PERGAMON
International Journal of Non-Linear Mechanics 36 (2001) 549–564
NON-LINEAR MECHANICS
www.elsevier.com/locate/ijnonlinmec

Thermoelasticity with thermomechanical constraints

N.H. Scott*

School of Mathematics, University of East Anglia, Norwich NR4 7TJ, UK

Received 24 July 2000; accepted 4 August 2000

Abstract

Equations are derived governing the behaviour of small disturbances superimposed on an underlying equilibrium configuration of a thermoelastic body. The body may be materially inhomogeneous, non-homogeneously prestrained and be subjected to a non-uniform temperature resulting in non-constant-coefficient partial differential equations. These equations are generalized to the case where thermomechanical constraints are present, both deformation-temperature and deformation-entropy constraints. It is known that the first of these types of constraints leads to material instabilities and the second does not. By examining nearly constrained materials, and taking an appropriate limit, we find that the instabilities associated with deformation-temperature constraints arise because the heat capacity at constant deformation becomes negative whilst deformation-entropy constraints are stable because the same heat capacity remains positive, though tending to zero in the limit of the constraint holding exactly. The ten important moduli of thermoelasticity are examined in the limit of each constraint holding exactly. It is found, for example, that for each type of constraint the heat capacity at constant stress remains positive and bounded away from zero. Results on wave propagation are also presented. © 2000 Published by Elsevier Science Ltd.

Keywords: Deformation-temperature constraints; Deformation-entropy constraints; Stability; Negative heat capacity; Waves

1. Introduction

Truesdell and Noll [1] placed the theory of purely mechanical constraints on a firm theoretical basis by postulating that the stress is constitutively determined only to within a reaction stress that does no work in any motion satisfying the constraint. For example, there is an arbitrary spherical pressure present as a component of the stress in any incompressible material. For materials in which temperature variation and heat conduction play a role, on the other hand, there is as yet no firmly established theory of thermomechanical constraints. Green et al. [2] regard a thermomechanical constraint as a restriction on allowable values of deformation, temperature and temperature gradient and postulate that stress, entropy and heat flux are constitutively determined only to within a reaction stress, entropy and heat flux which collectively give rise to zero entropy production in any process satisfying the thermomechanical constraint, see also [3]. We shall assume that the temperature gradient is unconstrained and refer to a constraint connecting the deformation and temperature as a deformation-temperature constraint.

A different theory for deformation-temperature constraints was developed by Casey and Krishanaswamy [4], generalizing a purely mechanical theory of Casey [5,6]. Expressions for stress

* Tel.: +44-1603-592572; fax: +44-1603-593868.
E-mail address: n.scott@uea.ac.uk (N.H. Scott).

0020-7462/00/$ - see front matter
PII: S0020-7462(00)00084-6

and entropy in the constrained material are derived by considering a related family of unconstrained thermomechanical materials and extending the domain of definition of the Helmholtz free energy in a differentiable manner away from the constraint manifold.

Manacorda [7] and Beevers [8] used the approach of [2] to demonstrate material instabilities for wave propagation in an isotropic thermoelastic material constrained to be incompressible at uniform temperature. Chadwick and Scott [9] further demonstrated material instabilities for a fully anisotropic thermoelastic material suffering an arbitrary deformation-temperature constraint. By considering a nearly constrained material they were able to show that the lack of stability may be associated with the heat capacity at constant deformation becoming negative as the limit in which the constraint holds exactly is approached. This property is explored further in the present paper. Negative heat capacity is well known to imply a lack of stability, see, for example, [10, pp. 193, 201; 11].

In order to overcome this lack of stability a new type of constraint has been introduced [12] in which the deformation has been linked with the entropy rather than the temperature. With this type of deformation-entropy constraint it can be shown that stability is maintained. These ideas have been developed further [13], partly in the context of wave propagation. By considering a nearly constrained material it was shown in [13] that as the limit of the deformation-entropy constraint is approached the heat capacity at constant deformation tends to zero through positive values. Although a zero heat capacity is anomalous, stability is maintained because the heat capacity never becomes negative. This behaviour is discussed further below.

Despite the anomalous behaviour of the heat capacity at constant deformation for either type of constraint, we show in this paper that, for either type of constraint, the heat capacity at constant stress remains positive and bounded away from zero. It is this heat capacity which is more commonly measured.

Leslie and Scott [14,15] have applied the ideas of [9,12,13] to the theory of longitudinal wave propagation in an isotropic thermoelastic material that is either incompressible [14], or nearly incompressible [15], at either uniform temperature or uniform entropy. These ideas are developed further here, largely outside the context of wave propagation, for arbitrary material anisotropy and thermomechanical constraint.

In Section 2 we outline the basic concepts of non-linear thermoelasticity which will be needed. In Section 3 we derive the field equations of unconstrained thermoelasticity. The equations are linearized about an equilibrium state which may be non-homogeneously deformed and have non-uniform temperature. After an initial discussion we assume Fourier's law of heat conduction for simplicity. The displacement-temperature form of the field equations is given, partly because of its later utility in considering deformation-temperature constraints and near constraints, and the displacement-entropy form is also given for its utility with deformation-entropy constraints and near constraints. The field equations were derived in referential form for convenience but are transformed to spatial form as this is the most frequently used. In all cases the linearized field equations with constant coefficients (i.e. for a homogeneous material, homogeneously deformed, and at uniform temperature) are deduced because this is an important special case. Section 3 is concluded with expressions for the internal energy and Helmholtz free energy as far as terms quadratic in the small quantities.

The field equations of constrained thermoelasticity are derived in Section 4 for an inhomogeneous material, non-homogeneously deformed at non-uniform temperature and then specialized to the case of constant coefficients. These equations reduce to those of [16] in the purely mechanical situation.

In Section 5 a deformation-entropy constraint is modelled by adding to the internal energy of the unconstrained material a large multiple of the square of the linearized deformation-entropy constraint, see (5.1) below. This large multiple, $\hat{\chi}'$, corresponds to an elastic modulus which becomes infinite in the limit $\hat{\chi}' \to \infty$ of the constraint holding exactly. (For example, in the case of near incompressibility at uniform entropy, $\hat{\chi}'$ would be the bulk modulus). Expressions for all ten thermoelastic moduli are evaluated in the deformation-entropy case. Whilst the heat capacity at constant deformation tends to zero that at constant stress remains finite and bounded away from zero. The limit $\hat{\chi}' \to \infty$ of the constraint is explored for all ten thermoelastic moduli. The limit of the heat capacity at constant deformation tending to zero, without the constraint holding, is shown to be equivalent to the constraint and the consequences are explored. Sinusoidal wave propagation is briefly considered in this limit and results unifying those of [13] are obtained.

In Section 6 the deformation-temperature constraint is modelled in a manner similar to that of Section 5. Results broadly comparable with those of Section 5 are obtained.

2. The basic concepts of thermoelasticity

We consider a body $\mathscr{B}$ consisting of thermoelastic solid material occupying a reference configuration B_r at time $t = 0$ and a spatial configuration B_t at time t. A material particle has Cartesian coordinates $\mathbf{X}$ in B_r and $\mathbf{x}$ in B_t. The functions $x_i(X_A, t)$, $i, A = 1,2,3$, describe the motion of $\mathscr{B}$ and the quantities

$$F_{iA} = \partial x_i / \partial X_A$$

are the components of the deformation gradient tensor $\mathbf{F}$ with determinant $J = \det \mathbf{F}$. The densities ρ_r in B_r and ρ in B_t are related by

$$\rho_r = \rho J$$

expressing the conservation of mass. The balance of momentum and that of energy are expressed in B_t by equations

$$\sigma_{ij,j} + \rho b_i = \rho \ddot{x}_i, \quad J^{-1} T \dot{S} + q_{i,i} = \rho r, \tag{2.1}$$

respectively, in which σ_{ij} are components of the Cauchy stress $\boldsymbol{\sigma}$, b_i are components of the body force per unit mass $\mathbf{b}$, q_i are components of the heat flux $\mathbf{q}$, and r is the heat supply per unit mass. T is the absolute temperature and S is the entropy per unit volume of B_r. A superposed dot denotes the material time derivative, so that $\ddot{\mathbf{x}}$ is the particle acceleration, and $(\ )_{,i}$ denotes current space derivative $\partial(\ )/\partial x_i$ with repeated suffixes being summed over from 1 to 3.

The components of the first Piola–Kirchhoff stress $\mathbf{P}$ and the referential heat flux $\mathbf{Q}$ are given by

$$P_{iA} = J \sigma_{ij} F^{-1}_{Aj}, \quad Q_A = J F^{-1}_{Aj} q_j \tag{2.2}$$

in terms of which balance equations (2.1) become

$$P_{iA,A} + \rho_r b_i = \rho_r \ddot{x}_i, \quad T\dot{S} + Q_{A,A} = \rho_r r \tag{2.3}$$

where $(\ )_{,A}$ denotes the referential space derivative $\partial(\ )/\partial X_A$.

The internal energy per unit volume of B_r is denoted by $U(\mathbf{F}, S, \mathbf{X})$ and depends on the deformation gradient $\mathbf{F}$, the entropy S and the material particle $\mathbf{X}$. The explicit dependence on $\mathbf{X}$ corresponds to material inhomogeneity and is usually suppressed. The internal energy is a potential function for stress and temperature:

$$\mathbf{P} = \frac{\partial U}{\partial \mathbf{F}}, \quad T = \frac{\partial U}{\partial S}. \tag{2.4}$$

The Helmholtz free energy $A(\mathbf{F}, T, \mathbf{X})$, per unit volume of B_r, is defined by the Legendre transformation

$$A(\mathbf{F}, T) := U - ST, \tag{2.5}$$

again suppressing explicit dependence on $\mathbf{X}$, and is also a potential function:

$$\mathbf{P} = \frac{\partial A}{\partial \mathbf{F}}, \quad S = -\frac{\partial A}{\partial T}. \tag{2.6}$$

The isothermal elasticity tensor is defined by

$$\tilde{c}_{iAjB} = \left(\frac{\partial P_{iA}}{\partial F_{jB}}\right)_T = \frac{\partial^2 A(\mathbf{F}, T)}{\partial F_{iA} \partial F_{jB}} \tag{2.7}$$

and the isentropic elasticity tensor is defined by

$$\hat{c}_{iAjB} = \left(\frac{\partial P_{iA}}{\partial F_{jB}}\right)_S = \frac{\partial^2 U(\mathbf{F}, S)}{\partial F_{iA} \partial F_{jB}}. \tag{2.8}$$

