Kazhdan-Lusztig cells in affine Weyl groups with unequal parameters

Jérémie Guilhot

Acknowledgments

First of all I wish to thank my supervisor Meinolf Geck who guided me through my Ph.D. studies. His patience, generosity and support made this thesis possible. I feel very fortunate to have been his student.

My thoughts are going to Fokko du Cloux who was my supervisor in Lyon during the second year of my Ph.D. I would also like to thank Philippe Caldéro for being my supervisor the past two years.

I am very grateful to George Lusztig and Cédric Bonnafé who I feel honoured to have had as referees. I would also like to thank Radha Kessar, Iain Gordon and Kenji Iohara who kindly accepted to be part of the jury.

I would like to thank all the staff of the Department of Mathematical Sciences in Aberdeen, I really enjoyed being there and after all this time I am still amazed at how good the atmosphere was. Many thanks to Lacri who read and corrected many parts of my Ph.D (including the acknowledgements!) and whose office door was always open for me. Thanks to all the postgraduates and postdocs for all the good times we had and for supporting me all these years. A special thanks to Fabien and Meadhbh who respectively had to share an office and a flat with me, I know it was not always easy! To be complete I should thank Christian and Paolo who are not in the Math Department, but nearly!

I wish to thank all the staff of the Camille Jordan Institute in Lyon, especially all the Ph.D. students for their warm welcome and all the nice times we had together.

During this Ph.D. I went to numerous conferences and workshops and I would like to thank all the people I have met there for making these events very lively and enjoyable. I'm thinking in particular of Maria, Daniel, Nicolas, Olivier B. and Olivier D. A special thanks to Jean-Baptiste for all the great times we had at these events and in Aberdeen.

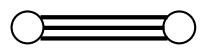
Last but not least, I am very grateful to my family and friends who were always there for me.

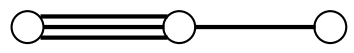
Introduction

This thesis is concerned with the theory of Hecke algebras, whose origin lies in a paper by Iwahori in 1964; see [**23**]. These algebras naturally arise in the representation theory of reductive algebraic groups over finite or p-adic fields, as endomorphism algebras of certain induced representations. The overall philosophy is that a significant amount of the representation theory of the group is controlled by the representation theory of those endomorphism algebras.

A standard situation can be described as follows. Let G be a finite group with a BN-pair with associated Weyl group W. Let $\mathcal{H}$ be the endomorphism algebra of the permutation representation of G on the cosets of B. By standard results, the irreducible representations of $\mathcal{H}$ are in bijection with the irreducible representations of G which admit non-zero vectors fixed by B. Now $\mathcal{H}$ has a standard basis indexed by the elements of W, usually denoted by $\{T_w | w \in W\}$. The multiplication can be described in purely combinatorial terms. Let S be a set of Coxeter generators of W. For any $w \in W$, we have $T_w = T_{s_1} ... T_{s_l}$ if $w = s_1 ... s_l$ ($s_i \in S$) is a reduced expression of w. Furthermore, we have $T_s^2 = q_s T_1 + (q_s - 1) T_s$ for any $s \in S$, where $q_s = |BsB/B|$. Now assume that G is the set of $\mathbf{F}_q$-rational points of a connected reductive algebraic defined over $\mathbf{F}_q$. Then we have $q_s = q^{c_s}$ where the numbers c_s are positive integers; they are called the parameters of $\mathcal{H}$. They extend to a weight function $L : W \to \mathbb{Z}$ in the sense of Lusztig [**38**], where $L(s) = c_s$ for all $s \in S$. Then it turns out that the above rules for the multiplication can be used to give an abstract definition of $\mathcal{H}$ without reference to the underlying group G, namely by explicit generators and relations in terms of W and the weight function L.

More generally, one can consider endomorphism algebras of representations obtained by Harish-Chandra induction of cuspidal representations of Levi subgroups of G. In another direction, one can consider p-adic groups instead of finite groups, in which case we obtain Hecke algebras associated with affine Weyl groups. Thus, it is an interesting and important problem to study the representation theory of abstract "Iwahori-Hecke algebras" associated with a finite or an affine Weyl group W and a weight function L. One should note, however, that not all possible weight functions actually arise "in nature", i.e., in the framework of representations of reductive groups over finite or p-adic fields. For example, consider the finite Weyl group W of type G_2 and the corresponding affine Weyl groups $\tilde{G}_2$, with diagrams

G_2 : 

$\tilde{G}_2$: 

The only weight functions on G_2 arising "in nature" are those with the following values on the simple reflections $(1,1)$, $(3,1)$, $(9,1)$; see [**26**, Table II, p35].

The only weight functions on $\tilde{G}_2$ arising "in nature" are those with the following values on the simple reflections $(9,1,1)$, $(3,1,1)$, $(1,1,1)$, $(1,3,3)$; see [**37**, 7.9, 7.23, 7.36, 8.14].

A major breakthrough in the study of representations of Hecke algebras with equal parameters was achieved in the celebrated paper "Representation of Coxeter groups and Hecke algebras" by Kazhdan and Lusztig (see [**24**]) where they first introduced the notion of left, right and two-sided cells of an arbitrary Coxeter group. The definition involves a new, canonical basis of the Iwahori-Hecke algebra $\mathcal{H}$. In a following paper ([**25**]), they showed that the Kazhdan-Lusztig basis of a Hecke algebra associated to a Weyl group has a geometric interpretation in terms of intersection cohomology of algebraic varieties. This connection has been of crucial importance to solve a number of problems in different aspects of representation theory; see [**36**].

From then on, cells have been intensively studied. Not only they give rise to representations of the Coxeter group W but also of the corresponding Iwahori-Hecke algebra $\mathcal{H}$. In type A, it turns out that the representations afforded by left cells give all the irreducible representations of $\mathcal{H}$. This is not true in general, however. In the general case of a Weyl group W, we say that two irreducible representations of W are linked if they both appear as constituents in a representation afforded by a left cell. By taking the transitive closure of this relation, we obtain a partition of the irreducible representations of W into so-called "families". These are in a natural bijection with the two-sided cells of W and play a crucial role in the classification of unipotent representations of reductive groups over finite fields; see Lusztig [**29**]. The decomposition for Weyl groups of the left cell representations into irreducible representations is completely known, see [**32**].

The cell theory of affine Weyl groups in the equal parameter case was first studied by Lusztig. In a series of papers, he studied the representations of the corresponding Hecke algebra afforded by cells (see [**30, 33, 34, 35**]). In particular, he described all the cells of the affine Weyl groups of ranks less than 2. The decomposition into cells have been explicitly described for type $\tilde{A}_r, r \in \mathbb{N}$ (see [**31, 40**]), ranks 2, 3 (see [**2, 14, 30**]) and types $\tilde{B}_4$, $\tilde{C}_4$ and $\tilde{D}_4$ (see [**10, 43, 44**]).

A special feature of affine Weyl groups is that there is a distinguished two-sided cell, the so called "lowest two-sided cell", which contains, roughly speaking, most of the elements of the group. This cell has been thoroughly studied by Shi ([**41, 42**]). In particular, he described the left cells lying in the lowest two-sided cell.

In 1983, Lusztig [**28**] generalized the definition of cells in the case where the simple reflections of the Coxeter groups are given different weights. This generalization of cells give rise to representations of Iwahori-Hecke algebras with unequal parameters.

Many of the problems that have been studied in the equal parameter case have natural extensions to the general case of unequal parameters. However, the knowledge in that case in nowhere near the one in the equal parameter case. The main reason is that a crucial ingredient in the proofs of the above-mentioned results in the equal parameter case is the geometric interpretation of the Kazhdan-Lusztig basis and the resulting "positivity properties", such as the positivity of the coefficient of the structure constants with respect to the Kazhdan-Lusztig basis. Simple examples show that these "positivity properties" definitely do no longer hold in the case of unequal parameters. Hence, the need to develop new methods for dealing with Kazhdan-Lusztig cells without reference to those "positivity properties". Ideally, these methods should work uniformly for all choices of parameters.

A major step in this direction is achieved by Lusztig's formulation of 15 conjectural properties **P1**–**P15** in [**38**, Chap. 14], which capture essential properties of cells for all choices of parameters. These properties can be used as an axiomatic basis for studying the structure and representations of Hecke algebras. See, for example, [**38**, Chap. 22] where Lusztig develops the representation theory of Hecke algebras associated with finite Weyl groups on the basis of **P1**–**P15**. These conjectures are known to hold for finite and affine Weyl groups in the equal parameter case, thanks to the above-mentioned geometric interpretation. As far as unequal parameters are concerned, **P1**–**P15** are only known to hold in some special situation, including:

- type B_n in the "asymptotic case", see [**7, 18**];
- infinite dihedral type, see [**38**, Chap. 17].

However, a general proof of **P1**–**P15** seems far out of reach at present.

In this context, our thesis forms a contribution to the programme of developping methods for dealing with cells which a) work uniformly for all choices of parameters and b) do not refer to a geometric interpretation. More precisely, we are mostly concerned with affine Weyl groups; the starting point is a thorough study of the cells in the affine Weyl group of type G_2 with unequal parameters. If **P1**–**P15** were known to hold then, for example, we could deduce that there are only finitely many left cells in each case. One of the results of this thesis shows that this conclusion is true, without using **P1**–**P15**. We also show that, in fact, there are only finitely many partitions of $\tilde{G}_2$ into left cells. The main ingredients for the proof of these results are, on the one hand, the invariance of the Kazhdan-Lusztig polynomials under "long enough" translations in an affine Weyl group and, on the other hand, explicit computations using GAP [**39**] and COXETER [**11**]. We will also determine the exact decomposition of $\tilde{G}_2$ into left cells for a certain class of weight functions.

The main theoretical results of this thesis concern the theory of the "lowest two-sided cell", which has been described by Xi ([**46**]) and Bremke ([**9**]) in the general case of unequal parameters. As mentioned before, the decomposition of this cell into left cells is known in the equal parameter case. It has also been determined is some specific cases of unequal parameters which still admit a geometric interpretation; see [**9**]. Our main result describes the decomposition of this lowest two-sided cell into left cells thus completing the work begun by Xi and Bremke. The proof uniformly works for all choices of parameters.

We now give an outline of the content of this thesis.

In Chapter 1, we present the theory of Coxeter groups. We give a classification of the Weyl groups and the affine Weyl groups.

In Chapter 2, we present the Kazhdan-Lusztig theory. In particular, we define left, right and two-sided cells and give some examples.

In Chapter 3, we introduce the geometric presentation in term of alcoves. Since it plays a key role in many results of this thesis, we give a number of examples. In the final section, we use this presentation to determine an upper bound on the degrees of the structure constants with respect to the standard basis.

In Chapter 4, we introduce the original setting for cells with unequal parameter, as defined in [**28**], where instead of a weight function, Lusztig defined the cells with respect to an abelian group and a total order on it. Then we show that this setting can be used to determine whether two weight functions give rise to essentially the same data on a given finite subset of an affine Weyl group. The main result of this chapter is the invariance of the Kazhdan-Lusztig polynomials of an affine Weyl group under "long enough" translations. We then apply both these results to $\tilde{G}_2$ to obtain some finiteness results about cells in this group.

In Chapter 5, we generalized an argument due to Geck ([**15**]) on the induction of Kazhdan-Lusztig cells (see also [**21**], where this idea was first developed). This result will be our main tool to "separate" cells. We give a first application where we show that under some specific condition on the parameters, the cells in a certain parabolic subgroup are still cells in the whole group.

In Chapter 6, we study the lowest two-sided cell of an affine Weyl group in the general case of unequal parameters. Using the generalized induction of Kazhdan-Lusztig cells, we determine its decomposition into left cells.

Finally, in Chapter 7, we give the decomposition of the affine Weyl group $\tilde{G}_2$ into left and two-sided cells for a whole class of weight functions. We also determine the partial left (resp. two-sided) order on the left (resp. two-sided) cells. Finally, we briefly discuss the "semicontinuity properties" of Kazhdan-Lusztig cells, recently conjectured by Bonnafé. We give some "conjectural" decompositions of $\tilde{G}_2$ into left cells for any weight functions, and show that it agrees with Bonnafé's conjecture.

Contents

CHAPTER 1

Reflection groups and Coxeter groups

We start this chapter with the study of finite reflection groups in a real Euclidean space. Each reflection determines a reflecting hyperplane (the set of fixed points) and a vector orthogonal to it (the "root"). This leads to the theory of root systems which will allow us to find simple presentations of finite reflection groups in terms of generators and relations; see Section 1.1.1. We will focus on finite reflection groups which arise naturally in Lie Theory, namely the "Weyl groups". Then, we will describe a class of infinite reflection groups, generated by affine reflections (the so-called "affine Weyl groups") which are closely related to Weyl groups; see Section 1.1.3. Such groups admit a simple presentation, similar to those of Weyl groups.

Motivated by the examples of finite reflection groups and affine Weyl groups, we will study a more general type of group, the Coxeter groups; see Section 1.2. Such groups are defined by a set of generators of order 2 subject only to relations which give the order of any product of two generators. This presentation leads to many combinatorial properties such as the Bruhat order; see Section 1.2.5.

The basic references for this chapter are [**8**],[**22**] and [**38**]

1.1. Reflection groups

Let V be a real Euclidean space with scalar product denoted by $\langle .,. \rangle$. We are interested in the study of reflection groups. First of all we should clarify what we mean by reflection. A reflection is a linear transformation on V which sends a non-zero vector $\alpha \in V$ to $-\alpha$ and fixes pointwise the hyperplane orthogonal to α. We denote such a reflection by σ_α and by H_α its set of fixed points. Let $x \in V$, we have

$$\sigma_\alpha(x) = x - \frac{2\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

1.1.1. Root system. A root system Φ is a finite set of non-zero vectors in V such that

(R1) for all $\alpha \in \Phi$ we have $\Phi \cap \mathbf{R}\alpha = \{\alpha, -\alpha\}$,
(R2) for all $\alpha \in \Phi$, we have $\sigma_\alpha \Phi = \Phi$.

The rank of Φ is the dimension of the vector space spanned by Φ. A root system is said to be "reducible" if there exist two orthogonal subspaces V_1, V_2 of V and two root systems Φ_1 (resp. Φ_2) of V_1 (resp. V_2) such that $\Phi = \Phi_1 \cup \Phi_2$.
We denote by W the group generated by the reflections σ_α, $\alpha \in \Phi$.

REMARK 1.1.1. The interest of this definition is that any finite reflection group can be realized in this way. Conversely, any reflection group arising from a root system is finite.

We fix a root system Φ in V. A set of positive roots (there can be many choices) is a subset Φ^+ of Φ such that

(1) for each root α exactly one of the roots $\alpha, -\alpha$ is contained in Φ^+,
(2) for any $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta \in \Phi$, we have $\alpha + \beta \in \Phi^+$.

Note that such a set exists. A simple system is a subset $\Delta \subset \Phi$ such that

(1) Δ is a basis of the $\mathbf{R}$-vector space spanned by Φ.
(2) any $\alpha \in \Phi$ is a $\mathbb{Z}$-linear combination of elements of Δ with coefficients all of the same sign.

Note that simple systems exist and they are all conjugate in W. The elements of Δ are called simple roots.

REMARK 1.1.2. A set of positive roots contains a unique simple system. Conversely, every simple system is contained in a unique set of positive roots.

Fix a simple system Δ. The reflection group W associated to Φ is in fact generated by the set $\{\sigma_\alpha,\ \alpha \in \Delta\}$. Let $\alpha, \beta \in \Delta$, we denote by $m_{\alpha,\beta}$ the order of $\sigma_\alpha \sigma_\beta$ in W. It can be shown that a group W arising from a root system has a presentation of the form

$$\langle \sigma_\alpha, \alpha \in \Delta \mid (\sigma_\alpha \sigma_\beta)^{m_{\alpha,\beta}} = 1,\ \sigma_\alpha^2 = 1,\ \alpha, \beta \in \Delta \rangle.$$

EXAMPLE 1.1.3. Let $n \in \mathbb{N}$. Let V be the Euclidean plane. Then, the set

$$\Phi := \{\alpha_i := \big(\cos(\frac{2i\pi}{n}), \sin(\frac{2i\pi}{n})\big) \mid 0 \leq i \leq m-1\}$$

is certainly a root system (of rank 2).

In Figure 1, we consider the case $n = 8$. The plain arrows represent a choice of positive roots and the thick arrows represent a simple system.

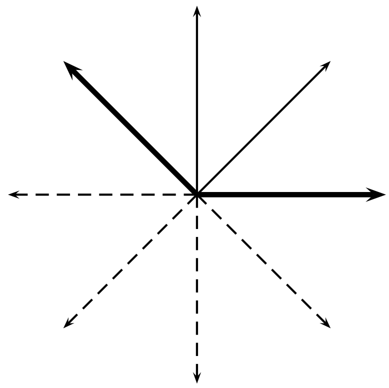


FIGURE 1. Root system of the Dihedral group of order 8

The reflection group associated to Φ is the Dihedral group with 8 elements. It is generated by σ_{α_0} and σ_{α_3} and has presentation

$$D_8 := \langle \sigma_{\alpha_0}, \sigma_{\alpha_3} \mid \sigma_{\alpha_0}^2 = \sigma_{\alpha_3}^2 = (\sigma_{\alpha_0}\sigma_{\alpha_3})^4 = 1 \rangle.$$

1.1.2. Weyl group. We say that a root system Φ is crystallographic if it satisfies the additional requirement

(R3) for all $\alpha, \beta \in \Phi$, we have $\frac{2\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle} \in \mathbb{Z}$.

The groups generated by crystallographic root systems are known as Weyl groups.

Let $\alpha, \beta \in \Phi$, then (R3) forces the angle between the hyperplanes H_α and H_β to be in $\{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$. Since the composition of the reflections σ_α and σ_β is a rotation of angle twice the angle between H_α and H_β, we see that if Φ is crystallographic then $m_{\alpha,\beta} \in \{2, 3, 4, 6\}$.

EXAMPLE 1.1.4. On Figure 2, we show all the possible crystallographic root systems of rank two. Note that the root system $A_1 \times A_1$ is reducible and the root systems A_2, B_2 and G_2 are irreducible.

REMARK 1.1.5. We shall not give details of the classification of irreducible crystallographic root systems. It can be found in [**8**]. One should only know that if Φ is an irreducible crystallographic root system, then at most two root lengths are possible. Moreover, there exist two root systems B_n and C_n ($n \geq 3$) which differ only by the length of their roots. They give rise to the same Weyl group. If $n = 2$, then the roots system B_2 and C_2 are the same.

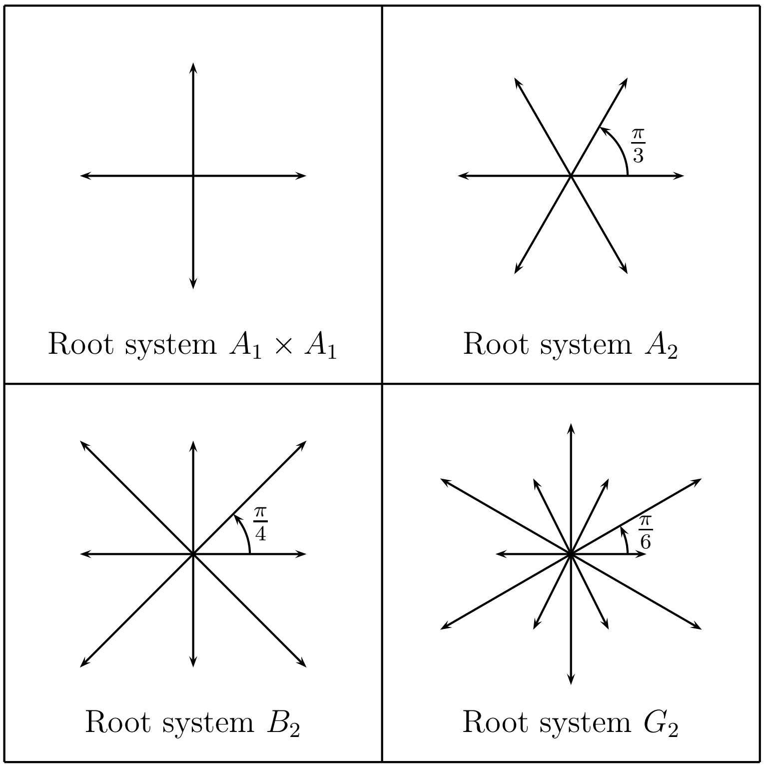


FIGURE 2. Crystallographic Root systems of rank 2

1.1.3. Affine Weyl group. Let Φ be a crystallographic root system with Weyl group W. We assume that Φ spans V. Let Q be the root lattice

$$Q := \{n_1\alpha_1 + \ldots + n_k\alpha_k \mid n_i \in \mathbb{Z},\ \alpha_i \in \Phi\}.$$

The Weyl group W acts on Q (see (R2)), thus we can form the semi-direct product

$$\tilde{W} := W \ltimes Q.$$

The group $\tilde{W}$ is called the affine Weyl group of Φ.

We give a geometric interpretation of $\tilde{W}$. Basically, instead of considering only linear transformations on V, we also consider affine transformations. Let $\alpha \in \Phi$. We denote by $\check{\alpha}$ the coroot of α defined by $\check{\alpha} = \frac{2\langle\alpha,.\rangle}{\langle\alpha,\alpha\rangle} \in V^*$ (where V^* is the dual space of V). One can identify $\check{\alpha} \in V^*$ with $\frac{2\alpha}{\langle\alpha,\alpha\rangle} \in V$. Then the set $\check{\Phi} = \{\check{\alpha} \mid \alpha \in \Phi\}$ (called the dual root system of Φ) is a crystallographic root system of V with Weyl group W. For any $\alpha \in \Phi$ and $k \in \mathbb{Z}$ let

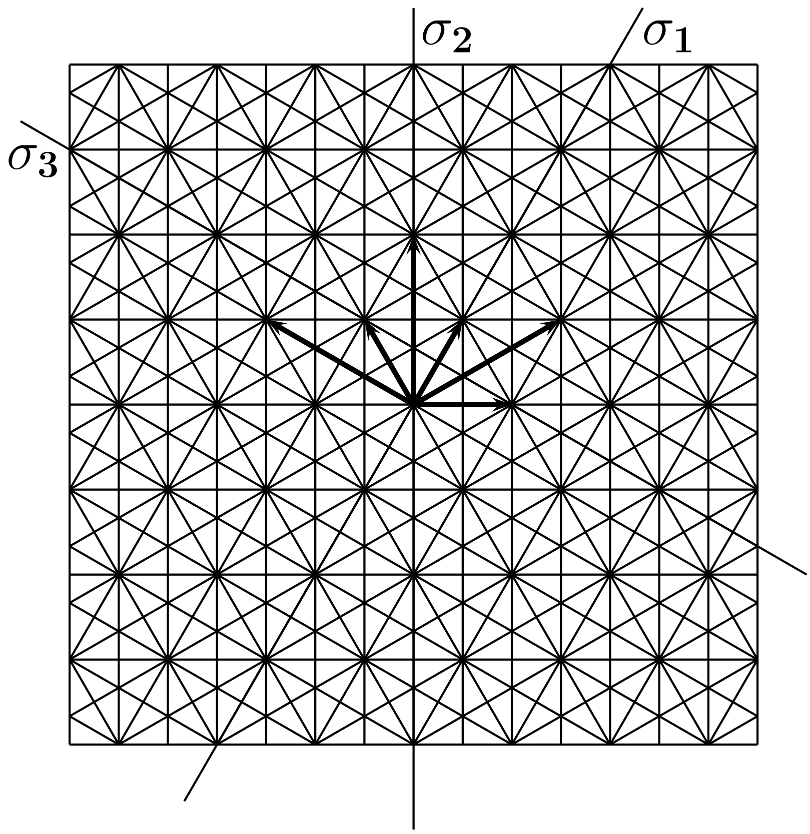
$$H_{\alpha,k} = \{x \in V \mid \check{\alpha}(x) = \frac{2\langle\alpha, x\rangle}{\langle\alpha,\alpha\rangle} = k\}.$$

The group $\tilde{W}$ is the group generated by all the reflections with fixed point sets the hyperplanes $H_{\alpha,k}$. We denote such a reflection by $\sigma_{\alpha,k}$. Note that W is generated by $\sigma_{\alpha,0}$ for $\alpha \in \Phi$. Assume that Φ is irreducible and let n be the rank of Φ, then it can be shown that $\tilde{W}$ is generated by $n+1$ reflections and has a presentation similar to the presentation of the Weyl group W.

