k,i+1} + \beta^{n-1}_{k-1,i-1}$ and this corresponds directly to an inductive decomposition of the corresponding homology module, see Theorem 6.2 in [2]. There are q-analogues of such formulae for $q > 1$ but it is not yet clear if these relate to similar inductive decompositions. It may even be possible to derive relations for $\beta^G_{k,i}$ for a general group G of order coprime to p.

(4) The inequality $\beta^G_{k,i} \geq \beta^K_{k,i}$ for $|G|$ and $|K|$ co-prime to p implies for large $p > n$ a Fisher type inequality $n^G_k - n^G_{k-i} \geq n^K_k - n^K_{k-i}$ which may be of independent interest.

When $\pi(p,q) = 2$ then π-exact sequences are exact in the usual sense and so this case warrants special attention:

Theorem 4.7. *For a coefficient field of characteristic $p > 0$ let $\mathscr{M}$ be the sequence for the subspace lattice of $V = GF(q)^n$ and suppose that $1 + q \equiv 0 \bmod p$. Further, let $G \subseteq P\Gamma L(V)$ have order co-prime to p.*

(1) *If n is odd then $\mathscr{M}$ and $\mathscr{M}^G$ are exact,*
(2) *If $n = 2m$ is even then* $\beta^n_{m,1} = (q-1)(q^3-1)\cdot\ldots\cdot(q^{n-1}-1) \geq \beta^G_{m,1} = n^G_m - 2\sum_{k=1..m}(-1)^{k+1}n^G_{m-k}$.

Proof. The condition $1+q \equiv 0 \bmod p$ is equivalent to $\pi = 2$ and for odd n there is no k with $n < 2k+1 < n+2$. Thus (1) follows from Theorems 3.1 and 4.5. If n is even, then $(m,1)$ is the only middle term and $\beta^n_{m,1}$ can be evaluated from Corollary 4.6. The formula for $\beta^n_{m,1}$ is Theorem 3.4 of [1]. □

Remark. This result has recently also been proved by Fisk in [5].

We conclude this section by giving an application which allows to compute the rank of *orbit inclusion matrices*. For this let $t \leq k$ and $t+k \leq n$ (so that $n^G_t \leq n^G_k$). Define the matrix $W^G_{t,k}$ whose columns are indexed by G-orbits on L_k, rows indexed by G-orbits on L_t, such that the (i,j)-entry, for a fixed $X \in L_k$ in the jth orbit, counts the number of $Y \in L_t$ with $Y \subseteq X$ belonging to the ith orbit.

Viewing $W^G_{k-i,k}$ as a matrix over R it is clear that up to a constant $W^G_{k-i,k}$ is the matrix of $\partial^i : M^G_k \to M^G_{k-i}$. As the initial section of the sequence $\mathscr{M}^G_{k^*,i^*}$ is exact by Theorem 4.5 it is easy find the rank of the inclusion maps in that section in the sequence. The following generalizes a rank formula of [6].

Corollary 4.8. *If p does not divide the order of G and if $k, 0 < i < \pi$ satisfy $2k-i \leq n$ then $n^G_{k-i} - n^G_{k-\pi} + n^G_{k-\pi-i} - \cdots$ is the p-rank of $W^G_{k-i,k}$.*

4.3. Infinite-dimensional spaces

Now suppose that L has infinite dimension. Let $RP\Gamma L(V)$ be the group ring of $P\Gamma L(V)$ over R. Here we will use the convention that a subring of $RP\Gamma L(V)$ may not contain the identity of $RP\Gamma L(V)$. So if A is such a subring of $RP\Gamma L(V)$ we put

$$A_k := \langle f^a\colon a \in A, f \in M_k\rangle.$$

As the inclusion map naturally restricts to $\partial : A_k \to A_{k-1}$ we have a sequence of M-submodules

$$\mathscr{A} :\ 0 \xleftarrow{\partial} A_0 \xleftarrow{\partial} A_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} A_{k-1} \xleftarrow{\partial} A_k \xleftarrow{\partial} \cdots$$

which is always π-homological. So consider π-exactness:

Definition 4.1. A subring A of $RP\Gamma L(V)$ is *ample* if $f \in A_k$ with $\partial(f) = 0$ implies that there is some $F \in A_{k-1+\pi}$ with $f = \partial^{\pi-1}(F)$.