The symmetry properties

$$\tilde{c}_{jBiA} = \tilde{c}_{iAjB}, \quad \hat{c}_{jBiA} = \hat{c}_{iAjB} \tag{2.9}$$

follow from (2.7) and (2.8). The temperature coefficient of stress is

$$\tilde{\beta}_{iA} = -\left(\frac{\partial P_{iA}}{\partial T}\right)_{\mathbf{F}} = -\frac{\partial^2 A}{\partial F_{iA} \partial T} = \left(\frac{\partial S}{\partial F_{iA}}\right)_T \tag{2.10}$$

and the entropy coefficient of stress is

$$\hat{\beta}_{iA} = -\left(\frac{\partial P_{iA}}{\partial S}\right)_{\mathbf{F}} = -\frac{\partial^2 U}{\partial F_{iA}\partial S} = -\left(\frac{\partial T}{\partial F_{iA}}\right)_S. \tag{2.11}$$

Two further thermodynamic potential functions, per unit volume of B_r, may be defined by means of Legendre transformations. They are the Gibb's function

$$G(\mathbf{P}, T) := A - \mathbf{P}\cdot\mathbf{F}, \tag{2.12}$$

in which the scalar product of two tensors is given by

$$\mathbf{P}\cdot\mathbf{F} = \operatorname{tr}\mathbf{P}\mathbf{F}^{\mathrm{T}} = P_{iA}F_{iA},$$

and the enthalpy

$$H(\mathbf{P}, S) := U - \mathbf{P}\cdot\mathbf{F} \tag{2.13}$$

which act as potential functions according to

$$\mathbf{F} = -\frac{\partial G}{\partial \mathbf{P}}, \quad S = -\frac{\partial G}{\partial T} \tag{2.14}$$

and

$$\mathbf{F} = -\frac{\partial H}{\partial \mathbf{P}}, \quad T = \frac{\partial H}{\partial S}. \tag{2.15}$$

The (temperature) coefficient of thermal expansion, at constant stress, is

$$\tilde{\alpha}_{iA} = \left(\frac{\partial F_{iA}}{\partial T}\right)_{\mathbf{P}} = -\frac{\partial^2 G}{\partial P_{iA}\partial T} = \left(\frac{\partial S}{\partial P_{iA}}\right)_T \tag{2.16}$$

and the entropy coefficient of thermal expansion, again at constant stress, is

$$\hat{\alpha}_{iA} = \left(\frac{\partial F_{iA}}{\partial S}\right)_{\mathbf{P}} = -\frac{\partial^2 H}{\partial P_{iA}\partial S} = -\left(\frac{\partial T}{\partial P_{iA}}\right)_S. \tag{2.17}$$

From (2.10), (2.16), (2.7) and (2.9) we have

$$\tilde{\beta}_{iA} = \left(\frac{\partial S}{\partial P_{jB}}\right)_T\left(\frac{\partial P_{jB}}{\partial F_{iA}}\right)_T = \tilde{c}_{iAjB}\tilde{\alpha}_{jB} \tag{2.18}$$

and similarly

$$\hat{\beta}_{iA} = \hat{c}_{iAjB}\hat{\alpha}_{jB}. \tag{2.19}$$

The isothermal compliance is

$$\tilde{s}_{iAjB} = \left(\frac{\partial F_{iA}}{\partial P_{jB}}\right)_T = -\frac{\partial^2 G}{\partial P_{iA}\partial P_{jB}} \tag{2.20}$$

and the isentropic compliance is

$$\hat{s}_{iAjB} = \left(\frac{\partial F_{iA}}{\partial P_{jB}}\right)_S = -\frac{\partial^2 H}{\partial P_{iA}\partial P_{jB}}. \tag{2.21}$$

Since

$$\left(\frac{\partial F_{iA}}{\partial P_{jB}}\right)_T\left(\frac{\partial P_{jB}}{\partial F_{kC}}\right)_T = \frac{\partial F_{iA}}{\partial F_{kC}} = \delta_{ik}\delta_{AC} \tag{2.22}$$

we may conclude from (2.7) and (2.20) that

$$\tilde{s}_{iAjB}\tilde{c}_{jBkC} = \delta_{ik}\delta_{AC} = \tilde{c}_{iAjB}\tilde{s}_{jBkC} \tag{2.23}$$

and similarly from (2.8) and (2.21) that

$$\hat{s}_{iAjB}\hat{c}_{jBkC} = \delta_{ik}\delta_{AC} = \hat{c}_{iAjB}\hat{s}_{jBkC}.$$

This equation and (2.23) may be written in direct notation as

$$\tilde{\mathbf{s}}\tilde{\mathbf{c}} = \tilde{\mathbf{c}}\tilde{\mathbf{s}} = \mathbf{I} = \hat{\mathbf{s}}\hat{\mathbf{c}} = \hat{\mathbf{c}}\hat{\mathbf{s}} \tag{2.24}$$

in which the unit tensor $\mathbf{I}$ has components $I_{iAjB} = \delta_{ij}\delta_{AB}$ and the compliance $\tilde{\mathbf{s}}$ has components $\tilde{s}_{iAjB}$, etc. Thus tensors $\tilde{\mathbf{s}}$ and $\tilde{\mathbf{c}}$ are mutually inverse, as are $\hat{\mathbf{s}}$ and $\hat{\mathbf{c}}$, so that it follows from (2.18) and (2.19) that

$$\tilde{\alpha}_{iA} = \tilde{s}_{iAjB}\tilde{\beta}_{jB}, \quad \hat{\alpha}_{iA} = \hat{s}_{iAjB}\hat{\beta}_{jB}. \tag{2.25}$$

In terms of direct notation these relations become

$$\tilde{\boldsymbol{\beta}} = \tilde{\mathbf{c}}\tilde{\boldsymbol{\alpha}}, \quad \hat{\boldsymbol{\beta}} = \hat{\mathbf{c}}\hat{\boldsymbol{\alpha}} \quad \text{and} \quad \tilde{\boldsymbol{\alpha}} = \tilde{\mathbf{s}}\tilde{\boldsymbol{\beta}}, \quad \hat{\boldsymbol{\alpha}} = \hat{\mathbf{s}}\hat{\boldsymbol{\beta}}\cdot$$

The heat capacity at constant deformation per unit volume of B_r is, from $(2.6)_2$,

$$c_{\mathbf{F}} = T\left(\frac{\partial S}{\partial T}\right)_{\mathbf{F}} = -T\frac{\partial^2 A}{\partial T^2} \tag{2.26}$$

(so that $c_{\mathbf{F}}/\rho_r$ is the specific heat) and the heat capacity at constant stress per unit volume of B_r is, from $(2.14)_2$,

$$c_{\mathbf{P}} = T\left(\frac{\partial S}{\partial T}\right)_{\mathbf{P}} = -T\frac{\partial^2 G}{\partial T^2}. \tag{2.27}$$

From (2.10), (2.11) and (2.26)

$$\tilde{\beta}_{iA} = -\left(\frac{\partial P_{iA}}{\partial T}\right)_{\mathbf{F}} = -\left(\frac{\partial P_{iA}}{\partial S}\right)_{\mathbf{F}}\left(\frac{\partial S}{\partial T}\right)_{\mathbf{F}} = \hat{\beta}_{iA}\frac{c_{\mathbf{F}}}{T}$$

so that

$$\hat{\beta}_{iA} = \frac{T}{c_{\mathbf{F}}}\tilde{\beta}_{iA}, \quad \hat{\alpha}_{iA} = \frac{T}{c_{\mathbf{P}}}\tilde{\alpha}_{iA}, \tag{2.28}$$

the second equation following similarly from (2.16), (2.17) and (2.27). Eqs. (2.28) furnish connections between the temperature and entropy coefficients of stress and between the temperature and entropy expansion coefficients, involving the specific heats $c_{\mathbf{F}}$ and $c_{\mathbf{P}}$, respectively. These quantities are all second derivatives of $U(\mathbf{F}, S)$, $A(\mathbf{F}, T)$ or $G(\mathbf{P}, T)$.

Other relationships between the second derivatives of thermodynamic potentials may be derived as follows. From (2.8) and (2.11) we have

$$\begin{aligned}\hat{c}_{iAjB} &= \left(\frac{\partial P_{iA}}{\partial F_{jB}}\right)_T + \left(\frac{\partial P_{iA}}{\partial T}\right)_{\mathbf{F}}\left(\frac{\partial T}{\partial F_{jB}}\right)_S \\ &= \left(\frac{\partial P_{iA}}{\partial F_{jB}}\right)_T + \left(\frac{\partial P_{iA}}{\partial T}\right)_{\mathbf{F}}\left(\frac{\partial P_{jB}}{\partial S}\right)_{\mathbf{F}},\end{aligned}$$

so that from (2.10), (2.11) and $(2.28)_1$

$$\hat{c}_{iAjB} = \tilde{c}_{iAjB} + \frac{T}{c_{\mathbf{F}}}\tilde{\beta}_{iA}\tilde{\beta}_{jB}. \tag{2.29}$$

Heat capacities at constant stress and deformation are related, from (2.27) and $(2.6)_2$, by

$$c_{\mathbf{P}} = -T\frac{\partial^2 A}{\partial T^2} - T\frac{\partial^2 A}{\partial F_{iA}\partial T}\left(\frac{\partial F_{iA}}{\partial T}\right)_{\mathbf{P}},$$

so that

$$c_{\mathbf{P}} = c_{\mathbf{F}} + T\tilde{\boldsymbol{\alpha}}\cdot\tilde{\boldsymbol{\beta}} = c_{\mathbf{F}} + T\tilde{c}_{iAjB}\tilde{\alpha}_{iA}\tilde{\alpha}_{jB}, \tag{2.30}$$

in which (2.26), (2.10), (2.16) and (2.18) have also been used. Using Lemma A.1 of the Appendix we can invert (2.29) to obtain

$$\hat{s}_{iAjB} = \tilde{s}_{iAjB} - \frac{T}{c_{\mathbf{P}}}\tilde{\alpha}_{iA}\tilde{\alpha}_{jB}, \tag{2.31}$$

where (2.25) and (2.30) have also been used.

It is usual to assume in thermoelasticity, on grounds of stability, that the isothermal elasticity tensor $\tilde{\mathbf{c}}$ is positive definite and that the heat capacity at constant deformation is positive:

$$c_{\mathbf{F}} > 0. \tag{2.32}$$

It follows that $\tilde{\mathbf{s}}$ and $\hat{\mathbf{c}}$, and hence $\hat{\mathbf{s}}$ (despite the negative sign in (2.31)), are also positive definite. It also follows from (2.30) that

$$c_{\mathbf{P}} > c_{\mathbf{F}}. \tag{2.33}$$

An alternative expression for $c_{\mathbf{F}}$ may be derived using (2.26) and $(2.4)_2$:

$$c_{\mathbf{F}} = T\bigg/\left(\frac{\partial T}{\partial S}\right)_{\mathbf{F}} = T\bigg/\frac{\partial^2 U}{\partial S^2}. \tag{2.34}$$

Also, from (2.19) and (2.28) we may deduce that

$$\frac{c_{\mathbf{P}}}{c_{\mathbf{F}}}\tilde{\beta}_{iA} = \hat{c}_{iAjB}\tilde{\alpha}_{jB} \tag{2.35}$$

which may be compared with (2.18).