Remark 1.1.6. The root systems B_n and C_n introduced in Remark 1.1.5 are dual. They give rise to the same Weyl group but not the same affine Weyl group.

Example 1.1.7. In the following figure, we give the example of $\tilde{G}_2$, the affine Weyl group associated to the root system G_2. The thick arrows represent a choice of positive roots. One can check that $\tilde{G}_2$ has the following presentation

$$\langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1,\ (\sigma_1\sigma_2)^6 = (\sigma_2\sigma_3)^3 = (\sigma_1\sigma_3)^2 = 1 \rangle.$$



1.2. Coxeter groups

A Coxeter system (W, S) consists of a group W and a finite set of generators $S \subset W$ subject (only) to the relations

$$(ss')^{m_{s,s'}} = 1$$

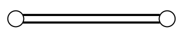
where $m_{s,s} = 1$ and $m_{s,s'} = m_{s',s} \geq 2$ for $s \neq s'$. As we have seen in the previous section, finite reflection groups and affine Weyl groups are Coxeter systems. That is why the elements of S are often called simple reflections. We will sometime refer to (W, S) as a Coxeter group.

In this section, (W, S) denotes a Coxeter system.

1.2.1. Coxeter graph. A convenient way to encode the informations in the presentation of W is in the so-called "Coxeter graph". The vertices of the Coxeter graph (say Γ) are in one to one correspondence with the set of generators S; if $m_{s,s'} = 2$ (i.e. s and s' commute) we do not join the corresponding vertices. We join the other vertices as follows

- If $m_{s,s'} = 3$, we join the vertices by 1 edge;
- If $m_{s,s'} = 4$, we join the vertices by 2 edges;
- If $m_{s,s'} = 6$, we join the vertices by 3 edges;
- In any other cases we label the edge with $m_{s,s'}$.

Example 1.2.1. The Dihedral group of order 8 (see Example 1.1.3) has the following graph



1.2.2. Standard parabolic subgroup and irreducible Coxeter group. Let I be a subset of S. Then one can consider the group generated by the generators $s \in I$. It is called a standard parabolic subgroup and we denote it by W_I.

Proposition 1.2.2. *Let (W, S) be a Coxeter system with graph Γ. Denote by Γ_i $(1 \leq i \leq n)$ the connected components of Γ and by S_i the corresponding subset of S. Then W is the direct product of the parabolic subgroups W_{S_i}, for $1 \leq i \leq n$.*

When the graph of (W, S) is connected, we say that (W, S) is irreducible. The above proposition shows that the study of Coxeter groups can be reduced to the case where W is irreducible.

Remark 1.2.3. It is readily checked that if Φ is an irreducible root system, then the associated reflection group W have a connected graph and hence is irreducible.

1.2.3. Geometric representation of W. As noticed before, finite reflection groups are Coxeter groups. Of course not all Coxeter groups will be reflection groups (in the sense of Section 1.1), however, if we redefine a reflection to be merely a linear transformation which fixes pointwise an hyperplane and send some non-zero vector to its opposite, we obtain a reasonable substitute...

Let (W, S) be a Coxeter system. Let V be a vector space with basis $\{e_s, s \in S\}$. Let B be the bilinear form on V defined by

$$B(e_s, e_{s'}) = -cos(\frac{\pi}{m_{s,s'}}).$$

We clearly have $B(e_s, e_s) = 1$ and $B(e_s, e_{s'}) \leq 0$ if $s \neq s'$. Now, to any $s \in S$ we can associate a "reflection" σ_s defined by

$$\sigma_s(x) = x - 2B(e_s, x)e_s \quad \text{for all } x \in V.$$

One can easily check that $\sigma_s(e_s) = -e_s$ and that H_s, the hyperplane orthogonal to e_s (with respect to B) is fixed pointwise. Furthermore, σ_s (for all $s \in S$) preserves the form B.

Definition 1.2.4. We say that

(1) W is tame if the bilinear form B is positive, that is $B(e, e) \geq 0$ for all $e \in V$;
(2) W is integral if $m_{s,s'} \in \{2, 3, 4, 6, \infty\}$.

The linear form B gives some informations about the Coxeter group W. For instance, we have

Theorem 1.2.5. *Let W be a Coxeter group. The following are equivalent*

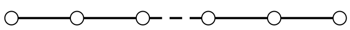
(i) *W is finite.*
(ii) *The bilinear form B is positive definite.*
(iii) *W is a finite reflection group.*

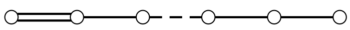
1.2.4. Classification. In this thesis, we are primarily concerned with tame Coxeter groups. There are 3 different kinds of tame, irreducible Coxeter groups:

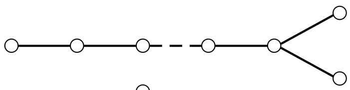
(1) finite and integral (i.e. the Weyl groups);
(2) finite and non integral;
(3) infinite and automatically integral (i.e. affine Weyl groups);

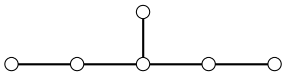
We give a classification of such groups in the following theorem.

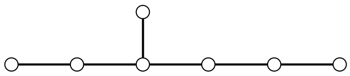
THEOREM 1.2.6. *Let W be an irreducible, finite, tame and integral Coxeter group. Then W has one of the following graphs (where the index denotes the number of vertices in the graph)*

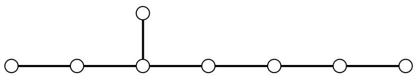
A_n, $(n \geq 1)$ 

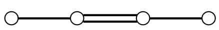
B_n, $(n \geq 2)$ 

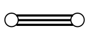
D_n, $(n \geq 4)$ 

E_6 

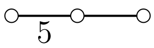
E_7 

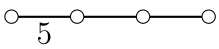
E_8 

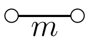
F_4 

G_2 

Let W be an irreducible, finite, tame and non-integral Coxeter group, then W has one of the following graphs

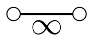
H_3 

H_4 

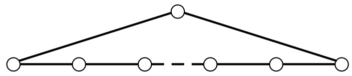
$I_2(m)$
$m \notin \{2, 3, 4, 6\}$ 

Let W be an irreducible, tame and infinite Coxeter group. Then W has one of the following graphs (where the index n indicates that the graph has $n+1$ vertices)

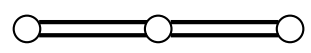
$\tilde{A}_1$



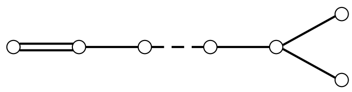
$\tilde{A}_n, \ (n \geq 2)$



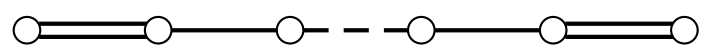
$\tilde{B}_2 = \tilde{C}_2$



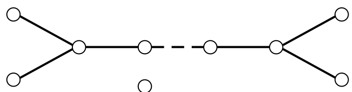
$\tilde{B}_n, \ (n \geq 3)$



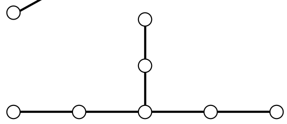
$\tilde{C}_n, \ (n \geq 3)$



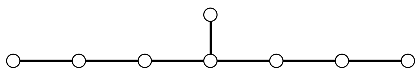
$\tilde{D}_n, \ (n \geq 4)$



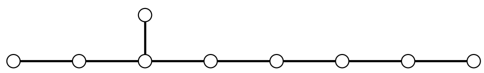
$\tilde{E}_6$



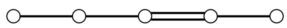
$\tilde{E}_7$



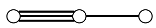
$\tilde{E}_8$



$\tilde{F}_4$



$\tilde{G}_2$



REMARK 1.2.7. One can check that any proper standard parabolic subgroup of an irreducible affine Weyl group is finite.

1.2.5. Bruhat order. The specific presentation of a Coxeter group gives rise to many combinatorial properties. We give a brief overview of these properties.

Let $w \in W$. The length of w (denoted $\ell(w)$) is the smallest integer $n \in \mathbb{N}$ such that w can be written $s_1 s_2 \dots s_n$ where $s_i \in S$ for all $1 \leq i \leq n$. In that case, $s_1 \dots s_n$ is called a reduced expression of w. The length is well defined and unique, however, given $w \in W$, there might be many different reduced expressions. It is readily checked that, for $s \in S$ and $w \in W$, we have either $\ell(sw) = \ell(w) - 1$ or $\ell(sw) = \ell(w) + 1$. This leads to the following definition.

Definition 1.2.8. Let $w \in W$. We set

$$\mathcal{L}(w) := \{s \in S \mid \ell(sw) = \ell(w) - 1\} \quad \text{and} \quad \mathcal{R}(w) = \{s \in S \mid \ell(ws) = \ell(w) - 1\}.$$

The set $\mathcal{L}(w)$ (resp. $\mathcal{R}(w)$) is called the left descent (resp. the right descent) set of w.

We will also need the following definition.

Definition 1.2.9. Let $x, y, z \in W$. We write $x.y$ if and only if $\ell(xy) = \ell(x) + \ell(y)$. Similarly, we write $x.y.z$ if and only if $\ell(xyz) = \ell(x) + \ell(y) + \ell(z)$.

The following result is the key fact about reduced expressions and is a very powerful tool in the study of Coxeter groups.

Theorem 1.2.10. **Exchange Condition**
Let $w \in W$ and $s \in S$ be such that $\ell(sw) = \ell(w) - 1$. Let $w = s_1 \dots s_n$ be a reduced expression of W. Then there exists $j \in \{1, \dots, n\}$ such that

$$ss_1 \dots s_{j-1} = s_1 \dots s_j$$

Remark 1.2.11. In fact, it can be shown that a group which is generated by elements of order 2 and which satisfy the exchange condition is a Coxeter group (see [**8**]).

Let $w \in W$. Let X_w be the set which consists of all sequences $(s_1, s_2, \dots, s_n)$ in W such that $s_1 s_2 \dots s_n$ is a reduced expression of w. We look at X as a graph where two vertices are joined if one is obtained by the other replacing a subsequence of the form $(s, s', s, s' \dots)$ of length $m_{s,s'} < \infty$ by $(s', s, s', s \dots)$. The next result is due to Matsumoto and Tits.

Theorem 1.2.12. *Let $w \in W$. The graph X_w defined above is connected.*

We are now ready to define the Bruhat order.

Definition 1.2.13. Let $y, w \in W$. We write $y \leq w$ if there exist a reduced expression $w = s_1 \dots s_n$ and a subsequence $i_1 < i_2 < \dots < i_r$ of $1, \dots, n$ such that

$$y = s_{i_1} \dots s_{i_r} \quad \text{and} \quad \ell(y) = r$$

We have the following characterization of the Bruhat order.

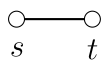
Lemma 1.2.14. *The following are equivalent*

(1) $y \leq w$;
(2) *for any reduced expression* $w = s_1 \dots s_n$ *there exists a subsequence* $i_1 < i_2 < \dots < i_r$ *of* $1, \dots, n$ *such that*

$$y = s_{i_1} \dots s_{i_r} \quad \textit{and} \quad \ell(y) = r$$

(3) *There exists a sequence* $y = y_0, y_1, \dots, y_n = w$ *such that* $\ell(y_i) - \ell(y_{i-1}) = 1$ *for all* $1 \leq i \leq n$ *and* y_{i-1} *is obtained from* y_i *by deleting a simple reflection in a reduced expression of* y_i.

Example 1.2.15. Let W be a Weyl group of type A_2 with graph as follows



Then W contains 6 elements and the Bruhat order can be described by the following Hasse diagram

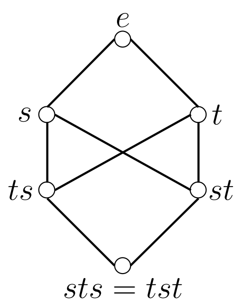


Figure 3. Bruhat order on A_2

1.2.6. Finite Coxeter group. A special feature in finite Coxeter groups is that there exists a unique element of maximal length which has many nice properties.

Let (W, S) be a finite Coxeter group and let w_0 be the unique element of maximal length. Since $\ell(w_0) = \ell(w_0^{-1})$, by unicity of the longest element, we must have $w_0 = w_0^{-1}$. Furthermore we have

$$\ell(w_0 y) = \ell(y w_0) = \ell(w_0) - \ell(y) \quad \text{for all } y \in W.$$

In fact, w_0 is characterized by the fact that $sw_0 < w_0$ for all $s \in S$. One can easily check that for all $y \in W$ we have $y \leq w_0$. Moreover, for all $y, w \in W$, we have

$$y \leq w \iff w_0 w \leq w_0 y \iff w w_0 \leq y w_0.$$

It can be shown that, if W is a Weyl group with root system Φ, then the length of the longest element is equal to the cardinal of any set of positive roots.

EXAMPLE 1.2.16. Let W be the Weyl group of type G_2 generated by s, t. The longest element is

$$ststst = tststs.$$

1.2.7. Coset of parabolic subgroups. Let I be a subset of S. We denote by W_I the subgroup of W generated by I.

PROPOSITION 1.2.17. *Let $z \in W$ and zW_I be a coset in W.*

(1) *This coset has a unique element w of minimal length.*
(2) *Let $y \in W_I$. We have $\ell(wy) = \ell(w) + \ell(y)$.*
(3) *The element w is characterized by the fact that $w < ws$ for all $s \in I$.*

We denote by X_I the set which consists of all the elements $z \in W$ which have minimal length in their coset zW_I; it is called the set of minimal coset representatives with respect to I.

Assume that W_I is finite.

(1) *The coset zW_I has a unique element x of maximal length.*
(2) *Let $y \in W_I$. We have $\ell(xy) = \ell(x) - \ell(y)$.*
(3) *The element x is characterized by the fact that $xs < x$ for all $s \in I$.*

REMARK 1.2.18. Let $I \subset S$ and $z \in W$. Then z can be written uniquely under the form xw where $x \in X_I$, $w \in W_I$ and $\ell(xw) = \ell(x) + \ell(w)$.

LEMMA 1.2.19. **Deodhar's lemma**
Let $I \subset S$ and X_I be the set of minimal coset representatives of W_I. Let $x \in X_I$ and $s \in S$. One of the following statement holds:

(i) *$sx \in X_I$ and $\ell(sx) = \ell(x) + 1$;*
(ii) *$sx \in X_I$ and $\ell(sx) = \ell(x) - 1$;*
(iii) *$sx \notin X_I$, in which case there exists $t \in I$ such that $sx = xt$ and*

$$\ell(sx) = \ell(x) + 1 = \ell(xt).$$

CHAPTER 2

Iwahori-Hecke algebras and Kazhdan-Lusztig cells

This chapter is an introduction to the fundamental Kazhdan-Lusztig theory. Following Lusztig ([**38**]), we start with the definition of weight functions on a Coxeter group (W, S). We then define the Iwahori-Hecke algebra associated to a Coxeter group and a weight function. Next we define the Kazhdan-Lusztig basis of an Iwahori-Hecke algebra. Using this "new basis" we introduce the notion of (left, right and two-sided) cells of a Coxeter group and the representations associated to them; see Section 2.6. Then we state a number of conjectures due to Lusztig, which are known to be true in the equal parameter case but for which no elementary proofs are known (see Section 2.9). Finally, we introduce the asymptotic algebra $\mathbf{J}$, which plays an important part in the study of Iwahori-Hecke algebras; see Section 2.10.

In this chapter, (W, S) denotes an arbitrary Coxeter system. We follow the exposition of Lusztig in [**38**] and we refer to this publication for more details and proofs.

2.1. Weight functions

DEFINITION 2.1.1. A weight function L is a function $L : W \to \mathbb{Z}$ such that

$$L(ww') = L(w) + L(w') \quad \text{whenever} \quad \ell(ww') = \ell(w) + \ell(w').$$

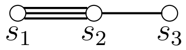
Note that the length function is a weight function.

Let L be a weight function on W. We see that

(1) L is completely determined by its values on S;
(2) if $s, t \in S$ are conjugate, then $L(s) = L(t)$.

The set $\{L(s))|s \in S\}$ is called the set of parameters. When $L = \ell$ we say that we are in the equal parameter case. In this thesis, we will only consider positive weight functions, that is $L(s) > 0$ for all $s \in S$ (except for Section 7.3). Note that $L(e) = 0$ where e denotes the identity element of W.
When a weight function L is fixed, we say that W, L is a weighted Coxeter group.

Example 2.1.2. Let W be an affine Weyl group of type $\tilde{G}_2$ with graph given by



Let L be a weight function on W. The order of s_2s_3 is odd, thus they are conjugate and we must have $L(s_2) = L(s_3) \in \mathbb{N}^*$. Hence a positive weight function on W is completely determined by the values $L(s_1) = a \in \mathbb{N}^*$ and $L(s_2) = b \in \mathbb{N}^*$.

If W is a finite irreducible Weyl group, unequal parameters can only arise in types B_n, F_4 and G_2. Note that we have at most two distinct parameters.

If W is an irreducible affine Weyl group, unequal parameters can only arise in types $\tilde{A}_1, \tilde{B}_n, \tilde{C}_n, \tilde{F}_4$ and $\tilde{G}_2$. In type $\tilde{C}_n$ there can be 3 distinct parameters. In the other cases, we have at most 2 distinct parameters.

2.2. Iwahori-Hecke algebras

From now and until the end of this chapter, we fix a weight function L on W. Let $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$ where v is an indeterminate. For $s \in S$ we set $v_s = v^{L(s)}$ and $\xi_s = v_s - v_s^{-1}$. We keep this setting until the end of this chapter. Recall that we always assume that $L(s) > 0$ for all $s \in S$.

Let $\mathcal{H}$ be the free $\mathcal{A}$-algebra with basis $\{T_w \mid w \in W\}$, identity element T_e (where e is the identity element of W) and multiplication given by

$$T_sT_w = \begin{cases} T_{sw}, & \text{if } sw > w, \\ T_{sw} + (v_s - v_s^{-1})T_w, & \text{if } sw < w, \end{cases}$$

for all $s \in S$, $w \in W$. The algebra $\mathcal{H}$ is called the Iwahori-Hecke algebra associated to the weighted Coxeter group W, L.

Remark 2.2.1. The definition of $\mathcal{H}$ depends on the weight function L, thus from now on, everything we say depends on L.

From the definition, one can check that T_s is invertible for all $s \in S$ with inverse

$$T_s^{-1} = T_s - \xi_s T_e.$$

Let $w \in W$ and $w = s_1 \dots s_n$ be a reduced expression of w. We have

$$T_w = T_{s_1}T_{s_2}\dots T_{s_n},$$

thus, since each T_{s_i} is invertible, we see that T_w is invertible for all $w \in W$.

The basis $\{T_w\}_{w \in W}$ is called the standard basis. For $x, y \in W$ we set

$$T_x T_y = \sum_{z \in W} f_{x,y,z} T_z$$

where $f_{x,y,z} \in \mathcal{A}$ are the structure constants with respect to the standard basis. We define an $\mathcal{A}$-linear map $\tau : \mathcal{H} \to \mathcal{A}$ by $\tau(T_w) = \delta_{w,1}$ for all $w \in W$.

PROPOSITION 2.2.2. *Let $x, y, z \in W$ and $h, h' \in \mathcal{H}$. We have*

(1) $\tau(T_x T_y) = \delta_{xy,1}$;
(2) $\tau(hh') = \tau(h'h)$;
(3) $\tau(T_x T_y T_z) \in v^M \mathbb{Z}[v^{-1}]$ *where* $M = \min(L(x), L(y), L(z))$.

REMARK 2.2.3. The form τ is symmetric. The dual basis of $\{T_w \mid w \in W\}$ is clearly $\{T_{w^{-1}} \mid w \in W\}$.

2.3. The $^{-}$ operator

There exists a unique ring involution on $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ such that $\bar{v} = v^{-1}$. We can extend this map to a ring involution $^{-} : \mathcal{H} \to \mathcal{H}$ such that

$$\overline{\sum_{w \in W} a_w T_w} = \sum_{w \in W} \bar{a}_w T_{w^{-1}}^{-1} \quad \text{where } a_w \in \mathcal{A}.$$

Let $w \in W$. We can write uniquely

$$\bar{T}_w = T_{w^{-1}}^{-1} = \sum_{y \in W} \bar{R}_{y,w} T_y$$

where $R_{y,w} \in \mathcal{A}$ are zero for all but finitely many $y \in W$. Note that these elements are denoted by $r_{y,w}$ in [**38**].

PROPOSITION 2.3.1. *Let $y, w \in W$. The R-polynomials satisfy the following properties.*

(1) *Let $s \in S$ be such that $sw < w$. We have*

$$R_{y,w} = \begin{cases} R_{sy,sw}, & \text{if } sy < y, \\ R_{sy,sw} + \xi_s R_{y,sw}, & \text{if } sy > y. \end{cases}$$

(2) *If $R_{y,w} \neq 0$ then $y \leq w$.*
(3) *We have*

$$\sum_{z \in W} \bar{R}_{y,z} R_{z,w} = \delta_{y,w}.$$

EXAMPLE 2.3.2. Let $y \leq w \in W$. One can show that

- $R_{y,y} = 1$;
- if $\ell(w) = \ell(y) + 1$ then y is obtained by deleting a simple reflection $s \in S$ in a reduced expression of w and $R_{y,w} = \xi_s$.
- if $\ell(w) = \ell(y) + 2$ then y is obtained by deleting two simple reflections $s, t \in S$ in a reduced expression of w and $R_{y,w} = \xi_s \xi_t$.

There exists a unique involutive antiautomorphism $\flat$ of $\mathcal{H}$ which sends T_s to T_s for any $s \in S$. It carries T_w to $T_{w^{-1}}$, for any $w \in W$. This antiautomorphism will be useful later on.

2.4. Kazhdan-Lusztig basis

We first introduce some notation. Let

$$A_{<0} := v^{-1}\mathbb{Z}[v^{-1}] \quad \text{and} \quad A_{\leq 0} := \mathbb{Z}[v^{-1}]$$

and

$$\mathcal{H}_{<0} := \sum_{w \in W} \mathcal{A}_{<0} T_w \quad \text{and} \quad \mathcal{H}_{\leq 0} := \sum_{w \in W} \mathcal{A}_{\leq 0} T_w.$$

The following theorem is due to Kazhdan and Lusztig ([**24**]) in the equal parameter case (i.e $L = \ell$) and to Lusztig ([**28, 38**]) in the unequal parameter case. It is the cornerstone of this theory.

THEOREM 2.4.1. *Let $w \in W$. There exists a unique element $C_w \in \mathcal{H}_{\leq 0}$ such that*

(1) $\overline{C_w} = C_w$ *and*
(2) $C_w \equiv T_w \mod \mathcal{H}_{<0}$

The elements $\{C_w \mid w \in W\}$ form an $\mathcal{A}$-basis of $\mathcal{H}$ known as the Kazhdan-Lusztig basis.

These elements were formerly denoted by C'_w in [**24**] and [**28**] ; they are denoted by c_w in [**38**].

For any $w \in W$ we set

$$C_w = \sum_{y \in W} P_{y,w} T_w$$

where $P_{y,w} \in \mathcal{A}_{\leq 0}$. These polynomials are called the Kazhdan-Lusztig polynomials. Note that they are denoted by $p_{y,w}$ in [**38**].

PROPOSITION 2.4.2. *Let $y, w \in W$. The Kazhdan-Lusztig polynomials $P_{y,w}$ satisfy the following properties.*

(1) $P_{y,y} = 1$ *for all* $y \in W$.
(2) $P_{y,w} = 0$ *unless* $y \leq w$.
(3) $P_{y,w} \in \mathcal{A}_{<0}$ *if* $y < w$.
(4) *For all* $x, w \in W$ *we have*

$$\bar{P}_{x,w} = \sum_{y;x \leq y \leq w} R_{x,y} P_{y,w}.$$

EXAMPLE 2.4.3. Let $s \in S$. One can check that $T_s + v_s^{-1} T_e$ is stable under the $^-$ involution, thus we have $C_s = T_s + v_s^{-1} T_e$ and $P_{e,s} = v_s^{-1}$.