Lemma 4.9. *A is ample if and only if $\mathscr{A}$ is π-exact.*

Proof. One implication is obvious and $\mathscr{A}$ is π-homological in any case. We show the exactness of $A_{k-i} \leftarrow A_k \leftarrow A_{k-i+\pi}$ by induction on i. The definition of ampleness is the case $i=1$. So suppose that $A_{k-i} \leftarrow A_k \leftarrow A_{k-i+\pi}$ is exact for all k and suppose that $f \in A_k$ satisfies $f^{(i+1)}=0$. Then $[f']^{(i)}=0$ and by induction there is some $F_0 \in A_{k-i-s+\pi}$ with $F_0^{(\pi-i)}=f'$. Hence $[F_0^{(\pi-i-1)}-f]'=0$ and so there is some $F_1 \in A_{k-1-i+\pi}$ with $F_1^{(\pi-i-1)}=F_0^{(\pi-i-1)}-f$. Hence $f=[F_0-F_1]^{(\pi-i-1)}$. $\square$

We give some examples of ample rings. An element $g \in P\Gamma L(V)$ is *finitary* if its fixed-space $Fix(g):=\{v \in V\colon v^g=v\}$ has finite co-dimension in V. An element $a=\sum a_g g \in RP\Gamma L(V)$ is *finitary* if $a_g \neq 0$ implies that g is finitary and a subring of $RP\Gamma L(V)$ is *finitary* if it is the R-linear span of finitary elements in $RP\Gamma L(V)$.

Theorem 4.10. *If A is a finitary subring of $RP\Gamma L(V)$ then $\mathscr{A}$ is π-exact.*

Proof. Let $f=f_1^{a_1}+f_2^{a_2}+\cdots+f_t^{a_t} \in Ker\,\partial \cap A_k$ with $f_i \in M_k$ and $a_i \in A$. If $a_i=\sum_s a_{i,s}g_{i,s}$ let $W_{i,s}:=Fix(g_{i,s})$ and let S_i be the support of f_i. Then $\bigcap_{i,s} W_{i,s}$ has finite co-dimension in V and so there is a finite-dimensional subspace V^* which is invariant under all $g_{i,s}$ and which contains S_i for all $i \leq t$. Furthermore, enlarging V^* if necessary, we can assume that $dim\,V^* \equiv 2k-1 \bmod p$.

Let M_j^* for $0 \leq j$ denote the module relative to the lattice of subspaces of V^* and let $\varepsilon^*: M_j^* \to M_{j+1}^*$ be the map described in Section 3.2. By Lemma 3.4 we have $\partial^{\pi-1}(\varepsilon^*)^{\pi-1}(f)=(d_k)(d_k+d_{k+1})\cdots(d_k+d_{k+1}+\cdots+d_{k+\pi-2})\cdot f$ where $d:=(d_k)(d_k+d_{k+1})\cdots(d_k+d_{k+1}+\cdots+d_{k+\pi-2})$ is non-zero by Lemma 3.6. Hence $\partial^{\pi-1}[d^{-1}\cdot(\varepsilon^*)^{\pi-1}(f)]=f$. It remains to show that $(\varepsilon^*)^{\pi-1}(f) \in A_{k+\pi-1}$.

As all terms in $a_i=\sum_s a_{i,s}g_{i,s}$ leave V^* invariant we have $\varepsilon^*(f_i^{a_i})=[\varepsilon^*(f_i)]^{a_i}$. Thus $(\varepsilon^*)^{\pi-1}(f_1^{a_1}+f_2^{a_2}+\cdots+f_t^{a_t})=[(\varepsilon^*)^{\pi-1}(f_1)]^{a_1}+[(\varepsilon^*)^{\pi-1}(f_2)]^{a_2}+\cdots+[(\varepsilon^*)^{\pi-1}(f_t)]^{a_t}$ which belongs to $A_{k+\pi-1}$. $\square$

Here are some consequences of Theorem 4.10. Firstly, it gives a simple and self-contained proof for the fact that $\mathscr{M}$ is π-exact if V is infinite dimensional: take as A the subring generated by the identity of $P\Gamma L(V)$. In Section 4 of [11] we have considered orbit modules associated to groups acting on L. These are constructed as follows. For a subgroup G of $P\Gamma L(V)$ let $A:=\langle g_1-g_2\colon g_i \in G\rangle$ be the augmentation ideal of G so that $A_k=\langle X^{g_1}-X^{g_2}\colon X \in L_k, g_i \in G\rangle$.