3. The field equations of unconstrained thermoelasticity

We introduce an equilibrium configuration B_{e} intermediate between B_{r} and B_t. Field quantities defined in B_{e} are distinguished by a subscript, or superscript as convenient, e and are independent of t but may depend on $\mathbf{X}$. In B_{e} the position vector of the material point $\mathbf{X}$ in B_{r} is denoted by $\mathbf{x}^{\mathrm{e}}(\mathbf{X})$ so that, in component form, the particle displacement $\mathbf{u}(\mathbf{X}, t)$ is defined by

$$x_i = x_i^{\mathrm{e}}(\mathbf{X}) + u_i(\mathbf{X}, t).$$

We assume that the deformation $B_{\mathrm{e}} \to B_t$ is infinitesimal in the sense that

$$\varepsilon = (u_{i,A}u_{i,A})^{1/2}.$$

is a small quantity. The incremental deformation gradient, first Piola–Kirchhoff stress, temperature and entropy are defined respectively by

$$\begin{aligned}u_{i,A} &= F_{iA} - F_{iA}^{\mathrm{e}}, \\ p_{iA} &= P_{iA} - P_{iA}^{\mathrm{e}}, \\ \theta &= T - T_{\mathrm{e}}, \\ \phi &= S - S_{\mathrm{e}},\end{aligned} \tag{3.1}$$

and depend on $\mathbf{X}$ and t. Each is assumed to be $O(\varepsilon)$. Because the pre-deformation $B_{\mathrm{r}} \to B_{\mathrm{e}}$ need not be homogeneous the equilibrium deformation gradient $F_{iA}^{\mathrm{e}} = \partial x_i^{\mathrm{e}}/\partial X_A$ may depend on $\mathbf{X}$. In addition, the Helmholtz free energy A may depend explicitly on $\mathbf{X}$, i.e. the material may be inhomogeneous, and so the equilibrium values $P^{\rm e}_{iA}$ and $S_{\rm e}$ also may depend on $\mathbf{X}$. The equilibrium temperature $T_{\rm e}$ would depend on $\mathbf{X}$ if, for example, different parts of the boundary of $B_{\rm e}$ had different prescribed temperatures.

3.1. The displacement-temperature form of the field equations

We now derive the field equations for $\mathbf{u}(\mathbf{X},t)$ and $\theta(\mathbf{X},t)$ on the assumption that these are both $O(\varepsilon)$ by linearizing balance equations (2.3). By linearizing $(2.6)_1$ and using definitions (2.7), (2.10) and (3.1) we find that

$$P_{iA} = P^{\rm e}_{iA} + \tilde{c}^{\rm e}_{iAkC}u_{k,C} - \tilde{\beta}^{\rm e}_{iA}\theta + O(\varepsilon^2).$$

The $O(1)$ part of $(2.3)_1$ is

$$P^{\rm e}_{iA,A} + \rho_{\rm r}b_i = 0 \tag{3.2}$$

and the $O(\varepsilon)$ part is

$$\{\tilde{c}^{\rm e}_{iAkC}u_{k,C}\}_{,A} - \{\tilde{\beta}^{\rm e}_{iA}\theta\}_{,A} = \rho_{\rm r}\ddot{u}_i \tag{3.3}$$

under the assumption that the body force $\mathbf{b}(\mathbf{X})$ is not affected by the deformation $B_{\rm e} \to B_t$. Eqs. (3.2) are the equilibrium equations in $B_{\rm e}$ and Eqs. (3.3) are equations of motion in B_t representing balance of momentum.

It remains to linearize $(2.3)_2$. From $(2.6)_2$ together with (2.10) and (2.26) we have

$$\dot{S} = \tilde{\beta}_{iA}\dot{F}_{iA} + \frac{c_F}{T}\dot{T} \tag{3.4}$$

so that $T\dot{S}$ linearizes using (3.1) to give

$$T_{\rm e}\dot{\phi} = T_{\rm e}\tilde{\beta}^{\rm e}_{iA}\dot{u}_{i,A} + c^{\rm e}_{\rm F}\dot{\theta}. \tag{3.5}$$

The heat flux $\mathbf{Q}$ is assumed to take the constitutive form $\mathbf{Q} = \mathbf{Q}(\mathbf{F}, T, \mathbf{G}, \mathbf{X})$ where the referential temperature gradient $\mathbf{G}$ is defined in component form by $G_A = T_{,A}$. Then linearizing gives

$$Q_A = Q^{\rm e}_A + \left(\frac{\partial Q_A}{\partial F_{jB}}\right)_{\rm e} u_{j,B} + \left(\frac{\partial Q_A}{\partial T}\right)_{\rm e}\theta + \left(\frac{\partial Q_A}{\partial G_B}\right)_{\rm e}\theta_{,B} + O(\varepsilon^2) \tag{3.6}$$

in which $Q^{\rm e}_A = Q_A(\mathbf{F}^{\rm e}, T_{\rm e}, \mathbf{G}^{\rm e}, \mathbf{X})$. It is assumed that the heat flux vanishes if the temperature gradient does, i.e.

$$\mathbf{Q}(\mathbf{F}, T, \mathbf{0}, \mathbf{X}) = \mathbf{0}, \quad T \text{ is any constant.} \tag{3.7}$$

If in $B_{\rm e}$ $\mathbf{G} = \mathbf{0}$ so that $T = T_{\rm e}$, a constant, then it follows that

$$\left(\frac{\partial Q_A}{\partial F_{jB}}\right)_{\rm e} = 0, \quad \left(\frac{\partial Q_A}{\partial T}\right)_{\rm e} = 0 \tag{3.8}$$

leading to the linear Fourier law of heat conduction

$$Q_A = -K^{\rm e}_{AB}\theta_{,B} \tag{3.9}$$

in which

$$K^{\rm e}_{AB} = -\left(\frac{\partial Q_A}{\partial G_B}\right)_{\rm e}$$

is the referential thermal conductivity tensor, see Chadwick [17]. But in $B_{\rm e}$ we may have $T_{\rm e} = T_{\rm e}(\mathbf{X})$ so that $\mathbf{G}^{\rm e} \neq \mathbf{0}$ and the heat flux $\mathbf{Q}^{\rm e}$ is non-vanishing, unlike that in (3.7). Then (3.8) may fail to hold leading to a linearization of (3.6) more complicated than (3.9). To simplify matters we shall make the common assumption that heat conduction is governed by Fourier's law (3.9) for all values of temperature:

$$Q_A = -K^{\rm e}_{AB}T_{,B} \tag{3.10}$$

in which the referential thermal conductivity tensor $K^{\rm e}_{AB}$ evaluated in $B_{\rm e}$ may depend on $\mathbf{X}$ because of material inhomogeneities or a non-homogeneous predeformation.

Using (3.10) residual energy equation $(2.3)_2$ may be written

$$T_{\rm e}\dot{\phi} - \{K^{\rm e}_{AB}T_{,B}\}_{,A} = \rho_{\rm r}r. \tag{3.11}$$

The $O(1)$ part of (3.11) is

$$-\{K^{\rm e}_{AB}T^{\rm e}_{,B}\}_{,A} = \rho_{\rm r}r \tag{3.12}$$

and the $O(\varepsilon)$ part is

$$T_{\rm e}\dot{\phi} - \{K^{\rm e}_{AB}\theta_{,B}\}_{,A} = 0$$

which on using (3.5) becomes

$$\{K^{\rm e}_{AB}\theta_{,B}\}_{,A} - T_{\rm e}\tilde{\beta}^{\rm e}_{iA}\dot{u}_{i,A} = c_{\rm F}\dot{\theta} \tag{3.13}$$

under the assumption that the heat supply $r(\mathbf{X})$ is not affected by the deformation $B_{\rm e} \to B_t$. The four equations (3.2) and (3.12) govern the equilibrium configuration B_e and the four equations (3.3) and (3.13) furnish four linear partial differential equations for the four small disturbances $\mathbf{u}(\mathbf{X},t)$ and $\theta(\mathbf{X},t)$ in which the coefficients may depend upon $\mathbf{X}$ but not upon t.

For a homogeneous body, homogeneously deformed in the absence of body force and heat supply, and with T_e constant, Eqs. (3.2) and (3.12) are satisfied trivially and the field equations (3.3) and (3.13) reduce to

$$\tilde{c}_{iAkC}u_{k,AC} - \tilde{\beta}_{iA}\theta_{,A} = \rho_r \ddot{u}_i,$$
$$K_{AB}\theta_{,AB} - T_e\tilde{\beta}_{iA}\dot{u}_{i,A} = c_F\dot{\theta} \tag{3.14}$$

in which coefficients are constants, independent of $\mathbf{X}$, understood to be evaluated in B_e so that superscripts e are omitted from now on.

3.2. The displacement-entropy form of the field equations

We now derive field equations for $\mathbf{u}(\mathbf{X},t)$ and $\phi(\mathbf{X},t)$ in the same way as the $\mathbf{u},\theta$ equations were derived in the previous subsection. Since we wish to work in terms of entropy, rather than temperature, we utilise the internal energy $U(\mathbf{F},S)$, rather than the Helmholtz free energy $A(\mathbf{F},T)$. By linearizing $(2.4)_1$ and using (2.8), (2.11) and (3.1) we find that

$$P_{iA} = P^e_{iA} + \hat{c}_{iAkC}u_{k,C} - \hat{\beta}_{iA}\phi + O(\varepsilon^2)$$

so that the $O(1)$ part of $(2.3)_1$ is given by (3.2) as before whilst the $O(\varepsilon)$ part, formerly (3.3), is replaced by

$$\{\hat{c}_{iAkC}u_{k,C}\}_{,A} - \{\hat{\beta}_{iA}\phi\}_{,A} = \rho_r\ddot{u}_i \tag{3.15}$$

where the coefficients $\hat{\mathbf{c}}$ and $\hat{\boldsymbol{\beta}}$ are understood to be evaluated in B_e so that superscripts e may be omitted.