REMARK 2.4.4. The antiautomorphism $\flat$ carries $\mathcal{H}_{\leq 0}$ to itself. Moreover, it commutes with $^-$. Thus one can check that $\flat(C_w) = C_{w^{-1}}$.

Let $w \in W$ and $s \in S$, we have the following multiplication formula

$$C_s C_w = \begin{cases} C_{sw} + \sum_{z;sz<z<w} M^s_{z,w} C_z, & \text{if } w < sw, \\ (v_s + v_s^{-1}) C_w, & \text{if } sw < w, \end{cases}$$

where $M^s_{y,w} \in \mathcal{A}$ satisfies

$$\overline{M^s_{y,w}} = M^s_{y,w}, \tag{2.1}$$

$$\Big(\sum_{z;y \leq z < w; sz < z} P_{y,z} M^s_{z,w} \Big) - v_s P_{y,w} \in \mathcal{A}_{<0}. \tag{2.2}$$

Let y, w and $s \in S$ be such that $ys < y < w < ws$. Using the anti-involution $\flat$ one gets similar formulas for right multiplication. We obtain some polynomials $M^{s,r}_{y,w} = M^s_{y^{-1},w^{-1}}$.

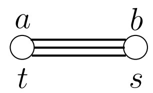
Since $C_s = T_s + v_s^{-1} T_e$, one can see that

$$T_s C_w = \begin{cases} C_{sw} - v^{-L(s)} C_w + \sum_{z;sz<z<w} M^s_{z,w} C_z, & \text{if } w < sw, \\ v^{L(s)} C_w, & \text{if } sw < w. \end{cases}$$

Let $y, w \in W$ and $s \in S$ be such that $sw < w$. The Kazhdan-Lusztig polynomials satisfy the following recursive formula

$$P_{y,w} = \begin{cases} v_s P_{y,sw} + P_{sy,sw} - \sum_{y \leq z < sw} P_{y,z} M^s_{z,sw} & \text{if } sy < y, \\ v_s^{-1} P_{sy,w}, & \text{if } y < sy. \end{cases}$$

EXAMPLE 2.4.5. Let W be a Weyl group of type G_2 with graph and weight function given by



where $a > b \in \mathbb{N}^*$. We want to compute the Kazhdan-Lusztig polynomials (see [**46**, Example 1.21]). First let w_0 be the longest element of W. Using the above recursive formula and the properties of the longest element, one can check that, for all $y \in W$, we have

$$P_{y,w_0} = v^{L(y)-L(w_0)}.$$

Doing some more computations we obtain

$$\begin{aligned}
P_{t,tst} &= P_{tst,tstst} = v^{-a-b} - v^{-a+b}\\
P_{e,tst} &= P_{ts,tstst} = P_{st,tstst} = v^{-2a-b} - v^{-2a+b}\\
P_{t,tstst} &= v^{-2a-2b} - v^{-2a} + v^{-2a+2b}\\
P_{e,tstst} &= v^{-3a-2b} - v^{-3a} + v^{-3a+2b}\\
P_{s,sts} &= P_{sts,ststs} = v^{-a-b} + v^{-a+b}\\
P_{e,sts} &= P_{ts,ststs} = P_{st,ststs} = v^{-a-2b} + v^{-a}\\
P_{t,ststs} &= v^{-a-3b} + v^{-a-b}\\
P_{s,ststs} &= v^{-2a-2b} + v^{-2a}\\
P_{e,ststs} &= v^{-2a-3b} + v^{-2a-b}.
\end{aligned}$$

and $P_{y,w} = v^{L(y)-L(w)}$ for all other pairs $y \leq w \in W$.
We have

$$\begin{aligned}
M^t_{tsts,ststs} &= M^t_{tst,stst} = M^t_{ts,sts} = M^t_{t,st} = v^{a-b} + v^{b-a},\\
M^t_{t,stst} &= M^t_{ts,ststs} = 1
\end{aligned}$$

and all the others are zero.

REMARK 2.4.6. Let $y, w \in W$ and $[y, w] := \{z \in W \mid y \leq z \leq w\}$. Using Proposition 2.4.2 (3) and the definition of the M-polynomials, one can see that the set of polynomials

$$\{M^s_{x,z},\ P_{x,z} \mid x, z \in [y, w]\}$$

is completely determined by the weight function L and the set of R-polynomials

$$\{R_{x,z} \mid x, z \in [y, w]\}.$$

Let $x, y \in W$. We set

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z$$

where $h_{x,y,z} \in \mathcal{A}$ are the structure constants with respect to the Kazhdan-Lusztig basis. It is clear that

$$h_{x,y,z} = \overline{h_{x,y,z}}.$$

2.5. Kazhdan-Lusztig cells

Let $y, w \in W$. We write $y \leftarrow_L w$ if there exists $s \in S$ such that C_y appears with a non-zero coefficient in the expression of $T_s C_w$ in the Kazhdan-Lusztig basis. The Kazhdan-Lusztig left pre-order $\leq_L$ on W is the transitive closure of this relation. One can see that

$$\mathcal{H} C_w \subseteq \sum_{y \leq_L w} \mathcal{A} C_y \text{ for any } w \in W.$$

The equivalence relation associated to $\leq_L$ will be denoted by $\sim_L$, that is

$$x \sim_L y \Longleftrightarrow x \leq_L y \text{ and } y \leq_L x \quad (x, y \in W).$$

The corresponding equivalence classes are called the left cells of W. Similarly, we define $\leq_R$, $\sim_R$ and right cells, multiplying on the right. In fact, using the antiautomorphism $\flat$ one can show that

$$x \leq_L y \Longleftrightarrow x^{-1} \leq_R y^{-1} \quad (x, y \in W).$$

We say that $x \leq_{LR} y$ if there exists a sequence

$$x = x_0, x_1, \ldots, x_n = y$$

such that for all $1 \leq i \leq n$ we have $x_{i-1} \leftarrow_L x_i$ or $x_{i-1} \leftarrow_R x_i$. We denote by $\sim_{LR}$ the associated equivalence relation and the equivalence classes are called the two-sided cells of W. One can see that

$$\mathcal{H} C_w \mathcal{H} \subseteq \sum_{y \leq_{LR} w} \mathcal{A} C_y \text{ for any } w \in W.$$

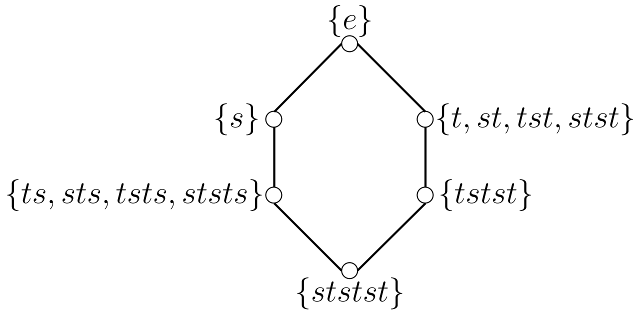
Remark 2.5.1. A two-sided cell is a union of left cells which is also a union of right cells. However, even if one knows the decomposition into left cells of W (and hence the decomposition into right cells via $\flat$) it is complicated to find the decomposition into two-sided cells (see for example [**7**] and [**4**]).

The pre-orders $\leq_L$, $\leq_R$ and $\leq_{LR}$ induce partial orders on the left, right and two-sided cells.

Example 2.5.2. Let W be of type G_2 with graph and weight function as in Example 2.4.5. The decomposition of W into left cells is as follows (see [**46**, Examples 1.21])

$$\{e\},\ \{s\},\ \{tstst\},\ \{t, st, tst, stst\},\ \{ts, sts, tsts, ststs\},\ \{ststst\}.$$

The partial order on the left cells can be described by the following Hasse diagram



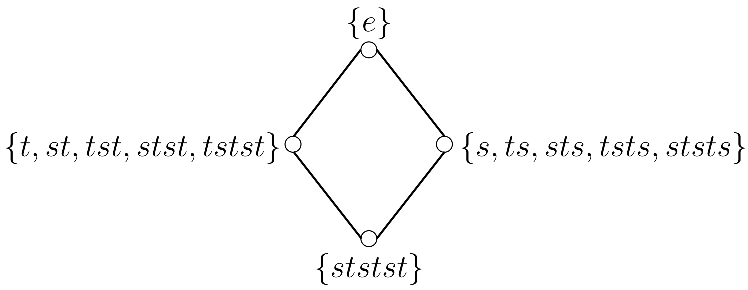
The two-sided cells are (in order)

$$\{e\} \leq_{LR} \{s\} \leq_{LR} \{t, st, ts, tst, sts, tsts, stst, ststs\} \leq_{LR} \{tstst\} \leq_{LR} \{ststst\}.$$

Now assume that $L = \ell$. In that case the left cells are

$$\{e\},\ \{t, st, tst, stst, tstst\},\ \{s, ts, sts, tsts, ststs\},\ \{ststst\}$$

and the order is as follows



The two-sided cells are given by

$$\{e\} \leq_{LR} \{t, st, tst, stst, tstst, s, ts, sts, tsts, ststs\} \leq_{LR} \{ststst\}.$$

The following result shows some connections between descent sets and cells.

Proposition 2.5.3. *Let $y, w \in W$. We have*

$$y \leq_L w \Rightarrow \mathcal{R}(w) \subseteq \mathcal{R}(y) \quad \textit{and} \quad y \leq_R w \Rightarrow \mathcal{L}(w) \subseteq \mathcal{L}(y).$$

In particular, if $y \sim_L w$ (resp. $y \sim_R w$) then $\mathcal{R}(y) = \mathcal{R}(w)$ (resp. $\mathcal{L}(y) = \mathcal{L}(w)$).

Definition 2.5.4. Let $\mathfrak{B}$ be a subset of W. We say that $\mathfrak{B}$ is a left ideal of W if and only if the $\mathcal{A}$-submodule generated by $\{C_w | w \in \mathfrak{B}\}$ is a left ideal of $\mathcal{H}$. Similarly one can define right and two-sided ideals of W.

REMARK 2.5.5. Here are some straightforward consequences of this definition
* Let $\mathfrak{B}$ be a left ideal and let $w \in \mathfrak{B}$. We have

$$\mathcal{H}C_w \subseteq \sum_{y \in \mathfrak{B}} \mathcal{A}C_y.$$

In particular, if $y \leq_L w$ then $y \in \mathfrak{B}$ and $\mathfrak{B}$ is a union of left cells.
* A union of left ideals is a left ideal.
* An intersection of left ideals is a left ideal.
* A left ideal which is stable by taking the inverse is a two-sided ideal. In particular it is a union of two-sided cells.

EXAMPLE 2.5.6. Let J be a subset of S. We set

$$\mathcal{R}^J := \{w \in W \mid J \subseteq \mathcal{R}(w)\} \quad \text{and} \quad \mathcal{L}^J := \{w \in W \mid J \subseteq \mathcal{L}(w)\}$$

Then the set $\mathcal{R}^J$ is a left ideal of W. Indeed let $w \in \mathcal{R}^J$ and $y \in W$ be such that $y \leq_L w$. Then we have $J \subseteq \mathcal{R}(w) \subseteq \mathcal{R}(y)$ and $y \in \mathcal{R}^J$. Similarly one can see that $\mathcal{L}^J$ is a right ideal of W.

2.6. Cell representations

In this section, we show how each cell gives rise to a representation of $\mathcal{H}$.

LEMMA 2.6.1. *Let $w \in W$.*

(1) $\mathcal{H}_{\leq_L w} = \bigoplus_{y; y \leq_L w} \mathcal{A}C_y$ *is a left ideal of* $\mathcal{H}$.
(2) $\mathcal{H}_{\leq_R w} = \bigoplus_{y; y \leq_R w} \mathcal{A}C_y$ *is a right ideal of* $\mathcal{H}$.
(3) $\mathcal{H}_{\leq_{LR} w} = \bigoplus_{y; y \leq_{LR} w} \mathcal{A}C_y$ *is a two-sided ideal of* $\mathcal{H}$.

Let $\mathcal{C}$ be a left cell of W and let $w \in \mathcal{C}$. The set

$$\bigoplus_{y; y \leq_L w} \mathcal{A}C_y \Big/ \bigoplus_{y; y <_L w} \mathcal{A}C_y$$

is a quotient of two left ideals (independent of the choice of $w \in \mathcal{C}$), hence it is a left $\mathcal{H}$-module. We denote this left module by $\mathcal{V}_\mathcal{C}$. It has an $\mathcal{A}$-basis which consists of the images of C_y, $y \in \mathcal{C}$ (via the canonical projection). More precisely, let e_y ($y \in \mathcal{C}$) be the image of C_y. The action of $\mathcal{H}$ on $\mathcal{V}_\mathcal{C}$ is given by the Kazhdan-Lusztig structure constants

$$C_w.e_y = \sum_{z \in \mathcal{C}} h_{w,y,z} e_z \quad \text{for all } y \in \mathcal{C} \text{ and } w \in W.$$

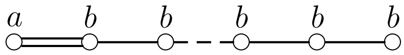
Let $\theta : \mathcal{A} \longrightarrow \mathbb{Q}$ be the unique ring homomorphism which sends v to 1. Then if we extend the scalars from $\mathcal{A}$ to $\mathbb{Q}$ (via θ) we obtain a representation of $\mathbb{Q} \otimes_{\mathcal{A}} \mathcal{H} = \mathbb{Q}[W]$ on $\mathbb{Q} \otimes_{\mathcal{A}} \mathcal{V}_{\mathcal{C}}$.

One can do similar constructions with right and two-sided cells. We obtain respectively right modules and two-sided modules.

We now give two examples where the left cells representations actually give rise to all the irreducible representations of $\mathcal{H}$.

Example 2.6.2. In the fundamental paper of Kazhdan and Lusztig ([**24**]), where they first introduced Kazhdan-Lusztig cells, they showed that if W is a Weyl group of type A_n (note that we are automatically in the equal parameter case), then the left cell representations are irreducible. Furthermore, any irreducible representation can be realized as a left cell representation.

Let W be a Weyl group of type B_n with weight function and diagram given by

$B_n,\ (n \geq 2)$ 

where $a, b \in \mathbb{N}^*$ satisfy $a/b > n-1$. Then we are in the so-called "asymptotic case" where the left cells have been described by Bonnafé and Iancu ([**4, 7**]). It turns out that in that case the left cell representations are irreducible. Conversely, any irreducible representation can be realized as a left cell representation. However, in the equal parameter case, it is not true anymore.

Note that if W is an affine Weyl group, some cells are infinite and thus give rise to infinite dimensional $\mathcal{H}$-modules.

2.7. On the structure constants

Let $y, w \in W$. We set

$$Q'_{y,w} := \sum (-1)^n P_{z_0,z_1} P_{z_1,z_2} \ldots P_{z_{n-1},z_n}$$

(where the sum runs over all the sequences $y = z_0 < z_1 < \ldots < z_n = w$ in W) and

$$Q_{y,w} = (-1)^{\ell(y)+\ell(w)} Q'_{y,w}.$$

In [**38**], these polynomials are denoted by $q'_{y,w}$ and $q_{y,w}$. For all $y, w \in W$, we have,

- $Q_{w,w} = 1$;
- $Q_{y,w} \in \mathcal{A}_{<0}$ if $y \neq w$;
- $Q_{y,w} = 0$ unless $y \leq w$;

Recall that, for $x, y \in W$, we have set

$$T_x T_y = \sum_{z \in W} f_{x,y,z} T_z \quad \text{and} \quad C_x C_y = \sum_{z \in W} h_{x,y,z} C_z.$$

Let

$$T_x T_y = \sum_{z \in W} f'_{x,y,z} C_z.$$

PROPOSITION 2.7.1. *Let $x, y, z \in W$. We have*

(1) $f_{x,y,z} = \sum_{z' \in W} P_{z,z'} f'_{x,y,z'}$;
(2) $f'_{x,y,z} = \sum_{z' \in W} Q_{z,z'} f_{x,y,z'}$;
(3) $h_{x,y,z} = \sum_{x',y' \in W} P_{x',x} P_{y',y} f'_{x',y',z}$.

All the above sums are finite.

Note that if $h_{x,y,z} \neq 0$ then $z \leq_R x$ and $z \leq_L y$.

DEFINITION 2.7.2. We say that W is bounded if there exists $N \in \mathbb{N}$ such that $v^{-N} f_{x,y,z} \in \mathcal{A}_{\leq 0}$ for all $x, y, z \in W$.

Let $\mathbf{I}$ be the set which consists of all subsets I of S such that W_I is finite. For $I \in \mathbf{I}$, we denote by w_I the longest element of W_I. Let

$$N := \max_{I \in \mathbf{I}} L(w_I).$$

Lusztig has conjectured that N should be a bound for W. One can easily see that N is reached. Indeed, $v^{L(w_I)}$ appears in f_{w_I,w_I,w_I} for all $I \in \mathbf{I}$. It is known that N is a bound for W in the following case

* W is finite (see [**38**]);
* W is an affine Weyl group (see [**9, 30**]).

Using Proposition 2.7.1, we see that if N is a bound for W, then, for all $x, y, z \in W$ we have

(1) $v^{-N} f'_{x,y,z} \in \mathcal{A}_{\leq} 0$,
(2) $v^{-N} h_{x,y,z} \in \mathcal{A}_{\leq} 0$.

2.8. The a-function

We now introduce Lusztig's **a**-function (see [**38**, Chap. 13]). In the remainder of this section, we assume that W is bounded by $N \in \mathbb{N}$.

DEFINITION-PROPOSITION 2.8.1. *Let $z \in W$. There exists a unique integer $\mathbf{a}(z) \in [0..N]$ such that*

(a) $h_{x,y,z} \in v^{\mathbf{a}(z)}\mathbb{Z}[v^{-1}]$ *for all $x, y \in W$,*
(b) $h_{x,y,z} \notin v^{\mathbf{a}(z)-1}\mathbb{Z}[v^{-1}]$ *for some $x, y \in W$.*

For any $x, y, z \in W$, we define $\gamma_{x,y,z^{-1}} \in \mathbb{Z}$ by the condition

$$h_{x,y,z} \equiv \gamma_{x,y,z^{-1}} v^{\mathbf{a}(z)} \mod v^{\mathbf{a}(z)-1}\mathbb{Z}[v^{-1}].$$

Note that, for any $z \in W$, there exist $x, y \in W$ such that $\gamma_{x,y,z^{-1}} \neq 0$.
For any $x, y, z \in W$ we have

$$f'_{x,y,z} = \gamma_{x,y,z^{-1}} v^{\mathbf{a}(z)} \mod v^{\mathbf{a}(z)-1}\mathbb{Z}[v^{-1}].$$

We now state some properties of the **a**-function.

PROPOSITION 2.8.2. *We have*

(1) $\mathbf{a}(e) = 1$;
(2) *If $z \in W - \{e\}$, then $\mathbf{a}(z) \geq \min_{s \in S} L(s) > 0$;*
(3) $\mathbf{a}(z) = \mathbf{a}(z^{-1})$;
(4) *for all $x, y, z \in W$, $\gamma_{x,y,z} = \gamma_{y^{-1},x^{-1},z^{-1}}$.*

Assume that W is finite and let w_0 be the longest element of W, then

(1) $\mathbf{a}(w_0) = L(w_0)$;
(2) *for any $w \in W - \{w_0\}$ we have $\mathbf{a}(w) < L(w_0)$.*

EXAMPLE 2.8.3. Let W be a Weyl group of type G_2 with graph and weight function as in Example 2.4.5. It turns out that the **a** function is constant on the two-sided cells (see next section). We have (see [**18**])

$$\begin{aligned}
\mathbf{a}(e) &= 0\\
\mathbf{a}(s) &= L(s)\\
\mathbf{a}(t) &= L(t)\\
\mathbf{a}(tstst) &= 3L(t) - 2L(s)\\
\mathbf{a}(ststst) &= 3L(s) + 3L(t).
\end{aligned}$$

We refer to [**18, 38**] for more examples.

2.9. Lusztig's conjectures

In this section we assume that W is bounded. We state a number of conjectures due to Lusztig (see [**38**, Chap. 14]), which are known in the equal parameter case (see Remark 2.9.2). For $z \in W$, let $\Delta(z) \in \mathbb{N}$ be such that

$$P_{e,z} = n_z v^{-\Delta(z)} + \text{strictly smaller powers of } v \ (n_z \neq 0).$$

One can check that $\Delta(z) = \Delta(z^{-1})$, $\Delta(e) = 0$ and $0 < \Delta(z) \leq L(z)$ for $z \neq e$. Finally let

$$\mathcal{D} := \{z \in W \mid \mathbf{a}(z) = \Delta(z)\}.$$

We have $z \in \mathcal{D} \Rightarrow z^{-1} \in \mathcal{D}$. We are now ready to state the conjectures.

CONJECTURE 2.9.1 (Lusztig). *The following properties hold.*

- **P1.** *For any $z \in W$ we have $\mathbf{a}(z) \leq \Delta(z)$.*
- **P2.** *If $d \in \mathcal{D}$ and $x, y \in W$ satisfy $\gamma_{x,y,d} \neq 0$, then $x = y^{-1}$.*
- **P3.** *If $y \in W$, there exists a unique $d \in \mathcal{D}$ such that $\gamma_{y^{-1},y,d} \neq 0$.*
- **P4.** *If $z' \leq_{LR} z$ then $\mathbf{a}(z') \geq \mathbf{a}(z)$. Hence, if $z' \sim_{LR} z$, then $\mathbf{a}(z) = \mathbf{a}(z')$.*
- **P5.** *If $d \in \mathcal{D}$, $y \in W$, $\gamma_{y^{-1},y,d} \neq 0$, then $\gamma_{y^{-1},y,d} = n_d = \pm 1$.*
- **P6.** *If $d \in \mathcal{D}$, then $d^2 = 1$.*
- **P7.** *For any $x, y, z \in W$, we have $\gamma_{x,y,z} = \gamma_{y,z,x}$.*
- **P8.** *Let $x, y, z \in W$ be such that $\gamma_{x,y,z} \neq 0$. Then $x \sim_L y^{-1}$, $y \sim_L z^{-1}$, $z \sim_L x^{-1}$.*
- **P9.** *If $z' \leq_L z$ and $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_L z$.*
- **P10.** *If $z' \leq_R z$ and $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_R z$.*
- **P11.** *If $z' \leq_{LR} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_{LR} z$.*
- **P12.** *Let $I \subseteq S$ and W_I be the parabolic subgroup generated by I. If $y \in W_I$, then $\mathbf{a}(y)$ computed in terms of W_I is equal to $\mathbf{a}(y)$ computed in terms of W.*

P13. *Any left cell $\mathcal{C}$ of W contains a unique element $d \in \mathcal{D}$. We have $\gamma_{x^{-1},x,d} \neq 0$ for all $x \in \mathcal{C}$.*

P14. *For any $z \in W$, we have $z \sim_{LR} z^{-1}$.*

P15. *Let v' be a second indeterminate and let $h'_{x,y,z} \in \mathbb{Z}[v', v'^{-1}]$ be obtained from $h_{x,y,z}$ by the substitution $v \mapsto v'$. If $x, x', y, w \in W$ satisfy $\mathbf{a}(w) = \mathbf{a}(y)$ then*

$$\sum_{y'} h'_{w,x',y'} h_{x,y',y} = \sum_{y'} h_{x,w,y'} h'_{y',x',y} \ .$$

Remark 2.9.2. Assume that $L = \ell$, that W is bounded and that we have

(1) $h_{x,y,z} \in \mathbb{N}[v, v^{-1}]$ for all $x, y, z \in W$;
(2) $P_{y,w} \in \mathbb{N}[v, v^{-1}]$ for all $y, w \in W$.

In the case where W is integral, using an interpretation of the Kazhdan-Lusztig polynomials in terms of intersection cohomology, one can show that (1) and (2) hold (see [**30, 38, 45**]). In the case where W is of type $I_2(m)$ ($m \notin \{2,3,4,6\}$), H_3 or H_4, (1) and (2) have been proved by Fokko du Cloux and Alvis, using explicit computations (see [**1, 13**]).

Under the assumptions (1) and (2), it can be shown that **P1–P15** hold (see [**38**, Chap. 15]). However, in the unequal parameter case, (1) and (2) do not hold anymore. For instance, even in a very small group like G_2, we have seen that negative coefficients arise in some Kazhdan-Lusztig polynomials (see Example 2.4.5).