Observe that $X+A_k=X^*+A_k$ for $X, X^* \in L_k$ iff X and X^* belong to the same G-orbit and further, that $X_1, X_2, \ldots$ is a system of orbit representatives iff $X_1+A_k, X_2+A_k, \ldots$ is a basis of M_k/A_k. For this reason we call

$$O_k := M_k/A_k$$

the *orbit module* of G on L_k and so we obtain an orbit module sequence induced from $\mathscr{M}$, that is

$$\mathscr{O}(G): 0 \xleftarrow{\partial} O_0 \xleftarrow{\partial} O_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} O_{k-1} \xleftarrow{\partial} O_k \xleftarrow{\partial} \cdots .$$

As $\mathscr{M}$ is π-exact we know that $\mathscr{O}(G)$ is π-exact if $\mathscr{A}$ has this property. Thus

Theorem 4.11. *If G is a finitary subgroup of $P\Gamma L(V)$ then $\mathscr{O}(G)$ is π-exact.*

Suppose now that the group G has finitely many orbits on some L_k. Then $t < k$ implies $n_t^G \leq n_k^G$ and one can define the $W_{k-i,k}^G$ as before. It is quite easy to see that $W_{k-i,k}^G$ is the matrix of $\partial^i : O_k^G \to O_{k-i}^G$ up to a constant. Therefore one can write down a formula for its rank just as in Corollary 4.8.

Theorem 4.12. *Let p be a prime and let G be a finitary subgroup of $P\Gamma L(V)$ with $n_k^G < \infty$ for some k. If $0 < i < \pi := \pi(p,q)$ then $n_{k-i}^G - n_{k-\pi}^G + n_{k-i-\pi}^G - n_{k-2\pi}^G \cdots$ is the p-rank of $W_{k-i,k}^G$.*

References

[1] G.E. Andrews, The Theory of Partitions, Addison-Wesley, Reading, MA, 1976.

[2] S. Bell, P. Jones, I.J. Siemons, On modular homology in the Boolean Algebra II, J. Algebra 199 (1998) 556–580.

[3] A. Björner, Topological methods, in: R.L. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics, North-Holland, Amsterdam, 1995 (Chapter 34).

[4] M. Dubois-Violette, $d^N = 0$: Generalized homology, K-Theory 14 (1998) 371–404.

[5] S. Fisk, Homology of projective space over finite fields, J. Combin. Theory, Ser. A 78 (1997) 309–312.

[6] A. Frumkin, A. Yakir, Rank of inclusion matrices and modular representation theory, Israel J. Math. 71 (1990) 309–320.

[7] G.D. James, Representations of General Linear Groups, LMS Lecture Note Series, vol. 94, Cambridge University Press, Cambridge, 1984.

[8] M.M. Kapranov, On the q-analogue of homological algebra, Cornell University Reprint, 1991.

[9] W. Mayer, A new homology theory Ann. Math. 48 (1947) 370–380, 594–605.

[10] V.B. Mnukhin, I.J. Siemons, The modular homology of inclusion maps and group actions, J. Combin. Theory 74 (1996) 287–300.

[11] V.B. Mnukhin, I.J. Siemons, On the modular homology in the Boolean algebra, J. Algebra 179 (1996) 191–199.

[12] V.B. Mnukhin, I.J. Siemons, Kernels of incidence maps, to appear.

[13] J.R. Munkres, Elements of Algebraic Topology, Addison-Wesley, Reading, MA, 1984.

[14] A.J.E. Ryba, Fibonacci representations of the symmetric groups, J. Algebra 170 (1994) 678–686.

[15] I.J. Siemons, On a class of partially ordered sets and their linear invariants, Geometriae Dedicata 41 (1992) 219–228.

[16] R.P. Stanley, Variations on differential posets, in: D. Stanton (Ed.), Invariant Theory and Tableaux, Springer, Berlin, 1990, pp. 145–165.

[17] Zhe-Xian Wan, Anzahl theorems in finite singular symplectic, unitary and orthogonal geometries, Discrete Math. 123 (1993) 131–150.

[18] Zhe-Xian Wan, Geometry of Classical Groups over Finite Fields, Chartwell Bratt, Lund, 1993.