With the definitions (2.11), (2.34) and (3.1), $(2.4)_2$ may be linearized to give

$$\theta = -\hat{\beta}_{iC}u_{i,C} + \frac{T_e}{c_F}\phi, \tag{3.16}$$

a useful connection between incremental temperature and entropy. By using $(2.28)_1$ it can be seen that (3.16) and (3.5) are equivalent.

Keeping Fourier's law in form (3.10) we find that the $O(1)$ part of $(2.3)_2$ is given by (3.12) as before. Using (3.16) we find that $O(\varepsilon)$ part, formerly (3.13), is replaced by

$$\left\{K_{AB}\left[\frac{T_e}{c_F}\phi - \hat{\beta}_{iC}u_{i,C}\right]_{,B}\right\}_{,A} = T_e\dot{\phi}. \tag{3.17}$$

Thus (3.15) and (3.17) furnish four linear partial differential equations for the four small disturbances $\mathbf{u}(\mathbf{X},t)$ and $\phi(\mathbf{X},t)$.

Using connections (2.29), $(2.28)_1$ and (3.16) it can be shown that different forms (3.3) and (3.15) of the field equations expressing the balance of momentum are, in fact equivalent. Using (3.16), (3.5) and $(2.28)_1$ it can be shown that residual energy equations (3.13) and (3.17) are also equivalent.

In the case of constant coefficients (3.15) and(3.17) reduce to

$$\hat{c}_{iAkC}u_{k,AC} - \hat{\beta}_{iA}\phi_{,A} = \rho_r\ddot{u}_i,$$
$$K_{AB}\left(\phi_{,AB} - \frac{c_F}{T_e}\hat{\beta}_{iC}u_{i,ABC}\right) = c_F\dot{\phi}, \tag{3.18}$$

which are equivalent to (3.14).

3.3. The field equations in spatial coordinates

It is often more convenient to express all quantities and derivatives in terms of position $\mathbf{x}^e$ in B_e, rather than $\mathbf{X}$, so that the displacements and increments in temperature and entropy are taken in the forms $\mathbf{u}(\mathbf{x}^e,t)$, $\theta(\mathbf{x}^e,t)$ and $\phi(\mathbf{x}^e,t)$. From conservation of mass we may write the density in B_e as

$$\rho_e = J_e^{-1}\rho_r \quad \text{where } J_e = \det \mathbf{F}^e \tag{3.19}$$

and from the chain rule

$$(\ )_{,A} = (\ )_{,j}F^e_{jA}, \tag{3.20}$$

where $(\ )_{,j}$ now denotes $\partial(\ )/\partial x^e_j$, we may deduce the Euler relation

$$(J_e^{-1}F^e_{iA})_{,i} = 0. \tag{3.21}$$

The spatial form of the thermal conductivity tensor is

$$k_{ij} = J_e^{-1}F^e_{iA}F^e_{jB}K_{AB} \tag{3.22}$$

in terms of which Fourier's law (3.10) becomes

$$q_i = -k_{ij}T_{,j} \tag{3.23}$$

where $\mathbf{q}$ is the spatial heat flux, see $(2.2)_2$. The Cauchy stress in B_e is given from $(2.2)_1$ by

$$\boldsymbol{\sigma}^e_{ij} = J_e^{-1}P^e_{iA}F^e_{jA}$$

so that using (3.21) and (3.19) enables equilibrium equation (3.2) to be written in the spatial form

$$\boldsymbol{\sigma}^e_{ij,j} + \rho_e b_i = 0. \tag{3.24}$$

On defining

$$\tilde{c}_{ijk\ell} = J_e^{-1}F^e_{jA}F^e_{\ell C}\tilde{c}_{iAkC}, \quad \tilde{\beta}_{ij} = J_e^{-1}F^e_{jA}\tilde{\beta}_{iA} \tag{3.25}$$

and using (3.21) and (3.19) we find that the spatial form of the momentum balance equation (3.3) is

$$\{\tilde{c}_{ijk\ell}u_{k,\ell}\}_{,j} - \{\tilde{\beta}_{ij}\theta\}_{,j} = \rho_e\ddot{u}_i. \tag{3.26}$$

Using (3.19)–(3.22) we find that residual energy equation (3.11) becomes in the spatial form

$$J_e^{-1}T_e\dot{\phi} - \{k_{ij}T_{,j}\}_{,i} = \rho_e r, \tag{3.27}$$

with $T_e = T(\mathbf{x}^e)$ denoting temperature in B_e. The $O(1)$ part of (3.27) is

$$-\{k_{ij}T^e_{,j}\}_{,i} = \rho_e r, \tag{3.28}$$

equivalent to (3.12), and the $O(\varepsilon)$ part is

$$\{k_{ij}\theta_{,j}\}_{,i} - \tilde{\beta}_{ik}\dot{u}_{i,k} = J_e^{-1}c_F\dot{\theta}, \tag{3.29}$$

equivalent to (3.13). Thus the spatial forms of the referential field equations (3.2), (3.3), (3.12) and (3.13) are given by (3.24), (3.26), (3.28) and (3.29), respectively.

We now derive spatial forms, involving $\mathbf{u}(\mathbf{x}^e, t)$ and $\phi(\mathbf{x}^e, t)$, of the $O(\varepsilon)$ field equations (3.15) and (3.17), the $O(1)$ equations still being given by (3.24) and (3.28). On defining

$$\hat{c}_{ijk\ell} = J_e^{-1}F^e_{jA}F^e_{\ell C}\hat{c}_{iAkC}, \quad \hat{\beta}_{ij} = J_e^{-1}F^e_{jA}\tilde{\beta}_{iA}$$

we find that the spatial form of (3.15) is

$$\{\hat{c}_{ijk\ell}u_{k,\ell}\}_{,j} - \{\hat{\beta}_{ij}\phi\}_{,j} = \rho_e\ddot{u}_i. \tag{3.30}$$

Similarly, we find from (3.11) and (3.16) that the spatial form of (3.17) is

$$\left\{k_{j\ell}\left[\frac{T_e}{c_F}\phi - J_e\hat{\beta}_{ik}u_{i,k}\right]_{,\ell}\right\}_{,j} = J_e^{-1}T_e\dot{\phi}. \tag{3.31}$$

For a homogeneous material homogeneously deformed with T_e a constant, (3.26) and (3.29) reduce to

$$\tilde{c}_{ijk\ell}u_{k,j\ell} - \tilde{\beta}_{ij}\theta_{,j} = \rho_e\ddot{u}_i,$$
$$k_{ij}\theta_{,ij} - \tilde{\beta}_{ik}\dot{u}_{i,k} = J_e^{-1}c_F\dot{\theta}, \tag{3.32}$$

spatial forms of (3.14), and (3.30) and (3.31) reduce to

$$\hat{c}_{ijk\ell}u_{k,j\ell} - \hat{\beta}_{ij}\phi_{,j} = \rho_e\ddot{u}_i,$$
$$k_{j\ell}\left(\phi_{,j\ell} - J_e\frac{c_F}{T_e}\hat{\beta}_{ik}u_{i,jk\ell}\right) = J_e^{-1}c_F\dot{\phi}, \tag{3.33}$$

the spatial forms of (3.18). In the special case where B_r and B_e coincide, so that $J_e = 1$, Eqs. (3.32) have been given by Chadwick [17, Eq. (19)] and Eqs. (3.33) by Scott [13, Eq. (2.15)] in different notations.

3.4. Quadratic forms of the thermodynamic potentials U and A

Expanding $U(\mathbf{F}, S)$ as far as $O(\varepsilon^2)$ terms in the small quantities (3.1) allows the internal energy to be expressed as

$$U(u_{i,A},\phi) = U_e + P^e_{iA}u_{i,A} + T_e\phi + \tfrac{1}{2}\left\{\hat{c}_{iAkC}u_{i,A}u_{k,C} - 2\hat{\beta}_{iA}u_{i,A}\phi + \frac{T_e}{c_F}\phi^2\right\} \tag{3.34}$$

which acts as a potential for the stress increment p_{iA} and temperature increment θ:

$$p_{iA} = -P^e_{iA} + \frac{\partial U}{\partial u_{i,A}}, \quad \theta = -T_e + \frac{\partial U}{\partial \phi}, \tag{3.35}$$

as can be verified using equations of Section 3.2. Coefficients in (3.34) are all evaluated in B_e and at this stage may depend on $\mathbf{X}$. The $O(\varepsilon)$ parts of the momentum balance equations $(2.3)_1$ and the residual energy balance $(2.3)_2$ may be written

$$p_{iA,A} = \rho_r\ddot{u}_i, \quad \{K_{AB}\theta_{,B}\}_{,A} = T_e\dot{\phi}. \tag{3.36}$$

On substituting (3.34) into (3.35) and then substituting the resulting explicit expressions for p_{iA} and θ into (3.36) we obtain the field equations (3.15) and (3.17) already derived.

Similarly, we find that the Helmholtz free energy

$$A(u_{i,A},\theta) = A_e + P^e_{iA} u_{i,A} - S_e \theta + \tfrac{1}{2}\left\{ \tilde{c}_{iAkC} u_{i,A} u_{k,C} - 2\tilde{\beta}_{iA} u_{i,A} \theta - \frac{c_F}{T_e}\theta^2 \right\} \tag{3.37}$$

acts as a potential:

$$p_{iA} = -P^e_{iA} + \frac{\partial A}{\partial u_{i,A}}, \quad \phi = -S_e - \frac{\partial A}{\partial \phi}, \tag{3.38}$$

so that balance equations (3.36) lead to field equations (3.3) and (3.13) already derived.

4. The field equations of constrained thermoelasticity

The most natural extension of a purely mechanical constraint $f(\mathbf{F}) = 0$ to thermoelasticity would seem to be a constraint of the form

$$f(\mathbf{F}, T) = 0 \tag{4.1}$$

involving the temperature T as well as the deformation gradient $\mathbf{F}$. However, Chadwick and Scott [9] and Leslie and Scott [14], amongst others, found that an arbitrary constraint of the form (4.1) results in instabilities in wave propagation. Scott [12] was therefore led to consider a constraint of the form

$$g(\mathbf{F}, S) = 0 \tag{4.2}$$

involving the entropy S in place of T. The instabilities caused by constraint (4.1) are absent if constraint (4.2) is employed, see [12]. The physical motivation for (4.2) is that it relates changes in deformation directly to the introduction of heat through the presence of the entropy S, see $(2.3)_2$.