For unequal parameters, these conjectures are known to be true in the following cases

(1) in the "quasi-split case" (see [**38**, Chap. 16]);
(2) infinite dihedral group (see [**38**, Chap. 17]);
(3) finite dihedral groups for any parameters (see [**18**]);
(4) type F_4 for any parameters (see [**16**]);
(5) type B_n in the "asymptotic case" (see [**7, 18**] and the references there in).

For instance, these conjectures yield the following.

Theorem 2.9.3. *Let W be a tame Coxeter group and assume that* **P1**–**P15** *hold. We have*

(1) *Any two-sided cell in W meets a finite parabolic subgroup.*
(2) *W has only finitely many left (resp. right, two-sided) cells.*
(3) *$\mathcal{D}$ is a finite set.*
(4) *A two-sided cell is a minimal union of left cells which is also a union of right cells.*

2.10. The asymptotic algebra J

Following Lusztig, we introduce the asymptotic algebra **J**. Even though we will not study this algebra in this thesis, it is important to mention it, since it plays a crucial role in the study of Iwahori-Hecke algebras associated to Weyl groups. In this section we assume that W is tame and that **P1–P15** hold.

DEFINITION 2.10.1. Let **J** be the free $\mathbb{Z}$-module with basis $\{t_w, w \in W\}$. We define a bilinear product on **J** by

$$t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z.$$

THEOREM 2.10.2 (Lusztig [**38**, Chap. 18]). *The free module* **J** *with the above multiplication is an associative ring with identity element* $1_{\mathbf{J}} = \sum_{d \in \mathcal{D}} n_d t_d$. *Let* $\mathbf{J}_{\mathcal{A}} = \mathcal{A} \otimes_{\mathbb{Z}} \mathbf{J}$. *Then we have a unital homomorphism of* $\mathcal{A}$*-algebras*

$$\phi\colon \mathcal{H} \to \mathbf{J}_{\mathcal{A}}, \qquad C_w \mapsto \sum_{\substack{z \in W, d \in \mathcal{D} \\ \mathbf{a}(z) = \mathbf{a}(d)}} h_{w,d,z}\, \hat{n}_z\, t_z,$$

where $\hat{n}_z$ *is defined as follows. Given* $z \in W$*, let* d *be the unique element of* $\mathcal{D}$ *such that* $d \sim_{\mathcal{L}} z^{-1}$*; then* $\hat{n}_z = n_d = \pm 1$*. (See* **P5**, **P13***.) Note that the function* $z \mapsto \hat{n}_z$ *is constant on the right cells of* W*.*

This ring **J** is called the asymptotic Hecke algebra. It plays a very important part in the proof of the following theorem.

THEOREM 2.10.3 (Geck [**17**]). *Assume that* W *is a finite Weyl group. Let* $\mathcal{H}_{\mathbb{Q}}$ *be the Iwahori-Hecke algebra defined over* $\mathbb{Q}[v, v^{-1}]$ *(instead of* $\mathbb{Z}[v, v^{-1}]$*). Then* $\mathcal{H}_{\mathbb{Q}}$ *is cellular in the sense of Graham-Lehrer (*[**19**]*).*

This theorem provides the general theory of Specht modules for Iwahori-Hecke algebras.

CHAPTER 3

Geometric presentation of an affine Weyl group

In this section we give another geometric presentation of an irreducible affine Weyl group in terms of alcoves (see [**9, 27, 46**]). This presentation is a very convenient way to "picture" an affine Weyl group. For instance, sets such as descent sets or the minimal left coset representatives (with respect to a parabolic subgroup) are easily "seen" in this presentation; see Examples 3.2.1 and 3.2.2. On a deeper level, it was conjectured by Lusztig that a left cell should be a connected set in this presentation.

Within this chapter, we give a number of examples to get used to this presentation. In the final section, we prove a result (using this presentation) which give a "local" bound on the degree of the structure constants associated to the standard basis. This theorem will play a crucial role in the study of the lowest two-sided cell; see Chapter 6.

3.1. Geometric presentation of an affine Weyl group

We now present a geometric presentation which will be of great use along this thesis. The basic references for this section are [**9, 27, 46**].

Let V be a Euclidean space of dimension $r \in \mathbb{N}^*$. Let Φ be an irreducible crystallographic root system of rank r and Φ^+ a fixed set of positive roots. Note that Φ spans V. We denote by $\check{\alpha}$ the coroot associated to α and we write $\langle x, \check{\alpha}\rangle$ for the value of $\check{\alpha} \in V^*$ at x, that is $\langle x, \check{\alpha}\rangle = \frac{2\langle x,\alpha\rangle}{\langle \alpha,\alpha\rangle}$. As in Section 1.3, for $\alpha \in \Phi^+$ and $k \in \mathbf{Z}$, we define the hyperplane

$$H_{\alpha,k} = \{x \in V \mid \langle x, \check{\alpha}\rangle = k\}.$$

We denote by $\mathcal{F}$ the set of all such hyperplanes and by $\sigma_{\alpha,k}$ the reflection with fixed point set $H_{\alpha,k}$. Let Ω be the group generated by all these reflections (it is the affine Weyl group generated by Φ). An alcove is a connected component of the set

$$V - \Big(\bigcup_{H\in\mathcal{F}} H\Big).$$

The group Ω acts simply transitively on the set of alcoves X. We regard Ω as acting on the right on X.

Let S be the set of Ω-orbits in the set of faces (codimension 1 facets) of alcoves. Then S consists of $r+1$ elements which can be represented as the $r+1$ faces of an alcove. If a face f is contained in the orbit $t \in S$, we say that f is of type t.

Let $s \in S$. We define an involution $A \mapsto sA$ of X as follows. Let $A \in X$; then sA is the unique alcove distinct from A which shares with A a face of type s. The set of such maps generates a group of permutations of X which is a Coxeter group (W, S). In fact, it is the affine Weyl group associated to Φ and we have $W \simeq \Omega$. We regard W as acting on the left on X.

PROPOSITION 3.1.1. *W acts simply transitively on X. Furthermore the action of W on X commutes with the action of Ω.*

Let A_0 be the fundamental alcove defined by

$$A_0 = \{x \in V \mid 0 < \langle x, \check{\alpha} \rangle < 1 \text{ for all } \alpha \in \Phi^+\}.$$

We associate to any alcove $A \in X$ the element $w \in W$ such that $A = wA_0$. Conversely, to any $w \in W$ we associate the alcove wA_0.

One can easily check that for $w \in W$ and any alcove $A \in W$ we have

$$\ell(w) = \text{ number of hyperplanes which separate } A \text{ and } wA.$$

Let $H = H_{\alpha,n} \in \mathcal{F}$. Then H divides $V - H$ into two half-spaces

$$V_H^+ = \{x \in V \mid \langle x, \check{\alpha} \rangle > n\},$$
$$V_H^- = \{x \in V \mid \langle x, \check{\alpha} \rangle < n\}.$$

3.2. Some examples

EXAMPLE 3.2.1. In Figure 1, we consider an affine Weyl group of type $\tilde{G}_2$

$$W := \langle s_1, s_2, s_3 \mid (s_1s_2)^6 = 1, (s_2s_3)^3 = 1, (s_1s_3)^2 = 1 \rangle.$$

The thick arrows represent a set Φ^+ of positive roots. The alcove zA_0 is the image of the fundamental alcove A_0 under the action of $z = s_3s_2s_1s_2s_1s_2 \in W$.
Let $W_{1,2}$ be the parabolic subgroup generated by s_1 and s_2. We denote by $X_{1,2}$ the set of minimal left coset representatives with respect to $W_{1,2}$. Finally, let

$$\mathcal{C} := \{xs_2s_1s_2s_1A_0 \mid x \in X_{1,2}\}.$$

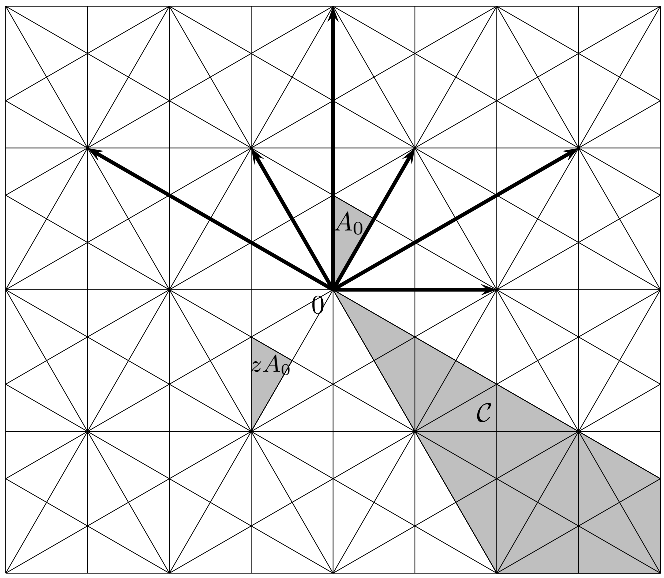
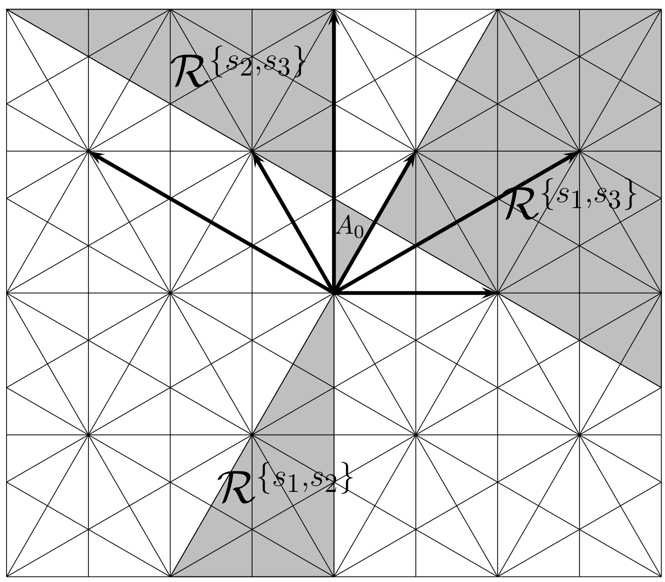


FIGURE 1. Geometric presentation of $\tilde{G}_2$

EXAMPLE 3.2.2. Again let W be of type $\tilde{G}_2$ as in the previous example. Let H_i $(1 \le i \le 3)$ be the hyperplane which contains the face of A_0 of type s_i. Let $\alpha_i \in \Phi^+$ be such that $H_i = H_{\alpha_i,0}$. We have

$$\begin{aligned} s_i \in \mathcal{R}(w) &\iff wA_0 \in V^-_{H_i} = \{x \in V | \langle x, \check{\alpha}_i\rangle < 0\} \quad \text{if } i = 1,2 \\ s_3 \in \mathcal{R}(w) &\iff wA_0 \in V^+_{H_3} = \{x \in V | \langle x, \check{\alpha}_i\rangle > 1\} \end{aligned}$$

In the next figure, we show the shape of the sets $\mathcal{R}^{\{s_1,s_2\}}$, $\mathcal{R}^{\{s_2,s_3\}}$ and $\mathcal{R}^{\{s_1,s_3\}}$ (where $\mathcal{R}^J = \{w \in W | J \subset \mathcal{R}(w)\}$).



Now let W be an arbitrary irreducible affine Weyl group.

We have seen in the previous example how one can describe the right descent sets in the geometric presentation. The next natural question is how can we characterized in this presentation the fact that $x.y$ (see Definition 1.2.9) for $x, y \in W$? To this end we introduce a new definition.

DEFINITION 3.2.3. Let $z \in W$ and $A \in X$. Let $H_1, \ldots, H_n$ be the set of hyperplanes which separate A and zA. For $1 \leq i \leq n$, let $E_{H_i}(zA)$ be the half-space defined by H_i which contains zA. Let

$$h_A(z) = \bigcap_{i=1}^{n} E_{H_i}(zA).$$

Recall that for any $w \in W$ and any $A \in X$, $\ell(w)$ is the number of hyperplanes which separate A and wA. Therefore one can see that $x.y$ if and only if

$$\{H \mid H \text{ separates } A \text{ and } yA\} \cap \{H \mid H \text{ separates } yA \text{ and } xyA\} = \emptyset,$$

or in other words

LEMMA 3.2.4. *Let $x, y \in W$ and $A \in X$. We have*

$$x.y \Leftrightarrow x(yA) \subset h_A(y).$$

See Example 4.2.4 for examples of such sets.

3.3. Weight function and geometric presentation

Let W be an irreducible affine Weyl group and let L be a weight function on W; we want to introduce L "into the picture". The following result will allow us to do so (see [**9**, Lemma 2.1]).

LEMMA 3.3.1. *Let $H \in \mathcal{F}$ and suppose that H supports a face of type $s \in S$ and a face of type $t \in S$. Then s and t are conjugate in W.*

As a consequence of this lemma, we can associate a weight c_H to any $H \in \mathcal{F}$, where $c_H = L(s)$ if H supports a face of type s.

Let λ be a 0-dimensional facet of an alcove. We denote by W_λ the stabilizer in W of the set of alcoves containing λ in their closure. It is a maximal parabolic subgroup of W with generating set $S_\lambda = S \cap W_\lambda$, thus it is finite and we denote by w_λ its longest element.

We now introduce the notion of special points.

DEFINITION 3.3.2. For a 0-dimensional facet λ of an alcove, we set

$$m(\lambda) := \sum_{H, \lambda \in H \in \mathcal{F}} c_H.$$

We say that λ is a special point if $m(\lambda)$ is maximal (among the values of m on 0-dimensional facets of alcoves). We denote by T the set of special points.

REMARK 3.3.3. One can see that for any special point λ, we have

$$m(\lambda) = \max_{I \subsetneq S} L(w_I).$$

It can be shown that it is enough to take the maximum where I runs over the subsets of S such that W_I is isomorphic to the underlying Weyl group. In the equal parameter case, we have $m(\lambda) = |\Phi^+|$ for any special point λ; see [**27**].

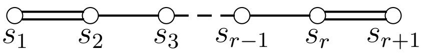
Let λ be a special point. A quarter with vertex λ is a connected component of the set

$$V - \Big(\bigcup_{\lambda \in H \in \mathcal{F}} H \Big)$$

Following Bremke, we now want to determine the set of special points for all affine Weyl groups. In order to do so, the following lemma is crucial (see [**9**, Lemma 2.2]).

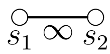
LEMMA 3.3.4. *Let H, H' be two parallel hyperplanes in $\mathcal{F}$ and let $s \neq s' \in S$. Assume that H supports a face of type s and H' a face of type s'. One (and only one) of the following statements holds.*

(1) *W is of type $\tilde{C}_r$ ($r \geq 2$) with graph*



and $(s, s') = (s_1, s_{r+1})$;

(2) *W is of type $\tilde{A}_1$ with graph*



and $(s, s') = (s_1, s_2)$;

(3) *s and s' are conjugate in W.*

Thus, if W is not of type $\tilde{C}_r$ ($r \geq 2$) or $\tilde{A}_1$, then any two parallel hyperplanes have the same weight.

We now fix some conventions about $\tilde{C}_r$ and $\tilde{A}_1$. We keep the notation of the above lemma. If W is of type $\tilde{C}_r$, we assume that the Weyl group associated to Φ is generated by $\{s_1, \ldots, s_r\}$ and that $L(s_1) \geq \ell(s_{r+1})$ (this is possible because of the symmetry of the graph). Similarly if W is of type $\tilde{A}_1$, we assume that the Weyl group associated to Φ is generated by $\{s_1\}$ and that $L(s_1) \geq L(s_2)$.

We now give the classification of special points (see **[9]**). Let P be the weight lattice

$$P := \{x \in V | \langle x, \check{\alpha} \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi^+\}.$$

Note that P is the set of points which lie in the intersection of $|\Phi^+|$ hyperplanes. Let T be the set of special points. We have

(1) If W is not of type $\tilde{C}_r$ $(r \geq 2)$ or $\tilde{A}_1$ then $T = P$.
(2) If W is of type $\tilde{C}_r$ $(r \geq 2)$ with $L(s_1) = L(s_{r+1})$ or of type $\tilde{A}_1$ with $L(s_1) = L(s_2)$ then $T = P$.
(3) If W is of type $\tilde{C}_n$ $(r \geq 2)$ with $L(s_1) > L(s_{r+1})$ or of type $\tilde{A}_1$ with $L(s_1) > L(s_2)$ then T is equal to the root lattice.

In all cases (with our convention for type $\tilde{C}_r$ and $\tilde{A}_1$), the point 0 is always a special point and W_0 is the Weyl group associated to the root system Φ.

Remark 3.3.5. The group Ω acts on the set of special points T. If $\lambda_1, \lambda_2 \in T$ lie in the same orbit then $W_{\lambda_1} = W_{\lambda_2}$. If $\lambda, \lambda' \in T$ do not lie in the same orbit then W_λ and $W_{\lambda'}$ are isomorphic but they are not generated by the same simple reflections in S. Let $\tilde{\nu} = m(\lambda)$ for $\lambda \in T$. The number N of orbits in T is

$$N := |\{J \subset S | W_J \simeq W_0 \text{ and } L(w_J) = \tilde{\nu}\}|.$$

For instance, in type $\tilde{A}_n$ there are $n+1$ orbits and in type $\tilde{G}_2$ only 1.

Example 3.3.6. Let W be an affine Weyl group of type $\tilde{C}_2$. We keep the notation of Lemma 3.3.4. The next figure describes the special points of $\tilde{C}_2$. In the case $L(s_1) = L(s_3)$, all the "circled" points are special points. There are two orbits under the action of Ω representated by the "white" and the "gray" points. In the case $L(s_1) > L_(s_3)$, only the "gray" points are special points and there is only one orbit.

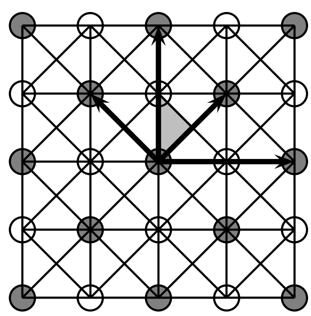


Figure 3. Special points of $\tilde{C}_2$.

3.4. Strips

We keep the setting of the previous sections. We say that two hyperplanes in $\mathcal{F}$ have the same direction if they are orthogonal to the same positive root. This defines an equivalence relation on $\mathcal{F}$. We denote by $\overline{\mathcal{F}}$ the set of directions (i.e. the equivalence classes of the above relation). We denote by $\overline{H}$ the direction of $H \in \mathcal{F}$. For $i \in \overline{\mathcal{F}}$, we set

$$c_i := \max_{H \in \mathcal{F}, \overline{H} = i} c_H \quad \text{for all } i \in \overline{\mathcal{F}}.$$

Remark 3.4.1. If W is not of type $\tilde{C}_n$ $(n \geq 2)$ or $\tilde{A}_1$ then for every $i \in \overline{\mathcal{F}}$ and for any hyperplane H of direction i we have $c_i = c_H$; see Lemma 3.3.4.

Following [**3, 9**], we introduce the notion of strips.

Definition 3.4.2. Let $i \in \overline{\mathcal{F}}$. The strips of direction i are the connected components of the set

$$V - \bigcup_{H \in \mathcal{F},\ \overline{H} = i} H.$$

For $A \in X$, we denote by $U_i(A)$ the unique strip of direction i which contains A.
The maximal strips of direction i are the connected components of

$$V - \bigcup_{\substack{H \in \mathcal{F},\ \overline{H} = i \\ c_H = c_i}} H.$$

Note that the strips as defined in [**9**] correspond to our maximal strips.

Remark 3.4.3. Let $A \in X$ and $i \in \overline{\mathcal{F}}$ and consider the strip $U_i(A)$. There exists a unique $\alpha \in \Phi^+$ and a unique $n \in \mathbb{Z}$ such that

$$U_i(A) = \{x \in V \mid n < \langle x, \check{\alpha} \rangle < n+1\},$$

in other words

$$U_i(A) = V^+_{H_{\alpha,n}} \cap V^-_{H_{\alpha,n+1}}.$$

We say that $U_i(A)$ is defined by $H_{\alpha,n}$ and $H_{\alpha,n+1}$.
Now let $i, j \in \overline{\mathcal{F}}$ and $\sigma \in \Omega$ be such that $\sigma(i) = j$. Then we have, for every $A \in X$, $\sigma(U_i(A)) = U_j(A\sigma)$ and the strip $U_j(A\sigma)$ is defined by the hyperplanes $\sigma(H_{\alpha,n})$ and $\sigma(H_{\alpha,n+1})$

EXAMPLE 3.4.4. Let W be an affine Weyl group of type $\tilde{C}_2$ generated by s_1, s_2, s_3 where s_1 and s_3 commute. If $L(s_1) = L(s_3)$ then the set of maximal strips and the set of strips coincide. In Figure 4, we show the strips of direction $\overline{s_1}$ which contain A_0 (where $\overline{s_1}$ is the direction of the hyperplane containing the face of type s_1 of A_0). If $L(s_1) > L(s_3)$ then the maximal strips are different from the strips. In Figure 5, we show the maximal strips of direction $\overline{s_1}$ in this case.

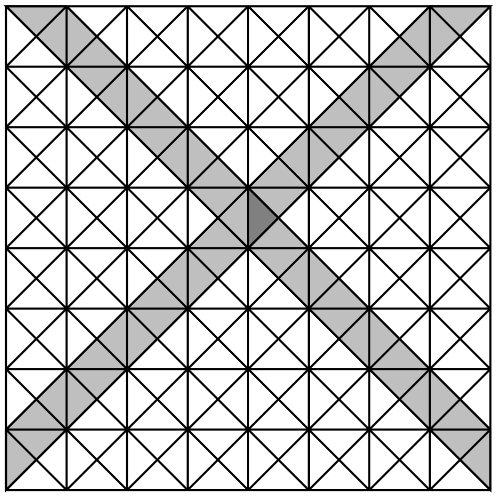


FIGURE 4. Strips of direction $\overline{s_1}$ containing A_0.

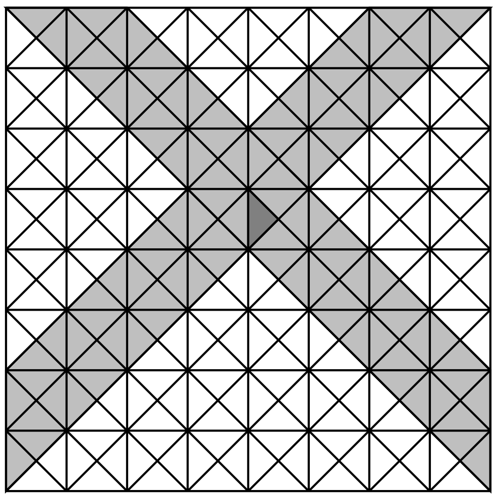


FIGURE 5. Maximal strips of direction $\overline{s_1}$ containing A_0.

3.5. Multiplication of the standard basis

In this section we give a result which gives an upper bound for the degree of the structure constants with respect to the standard basis.

For two alcoves $A, B \in X$, let

$$H(A,B) = \{H \in \mathcal{F} \mid H \text{ separates } A \text{ and } B\}.$$

Let $x, y \in W$; we define

$$\begin{aligned} H_{x,y} &= \{H \in \mathcal{F} \mid H \in H(A_0, yA_0) \cap H(yA_0, xyA_0)\}, \\ I_{x,y} &= \{i \in \overline{\mathcal{F}} \mid \exists H; \overline{H} = i, H \in H_{x,y}\}. \end{aligned}$$

For $i \in I_{x,y}$, let

$$c_{x,y}(i) = \max_{H \in H_{x,y}, \overline{H} = i} c_H$$

and

$$c_{x,y} = \sum_{i \in I_{x,y}} c_{x,y}(i).$$

We have

THEOREM 3.5.1. *Let $x, y \in W$ and*

$$T_x T_y = \sum_{z \in W} f_{x,y,z} T_z \quad \textit{where } f_{x,y,z} \in \mathcal{A}.$$

Then, the degree of $f_{x,y,z}$ in v is at most $c_{x,y}$.

REMARK 3.5.2. Note that this theorem implies that an affine Weyl group is bounded; see Section 2.7.

In order to prove this theorem we need a number of preliminary lemmas.

LEMMA 3.5.3. *Let $x, y \in W$ and $s \in S$ be such that $x < xs$ and $y < sy$. We have*

$$c_{xs,y} = c_{x,sy}.$$

PROOF. Let H_s be the unique hyperplane which separates yA_0 and syA_0. Since $x < xs$ and $y < sy$, one can see that

$$\begin{aligned} &H(A_0, yA_0) \cup \{H_s\} = H(A_0, syA_0), \\ &H(A_0, yA_0) \cap \{H_s\} = \emptyset, \end{aligned}$$

and

$$\begin{aligned}H(syA_0, xsyA_0) \cup \{H_s\} &= H(yA_0, xsyA_0),\\ H(syA_0, xsyA_0) \cap \{H_s\} &= \emptyset.\end{aligned}$$

Therefore we have

$$\begin{aligned}H_{x,sy} &= H(A_0, syA_0) \cap H(syA_0, xsyA_0)\\ &= (H(A_0, yA_0) \cup \{H_s\}) \cap H(syA_0, xsyA_0)\\ &= (H(A_0, yA_0) \cap H(syA_0, xsyA_0)) \cup (\{H_s\} \cap H(syA_0, xsyA_0))\\ &= H(A_0, yA_0) \cap H(syA_0, xsyA_0)\end{aligned}$$

and

$$\begin{aligned}H_{xs,y} &= H(yA_0, xsyA_0) \cap H(A_0, yA_0)\\ &= (H(syA_0, xsyA_0) \cup \{H_s\}) \cap H(A_0, yA_0)\\ &= (H(syA_0, xsyA_0) \cap H(A_0, yA_0)) \cup (\{H_s\} \cap H(A_0, yA_0))\\ &= H(syA_0, xsyA_0) \cap H(A_0, yA_0)\\ &= H_{x,sy}.\end{aligned}$$

Thus $c_{x,sy} = c_{xs,y}$. $\square$

LEMMA 3.5.4. *Let $x, y \in W$ and $s \in S$ be such that $xs < x$ and $sy < y$. We have*

$$c_{xs,sy} \leq c_{x,y}.$$

PROOF. Let H_s be the unique hyperplane which separate yA_0 and syA_0. One can see that

$$H_{xs,sy} = H_{x,y} - \{H_s\}.$$

The result follows. $\square$

LEMMA 3.5.5. *Let $x, y \in W$ and $s \in S$ be such that $xs < x$ and $sy < y$. Let H_s be the unique hyperplane which separates yA_0 and syA_0. Then we have*

$$\overline{H_s} \notin I_{xs,y} \quad \text{and} \quad \overline{H_s} \in I_{x,y}.$$

PROOF. We have

$$\begin{aligned}sy < y &\Longrightarrow H_s \in H(A_0, yA_0),\\ xs < x &\Longrightarrow H_s \in H(yA_0, xyA_0).\end{aligned}$$

Thus $H_s \in H_{x,y}$ and $\overline{H}_s \in I_{x,y}$.

Let $\alpha_s \in \Phi^+$ and $n_s \in \mathbb{Z}$ be such that $H_s = H_{\alpha_s,n_s}$. Assume that $n_s \geq 1$ (the case where $n_s \leq 0$ is similar).

Since $H_s \in H(A_0, yA_0)$ and yA_0 has a facet contained in H_s, we have

$$n_s < \langle x, \alpha_s \rangle < n_s + 1 \text{ for all } x \in yA_0.$$

Therefore, for all $m > n_s$, we have $H_{\alpha_s,m} \notin H(A_0, yA_0)$.

Now, since $xs < x$, we have

$$xsyA_0 \subset \{\lambda \in V \mid n_s < \langle \lambda, \alpha_s \rangle\}.$$

Therefore, for all $m \leq n_s$, we have $H_{\alpha_s,m} \notin H(yA_0, xsyA_0)$. Thus, there is no hyperplane parallel to H_s in $H_{xs,y}$, as required. $\square$

Let $x, y \in W$ and $s \in S$ be such that $xs < x$ and $sy < y$. Let H_s be the unique hyperplane which separates yA_0 and syA_0 and let σ_s be the corresponding reflection. Assume that $I_{xs,y} \neq \emptyset$ and let $i \in I_{xs,y}$. Recall that $U_i(yA_0)$ is the unique strip of direction i which contains yA_0. Since $i \in I_{xs,y}$ we have

$$A_0 \not\subset U_i(yA_0) \text{ and } xsyA_0 \not\subset U_i(yA_0).$$

One can see that one and only one of the hyperplanes which defines $U_i(yA_0)$ lies in $H_{xs,y}$. We denote by $H^{(i)}$ this hyperplane.

Let $H \in H_{xs,y}$. By the previous lemma we know that $H \neq H_s$. Consider the 4 connected components of $V - \{H, H_s\}$. We denote by E_{A_0}, E_{yA_0}, E_{syA_0} and E_{xsyA_0} the connected component which contains, respectively, A_0, yA_0, syA_0 and $xsyA_0$. Assume that $\sigma_s(H) \neq H$. Then, we have either

$$\sigma_s(H) \cap E_{yA_0} \neq \emptyset \text{ and } \sigma_s(H) \cap E_{A_0} \neq \emptyset$$

or

$$\sigma_s(H) \cap E_{xsyA_0} \neq \emptyset \text{ and } \sigma_s(H) \cap E_{syA_0} \neq \emptyset.$$

Furthermore, in the first case, $\sigma_s(H)$ separates E_{xsyA_0} and E_{syA_0}, and, in the second case, $\sigma_s(H)$ separates E_{yA_0} and E_{A_0}. In particular, we have

$$\begin{aligned} \sigma_s(H) \cap E_{yA_0} \neq \emptyset \quad &\Longrightarrow \quad \sigma_s(H) \in H(syA_0, xsyA_0) \\ \sigma_s(H) \cap E_{xsyA_0} \neq \emptyset \quad &\Longrightarrow \quad \sigma_s(H) \in H(A_0, yA_0). \end{aligned}$$

Moreover, we see that

$$\begin{aligned} \sigma_s(H) \cap E_{yA_0} \neq \emptyset \quad &\Longrightarrow \quad \sigma_s(H) \in H(syA_0, xsyA_0) \\ &\Longrightarrow \quad \sigma_s(H) \in H(yA_0, xsyA_0), \end{aligned}$$

We will say that $H \in H_{xs,y}$ is of s-type 1 if $\sigma_s(H) \cap E_{yA_0} \neq \emptyset$, and of s-type 2 if $\sigma_s(H) \cap E_{xsyA_0} \neq \emptyset$. Note that the type depends on element $s \in S$ that we consider. To sum up, we have

- if H is of s-type 1 then $\sigma_s(H) \in H(yA_0, xsyA_0)$;
- if H is of s-type 2 then $\sigma_s(H) \in H(A_0, yA_0)$.

We illustrate this result in Figure 6. Note that if $H, H' \in H_{xs,y}$ are parallel, then they have the same type.

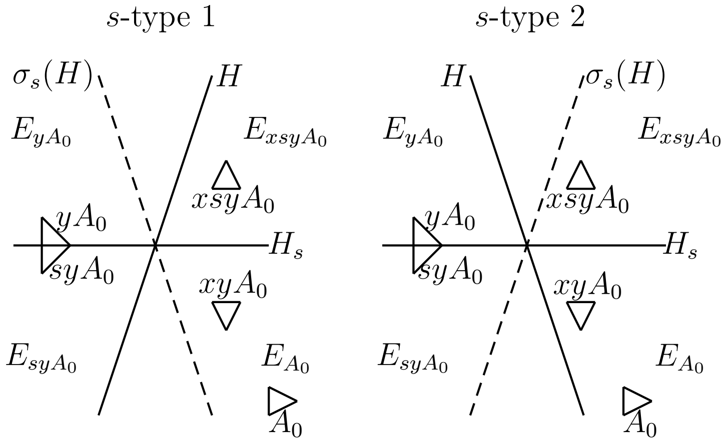


FIGURE 6. s-type 1 and s-type 2 hyperplanes

LEMMA 3.5.6. *Let $x, y \in W$ and $s \in S$ be such that $xs < x$ and $sy < y$. Let H_s be the unique hyperplane which separates yA_0 and syA_0 and let σ_s be the corresponding reflection. We have the following.*

a) *Let $H \in \mathcal{F}$. We have*

$$H \in H(yA_0, xsyA_0) \Rightarrow \sigma_s(H) \in H(yA_0, xyA_0).$$

b) *Let $H \in H_{xs,y}$ be of s-type 1; then $H \in H_{x,y}$.*

c) *Let $H \in H_{xs,y}$ be of s-type 2; then $\sigma_s(H) \in H_{x,y}$.*

d) *Let $H \in H_{xs,y}$ such that $\sigma_s(H) = H$; then $H \in H_{x,y}$.*

PROOF. We prove (a). Let $H \in H(yA_0, xsyA_0)$. Then $\sigma_s(H)$ separates $yA_0\sigma_s$ and $xsyA_0\sigma_s$. But we have

$$yA_0\sigma_s = syA_0 \quad \text{and} \quad xsyA_0\sigma_s = xssyA_0 = xyA_0.$$

Since $H \neq H_s$, we have $\sigma_s(H) \neq H_s$ and this implies that $\sigma_s(H)$ separates yA_0 and xyA_0.

We prove (b). We have $H \in H_{xs,y} = H(A_0, yA_0) \cap H(yA_0, xsyA_0)$. The hyperplane H is of s-type 1 thus $\sigma_s(H) \in H(yA_0, xsyA_0)$. Using (a) we see that $H \in H(yA_0, xyA_0)$. Therefore, $H \in H_{x,y}$.

We prove (c). Since H is of s-type 2 we have $\sigma_s(H) \in H(A_0, yA_0)$. Moreover, $H \in H(yA_0, xsyA_0)$ thus, using (a), we see that $\sigma_s(H) \in H(yA_0, xyA_0)$. Therefore, $\sigma_s(H) \in H_{x,y}$.

We prove (d). Using (a), we see that $\sigma_s(H) = H \in H(yA_0, xyA_0)$ and since $H \in H_{x_0,y} \subset H(A_0, yA_0)$, we get $H \in H_{x,y}$.

□

LEMMA 3.5.7. *Let $x, y \in W$ and $s \in S$ be such that $xs < x$ and $sy < y$. Let H_s be the unique hyperplane which separates yA_0 and syA_0. There is an injective map φ from $I_{xs,y}$ to $I_{x,y} - \{\overline{H_s}\}$.*

PROOF. Let σ_s be the reflection with fixed point set H_s. If $I_{xs,y} = \emptyset$ then the result is clear. We assume that $I_{xs,y} \neq \emptyset$. We define φ as follows.

(1) If $\sigma_s(H^{(i)}) \in H(A_0, yA_0)$ then set $\varphi(i) = \sigma_s(i)$;
(2) $\varphi(i) = i$ otherwise.

We need to show that $\varphi(i) \in I_{x,y} - \{\overline{H}_s\}$. The fact that $\varphi(i) \neq \overline{H}_s$ is a consequence of Lemma 3.5.5, where we have seen that $\overline{H}_s \notin I_{xs,y}$. Indeed, since $\varphi(i)$ is either i or $\sigma_s(i)$ and $i \neq \overline{H_s}$ we cannot have $\varphi(i) = \overline{H}_s$.
Let $i \in I_{xs,y}$ be such that $\sigma_s(H^{(i)}) \in H(A_0, yA_0)$. By Lemma 3.5.6 (a), we have $\sigma_s(H^{(i)}) \in H(yA_0, xyA_0)$. It follows that $\sigma_s(H^{(i)}) \in H_{x,y}$ and $\sigma_s(i) \in I_{x,y}$ as required.
Let $i \in I_{xs,y}$ be such that $\sigma_s(H^{(i)}) \notin H(A_0, yA_0)$. Then $H^{(i)}$ is of s-type 1. By the previous lemma we have $H^{(i)} \in H_{x,y}$ and $i \in I_{x,y}$.
We show that φ is injective. Let $i \in I_{xs,y}$ be such that $\varphi(i) = \sigma_s(i)$ and assume that $\sigma_s(i) \in I_{xs,y}$. We have

$$\sigma_s(U_i(yA_0)) = U_{\sigma_s(i)}(syA_0) = U_{\sigma_s(i)}(yA_0)$$

and $\sigma_s(H^{(i)})$ is one of the hyperplane which defines $U_{\sigma_s(i)}(yA_0)$. Furthermore since $\sigma_s(H^{(i)}) \in H(A_0, yA_0)$ we must have $\sigma_s(H^{(i)}) = H^{(\sigma_s(i))}$. It follows that $\sigma_s(H^{(\sigma_s(i))}) \in H(A_0, yA_0)$ and $\varphi(\sigma_s(i)) = i$. The result follows.

□

PROPOSITION 3.5.8. *Let $x, y \in W$ and $s \in S$ be such that $xs < x$ and $sy < y$. Let H_s be the unique hyperplane which separates yA_0 and syA_0. We have*

$$c_{xs,y} \leq c_{x,y} - c_{x,y}(\overline{H_s}).$$

PROOF. Let φ be as in the proof of the previous lemma. We keep the same notation. If $I_{xs,y} = \emptyset$ then the result is clear, thus we may assume that $I_{xs,y} \neq 0$.

First assume that W is not of type $\tilde{C}_r$ ($r \geq 2$) or $\tilde{A}_1$. Then any two parallel hyperplanes have the same weight, therefore we obtain, for $i \in I_{xs,y}$

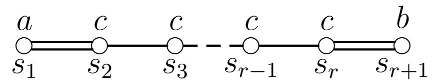
$$c_{xs,y}(i) = c_{H^{(i)}},$$

Moreover, since $c_H = c_{\sigma(H)}$ for any $H \in \mathcal{F}$ and $\sigma \in \Omega$, one can see that

$$c_{xs,y}(i) = c_{x,y}(\varphi(i)),$$

and the result follows using Lemma 3.5.7.

Now, assume that W is of type $\tilde{C}_r$, with graph and weight function given by



We have seen that the only case where two parallel hyperplanes H, H' do not have the same weight is when one of them, say H, supports a face of type s_1 and H' supports a face of type s_{r+1}.

If $a = b$, then parallel hyperplanes have the same weight and we can conclude as before.

Now assume that $a > b$. Let $i \in \overline{\mathcal{F}}$ be such that not all the hyperplanes with direction i have the same weight. Let $H = H_{\alpha,n}$ be a hyperplane with direction i and weight a. Then $H_{\alpha,n-1}$ and $H_{\alpha,n+1}$ have weight b because otherwise all the hyperplanes with direction i would have weight a.

CLAIM 3.5.9. Let $i \in I_{xs,y}$. We have

(1) if $H^{(i)}$ is of s-type 2 then $c_{x,y}(\varphi(i)) \geq c_{xs,y}(i)$;
(2) if $\sigma_s(H^{(i)}) = H^{(i)}$ then $c_{x,y}(\varphi(i)) \geq c_{xs,y}(i)$;
(3) if $H^{(i)}$ is of s-type 1 and $\sigma_s(H^{(i)}) \notin H(A_0, yA_0)$ then $c_{x,y}(\varphi(i)) \geq c_{xs,y}(i)$;

PROOF. We prove (1). Since $H^{(i)}$ is of s-type 2 we have $\sigma_s(H^{(i)}) \in H(A_0, yA_0)$ and $\varphi(i) = \sigma_s(i)$. Let $H \in H_{xs,y}$ be such that $\overline{H} = i$. Then H is of s-type 2 and $\sigma_s(H) \in H_{x,y}$ (see Lemma 3.5.6 (c)). The result follows.

We prove (2). Since $\sigma_s(H^{(i)}) = H^{(i)}$ we have $H^{(i)} \in H_{x,y}$ (see Lemma 3.5.6 (d)) and $\varphi(i) = i$. Let $H \in H_{xs,y}$ be such that $\overline{H} = i$. Then $\sigma_s(H) = H$ and $H \in H_{x,y}$. The result follows.
We prove (3). In that case we have $\varphi(i) = i$. Let $H \in H_{xs,y}$ be such that $\overline{H} = i$. Then H is of s-type 1 and $H \in H_{x,y}$ (see Lemma 3.5.6 (b)). The result follows. □

CLAIM 3.5.10. Let $i \in I_{xs,y}$ be such that

$$c_{xs,y}(i) = a \quad \text{and} \quad c_{x,y}(\varphi(i)) = b$$

Then we have $\sigma_s(i) \in I_{xs,y}$, $\varphi(\sigma_s(i)) = i$ and

$$c_{xs,y}(\sigma_s(i)) = b \quad \text{and} \quad c_{x,y}(i) = a$$

PROOF. By the previous claim, we know that $H^{(i)}$ is of s-type 1 and $\sigma_s(H^{(i)}) \in H(A_0, yA_0)$. Thus $\sigma_s(H^{(i)}) \in H_{x,y}$ and $\varphi(i) = \sigma_s(i)$. In particular, since $c_{x,y}(\varphi(i)) = b$, we must have $c_{\sigma_s(H^{(i)})} = b$, which implies that $c_{H^{(i)}} = b$.
Since $H^{(i)}$ is of s-type 1 we have $\sigma_s(H^{(i)}) \in H(yA_0, xsyA_0)$ which implies that $\sigma_s(H^{(i)}) \in H_{xs,y}$. Thus $\sigma_s(i) \in I_{xs,y}$. Arguying as in the proof of Lemma 3.5.7, we obtain $\sigma_s(H^{(i)}) = H^{(\sigma_s(i))}$ and $\varphi(\sigma_s(i)) = i$.
Let $\alpha \in \Phi^+$ and $n \in \mathbb{Z}$ be such that $H^{(i)} = H_{\alpha,n}$. Since $c_{xs,y}(i) = a$, one can see that one of the hyperplanes $H_{\alpha,n-1}$, $H_{\alpha,n+1}$ lies in $H_{xs,y}$. We denote this hyperplane by H. Note that $c_H = a$. Thus, since $c_{x,y}(\sigma_s(i)) = b$ we have $\sigma_s(H) \notin H_{x,y}$. Both hyperplanes $\sigma_s(H)$ and $\sigma_s(H^{(i)})$ separate yA_0 and xyA_0 but only $\sigma_s(H^{(i)})$ lies in $H_{x,y}$. This implies that A_0 lies in the strip defined by $\sigma_s(H)$ and $\sigma_s(H^{(i)})$. Since $\sigma_s(H^{(i)}) = H^{(\sigma_s(i))}$ this shows that the only hyperplane of direction $\sigma_s(i)$ which lies in $H_{xs,y}$ is $H^{(\sigma_s(i))}$. Thus we have $c_{xs,y}(\sigma_s(i)) = b$. Moreover $\varphi(\sigma_s(i)) = i$ and H is of s-type 1, thus $H \in H_{x,y}$ (see Lemma 3.5.6 (2)) and $c_{x,y}(i) = a$.

□

We now go back to the proof of Proposition 3.5.8. Let $I_>$ be the subset of $I_{xs,y}$ which consists of the directions i such that $c_{xs,y}(i) = a$ and $c_{x,y}(\varphi(i)) = b$. Using the previous claim, we see that the set $\sigma_s(I_>)$ is a subset of $I_{xs,y}$ such that for all $i \in \sigma_s(I_>)$ we have $c_{xs,y}(i) = b$ and $c_{x,y}(\varphi(i)) = a$. The proposition follows in the case where W is of type $\tilde{C}_r$ $(r \geq 2)$.

In the case where W is of type $\tilde{A}_1$, the result is clear, since we always have $I_{xs,y} = \emptyset$. The proposition is proved.

Proof of Theorem 3.5.1. Let $x, y \in W$ and

$$T_x T_y = \sum_{z \in W} f_{x,y,z} T_z \quad \text{where } f_{x,y,z} \in \mathcal{A}.$$

We want to prove that the degree of $f_{x,y,z}$ in v is less than or equal to $c_{x,y}$. We proceed by induction $\ell(x) + \ell(y)$.
If $\ell(x) + \ell(y) = 0$ the result is clear.
If $c_{x,y} = 0$ then $H_{x,y} = \emptyset$ and $x.y$. Thus $T_x T_y = T_{xy}$ and the result follows.
We may assume that $H_{x,y} \neq \emptyset$, which implies that $\ell(x) > 0$ and $\ell(y) > 0$. Let $x = s_k \dots s_1$ be a reduced expression of x. There exists $1 \leq i \leq k$ such that $(s_{i-1} \dots s_1).y$ and $s_i s_{i-1} \dots s_1 y < s_{i-1} \dots s_1 y$. Let $x_0 = s_k \dots s_i$ and $y_0 = s_{i-1} \dots s_1.y$. Let H_{s_i} be the unique hyperplane which separates $y_0 A_0$ and $s_i y_0 A_0$. Note that $c_{H_{s_i}} = L(s_i)$. We have

$$T_x T_y = T_{x_0} T_{y_0}$$

Using Lemma 3.5.3, we obtain $c_{x,y} = c_{x_0,y_0}$. We have

$$\begin{aligned}
T_{x_0,y_0} &= T_{s_k \dots s_{i+1}} T_{s_i} T_{y_0} \\
&= T_{s_k \dots s_{i+1}} (T_{s_i y_0} + \xi_{s_i} T_{y_0}) \\
&= T_{s_k \dots s_{i+1}} T_{s_i y_0} + \xi_{s_i} T_{s_k \dots s_{i+1}} T_{y_0} \\
&= T_{x_0 s_i, s_i y_0} + \xi_{s_i} T_{x_0 s_i, y_0}
\end{aligned}$$

By induction, $T_{x_0 s_i} T_{s_i y_0}$ is an $\mathcal{A}$-linear combination of T_z with coefficients of degree less than or equal to $c_{x_0 s_i, s_i y_0}$. Using Lemma 3.5.4, we have $c_{x_0 s_i, s_i y_0} \leq c_{x_0,y_0} = c_{x,y}$.
By induction, $T_{x_0 s_i} T_{y_0}$ is an $\mathcal{A}$-linear combination of T_z with coefficients of degree less than or equal to $c_{x_0 s_i, y_0}$. Therefore the degree of the polynomials occuring in $\xi_{s_i} T_{x_0 s_i} T_{y_0}$ is less than or equal to $L(s_i) + c_{x_0 s_i, y_0}$. Applying Proposition 3.5.8 to x_0 and y_0 we obtain

$$c_{x_0 s_i, y_0} \leq c_{x_0,y_0} - c_{x_0,y_0}(\overline{H_{s_i}})$$

Since $c_{x_0,y_0}(\overline{H_{s_i}}) \geq c_{H_{s_i}} = L(s_i)$ we obtain

$$L(s_i) + c_{x_0 s_i, y_0} \leq c_{x_0,y_0} = c_{x,y}.$$

The theorem is proved.

□

CHAPTER 4

On the determination of cells in affine Weyl groups

In this chapter, we introduce the original setting for cells with unequal parameters, where instead of a weight function, Lusztig ([**28**]) defined the cells with respect to an abelian group and a total order on it. Then following Geck ([**16**]), we find a criterion in order to determine whether two weight functions give rise to essentially the same data (i.e R-polynomials, Kazhdan-Lusztig polynomials...) on a given subset of W, namely a Bruhat interval. In Section 4.3.1, we prove that, in an affine Weyl group, the Kazhdan-Lusztig polynomials are invariant under "long enough" translations. Finally, applying these results to the case where W is of type $\tilde{G}_2$, we will show that

(1) there are only finitely many possible decompositions of W into left cells;
(2) the number of left cells is finite in each case.

We use the geometric presentation of an affine Weyl group and keep the same notation as in the previous chapter.