4.1. *The deformation-entropy constraint* $g(\mathbf{F}, S) = 0$

Constraint (4.2) may appear in an arbitrary manner in the internal energy so we assume

$$U^*(\mathbf{F}, S) = \bar{U}(\mathbf{F}, S, g(\mathbf{F}, S)) \tag{4.3}$$

following the method of Chadwick et al. [18] in the purely mechanically constrained case. The stress and temperature are given by

$$\mathbf{P}^* = \frac{\partial \bar{U}}{\partial \mathbf{F}} + \frac{\partial \bar{U}}{\partial g}\frac{\partial g}{\partial \mathbf{F}},$$

$$T^* = \frac{\partial \bar{U}}{\partial S} + \frac{\partial \bar{U}}{\partial g}\frac{\partial g}{\partial S} \tag{4.4}$$

in which $\partial \bar{U}/\partial g$ may take any value. In $(4.4)_1$ $\partial \bar{U}/\partial g$ plays the role of the arbitrary stress which exists for any purely mechanical constraint. Without loss of generality, therefore, we may replace the arbitrary dependence of U^* upon the constraint function $g(\mathbf{F}, S)$ by the linear form

$$U^* = U + \hat{\eta} g(\mathbf{F}, S), \tag{4.5}$$

in which the function $\hat{\eta}(\mathbf{x}, t)$ is arbitrary, and obtain in place of (4.4)

$$\mathbf{P}^* = \mathbf{P} + \hat{\eta}\hat{\mathbf{N}}, \quad T^* = T - \hat{\eta}\nu, \tag{4.6}$$

where $\mathbf{P}$ and T are the stress and temperature obtained from (2.4) in the unconstrained material, with the definitions

$$\hat{\mathbf{N}} = \frac{\partial g}{\partial \mathbf{F}}, \quad \nu = -\frac{\partial g}{\partial S}. \tag{4.7}$$

For simplicity we assume that $\mathbf{P}^{*e} = \mathbf{P}^e$ and $T^*_e = T_e$, so that $\hat{\eta}(\mathbf{x}^e, t)$ is $O(\varepsilon)$, the $O(1)$ equations (3.2) and (3.12) remain unchanged, the incremental stress and temperature become

$$p^*_{iA} = \hat{c}_{iAkC} u_{k,C} - \hat{\beta}_{iA}\phi + \hat{N}_{iA}\hat{\eta}, \tag{4.8}$$

$$\theta^* = -\hat{\beta}_{iA} u_{i,A} + \frac{T_e}{c_F}\phi - \nu\hat{\eta}, \tag{4.9}$$

and balance equations (3.36) become

$$p^*_{iA,A} = \rho_r \ddot{u}_i, \quad \{K_{AB}\theta^*_{,B}\}_{,A} = T_e\dot{\phi}$$

leading to the field equations of deformation-entropy constrained thermoelasticity

$$\{\hat{c}_{iAkC} u_{k,C}\}_{,A} - \{\hat{\beta}_{iA}\phi\}_{,A} + \{\hat{N}_{iA}\hat{\eta}\}_{,A} = \rho_r \ddot{u}_i$$

$$\left\{ K_{AB}\left[\frac{T_e}{c_F}\phi - \hat{\beta}_{iC} u_{i,C} - \nu\hat{\eta} \right]_{,B} \right\}_{,A} = T_e\dot{\phi}, \tag{4.10}$$

which replace the field equations (3.15) and (3.17) of unconstrained thermoelasticity.

Constraint (4.2) is satisfied in B_r and B_e so that

$$g(\mathbf{I}, S_r) = 0 = g(\mathbf{F}^e, S_e)$$

and may be linearized in B_t to give

$$\hat{N}_{iA} u_{i,A} - v\phi = 0 \tag{4.11}$$

where now $\hat{\mathbf{N}}$ and v, defined by (4.7), may depend upon $\mathbf{X}$ but not upon $\mathbf{F}$, S or t.

Eqs. (4.10) and (4.11) constitute a set of five equations for the five unknown functions u_i, ϕ and $\hat{\eta}$ of deformation-entropy constrained thermoelasticity. If the body is homogeneous and homogeneously deformed then the coefficients of the field quantities in (4.10) and (4.11) are constant and the equations reduce to

$$\hat{c}_{iAkC} u_{k,AC} - \hat{\beta}_{iA} \phi_{,A} + \hat{N}_{iA} \hat{\eta}_{,A} = \rho_r \ddot{u}_i,$$
$$K_{AB}\left(\phi_{,AB} - \frac{c_F}{T_e}\hat{\beta}_{iC} u_{i,ABC} - \frac{v c_F}{T_e}\hat{\eta}_{,AB}\right) = c_F \dot{\phi}, \tag{4.12}$$

the spatial forms of which have been given by Scott [13, (3.13)].

4.2. The deformation-temperature constraint $f(\mathbf{F}, T) = 0$

The argument follows closely that of Section 4.1 in the deformation-entropy case. Instead of the internal energy we work with the Helmholtz free energy

$$A^\dagger = A + \tilde{\eta} f(\mathbf{F}, T), \tag{4.13}$$

where $\tilde{\eta} = \tilde{\eta}(\mathbf{X}, t)$, in place of (4.5), leading to

$$\mathbf{P}^\dagger = \mathbf{P} + \tilde{\eta}\tilde{\mathbf{N}}, \quad S^\dagger = S + \tilde{\eta}\alpha, \tag{4.14}$$

with

$$\tilde{\mathbf{N}} = \frac{\partial f}{\partial \mathbf{F}}, \quad \alpha = -\frac{\partial f}{\partial T} \tag{4.15}$$

in place of (4.6) and (4.7). Eqs. (4.8) and (4.9) are replaced by

$$p^\dagger_{iA} = \tilde{c}_{iAkC} u_{k,C} - \tilde{\beta}_{iA} u_{i,A} + \tilde{N}_{iA}\tilde{\eta}, \tag{4.16}$$

$$\phi^\dagger = \tilde{\beta}_{iA} u_{i,A} + \frac{c_F}{T_e}\theta - \alpha\tilde{\eta}, \tag{4.17}$$

so that the balance equations (3.36), taken in the form

$$p^\dagger_{iA,A} = \rho_r \ddot{u}_i, \quad \{K_{AB}\theta_{,B}\}_{,A} = T_e \dot{\phi}^\dagger, \tag{4.18}$$

lead to the field equations

$$\{\tilde{c}_{iAkC} u_{k,C} - \tilde{\beta}_{iA} u_{i,A} + \tilde{N}_{iA}\tilde{\eta}\}_{,A} = \rho_r \ddot{u}_i,$$
$$\{K_{AB}\theta_{,B}\}_{,A} - T_e \tilde{\beta}_{iA} \dot{u}_{i,A} - T_e \alpha \dot{\tilde{\eta}} = c_F \dot{\theta} \tag{4.19}$$

in place of (3.3) and (3.12).

Constraint (4.1) may be linearized in B_t to give

$$\tilde{N}_{iA} u_{i,A} - \alpha\theta = 0 \tag{4.20}$$

so that Eqs. (4.19) and (4.20) furnish five field equations for the five unknown functions u_i, θ and $\tilde{\eta}$ of deformation-temperature constrained thermoelasticity. For a homogeneous body homogeneously deformed equations (4.19) reduce to

$$\tilde{c}_{iAkC} u_{k,AC} - \tilde{\beta}_{iA} u_{i,AC} + \tilde{N}_{iA}\tilde{\eta}_{,A} = \rho_r \ddot{u}_i,$$
$$K_{AB}\theta_{,AB} - T_e \tilde{\beta}_{iA} \dot{u}_{i,A} - T_e \alpha \dot{\tilde{\eta}} = c_F \dot{\theta}, \tag{4.21}$$

equivalent to the spatial forms [13, (3.5)].

5. The deformation-entropy near constraint

5.1. Expressions for the ten thermoelastic moduli in the nearly constrained case

We take the internal energy in the form

$$U^{**} = U + \tfrac{1}{2}\hat{\chi}'\{g(\mathbf{F}, S)\}^2 \tag{5.1}$$

so that $\hat{\chi}' = 0$ corresponds to the original unconstrained material and the limit

$$\hat{\chi}' \to \infty, \quad g(\mathbf{F}, S) \to 0,$$
$$\hat{\eta} := \hat{\chi}' g(\mathbf{F}, S) \text{ remains bounded} \tag{5.2}$$

corresponds to the constraint holding exactly. The stress and temperature in the nearly constrained material (5.1) are given by

$$P^{**}_{iA} = P_{iA} + \hat{\chi}' g(\mathbf{F}, S)\hat{N}_{iA}, \quad T^{**} = T - \hat{\chi}' g(\mathbf{F}, S)v. \tag{5.3}$$

On making identification $(5.2)_3$ in (5.3) we may arrive at the equations appropriate if the constraint holds exactly, see (4.10).

In the linear approximation we must keep terms up to $O(\varepsilon^2)$ in (5.1) to obtain

$$U^{**} = U + \tfrac{1}{2}\hat{\chi}'\{\hat{N}_{iA}u_{i,A} - v\phi\}^2 \tag{5.4}$$

where U is given by (3.34). By regrouping the various quadratic terms we see that

$$U^{**} = U_e + P^e_{iA}u_{i,A} + T_e\phi + \tfrac{1}{2}\left\{\hat{c}^{**}_{iAkC}u_{i,A}u_{k,C} - 2\hat{\beta}^{**}_{iA}u_{i,A}\phi + \frac{T_e}{c^{**}_F}\phi^2\right\} \tag{5.5}$$

in which

$$\hat{c}^{**}_{iAjB} := \hat{c}_{iAjB} + \hat{\chi}'\hat{N}_{iA}\hat{N}_{jB}$$
$$\hat{\beta}^{**}_{iA} := \hat{\beta}_{iA} + \hat{\chi}'v\hat{N}_{iA}$$
$$c^{**}_F := c_F/(1+\hat{\chi}) \tag{5.6}$$

with

$$\hat{\chi} := \hat{\chi}'v^2c_F/T_e. \tag{5.7}$$

Now a nearly constrained material ($0 \leqslant \hat{\chi}' < \infty$) is, in fact, an unconstrained material and so the field equations can be read off from (3.18):

$$\hat{c}^{**}_{iAkC}u_{k,AC} - \hat{\beta}^{**}_{iA}\phi_{,A} = \rho_r\ddot{u}_i,$$
$$K_{AB}\left(\phi_{,AB} - \frac{c^{**}_F}{T_e}\hat{\beta}^{**}_{iC}u_{i,ABC}\right) = c^{**}_F\dot{\phi} \tag{5.8}$$

by replacing the elastic constants occurring in (3.18) by those defined at (5.6). From $(2.28)_2$ and $(5.6)_{2,3}$ we see that

$$\tilde{\beta}^{**}_{iA} = \frac{c^{**}_F}{T_e}\hat{\beta}^{**}_{iA} = \frac{\tilde{\beta}_{iA} + v^{-1}\hat{\chi}\hat{N}_{iA}}{1+\hat{\chi}}, \tag{5.9}$$

a combination of moduli occurring in $(5.8)_2$, whilst from (2.29) we have

$$\tilde{c}^{**}_{iAjB} = \hat{c}^{**}_{iAjB} - \frac{T_e}{c^{**}_F}\tilde{\beta}^{**}_{iA}\tilde{\beta}^{**}_{jB}, \tag{5.10}$$

leading by means of $(5.6)_{1,3}$ and (5.9) to

$$\tilde{c}^{**}_{iAjB} = \tilde{c}_{iAjB} + \frac{\hat{\chi}}{1+\hat{\chi}}.\frac{T_e}{c_F}(\tilde{\beta}_{iA} - v^{-1}\hat{N}_{iA}) \times(\tilde{\beta}_{jB} - v^{-1}\hat{N}_{jB}). \tag{5.11}$$

From $(5.6)_1$, (2.25) and Lemma A.1 we get

$$\hat{s}^{**}_{iAjB} = \hat{s}_{iAjB} - \frac{\hat{\chi}'\hat{s}_{iAkC}\hat{N}_{kC}\hat{s}_{jB\ell D}\hat{N}_{\ell D}}{1+\hat{\chi}'\hat{s}_{iAjB}\hat{N}_{iA}\hat{N}_{jB}}. \tag{5.12}$$

By using (2.28) and $(2.24)_2$ in (2.30) we obtain

$$\frac{1}{c_P} = \frac{1}{c_F} - \frac{1}{T_e}\hat{s}_{iAjB}\hat{\beta}_{iA}\hat{\beta}_{jB} \tag{5.13}$$

in which using $(5.6)_3$, (5.12) and $(5.6)_2$ gives

$$\frac{1}{c^{**}_P} = \frac{1}{c_P} + \frac{\hat{\chi}'}{1+\hat{\chi}'\hat{s}_{iAjB}\hat{N}_{iA}\hat{N}_{jB}}\cdot\frac{[v - \hat{\alpha}_{iA}\hat{N}_{iA}]^2}{T_e}. \tag{5.14}$$

By using similar arguments it can be shown, in direct notation, that

$$\hat{\boldsymbol{\alpha}}^{**} = \hat{\boldsymbol{\alpha}} + \frac{\hat{\chi}'(v - \hat{\boldsymbol{\alpha}}\cdot\hat{\mathbf{N}})}{1+\hat{\chi}'\hat{\mathbf{N}}\cdot\hat{\mathbf{s}}\,\hat{\mathbf{N}}}\hat{\mathbf{s}}\,\hat{\mathbf{N}},$$

$$\tilde{\boldsymbol{\alpha}}^{**} = \tilde{\boldsymbol{\alpha}} - \frac{\hat{\chi}\{1 + \frac{T_e}{c_F}\tilde{\boldsymbol{\beta}}\cdot\tilde{\mathbf{s}}(\tilde{\boldsymbol{\beta}} - v^{-1}\hat{\mathbf{N}})\}\tilde{\mathbf{s}}(\tilde{\boldsymbol{\beta}} - v^{-1}\hat{\mathbf{N}})}{1+\hat{\chi}+\hat{\chi}\frac{T_e}{c_F}(\tilde{\boldsymbol{\beta}} - v^{-1}\hat{\mathbf{N}})\cdot\tilde{\mathbf{s}}(\tilde{\boldsymbol{\beta}} - v^{-1}\hat{\mathbf{N}})},$$

$$\tilde{\mathbf{s}}^{**} = \tilde{\mathbf{s}} - \frac{\hat{\chi}\tilde{\mathbf{s}}(\tilde{\boldsymbol{\beta}} - v^{-1}\hat{\mathbf{N}})\otimes\tilde{\mathbf{s}}(\tilde{\boldsymbol{\beta}} - v^{-1}\hat{\mathbf{N}})}{\frac{c_F}{T_e} + \hat{\chi}\{\frac{c_F}{T_e} + (\tilde{\boldsymbol{\beta}} - v^{-1}\hat{\mathbf{N}})\cdot\tilde{\mathbf{s}}(\tilde{\boldsymbol{\beta}} - v^{-1}\hat{\mathbf{N}})\}}, \tag{5.15}$$

where $\otimes$ denotes the fourth-rank dyadic product of two second rank tensors.

Eqs. (5.6), (5.9), (5.11), (5.12), (5.14) and (5.15) constitute a set of expressions for the ten thermoelastic moduli of deformation-entropy nearly constrained thermoelasticity in terms of their counterparts in the unconstrained theory. In the limit $\hat{\chi}' = 0$, $\hat{\chi} = 0$ these two sets of moduli coincide, as is to be expected. We now explore the limit $\hat{\chi}' \to \infty$, $\hat{\chi} \to \infty$ in which the constraint holds exactly.

5.2. The limit $\hat{\chi}' \to \infty$, the constraint holds exactly

We shall discuss the behaviour in the limit of each of the ten thermoelastic moduli of Section 5.1 in turn. Considering the moduli of (5.6), we find that

$$\hat{\mathbf{c}}^{**} \to \infty, \quad \hat{\boldsymbol{\beta}}^{**} \to \infty, \quad c_{\mathrm{F}}^{**} \to 0. \tag{5.16}$$

In the context of wave propagation it has been shown [13, Eq. (4.32)] that the infinite limits $(5.16)_{1,2}$ do indeed result in the secular equation of deformation-entropy constrained thermoelasticity [13, Eq. (3.14)]. We may interpret zero limit $(5.16)_3$ of the heat capacity by differentiating the exact deformation-entropy constraint (4.11) to obtain, from definition $(2.26)_1$,

$$c_{\mathrm{F}}^{*} = T_{\mathrm{e}}\left(\frac{\partial\phi}{\partial\phi}\right)_{u_{i,A}} = 0, \tag{5.17}$$

c_{F}^{*} denoting the heat capacity of the constrained material. Taking the limit $\hat{\chi} \to \infty$ of (5.9) results in

$$\tilde{\beta}_{iA}^{**} \to \tilde{\beta}_{iA}^{*} := v^{-1}\hat{N}_{iA}, \tag{5.18}$$

so that the temperature coefficient of stress in the constraint limit is determined by the constraint itself. Alternatively, one may regard the nature of a possible constraint as being determined by knowledge of the temperature coefficient of stress. We may interpret the limit (5.18) by differentiating constraint (4.11) and using (2.10) to obtain

$$v^{-1}\hat{N}_{iA} = \left(\frac{\partial\phi}{\partial u_{i,A}}\right)_{\theta} = \tilde{\beta}_{iA}^{*}. \tag{5.19}$$

Taking the limit $\hat{\chi} \to \infty$ in the remaining thermoelastic moduli (5.11), (5.12), (5.14) and (5.15) results in limiting quantities that remain finite and do not vanish. We may not put $c_{\mathrm{F}} = 0$ and $\tilde{\boldsymbol{\beta}} - v^{-1}\hat{\mathbf{N}} = \mathbf{0}$ in these limiting quantities because here c_{F} and $\tilde{\boldsymbol{\beta}}$ refer to the underlying unconstrained material, before a constraint or near constraint is imposed, and may take on any (physically acceptable) values.

5.3. The limit $c_{\mathrm{F}} \to 0$, equivalent to the constraint

Connection (3.16) between temperature θ and entropy ϕ may be inverted to give

$$\phi = \tilde{\beta}_{iC}u_{i,C} + \frac{c_{\mathrm{F}}}{T_{\mathrm{e}}}\theta, \tag{5.20}$$

where $(2.28)_1$ has been used. The stress increment $\mathbf{p}$ is taken in the component form

$$p_{iA} = \tilde{c}_{iAkC}u_{k,C} - \tilde{\beta}_{iA}\theta. \tag{5.21}$$

We shall investigate the limit $c_{\mathrm{F}} \to 0$ in these equations, without employing constraint (4.11) directly, because this limit, in the form $c_{\mathrm{F}}^{**} \to 0$ of the nearly constrained material, see $(5.6)_3$, is inherent in the existence of the constraint. In this limit (5.20) becomes

$$\phi = \tilde{\beta}_{iC}u_{i,C} \tag{5.22}$$

and if consistency with constraint (4.11) is to be maintained we must have

$$\tilde{\beta}_{iA} = v^{-1}\hat{N}_{iA}, \tag{5.23}$$

which is precisely what (5.9) gives in the limit $\hat{\chi} \to \infty$, see (5.18) and (5.19). We may state that

$$\tilde{\boldsymbol{\beta}} - v^{-1}\hat{\mathbf{N}} \to \mathbf{0} \quad \text{as } c_{\mathrm{F}} \to 0. \tag{5.24}$$

On differentiating (5.20) with respect to ϕ at fixed $\mathbf{p}$, for $c_{\mathrm{F}} > 0$, and using (2.17) and (2.27), we find that

$$1 - \hat{\boldsymbol{\alpha}} \cdot \tilde{\boldsymbol{\beta}} = \frac{c_{\mathrm{F}}}{c_{\mathrm{P}}}, \tag{5.25}$$

so that in the limit $c_{\mathrm{F}} \to 0$, bearing in mind (5.24) and (5.25) gives

$$v - \hat{\boldsymbol{\alpha}} \cdot \hat{\mathbf{N}} \to 0 \quad \text{as } c_{\mathrm{F}} \to 0. \tag{5.26}$$

For any finite $\hat{\chi}' > 0$ we see that, for the nearly constrained material (5.4), Eq. (5.7) implies that

$$\hat{\chi} \to 0 \quad \text{as } c_{\mathrm{F}} \to 0, \tag{5.27}$$

in sharp contrast to the limit $\hat{\chi} \to \infty$ of constraint if $c_{\mathrm{F}} > 0$. In the limit (5.27) we see that (5.9) gives $\tilde{\boldsymbol{\beta}}^{**} \to \tilde{\boldsymbol{\beta}}$ which agrees with (5.18) since (5.24) holds in this limit. From the limits (5.24), (5.26) and (5.27) applied to the thermoelastic moduli (5.11), (5.14) and (5.15) we find that in the limit $c_F \to 0$,

$$\tilde{\mathbf{c}}, c_P, \hat{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\alpha}}, \tilde{\mathbf{s}}$$

retain the same values as for the unconstrained material if $\hat{\chi}' = 0$ and remain finite if $\hat{\chi}' \neq 0$.