4.1. Weight function and total order

4.1.1. Total ordering. Let W be an arbitrary Coxeter group. Following Lusztig ([**28**]), let Γ be an abelian group written multiplicatively and let $\mathbf{A} := \mathbb{Z}[\Gamma]$ be the group algebra of Γ over $\mathbb{Z}$. Let $\{v_s \mid s \in S\}$ be a subset of Γ such that $v_s = v_t$ whenever $s, t \in S$ are conjugate in W. Then we can define the corresponding generic Iwahori-Hecke algebra $\mathbf{H}$, with $\mathbf{A}$-basis $\{\mathbf{T}_w \mid w \in W\}$, identity element $\mathbf{T}_e$ and multiplication given by the rule

$$\mathbf{T}_s\mathbf{T}_w = \begin{cases} \mathbf{T}_{sw}, & \text{if } \ell(sw) > \ell(w), \\ \mathbf{T}_{sw} + (v_s - v_s^{-1})\mathbf{T}_w, & \text{if } \ell(sw) < \ell(w). \end{cases}$$

Let $a \to \bar{a}$ be the involution of $\mathbb{Z}[\Gamma]$ defined by $\overline{g} = g^{-1}$ for $g \in \Gamma$. We can extend it to a map from $\mathbf{H}$ to itself by

$$\overline{\sum_{w\in W} a_w \mathbf{T}_w} = \sum_{w\in W} \bar{a}_w \mathbf{T}_{w^{-1}}^{-1} \qquad (a_w \in \mathbb{Z}[\Gamma]).$$

Then $h \to \bar{h}$ is a ring involution.

Doing the same construction as in Chapter 2, we obtain the generic $\mathbf{R}$-polynomials, which satisfy similar properties as the R-polynomials.

Now we choose a total ordering of Γ. This is specified by a multiplicatively closed subset $\Gamma_+ \subset \Gamma - \{1\}$ such that $\Gamma = \Gamma_+ \cup \{1\} \cup \Gamma_-$ (disjoint union) where $\Gamma_- = \{g^{-1} \mid g \in \Gamma_+\}$. Moreover, assume that

$$\{v_s \mid s \in S\} \subset \Gamma_+.$$

Thus, in Chapter 2 one can replace $\mathcal{A}_{<0}$ by $\mathbb{Z}[\Gamma_-]$. We obtain

(1) the corresponding Kazhdan-Lusztig basis $\{\mathbf{C}_w\}_{w\in W}$;
(2) the Kazhdan-Lusztig polynomials $\mathbf{P}_{y,w} \in \mathbb{Z}[\Gamma_-]$ for all $y < w \in W$;
(3) the polynomials $\mathbf{M}^s_{y,w}$ whenever $sy < y < w < sw$ ($s \in S$, $y, w \in W$).

As before, these data determine a pre-order relation $\leq_L$ (resp. $\leq_{LR}$) on W and the corresponding partition of W into left cells (resp. two-sided cells).

To sum up, we have the following correspondences

$$\begin{array}{rcl}
\mathcal{A} = \mathbb{Z}[v, v^{-1}] & \longleftrightarrow & \mathbf{A} = \mathbb{Z}[\Gamma] \\
\text{weight function } L & \longleftrightarrow & \text{total order } \Gamma = \Gamma_+ \cup \{1\} \cup \Gamma_- \\
v^{L(s)} \in \{v^n | n \in \mathbb{N}^*\} & \longleftrightarrow & v_s \in \Gamma_+ \\
\mathcal{H} & \longleftrightarrow & \mathbf{H} \\
T_w & \longleftrightarrow & \mathbf{T}_w \\
R_{y,w} & \longleftrightarrow & \mathbf{R}_{y,w} \\
\mathcal{A}_{<0} = v^{-1}\mathbb{Z}[v^{-1}] & \longleftrightarrow & \mathbb{Z}[\Gamma_-] \\
C_w & \longleftrightarrow & \mathbf{C}_w \\
P_{y,w} \in \mathcal{A}_{<0} & \longleftrightarrow & \mathbf{P}_{y,w} \in \mathbb{Z}[\Gamma_-] \\
M^s_{y,w} & \longleftrightarrow & \mathbf{M}^s_{y,w}
\end{array}$$

EXAMPLE 4.1.1. Let W be a Weyl group of type G_2, generated by s, t. Let Q, q be independent indeterminates over $\mathbb{Z}$ and consider the abelian group

$$\Gamma = \{Q^i q^j \mid i, j \in \mathbb{Z}\}.$$

Consider the lexicographic order on Γ (with $Q > q$), i.e.:

$$\Gamma_+ := \{Q^i q^j \mid i > 0, j \in \mathbb{Z}\} \cup \{q^i \mid i > 0\}.$$

We set $v_t = Q$ and $v_s = q$. Doing some computations (see [**46**, Example 1.21]), we obtain

$$\begin{aligned}
\mathbf{P}_{t,tst} &= \mathbf{P}_{tst,tstst} = Q^{-1}q^{-1} - Q^{-1}q \\
\mathbf{P}_{e,tst} &= \mathbf{P}_{ts,tstst} = \mathbf{P}_{st,tstst} = Q^{-2}q^{-1} - Q^{-2}q \\
\mathbf{P}_{t,tstst} &= Q^{-2}q^{-2} - Q^{-2} + Q^{-2}q^{2} \\
\mathbf{P}_{e,tstst} &= Q^{-3}q^{-2} - Q^{-3} + Q^{-3}q^{2} \\
\mathbf{P}_{s,sts} &= \mathbf{P}_{sts,ststs} = Q^{-1}q^{-1} + Q^{-1}q \\
\mathbf{P}_{e,sts} &= \mathbf{P}_{ts,ststs} = \mathbf{P}_{st,ststs} = Q^{-1}q^{-2} + Q^{-1} \\
\mathbf{P}_{t,ststs} &= Q^{-1}q^{-3} + Q^{-1}q^{-1} \\
\mathbf{P}_{s,ststs} &= Q^{-2}q^{-2} + Q^{-2} \\
\mathbf{P}_{e,ststs} &= Q^{-2}q^{-3} + Q^{-2}q^{-1}.
\end{aligned}$$

For all other pairs $y \leq w$ we have $P_{y,w} = Q^{\ell_t(y)-\ell_t(w)}q^{\ell_s(y)-\ell_s(w)}$, where $\ell_t(z)$ (resp. ℓ_s) denotes the number of t's (resp. s's) which appears in a reduced expression of z.
The $\mathbf{M}$ polynomials are as follows

$$\begin{aligned}
\mathbf{M}^t_{tsts,ststs} &= \mathbf{M}^t_{tst,stst} = \mathbf{M}^t_{ts,sts} = \mathbf{M}^t_{t,st} = Qq^{-1} + Q^{-1}q \\
\mathbf{M}^t_{t,stst} &= \mathbf{M}^t_{ts,ststs} = 1
\end{aligned}$$

and all the others are zero.
Compare to the situation in Example 2.4.5.

4.1.2. Total order and weight function.

4.1.2.1. *Finite Coxeter Groups.* Let W be a finite Coxeter group. In [**16**], Geck has established a link between these two situations, where you have an abelian group Γ with a total order specified by $\Gamma_+ \subset \Gamma$ and a choice of parameters $\{v_s \mid s \in S\} \subset \Gamma_+$ on the one hand, and a weight function L on the other hand. We keep the setting of the previous section.

First, let $\Gamma^a_+(W)$ be the set of all elements $\gamma \in \Gamma_+$ such that γ^{-1} occurs with a non zero coefficient in a polynomial $\mathbf{P}_{y,w}$ for some $y < w$ in W. Next for any y, w in W and $s \in S$ such that $\mathbf{M}^s_{y,w} \neq 0$, we write $\mathbf{M}^s_{y,w} = n_1\gamma_1 + ... + n_r\gamma_r$ where $0 \neq n_i \in \mathbb{Z}$, $\gamma_i \in \Gamma$ and $\gamma_{i-1}^{-1}\gamma_i \in \Gamma_+$ for $2 \leq i \leq r$. Let $\Gamma^b_+(W)$ be the set of all elements $\gamma_{i-1}^{-1}\gamma_i \in \Gamma_+$ arising in this way, for any y, w, s such that $\mathbf{M}^s_{y,w} \neq 0$. Finally set $\Gamma_+(W) = \Gamma^a_+(W) \cup \Gamma^b_+(W)$.

PROPOSITION 4.1.2 (Geck [**16**, 2.10]). *Assume that we have a ring homomorphism*

$$\sigma : \mathbb{Z}[\Gamma] \to \mathbb{Z}[v, v^{-1}], \quad v_s \to v^{L(s)}$$

such that

$$\sigma(\Gamma_+(W)) \subseteq \{v^n \mid n > 0\}. \tag{$*$}$$

Then $\sigma(\mathbf{P}_{y,w}) = P_{y,w}$ *for all* $y < w$ *in* W *and* $\sigma(\mathbf{M}^s_{y,w}) = M^s_{y,w}$ *for any* $y, w \in W$ *such that* $sy < y < w < sw$. *Furthermore, the relation* $\leq_L$ *on* W *defined with respect to the weight function* L *is the same as the one defined with respect to* $\Gamma_+ \subset \Gamma$.

EXAMPLE 4.1.3. Let W be a Weyl group of type G_2 generated by s, t. On the one hand, let Q, q be independent indeterminates over $\mathbb{Z}$ and consider the abelian group

$$\Gamma = \{Q^i q^j \mid i, j \in \mathbb{Z}\}.$$

On the other hand, let L be a weight function L on W. It is determined by the values $L(s_1) = a \in \mathbb{N}^*$ and $L(s_2) = b \in \mathbb{N}^*$. We shall denote such a weight function by $L_{a,b}$. We denote by $\sigma_{a,b}$ the ring homomorphism

$$\begin{array}{cccc} \sigma_{a,b}: & \mathbb{Z}[\Gamma] & \longrightarrow & \mathbb{Z}[v, v^{-1}] \\ & Q & \longmapsto & v^a \\ & q & \longmapsto & v^b \end{array}$$

Consider the lexicographic order on Γ (with $Q > q$). In Example 4.1.1 we have computed all the Kazhdan-Lusztig polynomials and all the M-polynomials. Thus we see that

$$\Gamma_+(W) \subset \{Q^3q^{-1}, Q^3q^{-2}, Q^2q^{-1}, Q^2q^{-2}, Q^1q^{-1}, Q^cq^d (\text{where } c, d > 0)\}.$$

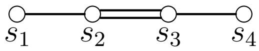
In other words, Condition $(*)$ in the previous proposition is satisfied for any ring homomorphism $\sigma_{a,b}$ such that $a > b$. Thus any weight function $L_{a,b}$ such that $a > b$ gives rise to the same left cell decomposition. Now by symmetry of the graph, the case $b > a$ is similar.

To sum up, there are just three distinct decompositions into left cells corresponding to the following weight functions

(1) $L_{a,b}$ such that $a > b$;
(2) $L_{a,b}$ such that $a = b$;
(3) $L_{a,b}$ such that $a < b$.

EXAMPLE 4.1.4. Let W be a Weyl group of type F_4 with graph as follows

F_4 :



We denote by $L_{a,b}$ the weight function on W such that $L(s_1) = L(s_2) = a \in \mathbb{N}^*$ and $L(s_3) = L(s_4) = b \in \mathbb{N}^*$. By symmetry of the graph we may assume that $a \geq b$. We have the following theorem.

THEOREM 4.1.5 (Geck [**16**, Theorem 4.8]). *Let $L = L_{a,b}$ and $L' = L_{a',b'}$ be two weight functions on W such that $b \geq a > 0$ and $b' \geq a' > 0$. Then L, L' define the same partition of W into left cells if and only if $L, L' \in \mathcal{L}_i$ for $i \in \{0, 1, 2, 3\}$, where $\mathcal{L}_i$ are defined as follows:*

$$\begin{aligned}
\mathcal{L}_0 &= \{(c, c, c, c) \quad \mid c > 0\},\\
\mathcal{L}_1 &= \{(c, c, 2c, 2c) \mid c > 0\},\\
\mathcal{L}_2 &= \{(c, c, d, d) \quad \mid 2c > d > c > 0\},\\
\mathcal{L}_3 &= \{(c, c, d, d) \quad \mid d > 2c > 0\}.
\end{aligned}$$

4.1.2.2. *Infinite Coxeter groups.* If W is an infinite Coxeter group, we cannot compute the set $\Gamma_+(W)$. However, if we restrict ourself to a finite subset of W, namely a Bruhat interval, we find a similar result.

Let $y, w \in W$, $s \in S$ and $I = [y, w]$. We now define three subsets $\Gamma_+^a(I)$, $\Gamma_+^{b,s}(I)$, $\Gamma_+^{c,s} \subset \Gamma_+$. First, let $\Gamma_+^a(I)$ be the set of all elements $\gamma \in \Gamma_+$ such that γ^{-1} occurs with a non-zero coefficient in a polynomial $\mathbf{P}_{z_1,z_2}$ for some $z_1 < z_2$ in I. Next for any z_1, z_2 in I such that $\mathbf{M}^s_{z_1,z_2} \neq 0$ we write $\mathbf{M}^s_{z_1,z_2} = n_1\gamma_1 + ... + n_r\gamma_r$ where $0 \neq n_i \in \mathbb{Z}$, $\gamma_i \in \Gamma$ and $\gamma_{i-1}^{-1}\gamma_i \in \Gamma_+$ for $2 \leq i \leq r$. Let $\Gamma_+^{b,s}(I)$ be set of all elements $\gamma_{i-1}^{-1}\gamma_i \in \Gamma_+$ arising in this way, for any $z_1, z_2 \in I$ such that $\mathbf{M}^s_{z_1,z_2} \neq 0$. Finally let $\Gamma_+^{c,s}$ be the set of all elements $\gamma \in \Gamma_+$ such that γ^{-1} occurs with a non-zero coefficient in a polynomial of the form

$$\sum_{z; z_1 \leq z < z_2; sz < z} \mathbf{P}_{z_1,z}\mathbf{M}^s_{z,z_2} - v_s\mathbf{P}_{z_1,z_2}$$

where $z_1, z_2 \in I$ and $sz_1 < z_1 < z_2 < sz_2$. We set $\Gamma_+^s(I) = \Gamma_+^a(I) \cup \Gamma_+^{b,s}(I) \cup \Gamma_+^{c,s}$.

PROPOSITION 4.1.6. *Let $y, w \in W$, $s \in S$ and $I = [y, w]$. Assume that we have a ring homomorphism*

$$\sigma : \mathbb{Z}[\Gamma] \to \mathbb{Z}[v, v^{-1}], \quad v_s \to v^{L(s)}$$

such that

$$\sigma(\Gamma_+^s(I)) \subseteq \{v^n \mid n > 0\}. \qquad (*)$$

Then $\sigma(\mathbf{P}_{z_1,z_2}) = P_{z_1,z_2}$ *for all* $z_1 < z_2$ *in* I *and* $\sigma(\mathbf{M}^s_{z_1,z_2}) = M^s_{z_1,z_2}$ *for any* $z_1, z_2 \in I$ *such that* $sz_1 < z_1 < z_2 < sz_2$.

PROOF. We have $\overline{\sigma(p)} = \sigma(\overline{p})$ for all $p \in \mathbf{Z}[\Gamma]$. Moreover, the $\mathbf{R}$-polynomials do not depend on the order, therefore we have $\sigma(\mathbf{R}_{z_1,z_2}) = R_{z_1,z_2}$ for any $z_1, z_2 \in W$.

We prove by induction on $\ell(z_2) - \ell(z_1)$ that $\sigma(\mathbf{P}_{z_1,z_2}) = P_{z_1,z_2}$ for all $z_1 \leq z_2$ in I. If $\ell(z_2) - \ell(z_1) = 0$ it is clear.

Assume that $\ell(z_2) - \ell(z_1) > 0$. Applying σ to the formula in Proposition 2.4.2 (3) using the induction hypothesis yields

$$\begin{aligned}\sigma(\overline{\mathbf{P}}_{z_1,z_2}) - \sigma(\mathbf{P}_{z_1,z_2}) &= \sum_{z_1 < z \leq z_2} \sigma(\mathbf{R}_{z_1,z})\sigma(\mathbf{P}_{z,z_2}) \\ &= \sum_{z_1 < z \leq z_2} R_{z_1,z} P_{z,z_2}.\end{aligned}$$

This relation and condition $(*)$ implies that $\sigma(\mathbf{P}_{z_1,z_2}) = P_{z_1,z_2}$.

Let $z_1, z_2 \in I$ and $s \in S$ be such that $sz_1 < z_1 < z_2 < sz_2$. We prove by induction on $\ell(z_2) - \ell(z_1)$ that $\sigma(\mathbf{M}^s_{z_1,z_2}) = M^s_{z_1,z_2}$.

Since $\mathbf{M}^s_{z_1,z_2} = \overline{\mathbf{M}^s_{z_1,z_2}}$, we have $\sigma(\mathbf{M}^s_{z_1,z_2}) = \overline{\sigma(\mathbf{M}^s_{z_1,z_2})}$. Furthermore, using $(*)$ and the definition of the $\mathbf{M}$-polynomials we have

$$\sigma(\mathbf{M}^s_{z_1,z_2}) + \sum_{z; z_1 < z < z_2; sz < z} P_{z_1,z} M^s_{z,z_2} - v^{L(s)} P_{z_1,z_2} \in v^{-1}\mathbb{Z}[v^{-1}].$$

This relation implies that $\sigma(\mathbf{M}^s_{z_1,z_2}) = M^s_{z_1,z_2}$. Moreover we can see that if $\mathbf{M}^s_{z_1,z_2} \neq 0$ then $M^s_{z_1,z_2}$ is a combination of pairwise different powers of v. Thus $M^s_{z_1,z_2} \neq 0$. □

If condition $(*)$ is satisfied for all $s \in S$, then we can conclude that $x, z \in I$ satisfy $x \leftarrow_L z$ with respect to the total order Γ_+ if and only if they satisfy $x \leftarrow_L z$ with respect to the weight function L. For an example, see Section 4.4.

4.2. On the translations in an affine Weyl group

Let (W, S) be an irreducible affine Weyl group.

DEFINITION 4.2.1. Let $u \in W$. We say that u is a translation if there exists a vector $\vec{u} \neq 0$ such that $t_{\vec{u}}$, the translation by the vector $\vec{u}$, is in Ω and

$$uA_0 = A_0 t_{\vec{u}}.$$

Note that $t_{\vec{u}}$ is uniquely determined by u.

REMARK 4.2.2. There can be elements in W which "translate" A_0 but which are not translations according to our definition. For instance, let W be an affine Weyl group of type $\tilde{C}_2$, generated by $S := \{s_1, s_2, s_3\}$ where $s_1 s_3 = s_3 s_1$. Let $u = s_1 s_2 s_3$. Then uA_0 is a translate of A_0 by a vector $\vec{u}$, however, $t_{\vec{u}}$ does not lie in Ω.

Let $u \in W$ be a translation and let B be an alcove. Let $\sigma \in \Omega$ be such that $A_0\sigma = B$. We have

$$uB = u(A_0\sigma) = A_0 t_{\vec{u}}\sigma = A_0\sigma t_{\sigma(\vec{u})} = Bt_{\sigma(\vec{u})}$$

Therefore uB is a translate of B. Note that we have used the fact that the action of Ω commutes with the action of W.

From now on and until the end of this section, we fix a translation $u \in W$. Consider the orbit of $\vec{u}$ under the action of Ω. It is finite since the group of linear transforms associated to Ω is isomorphic to Ω_0 (the stabilizer of the origin in V) which is finite. Let

$$Orb_{\Omega}(\vec{u}) = \{\vec{u_1} = \vec{u}, \vec{u_2}, ..., \vec{u_n}\}.$$

We denote by $u_i \in W$ the corresponding translations in W. Finally, let v_i be the special point $t_{\vec{u_i}}(0)$.

Recall the definition of $h_{A_0}(w)$ for $w \in W$ in 3.2.3 and Definition 1.2.9.

LEMMA 4.2.3. *Let $u \in W$ be a translation associated to $t_{\vec{u}} \in \Omega$.*

(a) *Let $r_1 \leq r_2 \in \mathbb{N}^*$. We have*

$$h_{A_0}(u^{r_2}) \subset h_{A_0}(u^{r_1}) \quad \text{and} \quad h_{A_0}(u^{r_2}) = t_{(r_2 - r_1)\vec{u}}(h_{A_0}(u^{r_1})).$$

(b) *Let $r \in \mathbb{N}^*$. We have*

$$z.u \Leftrightarrow z.u^r.$$

Before giving the proof, we give the example of $\tilde{G}_2$ to illustrate this lemma.

EXAMPLE 4.2.4. Let W be an affine Weyl group of type $\tilde{G}_2$ generated by $S = \{s_2, s_2, s_3\}$ where s_1 and s_3 commute. Let $u = u_1 = s_2 s_1 s_2 s_1 s_2 s_3$. Then one can check that u is a translation and its orbit under the action of Ω contains 6 elements, namely

$$\begin{aligned}
u_1 &= s_2 s_1 s_2 s_1 s_2 s_3 \\
u_2 &= s_1 s_2 s_1 s_2 s_3 s_2 \\
u_3 &= s_1 s_2 s_3 s_2 s_1 s_2 \\
u_4 &= s_3 s_2 s_1 s_2 s_1 s_2 \\
u_5 &= s_2 s_3 s_2 s_1 s_2 s_1 \\
u_6 &= s_2 s_1 s_2 s_3 s_2 s_1
\end{aligned}$$

In Figure 1, we show the shape of the set $h_{A_0}(u_i)$ $(1 \leq i \leq 6)$.

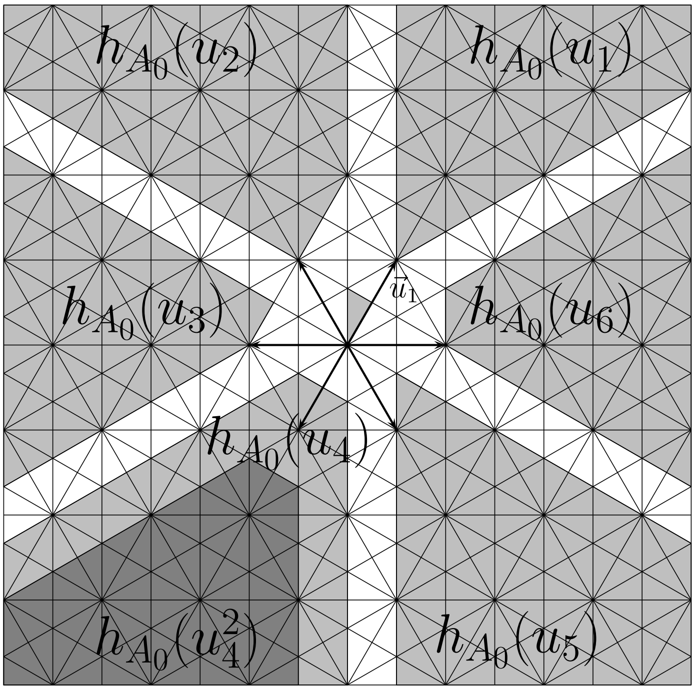


FIGURE 1. The sets $h_{A_0}(u_i)$ in $\tilde{G}_2$

REMARK 4.2.5. The sets $h_{A_0}(u_i)$ are disjoint. As we will see later on, it is true in general for any translation in an affine Weyl group. In fact, this will be the key point to prove the invariance of the Kazhdan-Lusztig polynomials by translation.

PROOF. (a) Let $\alpha \in \Phi^+$ and $k_\alpha = \langle \vec{u}, \check{\alpha} \rangle$. Since $t_{\vec{u}} \in \Omega$, one can see that $k_\alpha \in \mathbb{Z}$. For any $r \in \mathbb{N}$, we have

$$rk_\alpha < \langle x, \check{\alpha} \rangle < rk_\alpha + 1 \text{ for all } x \in u^r A_0.$$

Note that, if $k_\alpha = 0$, there is no hyperplane of the form $H_{\alpha,m}$ $(m \in \mathbb{Z})$ which separates A_0 and $u^r A_0$.
Let φ (resp. φ^+, φ^-) be the subset of Φ^+ which consists of all positive roots β such that $k_\beta \neq 0$ (resp. $k_\beta > 0$, $k_\beta < 0$). For $\beta \in \varphi$, we define

$$H^\beta = \begin{cases} H_{\beta, rk_\beta} & \text{if } \beta \in \varphi^+, \\ H_{\beta, rk_\beta + 1} & \text{if } \beta \in \varphi^-. \end{cases}$$

Then, one can check that

$$h_{A_0}(u^r) = \bigcap_{\beta \in \varphi} E_{H^\beta}(u^r A_0). \qquad (*)$$

Let $r_1 \leq r_2 \in \mathbb{N}^*$ and $\beta \in \varphi$. We suppose that $\beta \in \varphi^+$ (the case $\beta \in \varphi^-$ is similar). We have

$$E_{H^\beta}(u^{r_1}) = \{x \in V \mid \langle x, \check{\beta} \rangle > r_1 k_\beta\}$$

and

$$E_{H^\beta}(u^{r_2}) = \{x \in V \mid \langle x, \check{\beta} \rangle > r_2 k_\beta\}.$$

Thus

$$E_{H^\beta}(u^{r_2}) \subset E_{H^\beta}(u^{r_1}) \quad \text{and} \quad E_{H^\beta}(u^{r_2}) = t_{(r_2 - r_1)\vec{u}} E_{H^\beta}(u^{r_1})$$

and the result follows using relation $(*)$.