We now discuss the remaining three moduli $\hat{\boldsymbol{\beta}}, \hat{\mathbf{c}}$ and $\hat{\mathbf{s}}$ in the limit $c_F \to 0$. From $(2.28)_1$, (5.24) and $(5.6)_2$

$$\hat{\boldsymbol{\beta}}^{**} = v\left(\frac{T_e}{v^2 c_F} + \hat{\chi}'\right)\hat{\mathbf{N}} \tag{5.28}$$

and from (2.29), (5.24) and $(5.6)_1$

$$\hat{\mathbf{c}}^{**} = \tilde{\mathbf{c}} + \left(\frac{T_e}{v^2 c_F} + \hat{\chi}'\right)\hat{\mathbf{N}} \otimes \hat{\mathbf{N}}. \tag{5.29}$$

The factors $1/c_F$ are retained as it is not possible to take the limit $c_F \to 0$ in (5.28) and (5.29). Applying Lemma A.1 to (5.29) gives

$$\hat{\mathbf{s}}^{**} = \tilde{\mathbf{s}} - \frac{(\tilde{\mathbf{s}}\hat{\mathbf{N}})\otimes(\tilde{\mathbf{s}}\hat{\mathbf{N}})}{(\frac{T_e}{v^2 c_F} + \hat{\chi}')^{-1} + \hat{\mathbf{N}} \cdot \tilde{\mathbf{s}}\hat{\mathbf{N}}}. \tag{5.30}$$

In the limit $c_F \to 0$, (5.30) gives

$$\hat{\mathbf{s}}^{**} \to \hat{\mathbf{s}}^{*} := \tilde{\mathbf{s}} - \frac{(\tilde{\mathbf{s}}\hat{\mathbf{N}})\otimes(\tilde{\mathbf{s}}\hat{\mathbf{N}})}{\hat{\mathbf{N}} \cdot \tilde{\mathbf{s}}\hat{\mathbf{N}}} \tag{5.31}$$

which satisfies

$$\hat{\mathbf{s}}^{*}\hat{\mathbf{N}} = \mathbf{0}. \tag{5.32}$$

The second-rank tensor $\hat{\mathbf{N}}$ is an eigentensor of the fourth-rank tensor (5.29) corresponding to infinite eigenvalue. Similarly, $\hat{\mathbf{N}}$ is a null tensor of (5.30) in the limit $c_F \to 0$, see (5.32). Whether $\hat{\chi}' = 0$ or $\hat{\chi}' > 0$ makes no difference in the limit $c_F \to 0$ to Eqs. (5.28), (5.29) and (5.30).

5.3.1. Sinusoidal wave propagation

Let us consider the complex exponential wave forms

$$\{u_i, \theta\} = \{U_i, \Theta\}\exp\{\mathrm{i}\omega(\mathbf{N}\cdot\mathbf{X}/v - t)\} \tag{5.33}$$

in which U_i and Θ are (complex) constant wave amplitudes, i denotes $\sqrt{(-1)}$, ω is the real frequency, $\mathbf{N}$ is the real unit vector defining the direction of wave propagation and v is the (complex) wave speed. A non-zero imaginary part of v leads to attenuation or growth of wave amplitude.

Substituting wave forms (5.33) into the field equations (3.14), with $c_F = 0$, leads to the secular equation

$$w\tilde{\mathbf{b}} \cdot \{\tilde{\mathbf{Q}} - w\mathbf{I}\}^{\mathrm{adj}}\tilde{\mathbf{b}} + \mathrm{i}\omega k \rho_r T_e^{-1} \times \det(\tilde{\mathbf{Q}} - w\mathbf{I}) = 0 \tag{5.34}$$

in which

$$w := \rho_r v^2, \quad \tilde{b}_i := \tilde{\beta}_{iA} N_A,$$
$$\tilde{Q}_{ij} = \tilde{c}_{iAjB} N_A N_B, \quad k = K_{AB} N_A N_B \tag{5.35}$$

and adj denotes the adjugate. The secular equation (5.34) is the same as that obtained [13, Eq. (2.22)] by taking the limit $c_F \to 0$ directly in the secular equation [13, Eq. (2.17)] of unconstrained thermoelasticity. Using (5.23) it can further be shown that (5.34) is equivalent to [13, Eq. (3.14)], the secular equation of deformation-entropy constrained thermoelasticity.

6. The deformation-temperature near constraint

6.1. Expressions for the ten thermoelastic moduli in the nearly constrained case

We take the Helmholtz free energy in the form

$$A^{\dagger\dagger} = A + \tfrac{1}{2}\tilde{\chi}'\{f(\mathbf{F}, T)\}^2 \tag{6.1}$$

so that $\tilde{\chi}' = 0$ corresponds to the original unconstrained material and the limit

$$\tilde{\chi}' \to \infty, \quad f(\mathbf{F}, T) \to 0,$$
$$\tilde{\eta} := \tilde{\chi}' f(\mathbf{F}, T) \text{ remains bounded} \tag{6.2}$$

corresponds to the constraint holding exactly. The stress and entropy in the nearly constrained material (6.1) are given by

$$P_{iA}^{\dagger\dagger} = P_{iA} + \tilde{\chi}' f(\mathbf{F}, T)\tilde{N}_{iA}, \quad S^{\dagger\dagger} = S + \tilde{\chi}' f(\mathbf{F}, T)\alpha. \tag{6.3}$$

On making the identification $(6.2)_3$ in (6.3) we may arrive at the equations appropriate if the constraint holds exactly, see (4.19).

In the linear approximation we must keep terms up to $O(\varepsilon^2)$ in (6.1) to obtain

$$A^{\dagger\dagger} = A + \tfrac{1}{2}\tilde{\chi}'\{\tilde{N}_{iA}u_{i,A} - \alpha\theta\}^2 \tag{6.4}$$

where A is given by (3.37). By regrouping the various quadratic terms we see that

$$A^{\dagger\dagger} = A_e + P^e_{iA}u_{i,A} - S_e\theta + \tfrac{1}{2}\left\{\tilde{c}^{\dagger\dagger}_{iAkC}\, u_{i,A}u_{k,C} - 2\tilde{\beta}^{\dagger\dagger}_{iA}u_{i,A}\theta - \frac{c^{\dagger\dagger}_F}{T_e}\theta^2\right\} \tag{6.5}$$

in which

$$\tilde{c}^{\dagger\dagger}_{iAjB} := \tilde{c}_{iAjB} + \tilde{\chi}'\tilde{N}_{iA}\tilde{N}_{jB},$$

$$\tilde{\beta}^{\dagger\dagger}_{iA} := \tilde{\beta}_{iA} + \tilde{\chi}'\alpha\tilde{N}_{iA},$$

$$c^{\dagger\dagger}_F := c_F - \tilde{\chi}'\alpha^2 T_e = c_F(1-\tilde{\chi}) \tag{6.6}$$

with

$$\tilde{\chi} := \tilde{\chi}'\alpha^2 T_e / c_F. \tag{6.7}$$

Once again we observe that a nearly constrained material is an unconstrained material and so the field equations can be read off from (3.14):

$$\tilde{c}^{\dagger\dagger}_{iAkC}u_{k,AC} - \tilde{\beta}^{\dagger\dagger}_{iA}\theta_{,A} = \rho_r\ddot{u}_i,$$

$$K_{AB}\theta_{,AB} - T_e\tilde{\beta}^{\dagger\dagger}_{iA}\dot{u}_{i,A} = c^{\dagger\dagger}_F\dot{\theta} \tag{6.8}$$

by replacing the elastic constants occurring in (3.14) by those defined at (6.6).

Of the ten elastic moduli discussed in Section 5.1 in the case of a deformation-entropy near-constraint, we have found expressions for three in the deformation-temperature case, namely $(6.6)_{1,2,3}$. By using similar methods we may derive expressions for the other seven, in direct notation:

$$\hat{\boldsymbol{\beta}}^{\dagger\dagger} = (1-\tilde{\chi})^{-1}(\hat{\boldsymbol{\beta}} + \tilde{\chi}\alpha^{-1}\tilde{\mathbf{N}}), \tag{6.9}$$

$$\hat{\mathbf{c}}^{\dagger\dagger} = \hat{\mathbf{c}} + \tilde{\chi}(1-\tilde{\chi})^{-1}c_F T_e^{-1}(\hat{\boldsymbol{\beta}} + \alpha^{-1}\tilde{\mathbf{N}}) \otimes(\hat{\boldsymbol{\beta}} + \alpha^{-1}\hat{\mathbf{N}}), \tag{6.10}$$

$$\tilde{\mathbf{s}}^{\dagger\dagger} = \tilde{\mathbf{s}} - \frac{(\tilde{\mathbf{s}}\,\tilde{\mathbf{N}})\otimes(\tilde{\mathbf{s}}\,\tilde{\mathbf{N}})}{(\tilde{\chi}')^{-1} + \tilde{\mathbf{N}}\cdot\tilde{\mathbf{s}}\,\tilde{\mathbf{N}}}, \tag{6.11}$$

$$\tilde{\boldsymbol{\alpha}}^{\dagger\dagger} = \tilde{\boldsymbol{\alpha}} + \frac{\alpha - \tilde{\boldsymbol{\alpha}}\cdot\tilde{\mathbf{N}}}{(\tilde{\chi}')^{-1} + \tilde{\mathbf{N}}\cdot\tilde{\mathbf{s}}\tilde{\mathbf{N}}}\,\tilde{\mathbf{s}}\,\tilde{\mathbf{N}}, \tag{6.12}$$

$$\hat{\mathbf{s}}^{\dagger\dagger} = \tilde{\mathbf{s}} - \frac{\tilde{\chi}c_F\hat{\mathbf{s}}(\hat{\boldsymbol{\beta}} + \alpha^{-1}\tilde{\mathbf{N}})\otimes\hat{\mathbf{s}}(\hat{\boldsymbol{\beta}} + \alpha^{-1}\tilde{\mathbf{N}})}{T_e(1-\tilde{\chi}) + \tilde{\chi}c_F(\hat{\boldsymbol{\beta}} + \alpha^{-1}\tilde{\mathbf{N}})\cdot\hat{\mathbf{s}}(\hat{\boldsymbol{\beta}} + \alpha^{-1}\tilde{\mathbf{N}})}, \tag{6.13}$$

$$c^{\dagger\dagger}_P = c_P - \frac{T_e(\alpha - \tilde{\boldsymbol{\alpha}}\cdot\tilde{\mathbf{N}})^2}{(\tilde{\chi}')^{-1} + \tilde{\mathbf{N}}\cdot\tilde{\mathbf{s}}\,\tilde{\mathbf{N}}}, \tag{6.14}$$

$$\hat{\boldsymbol{\alpha}}^{\dagger\dagger} = \hat{\boldsymbol{\alpha}} + \frac{\frac{T_e}{c_P}(\alpha - \tilde{\boldsymbol{\alpha}}\cdot\tilde{\mathbf{N}})}{(\tilde{\chi}')^{-1} + \tilde{\mathbf{N}}\cdot\tilde{\mathbf{s}}\tilde{\mathbf{N}} - \frac{T_e}{c_P}(\alpha - \tilde{\boldsymbol{\alpha}}\cdot\tilde{\mathbf{N}})^2} \times\left\{\frac{T_e}{c_P}(\alpha - \tilde{\boldsymbol{\alpha}}\cdot\tilde{\mathbf{N}})\tilde{\boldsymbol{\alpha}} + \tilde{\mathbf{s}}\tilde{\mathbf{N}}\right\}. \tag{6.15}$$

6.2. The limit $\chi' \to \infty$, the constraint holds exactly

In the limit $\chi' \to \infty$, when the deformation-temperature constraint holds exactly, we find from (6.6) that

$$\tilde{\mathbf{c}}^{\dagger\dagger} \to \infty, \quad \tilde{\boldsymbol{\beta}}^{\dagger\dagger} \to \infty, \quad c^{\dagger\dagger}_F \to -\infty. \tag{6.16}$$

In the context of wave propagation it has been shown [13, Eq. (4.16)] that the infinite limits $(6.16)_{1,2}$ do indeed result in the secular equation of deformation-temperature constrained thermoelasticity [13, Eq. (3.6)]. The limit $(6.16)_3$ implies that for $\tilde{\chi}'$ large enough the heat capacity becomes negative. A negative heat capacity is well known to correspond to a lack of stability, see [10, pp. 193, 201; 11], and the secular equation already mentioned [13, Eq. (3.6)] does indeed have at least one unstable branch. The infinite limit $(6.16)_3$ may be interpreted by differentiating the exact deformation-temperature constraint (4.20) to give

$$\frac{1}{c^{\dagger}_F} = \frac{1}{T_e}\left(\frac{\partial\theta}{\partial\phi}\right)_{u_{i,A}} = 0, \tag{6.17}$$

$c^{\dagger}_F$ denoting the heat capacity of the constrained material.

Taking the limit $\tilde{\chi} \to \infty$ of (6.9), which corresponds to $\tilde{\chi}' \to \infty$ with c_F finite, gives

$$\tilde{\beta}^{\dagger\dagger}_{iA} \to \tilde{\beta}^{\dagger}_{iA} := -\alpha^{-1}\tilde{N}_{iA} \tag{6.18}$$

so that once again the temperature coefficient of stress in the constraint limit is determined by the constraint itself. Limit (6.18) may also be interpreted by differentiating constraint (4.20) and using (2.11) to obtain

$$\alpha^{-1}\tilde{N}_{iA} = \left(\frac{\partial\theta}{\partial u_{i,A}}\right)_{\phi} = -\hat{\beta}^{*}_{iA}. \tag{6.19}$$

6.3. The limit $c_{\mathbf{F}} \to -\infty$, equivalent to the constraint

We take the temperature in the form (3.16):

$$\theta = -\hat{\beta}_{iC}u_{i,C} + \frac{T_{\mathrm{e}}}{c_{\mathbf{F}}}\phi \tag{6.20}$$

and the stress in the form

$$p_{iA} = \hat{c}_{iAkC}u_{k,C} - \hat{\beta}_{iA}\phi \tag{6.21}$$

and investigate the limit $c_{\mathbf{F}} \to -\infty$ in these equations, without employing constraint (4.20) directly. This limit, in the form $c_{\mathbf{F}}^{\dagger\dagger} \to -\infty$ of the nearly constrained material, see $(6.6)_3$ is implied by the constraint. Then (6.20) becomes

$$\theta = -\hat{\beta}_{iC}u_{i,C} \tag{6.22}$$

and consistency with (4.20) requires

$$\hat{\boldsymbol{\beta}} + \alpha^{-1}\tilde{\mathbf{N}} \to \mathbf{0} \quad \text{as } c_{\mathbf{F}} \to -\infty, \tag{6.23}$$

consonant with (6.19). By arguing as in Section 5.3 we also find that

$$\alpha - \tilde{\boldsymbol{\alpha}}\cdot\tilde{\mathbf{N}} \to 0 \quad \text{as } c_{\mathbf{F}} \to -\infty. \tag{6.24}$$

For any finite $\tilde{\chi}' > 0$ Eq. (6.7) implies that

$$\tilde{\chi} \to 0 \quad \text{as } c_{\mathbf{F}} \to -\infty \tag{6.25}$$

for the nearly constrained material, whereas the limit is $\tilde{\chi} \to \infty$ if $c_{\mathbf{F}} > -\infty$. We note that $\tilde{\chi}c_{\mathbf{F}} = \tilde{\chi}'\alpha^2 T_{\mathrm{e}}$ remains finite as $c_{\mathbf{F}} \to -\infty$ and it follows that the moduli (6.10)–(6.15) also exist and are finite in this limit.

6.3.1. Sinusoidal wave propagation

On dividing $(3.18)_2$ by $c_{\mathbf{F}}$ and taking the limit $c_{\mathbf{F}} \to -\infty$ we obtain the field equation

$$-K_{AB}\hat{\beta}_{iC}u_{i,ABC} = T_{\mathrm{e}}\dot{\phi}. \tag{6.26}$$

Substituting wave forms (5.33), with θ and Θ replaced by ϕ and Φ respectively, into the field equations $(3.18)_1$ and (6.26) leads to the secular equation

$$w\det(\hat{\mathbf{Q}} - w\mathbf{I}) - \mathrm{i}\omega k\rho_{\mathrm{r}}T_{\mathrm{e}}^{-1}\hat{\boldsymbol{b}}\cdot\{\hat{\mathbf{Q}} - w\mathbf{I}\}^{\mathrm{adj}}\hat{\mathbf{b}} = 0 \tag{6.27}$$

in which all quantities are defined as before with the addition of

$$\hat{b}_i := \hat{\beta}_{iA}N_A, \quad \hat{Q}_{ij} = \hat{c}_{iAjB}N_AN_B. \tag{6.28}$$

From $(2.28)_1$, (2.29), $(5.35)_3$ and (6.28) we have

$$\tilde{\mathbf{Q}} = \hat{\mathbf{Q}} - c_{\mathbf{F}}T_{\mathrm{e}}^{-1}\hat{\mathbf{b}}\otimes\hat{\mathbf{b}}\cdot \tag{6.29}$$

Using (6.29) in the secular equation [13, Eq. (2.17)] of unconstrained thermoelasticity and taking the limit $c_{\mathbf{F}} \to -\infty$ results in the secular equation (6.27) already found for that limit. Using (6.23) it can also be shown that (6.27) is equivalent to the secular equation of deformation-temperature constrained thermoelasticity, see [13, Eq. (3.6)].

Appendix A

Lemma A.1. *If* **A** *and* **B** *are fourth-rank tensors,* **C** *a second-rank tensor and* λ *a scalar related by*

$$\mathbf{A} = \mathbf{B} + \lambda\mathbf{C}\otimes\mathbf{C},$$

where $\otimes$ *denotes the dyadic product, and if* **B** *is symmetric in the sense that* $B_{iAjB} = B_{jBiA}$, *then provided that* $\mathbf{B}^{-1}$ *exists,* $\mathbf{A}^{-1}$ *exists and is given by*

$$\mathbf{A}^{-1} = \mathbf{B}^{-1} - \frac{(\mathbf{B}^{-1}\mathbf{C})\otimes(\mathbf{B}^{-1}\mathbf{C})}{\lambda^{-1} + \mathbf{C}\cdot\mathbf{B}^{-1}\mathbf{C}},$$

if the denominator does not vanish.

The proof is by multiplication of these equations.

References

[1] C. Truesdell, W. Noll, The Non-Linear Field Theories of Mechanics, Handbuch der Physik, III/3, Springer, New York, 1965.

[2] A.E. Green, P.M. Naghdi, J.A. Trapp, Thermodynamics of a continuum with internal constraints, Int. J. Engng. Sci. 8 (1970) 891–908.

[3] M.E. Gurtin, P. Podio Guidugli, The thermodynamics of constrained materials, Arch. Rat. Mech. Anal. 51 (1973) 192–208.

[4] J. Casey, S. Krishanaswamy, On constrained thermoelastic materials, in: R.C. Batra, M.F. Beatty (Eds.), Contemporary Research in Mechanics and Mathematics of Materials, CIMNE, Barcelona, 1996, pp. 359–371.

[5] J. Casey, A treatment of internally constrained materials, J. Appl. Mech. 62 (1995) 542–544.

[6] J. Casey, A treatment of internally constrained materials, J. Appl. Mech. 63 (1996) 240.

[7] T. Manacorda, Onde elementari nella termoelasticità di solidi incomprimibili, Atti dell' Accademia delle scienze, Torino 101 (1967) 503–509.

[8] C.E. Beevers, Finite Waves in Thermoelasticity, Ph.D. Thesis, University of Manchester, 1969.

[9] P. Chadwick, N.H. Scott, Linear dynamical stability in constrained thermoelasticity: I. Deformation-temperature constraints, Quart. J. Mech. Appl. Math. 45 (1992) 641–650.

[10] R.J. Knops, E.W. Wilkes, Theory of elastic stability, in: C. Truesdell (Ed.), Mechanics of Solids III, Springer, Berlin, 1984, pp. 125–302.

[11] N.H. Scott, A theorem in thermoelasticity and its application to linear stability, Proc. Roy. Soc. London A 424 (1989) 143–153.

[12] N.H. Scott, Linear dynamical stability in constrained thermoelasticity: II. Deformation-entropy constraints, Quart. J. Mech. Appl. Math. 45 (1992) 651–662.

[13] N.H. Scott, Connections between deformation-temperature and deformation-entropy constraints and near-constraints in thermoelasticity, Int. J. Engng. Sci. 34 (1996) 1689–1704.

[14] D.J. Leslie, N.H. Scott, Incompressibility at uniform temperature or entropy in isotropic thermoelasticity, Quart. J. Mech. Appl. Math. 51 (1998) 191–211.

[15] D.J. Leslie, N.H. Scott, Near incompressibility at uniform temperature or entropy in isotropic thermoelasticity, Math. Mech. Solids 3 (1998) 243–275.

[16] N.H. Scott, Small vibrations of prestrained constrained elastic materials: the idealized fibre-reinforced material, Int. J. Solids Struct. 27 (1991) 1969–1980.

[17] P. Chadwick, Basic properties of plane harmonic waves in a prestressed heat-conducting elastic material, J. Thermal Stresses 2 (1979) 193–214.

[18] P. Chadwick, A.M. Whitworth, P. Borejko, Basic theory of small-amplitude waves in a constrained elastic body, Arch. Rat. Mech. Anal. 87 (1985) 339–354.