(b) The statement follows from (a) and Lemma 3.2.4. $\square$

Remark 4.2.6. Note that this Lemma implies that $\ell(u^r) = r\ell(u)$. Indeed, we have

$$u(uA_0) \in h_{A_0}(u^2) \subset h_{A_0}(u)$$

thus, using Lemma 3.2.4, we get the result.

From now on, we write $[1, n] := \{1, \ldots, n\}$.

Lemma 4.2.7. (a) *For any $i, j \in [1, n]$ we have $\ell(u_i) = \ell(u_j)$.*

(b) *Let $z_1, z_2 \in W$, $r \in \mathbb{N}^*$ and $i \in [1, n]$ be such that $z_1.u_i^r.z_2$. There exists $k, m \in [1, n]$ such that*

$$z_1.u_i^r.z_2 = z_1.z_2.u_m^r = u_k^r.z_1.z_2.$$

(c) *Let $z_1, z_2 \in W$, $r \geq 1$ and $i \in [1, n]$. We have the following equivalence*

$$z_1.u_i^r.z_2 \Leftrightarrow z_1.u_i^{r+1}.z_2.$$

Proof. (a) Let $A \in X$ and A' be a translate of A (by a translation in Ω). Then the number of hyperplanes which separate A and A' is equal to the number of hyperplanes which separate zA and zA' for any $z \in W$.

Let $i, j \in [1, n]$, $\sigma \in \Omega$ and $z \in W$ be such that $\vec{u}_i\sigma = \vec{u}_j$ and $zA_0 = A_0\sigma$.
We have

$$\begin{aligned} \ell(u_i) &= |\{H \mid H \text{ separates } A_0 \text{ and } A_{v_i}\}| \\ &= |\{H \mid H \text{ separates } z^{-1}A_0 \text{ and } z^{-1}A_{v_i}\}| \\ &= |\{H \mid H \text{ separates } z^{-1}A_0\sigma \text{ and } z^{-1}A_{v_i}\sigma\}|. \end{aligned}$$

Since

$$z^{-1}A_0\sigma = A_0$$

and

$$z^{-1}A_{v_i}\sigma = z^{-1}A_0 t_{\vec{u}_i}\sigma = z^{-1}A_0\sigma t_{\vec{u}_j} = A_{v_j},$$

we obtain

$$\begin{aligned}\ell(u_i) &= |\{H \mid H \text{ separates } z^{-1}A_0\sigma \text{ and } z^{-1}A_{v_i}\sigma\}| \\ &= |\{H \mid H \text{ separates } A_0 \text{ and } A_{v_j}\}| \\ &= \ell(u_j)\end{aligned}$$

as desired.
(b) Let $\sigma_{z_1}, \sigma_{z_2} \in \Omega$ and $k, m \in [1, n]$ be such that

$$z_1 A_0 = A_0\sigma_{z_1} \quad , \quad \sigma_{z_1}^{-1}(\vec{u}_i) = \vec{u}_k.$$

and

$$z_2 A_0 = A_0\sigma_{z_2} \quad , \quad \sigma_{z_2}(\vec{u}_i) = \vec{u}_m.$$

We have

$$z_1.u_i^r.z_2 A_0 = A_0\sigma_{z_1} t_{r\vec{u}_i}\sigma_{z_2} = A_0 t_{r\sigma_{z_1}^{-1}(\vec{u}_i)}\sigma_{z_1}\sigma_{z_2} = u_k^r z_1 z_2 A_0,$$

which implies that $z_1.u_i^r.z_2 = u_k^r z_1 z_2$. Now, since $\ell(u_i) = \ell(u_k)$, we must have $u_k^r.z_1.z_2$. Similarly, one can show that $z_1.u_i^r.z_2 = z_1.z_2.u_m^r$.

(c) The statement follows from (b) and Lemma 4.2.3 (b).

□

We now state the main result of this section.

THEOREM 4.2.8. *Let $i, j \in [1, n]$ and $r_1, r_2 \in \mathbb{N}^*$ be such that $i \neq j$. We have*

$$h_{A_0}(u_i^{r_1}) \cap h_{A_0}(u_j^{r_2}) = \emptyset.$$

PROOF. According to Lemma 4.2.3 (a), to prove the theorem, it is enough to show that, for any $i \neq j \in [1, n]$, we have

$$h_{A_0}(u_i) \cap h_{A_0}(u_j) = \emptyset.$$

Let

$$\mathcal{F}_0 := \{H \in \mathcal{F} \mid 0 \in H,\ v_i \notin H \text{ for all } i \in [1, n]\}.$$

Consider the connected component of

$$V - \bigcup_{H \in \mathcal{F}_0} H.$$

Since there exists $\sigma \in \Omega_0$ such that $\sigma(v_i) = v_j$, there is a hyperplane which separates v_i and v_j and which contains 0. Therefore A_{v_i} and A_{v_j} do not lie in the same connected component. For $i \in [1,n]$, let $\mathcal{C}_i$ be the connected component which contains A_{v_i}. To prove the theorem, it is enough to show that $h_{A_0}(u_i) \subset \mathcal{C}_i$ for all $i \in [1,n]$.

Let H be a wall of $\mathcal{C}_i$ and let $E_H(\mathcal{C}_i)$ be the half-space defined by H which contains $\mathcal{C}_i$. Since $0 \in H$ and $v_i \notin H$, one can see that $H' = t_{\vec{u}_i}(H) \neq H$. Thus either H separates A_0 and A_{v_i} or H' does.

If H separates A_0 and A_{v_i} then, as $A_{v_i} \subset \mathcal{C}_i$, we must have $h_{A_0}(u_i) \subset E_H(\mathcal{C}_i)$.

Now, assume that H does not separate A_0 and A_{v_i}. Let $\beta \in \Phi^+$ and $m \in \mathbb{Z}$ be such that $H = H_{\beta,0}$ and $H' = H_{\beta,m}$. In that case we have

$$A_0, A_{v_i} \in E_H(\mathcal{C}_i) = \{x \in V \mid \langle x, \check{\beta} \rangle > 0\}.$$

Thus one can see that we must have $m > 0$. Let $E_{H'}(A_{v_i})$ be the half-space defined by H' which contains A_{v_i}. We have

$$E_{H'}(A_{v_i}) = \{x \in V \mid \langle x, \check{\beta} \rangle > m\}$$

and

$$h_{A_0}(u_i) \subset E_{H'}(A_{v_i}) \subset E_H(\mathcal{C}_i).$$

We have shown that for every wall of $\mathcal{C}_i$, $h_{A_0}(u_i)$ lies on the same side of this wall as $\mathcal{C}_i$, thus $h_{A_0}(u_i) \subset \mathcal{C}_i$ as required. $\square$

Corollary 4.2.9. *(of Theorem 4.2.8)*

(a) *Let $z, z' \in W$, $r \in \mathbb{N}^*$, $m \in \mathbb{N}$ and $i, j \in [1,n]$. We have*

$$z.u_i^r = z'.u_j^{r+m} \Longrightarrow i = j \text{ and } z = z'.u_j^m.$$

(b) *Let $z_1, z_2, z_1', z_2' \in W$, $r \in \mathbb{N}^*$, $m \in \mathbb{N}$ and $i, j \in [1,n]$. For all $k \geq 0$ we have*

$$z_1.u_i^r.z_2 = z_1'.u_j^{r+m}.z_2' \Leftrightarrow z_1.u_i^{r+k}.z_2 = z_1'.u_j^{r+k+m}.z_2'.$$

Proof. (a) We have $z.u_i^r A_0 \in h_{A_0}(u_i^r)$ and $z'.u_j^{r+m} = z'.u_j^m.u_j^r \in h_{A_0}(u_j^r)$. Since $z.u_i^r = z'.u_j^{r+m}$, applying Theorem 4.2.8 yields $i = j$. The result follows.

(b) The statement follows from Lemma 4.2.7 and (a). $\square$

4.3. On the Kazhdan-Lusztig polynomials

Let $u \in W$ be a translation associated to $t_{\vec{u}} \in \Omega$ and let $M = \ell(u)$. One can easily see that $M \geq 2$. We keep the same notations as in the previous section. In this section we want to prove

THEOREM 4.3.1. *Let $z, z' \in W$ and $i, j \in [1,n]$ be such that $z.u_i$ and $z'.u_j$. Let $N = \ell(z') - \ell(z)$. Then for all $r > N(M+1)$ and for all $k \geq 0$ we have*

$$\mathbf{P}_{zu_i^r, z'u_j^r} = \mathbf{P}_{zu_i^{r+k}, z'u_j^{r+k}}$$

and if there exists $s \in S$ which satisfies $szu_i^r < zu_i^r < z'u_j^r < sz'u_j^r$, then

$$\mathbf{M}^s_{zu_i^r, z'u_j^r} = \mathbf{M}^s_{z.u_i^{r+k}, z'u_j^{r+k}} \ .$$

Our first task is to construct an isomorphism from the Bruhat interval $[z.u_i^r, z'.u_j^r]$ to $[z.u_i^{r+k}, z'.u_j^{r+k}]$ and then to show that the corresponding $\mathbf{R}$-polynomials are equal.

LEMMA 4.3.2. *Let $z \in W$ and $i \in [1,n]$ be such that $z.u_i$. Let $r \in \mathbb{N}^*$ and $y \in W$ be such that $r > \ell(z.u_i^r) - \ell(y)$. Then we have*

$$\begin{aligned} y \leq z.u_i^r \quad \Leftrightarrow \quad & \exists z_1, z_2 \in W, n_1, n_2 \in \mathbb{N} \text{ such that } z_1.u_i^{r-N}.z_2 = y \\ & z_1 \leq z.u_i^{n_1}, z_2 \leq u_i^{n_2} \text{ and } n_1 + n_2 = N. \end{aligned}$$

Furthermore, there exists a unique $z_y \in W$ and $m \in [1,n]$ such that $y = z_y.u_m^{r-N}$.

PROOF. "$\Leftarrow$" is clear.
"$\Rightarrow$" Let $N = \ell(z.u_i^r) - \ell(y)$. We proceed by induction on N.
If $N = 0$, it's clear.
Let $N > 0$. There exists $y' \in W$ such that $y \leq y' \leq z.u_i^r$ and

$$\ell(z.u_i^r) - \ell(y') = N - 1 \quad \text{and} \quad \ell(y') - \ell(y) = 1.$$

Applying the inductive assumption yields

$$\begin{aligned} &\exists z_1', z_2' \in W, n_1', n_2' \in \mathbb{N} \text{ such that } z_1'.u_i^{r-N+1}.z_2' = y' \\ &z_1' \leq z.u_i^{n_1'}, z_2' \leq u_i^{n_2'}, n_1' + n_2' = N - 1 \ . \end{aligned}$$

Let

$$y' = s_p...s_{m+1}(s_m...s_k)^{r-N+1}s_{k-1}...s_1 \qquad (p \geq m \geq k \geq 1)$$

be a reduced expression of y' such that

$$z_1' = s_p...s_{m+1}, \ u_i = s_m...s_k \text{ and } z_2' = s_{k-1}...s_1 \ .$$

We know that y can be obtained by deleting a simple reflection $s \in S$ in a reduced expression of y'. If there exists $l \in \mathbb{N}$ such that

$$y = s_p...\hat{s}_l...s_k(s_m...s_k)^{r-N}s_{k-1}...s_1 \qquad (p \geq l \geq k)$$

or

$$y = s_p ... s_{m+1}(s_m ... s_k)^{r-N} s_m ... \hat{s}_l ... s_1 \qquad (m \geq l \geq 1)$$

(where $\hat{s}$ means that we have deleted s) the result is straightforward.
Now assume that there exists $l_1, l_2 \in \mathbb{N}^*$ such that $l_1 + l_2 = r - N$ and

$$y = z'_1.u_i^{l_1}.\hat{u}_i.u_i^{l_2}.z'_2 \ .$$

where $\hat{u}_i$ is obtained by deleting a simple reflection in $s_m...s_k$.
Let $j \in [1, n]$ be such that $u_i^{l_1}.\hat{u}_i = \hat{u}_i.u_j^{l_1}$. We have $y = z'_1.\hat{u}_i.u_j^{l_2}.u_i^{l_1}.z'_2$ which implies that $u_j^{l_2}.u_i^{l_1}$. Furthermore, we have

$$u_j^{l_2}.u_i^{l_1} A_0 = A_0 t_{l_2 \vec{u}_j} t_{l_1 \vec{u}_i} = A_0 t_{l_1 \vec{u}_i} t_{l_2 \vec{u}_j} = u_i^{l_1}.u_j^{l_2} A_0.$$

Applying Corollary 4.2.9, we get $i = j$. Thus

$$y = z'_1.u_i^{l_1}.\hat{u}_i.u_i^{l_2}.z'_2 = z'_1.\hat{u}_i.u_i^{r-N}.z'_2.$$

Let

$$z_1 = z'_1.\hat{u}_i \quad n_1 = n'_1 + 1$$
$$z_2 = z'_2 \qquad n_2 = n'_2.$$

Then one can check that $z_1 = z'_1.\hat{u}_i \leq z.u_i^{n_1}$ and $z_2 \leq u_i^{n_2}$. Thus we get the result by induction.

Let $m \in [1, n]$ be such that $y = z_1.u_i^{r-N}.z_2 = z_1.z_2.u_m^{r-N}$. Let $z_y = z_1.z_2$. Assume that there exists $w \in W$ and $k \in [1, n]$ such that $y = w.u_k^{r-N}$. By Corollary 4.2.9, we have $k = m$ and $w = z_y$, which concludes the proof. $\square$

Lemma 4.3.3. *Let $z, z' \in W$ and $i, j \in [1, n]$ be such that $z.u_i$ and $z'.u_j$. Let $r_1, r_2 \in \mathbb{N}^*$ be such that $r_2 \geq \ell(z'.u_j^{r_2}) - \ell(z.u_i^{r_1})$. Then for all $k \geq 0$ we have*

$$z.u_i^{r_1} \leq z'.u_j^{r_2} \Leftrightarrow z.u_i^{r_1+k} \leq z'.u_j^{r_2+k}.$$

Proof. Let $N = \ell(z'.u_j^{r_2}) - \ell(z.u_i^{r_1})$. Applying the previous lemma and Corollary 4.2.9 yields the following equivalences, for any $k \geq 0$

$$\begin{aligned}
& z.u_i^{r_1} \leq z'.u_j^{r_2} \\
& \Leftrightarrow \exists z_1, z_2 \in W, n_1, n_2 \in \mathbb{N} \text{ such that } z_1.u_j^{r_2-N}.z_2 = z.u_i^{r_1} \\
& \quad z_1 \leq z'.u_j^{n_1}, z_2 \leq u_j^{n_2} \text{ and } n_1 + n_2 = N \\
& \Leftrightarrow \exists z_1, z_2 \in W, n_1, n_2 \in \mathbb{N} \text{ such that } z_1.u_j^{r_2-N+k}.z_2 = z.u_i^{r_1+k}, \\
& \quad z_1 \leq z'.u_j^{n_1}, z_2 \leq u_j^{n_2} \text{ and } n_1 + n_2 = N \\
& \Leftrightarrow z u_i^{r_1+k} \leq z' u_j^{r_2+k}.
\end{aligned}$$

□

PROPOSITION 4.3.4. *Let $z, z' \in W$ and $i, j \in [1, n]$ be such that $z.u_i$ and $z'.u_j$. Let $r \in \mathbb{N}^*$ be such that $r > \ell(z'.u_j^r) - \ell(z.u_i^r)$. Then for all $k \geq 0$, the Bruhat interval*

$$I_1 = [z.u_i^r, z'.u_j^r] = \{y \in W \mid z.u_i^r \leq y \leq z'.u_j^r\}$$

is isomorphic to $I_2 = [z.u_i^{r+k}, z'.u_j^{r+k}]$.

PROOF. Let $y \in I_1$ and $N_y = \ell(z'.u_j^r) - \ell(y)$. There exists a unique $z_y \in W$ and $m \in \mathbb{N}$ such that $y = z_y.u_m^{r-N_y}$.
Let

$$\begin{array}{cccc} \varphi: & I_1 & \longrightarrow & I_2 \\ & z_y.u_m^{r-N_y} & \longmapsto & z_y.u_m^{r+k-N_y}. \end{array}$$

We need to show that φ is an isomorphism of Bruhat intervals.
Let $y' \leq y \in I_1$. Let $N_{y'} = \ell(z'.u_j^r) - \ell(y')$. There exists a unique $z_{y'} \in W$ and $m' \in \mathbb{N}$ such that $y = z_{y'}.u_{m'}^{r-N_{y'}}$. One can check that we can apply Lemma 4.3.3, we obtain

$$\begin{aligned} & z.u_i^r \leq y = z_y.u_m^{r-N_y} \leq y' = z_{y'}.u_{m'}^{r-N_y'} \leq z'.u_j^r \\ \Longleftrightarrow \quad & z.u_i^{r+k} \leq \varphi(y) = z_y.u_m^{r-N_y+k} \leq \varphi(y') = z_{y'}.u_{m'}^{r-N_{y'}+k} \leq z'.u_j^{r+k}. \end{aligned}$$

By Corollary 4.2.9 we see that φ is injective. One can easily check that φ is surjective. The result follows. □

The next step is to show that the corresponding $\mathbf{R}$-polynomials are equal. Let Γ be an abelian group together with a total order specified by Γ_+. Let $\{v_s \mid s \in S\} \subset \Gamma_+$ be the set of parameters and $\xi_s = v_s - v_s^{-1}$.

Let $y, w \in W$ and $s \in S$ be such that $sw < w$. Recall that the $\mathbf{R}$-polynomials satisfy $\mathbf{R}_{y,w} = 0$ unless $y \leq w$, $\mathbf{R}_{y,y} = 1$ and the recursive relation

$$\mathbf{R}_{y,w} = \begin{cases} \mathbf{R}_{sy,sw}, & \text{if } sy < y, \\ \mathbf{R}_{sy,sw} + (v_s - v_s^{-1})\mathbf{R}_{y,sw}, & \text{if } sy > y\ . \end{cases}$$

PROPOSITION 4.3.5. *Let $z, z' \in W$ and $i, j \in [1, n]$ be such that $z.u_i$ and $z'.u_j$. Let $r \in \mathbb{N}^*$ be such that $r > (\ell(z') - \ell(z))M$. Then for all $k \geq 0$ we have*

$$\mathbf{R}_{z.u_i^r, z'.u_j^r} = \mathbf{R}_{z.u_i^{r+k}, z'.u_j^{r+k}}.$$

PROOF. Let $N = \ell(z') - \ell(z)$. We proceed by induction on N.
If $i = j$ or if $N < 0$ then the result is obvious.

If $N = 0$, since

$$\mathbf{R}_{z.u_i^r, z'.u_j^r} = \delta_{z.u_i^r, z'.u_j^r} \text{ and } \mathbf{R}_{z.u_i^{r+k}, z.u_j^{r+k}} = \delta_{z.u_i^{r+k}, z.u_j^{r+k}}$$

the result follows from Corollary 4.2.9.

Let $N \geq 1$ and $i \neq j$. Note that in this case $r > M$.

Let $u_j = s_M ... s_1$ be a reduced expression. There exists $1 \leq l \leq M$ such that $(z.u_i^r s_1 ... s_{l-1}) s_l > z.u_i^r s_1 ... s_{l-1}$. Indeed, if not, then $z.u_i^r = y.u_j$ for some $y \in W$. By Corollary 4.2.9, this implies that $i = j$, but we assumed that $i \neq j$. Let $l \in [1, m]$ be the smallest element with this property. The minimality of l implies that $\ell(z.u_i^r s_1 ... s_{l-1}) = \ell(z.u_i^r) - (l-1)$.

One can see that $zu_i^r s_1 ... s_{l-1} \leq z.u_i^r$. Let $y, w \in W$ and $m, q, p \in [1, n]$ be such that

$$\begin{aligned} zu_i^r s_1 ... s_{l-1} &= y.u_m^{r-l+1} \\ y.u_m^{r-l+1}.s_l &= y.s_l.u_p^{r-l+1} \\ z' u_j^r s_1 ... s_l &= z'.u_j^{r-1}.s_M ... s_{l+1} = w.u_q^{r-1} \end{aligned}$$

By Corollary 4.2.9 we see that

$$\begin{aligned} zu_i^{r+k} s_1 ... s_{l-1} &= y.u_m^{r+k-l+1} \\ y.u_m^{r+k-l+1}.s_l &= y.s_l.u_p^{r+k-l+1} \\ z' u_j^{r+k} s_1 ... s_l &= z'.u_j^{r+k-1}.s_M ... s_{l+1} = w.u_q^{r+k-1}. \end{aligned}$$

Applying the recursive formula for the $\mathbf{R}$-polynomials, we obtain

$$\begin{aligned} \mathbf{R}_{z.u_i^r, z.u_j^r} &= \mathbf{R}_{zu_i^r s_1 ... s_{l-1}, z_2 u_j^r s_1 ... s_{l-1}} \\ &= \mathbf{R}_{y.u_m^{r-l+1}, z_2 u_j^r s_1 ... s_{l-1}} \\ &= \mathbf{R}_{ys_l.u_p^{r-l+1}, w.u_q^{r-1}} + \xi_{s_l} \mathbf{R}_{y.u_m^{r-l+1}, w.u_q^{r-1}} \end{aligned}$$

and

$$\begin{aligned} \mathbf{R}_{z.u_i^{r+k}, z.u_j^{r+k}} &= \mathbf{R}_{zu_i^{r+k} s_1 ... s_{l-1}, z_2 u_j^{r+k} s_1 ... s_{l-1}} \\ &= \mathbf{R}_{y.u_m^{r-l+1+k}, z_2 u_j^{r+k} s_1 ... s_{l-1}} \\ &= \mathbf{R}_{ys_l.u_p^{r-l+1+k}, w.u_q^{r-1+k}} + \xi_{s_l} \mathbf{R}_{y.u_m^{r-l+1+k}, w.u_q^{r-1+k}} \quad . \end{aligned}$$

Therefore to prove the theorem it is enough to show that

$$\mathbf{R}_{ys_l.u_p^{r-l+1},w.u_q^{r-1}} = \mathbf{R}_{ys_l.u_p^{r-l+1+k},w.u_q^{r-1+k}},$$
$$\mathbf{R}_{y.u_m^{r-l+1},w.u_q^{r-1}} = \mathbf{R}_{y.u_m^{r-l+1+k},w.u_q^{r-1+k}}.$$

If $l = 1$ we have

$$\mathbf{R}_{ys_l.u_p^r,w.u_q^{r-1}} = \mathbf{R}_{(ys_l.u_p).u_p^{r-1},w.u_q^{r-1}},$$
$$\mathbf{R}_{y.u_m^r,w.u_q^{r-1}} = \mathbf{R}_{(y.u_m).u_m^{r-1},w.u_q^{r-1}}$$

and

$$\ell(w) - \ell(ys_lu_p) = N-2,$$
$$\ell(w) - \ell(yu_m) = N-1.$$

Moreover $r-1 > 0$ (we have seen that $r > M \geq 2$) and

$$r-1 > MN-1 \geq MN-N = M(N-1) > M(N-2).$$

Therefore in both cases we can apply the induction hypothesis which yields the desired equalities.
If $l > 1$, we have

$$\mathbf{R}_{ys_lu_p^{r-l+1},w.u_q^{r-1}} = \mathbf{R}_{ys_lu_p^{r-l+1},w.u_q^{l-2}.u_q^{r-l+1}},$$
$$\mathbf{R}_{y.u_m^{r-l+1},w.u_q^{r-1}} = \mathbf{R}_{y.u_m^{r-l+1},w.u_q^{l-2}u_q^{r-l+1}}$$

and

$$\ell(w.u_q^{l-2}) - \ell(ys_l) = N-2,$$
$$\ell(w.u_q^{l-2}) - \ell(y) = N-1.$$

Moreover $r-l+1 > 0$ and

$$r-l+1 > r-M > M(N-1) > M(N-2)$$

and once more the induction hypothesis gives the desired equalities. $\square$

We are now ready to prove Theorem 4.3.1.

PROOF. The intervals $I_1 = [z.u_i^r, z'.u_j^r]$, $I_2 = [z.u^{r+k}, z'.u^{r+k}]$ are isomorphic with respect to the Bruhat order via φ (as defined in Proposition 4.3.4).
Let $y_1, y_2 \in I$ $(\ell(y_1) \leq \ell(y_2))$, $z_1, z_2 \in W$, $N_1, N_2 \in \mathbb{N}$ and $m_1, m_2 \in [1, n]$ be such that

$$N_1 = \ell(z'.u^{r+k}) - \ell(y_1) \quad \text{and} \quad y_1 = z_1u_{m_1}^{r-N_1},$$

$$N_2 = \ell(z'.u^{r+k}) - \ell(y_2) \quad \text{and} \quad y_2 = z_2 u_{m_2}^{r-N_2}.$$

We have

$$r - N_1 \geq r - N > N(M+1) - N = MN \geq M(\ell(y_2) - \ell(y_1)).$$

Thus, by Proposition 4.3.5, we obtain

$$\begin{aligned}
\mathbf{R}_{y_1,y_2} &= \mathbf{R}_{z_1 u_{m_1}^{r-N_1}, z_2 u_{m_2}^{r-N_2}} \\
&= \mathbf{R}_{z_1 u_{m_1}^{r-N_1}, z_2 u_{m_2}^{N_1-N_2} u_{m_2}^{r-N_1}} \\
&= \mathbf{R}_{z_1 u_{m_1}^{r+k-N_1}, z_2 u_{m_2}^{N_1-N_2} u_{m_2}^{r+k-N_1}} \\
&= \mathbf{R}_{\varphi(y_1),\varphi(y_2)}.
\end{aligned}$$

Therefore, by Remark 2.4.6, we get the result. □

4.4. Application to $\tilde{G}_2$

The aim of this section is to use the invariance of the Kazhdan-Lusztig polynomials by translation and the methods presented in Section 4.1.2 to prove the following result.

THEOREM 4.4.1. *Let W be an affine Weyl group of type $\tilde{G}_2$. We have*

(1) *there are only finitely many possible decompositions of W into left cells;*
(2) *the number of left cells is finite in each case.*

The proof of this theorem involves some explicit computations. We have developed some program in GAP3 which given an interval I, $s \in S$ and a monomial order on Γ, compute the following data

(1) The Kazhdan-Lusztig polynomials $\mathbf{P}_{y,w}$ for all $y, w \in I$,
(2) $\mathbf{M}^s_{y,w}$ for all $y, w \in I$ such that $sy < y < w < sw$,
(3) $\sum\limits_{z; z_1 \leq z < z_2; sz < z} \mathbf{P}_{z_1,z}\mathbf{M}^s_{z,z_2} - v_s \mathbf{P}_{z_1,z_2}$ for all $z_1, z_2 \in I$,

so that we can compute the set $\Gamma^s_+(I)$ as described in Proposition 4.1.6.

To prove the theorem we proceed as follows. Using Proposition 4.1.6, Theorem 4.3.1 and our GAP3 program, we will find a collection of non-zero M-polynomials. We will then find a finite number of infinite sets such that each of these sets is included in a left cell for any choice of parameters and such that all the elements of W lie in one of these sets except for a finite number.

Throughout this section, let W be an affine Weyl group of type $\tilde{G}_2$ together with a positive weight function, with presentation as follows

$$W := \langle s_1, s_2, s_3 \mid (s_1s_2)^6 = 1, (s_2s_3)^3 = 1, (s_1s_3)^2 = 1\rangle.$$

A weight function L on W is uniquely determined by

$$L(s_1) = a \text{ and } L(s_2) = L(s_3) = b \quad a, b \in \mathbb{N}^*$$

We shall denote such a weight function by $L = L_{a,b}$.

4.4.1. Computations. In this section, we study an example in detail to show how one can prove that a M-polynomial is non zero for a whole class of weight functions.

Let Q, q be independent indeterminates over $\mathbb{Z}$ and consider the abelian group

$$\Gamma = \{Q^i q^j \mid i, j \in \mathbb{Z}\}.$$

Let v be another indeterminate. For all $a, b \in \mathbb{N}^*$ we have a ring homomorphism

$$\sigma_{a,b} : \mathbb{Z}[\Gamma] \to \mathbb{Z}[v, v^{-1}], \qquad Q^i q^j \to v^{ai+bj}.$$

We will need the following lemma.

LEMMA 4.4.2. *Let $y < w \in W$, $I = [y, w]$ and $s \in S$ be such that*

$$sy < y < w < sw.$$

(1) *Consider the total order given by*

$$\Gamma^1_+ = \{Q^i q^j \mid i > 0, j \in \mathbb{Z}\} \cup \{q^i \mid i > 0\}.$$

Suppose that, for $c, d \in \mathbb{N}^*$, we have

$$\Gamma^s_+(I) \subseteq \{q^j \mid j > 0\} \cup \{Q^i q^j \mid i > 0, ci + dj \geq 0\}.$$

Then condition (∗) in Proposition 4.1.6 holds for any $\sigma_{a,b}$ such that $a/b > c/d$.
Furthermore, if $\mathbf{M}^s_{y,w} \neq 0$, then for any weight function $L_{a,b}$ such that $a/b > c/d$, we have $M^s_{y,w} \neq 0$.

(2) *Let $c \geq d \in \mathbb{N}^*$. Consider the total order given by*

$$\Gamma^2_+ = \{Q^i q^j \mid ci + dj > 0\} \cup \{Q^{dj} q^{-cj} \mid j > 0\}.$$

Suppose that we have, for some $e > c/d \in \mathbb{Q}_{>0}$

$$\Gamma^s_+(I) \subseteq \{q^j \mid j > 0\} \cup \{Q^i q^j \mid i > 0, i + j \geq 0\}$$

$$\cup\{Q^i q^j \mid j > -i > 0, -j/i \geq e\} \cup \{Q^i q^j \mid -j > i > 0, -j/i \leq c/d\}.$$

Then condition $(*)$ in Proposition 4.1.6 holds for any $\sigma_{a,b}$ such that $e > a/b > c/d$.
Furthermore, if $\mathbf{M}^s_{y,w} \neq 0$, then for any weight function $L_{a,b}$ such that $e > a/b > c/d$, we have $M^s_{y,w} \neq 0$.

Proof. We prove 1. Let $i, j \in \mathbb{Z}$ be such that $Q^i q^j \in \Gamma^s_+(I)$. We must show that $ai + bj > 0$ provided that $a/b > c/d$.
If $i = 0$ then $j > 0$ and $ai + bj = bj > 0$.
If $i > 0$ and $ci + dj \geq 0$ then

$$ai + bj = b(ia/b + j) > b(ic/d + j) \geq 0$$

as required.

We prove 2. Let $i, j \in \mathbb{Z}$ be such that $Q^i q^j \in \Gamma^s_+(I)$. We must show that $ai + bj > 0$ provided that $e > a/b > c/d$.
If $i = 0$ then $j > 0$ and $ai + bj = bj > 0$.
If $i > 0$ and $i + j \geq 0$ then

$$ai + bj > b(c/d)i + bj = b(ic/d + j) > b(i + j) \geq 0.$$

If $j > -i > 0$ and $-j/i \geq e$ then

$$ai + bj = bj((a/b)(i/j) + 1) > bj(ei/j + 1) \geq 0.$$

Finally, if $-j > i > 0$ and $-j/i \leq c/d$ then

$$ai + bj = ai(1 + (b/a)j/i) > ai(1 + (d/c)(i/j)) \geq 0$$

as required.

□

Note that, in the situation of the above lemma, we will always have $a > b$. But similar results also hold for $b > a$.

We now study an example in detail. Let $u = s_1 s_2 s_1 s_2 s_3 s_1 s_2 s_1 s_2 s_3 \in W$. Let $x = s_3 u^6$ and $y = s_3 s_2 s_1 s_2 s_3 s_1 s_2 s_1 s_2 s_3 u^6$. We want to show that $M^{s_1}_{x,y} \neq 0$ for all parameters a, b such that $a \geq b$.

CLAIM 4.4.3. For any parameters a, b such that $a/b > 3$ we have $M^{s_1}_{x,y} \neq 0$.

PROOF. Consider the total order given by

$$\Gamma_+ = \{Q^i q^j \mid i > 0, j \in \mathbb{Z}\} \cup \{q^i \mid i > 0\}.$$

Using our GAP3 program to compute the set $\Gamma^{s_1}_+(I)$, we find $\mathbf{M}^{s_1}_{x,y} \neq 0$ and

$$\Gamma^s_+(I) \subseteq \{q^j \mid j > 0\} \cup \{Q^i q^j \mid 3i + j \geq 0\}.$$

Therefore, applying Lemma 4.4.2, we see that, for any parameters a, b such that $a/b > 3$, we have

$$M^{s_1}_{x,y} = \sigma_{a,b}(\mathbf{M}^{s_1}_{x,y}) \quad \text{and} \quad \mathbf{M}^{s_1}_{x,y} \neq 0 \Longrightarrow M^{s_1}_{x,y} \neq 0.$$

□

In order to deal with weight functions $L_{a,b}$ such that $a/b < 3$, we proceed as follows. Let

$$\mathcal{E} = \{x \in \mathbb{Q}_{>0} \mid x = \pm j/i \text{ where } j < 0,\ i \neq 0, Q^i q^j \in \Gamma^s_+(I)\}.$$

The largest element of $\mathcal{E}$ below 3 is 2. This leads us to consider the total order given by

$$\Gamma_+ = \{Q^i q^j \mid 2i + j > 0\} \cup \{Q^j q^{-2j} \mid j > 0\}.$$

CLAIM 4.4.4. For any parameters a, b such that $3 > a/b > 2$ we have $M^{s_1}_{x,y} \neq 0$.

PROOF. Consider the total order given by

$$\Gamma_+ = \{Q^i q^j \mid 2i + j > 0\} \cup \{Q^j q^{-2j} \mid j > 0\}.$$

Computing $\Gamma^s_+(I)$ gives $\mathbf{M}^{s_1}_{x,y} \neq 0$ and

$$\Gamma^s_+(I) \subseteq \{q^j \mid j > 0\} \cup \{Q^i q^j \mid i > 0, i + j \geq 0\}$$

$$\cup \{Q^i q^j \mid j > -i > 0, -j/i \geq 3\} \cup \{Q^i q^j \mid -j > i > 0, -j/i \leq 2\}.$$

Therefore, for any parameters a, b such that $3 > a/b > 2$, we have

$$M^{s_1}_{x,y} = \sigma_{a,b}(\mathbf{M}^{s_1}_{x,y}) \quad \text{and} \quad \mathbf{M}^{s_1}_{x,y} \neq 0 \Longrightarrow M^{s_1}_{x,y} \neq 0.$$

□

Again we look at the set

$$\mathcal{E} = \{x \in \mathbb{Q}_{>0} \mid x = \pm j/i \text{ where } j < 0,\ i \neq 0, Q^i q^j \in \Gamma^s_+(I)\}.$$

The largest element of $\mathcal{E}$ below 2 is 3/2.

CLAIM 4.4.5. For any parameters a, b such that $2 > a/b > 3/2$ we have $M_{x,y}^{s_1} \neq 0$.

PROOF. Consider the total order given by

$$\Gamma_+ = \{Q^i q^j \mid 3i + 2j > 0\} \cup \{Q^{2j} q^{-3j} \mid j > 0\}.$$

We find $\mathbf{M}_{x,y}^{s_1} \neq 0$ and

$$\Gamma_+^s(I) \subseteq \{q^j \mid j > 0\} \cup \{Q^i q^j \mid i > 0, i + j \geq 0\}$$
$$\cup \{Q^i q^j \mid j > -i > 0, -j/i \geq 2\} \cup \{Q^i q^j \mid -j > i > 0, -j/i \leq 3/2\}.$$

The result follows using Lemma 4.4.2. □

We look at the set $\mathcal{E}$ (defined as above), we find that the largest element of $\mathcal{E}$ below $3/2$ is $4/3$.

CLAIM 4.4.6. For any parameters a, b such that $3/2 > a/b > 4/3$ we have $M_{x,y}^{s_1} \neq 0$.

PROOF. Consider the total order given by

$$\Gamma_+ = \{Q^i q^j \mid 4i + 3j > 0\} \cup \{Q^{3j} q^{-4j} \mid j > 0\}.$$

We find $\mathbf{M}_{x,y}^{s_1} \neq 0$ and

$$\Gamma_+^s(I) \subseteq \{q^j \mid j > 0\} \cup \{Q^i q^j \mid i > 0, i + j \geq 0\}$$
$$\cup \{Q^i q^j \mid j > -i > 0, -j/i \geq 3/2\} \cup \{Q^i q^j \mid -j > i > 0, -j/i \leq 4/3\}.$$

The result follows using Lemma 4.4.2. □

We now continue the procedure.

CLAIM 4.4.7. For any parameters a, b such that $4/3 > a/b > 5/4$ we have $M_{x,y}^{s_1} \neq 0$.

PROOF. Consider the total order given by

$$\Gamma_+ = \{Q^i q^j \mid 5i + 4j > 0\} \cup \{Q^{4j} q^{-5j} \mid j > 0\}.$$

We find $\mathbf{M}_{x,y}^{s_1} \neq 0$ and

$$\Gamma_+^s(I) \subseteq \{q^j \mid j > 0\} \cup \{Q^i q^j \mid i > 0, i + j \geq 0\}$$
$$\cup \{Q^i q^j \mid j > -i > 0, -j/i \geq 4/3\} \cup \{Q^i q^j \mid -j > i > 0, -j/i \leq 5/4\}$$

The result follows using Lemma 4.4.2. □

CLAIM 4.4.8. For any parameters a, b such that $5/4 > a/b > 1$ we have $M_{x,y}^{s_1} \neq 0$.

Proof. Consider the total order given by

$$\Gamma_+ = \{Q^i q^j \mid i+j>0\} \cup \{Q^j q^{-j} \mid j>0\}.$$

We find $\mathbf{M}^{s_1}_{x,y} \neq 0$ and

$$\begin{aligned}\Gamma^s_+(I) \subseteq \{q^j \mid j>0\} \cup \{Q^i q^j \mid i>0, i+j \geq 0\}\\ \cup\{Q^i q^j \mid j > -i > 0, -j/i \geq 5/4\}\end{aligned}$$

The result follows using Lemma 4.4.2. □

Finally we compute $M^{s_1}_{x,y}$ for the parameters a, b where $a/b \in \{3, 2, 3/2, 4/3, 5/4, 1\}$, and we find that these are non-zero. Thus $M^{s_1}_{x,y}$ is non zero for all parameters such that $a \geq b$.

4.4.2. Proof of Theorem 4.4.1. Using the methods of the previous section, we find a collection of non-zero M-polynomials and some infinite sets which are included in a left cell for any parameters.

Let $u = s_1s_2s_1s_2s_3s_1s_2s_1s_2s_3 \in W$. One can check that u is a translation. Let

$$\begin{aligned}\Pi = \{&e,\ s_3,\ s_2s_3,\ s_1s_2s_3,\ s_2s_1s_2s_3,\\ &s_3s_2s_1s_2s_3,\ s_1s_2s_1s_2s_3,\ s_3s_1s_2s_1s_2s_3,\ s_2s_3s_1s_2s_1s_2s_3,\\ &s_1s_2s_3s_1s_2s_1s_2s_3,\ s_2s_1s_2s_3s_1s_2s_1s_2s_3,\ s_3s_2s_1s_2s_3s_1s_2s_1s_2s_3\}\\ W_1 = \{&e, s_1, s_1s_2, s_1s_2s_1, s_1s_2s_1s_2, s_1s_2s_1s_2s_1\}\\ = \{&w_1, w_2, w_3, w_4, w_5, w_6\}\end{aligned}$$

and $y = s_3s_2s_1s_2s_3s_1s_2s_1s_2s_3$.
For $w_i \in W_1$ let $\sigma_{w_i} \in \Omega$ be such that $w_iA_0 = A_0\sigma_{w_i}$. One can check that

$$Orb_\Omega(\vec{u}) = \{\vec{u}\sigma_{w_1}, ..., \vec{u}\sigma_{w_6}\} = \{\vec{u}_1, ..., \vec{u}_6\}.$$

For $1 \leq i \leq 6$, let

$$x^r_{1,i} = s_3.u^r.w_i \quad \text{and} \quad y^r_{1,i} = y.u^r.w_i$$

and

$$x^r_{2,i} = s_3s_1s_2s_1s_2s_3u^r.w_i \quad \text{and} \quad y^r_{2,i} = y.s_1s_2s_1s_2s_3.w_i.u^r_i.$$

Let $k = 1, 2$. We know that for r large enough we have

$$M^{s_1}_{x^r_{k,i}, y^r_{k,i}} = M^{s_1}_{x^{r+n}_{k,i}, y^{r+n}_{k,i}} \text{ for all } n \geq 0.$$

In fact using our GAP3 program one can show that this is true for all $r \geq 6$. Doing as in the previous section, one can show that

$$M^{s_1}_{x^r_{1,i},y^r_{1,i}} \quad \text{and} \quad M^{s_1}_{x^r_{2,i},y^r_{2,i}}$$

are non-zero for all $r \geq 6$ and for all parameters. This implies that the following sets are included in a left cell:

$$C_i = \{z.u_1^r.w_i \mid r \geq 7, z \in \Pi\},\ 1 \leq i \leq 6.$$

We show the shape of the sets C_i on Figure 2.

Now, let $u = s_2s_1s_2s_1s_2s_3 \in W$. One can check that u is a translation. Let

$$W_2 = \{e, s_2, s_2s_1, s_2s_1s_2, s_2s_1s_2s_1, s_2s_1s_2s_1s_2\} = \{w_1, w_2, w_3, w_4, w_5, w_6\}.$$

For $w_i \in W_2$ let $\sigma_{w_i} \in \Omega$ be such that $w_iA_0 = A_0\sigma_{w_i}$. One can check that

$$Orb_\Omega(\vec{u}) = \{\vec{u}\sigma_{w_1}, ..., \vec{u}\sigma_{w_6}\} = \{\vec{u}_1, ..., \vec{u}_6\}.$$

For $1 \leq i \leq 6$, let

$$x^r_{1,i} = s_2s_3.u^r.w_i \quad \text{and} \quad y^r_{1,i} = y.u_r.w_i$$

and

$$x^r_{2,i} = s_3s_2s_1s_2s_3u^r.w_i \quad \text{and} \quad y^r_{2,i} = s_3s_2s_1s_2s_1s_2s_3u^{r+1}.w_i.$$

We know that for r large enough we have

$$M^{s_1}_{x^r_{k,i},y^r_{k,i}} = M^{s_1}_{x^{r+n}_{k,i},y^{r+n}_{k,i}} \text{ for all } n \geq 0.$$

In fact using our GAP3 program one can show that this is true for all $r \geq 6$. Doing as in the previous section, one can show that

$$M^{s_2}_{x^r_{1,i},y^r_{1,i}} \quad \text{and} \quad M^{s_2}_{x^r_{2,i},y^r_{2,i}}$$

are non-zero for all $r \geq 6$ and for all parameters a, b such that $a/b \leq 2$. Therefore, for these parameters, the following sets are included in a left cell:

$$B_i = \{z.u^r.w_i \mid r \geq 7, z \in \Pi\},\ 1 \leq i \leq 6.$$

We show the shape of the sets B_i on Figure 2.

Let $a, b \in \mathbb{N}$ be such that $a/b > 2$. Arguing as before, we find that, for $r \geq 6$, the polynomials

$$M^{s_1}_{s_1s_2s_3u^rw_i,s_2s_3s_2s_1s_2s_1s_2s_3u^rw_i},\quad M^{s_2}_{s_2s_3u^rw_i,s_3u^{r+1}w_i}$$
$$M^{s_1}_{s_1s_2s_3u^rw_i,s_3s_2s_1s_2s_3u^rw_i}$$

are non-zero. Therefore, for these parameters, the following sets are included in a left cell:

$$B_i = \{z.u^r.w_i \mid r \geq 7, z \in \Pi\},\ 1 \leq i \leq 6$$

Let $w_0 = s_1s_2s_1s_2s_1s_2$. The set

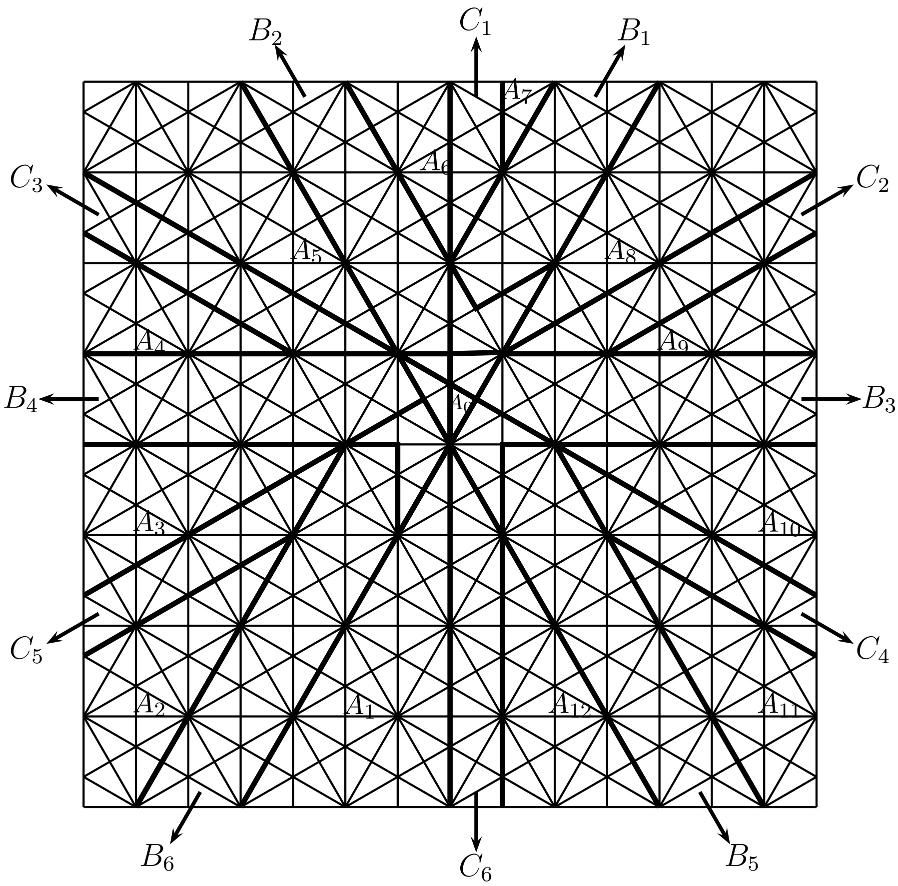
$$W_T = \{w \in W \mid w = z'.w_0.z, z' \in W\}$$

is known to contain finitely many left cells; see [**9, 46**] and Chapter 6. Now one can check that the set W_T together with the B_i's and the C_i's contain all the elements of W except for a finite number. The theorem follows.

Computing some more coefficients in the case where $a >> b$, we find a more precise decomposition of W which is included in the left cell decomposition. We show this decomposition on Figure 2. We identify $w \in W$ with the alcove wA_0. The sets which are included in a left cell are formed by the alcoves lying in the same connected component after removing the thick line. In fact, we will prove that this the actual decomposition of W into left cells; see Chapter 7.
We have

$$W_T = \bigcup_{i=1}^{12} A_i.$$

The figure also show the shape of the sets B_i and C_i.



B_2
C_1
B_1
A_7
A_6
C_3
C_2
A_5
A_8
A_4
A_9
B_4
A_0
B_3
A_3
A_{10}
C_5
C_4
A_2
A_1
A_{12}
A_{11}
B_6
C_6
B_5

FIGURE 2.

CHAPTER 5

Generalized induction of Kazhdan-Lusztig cells

In [**15**], Geck showed that the Kazhdan-Lusztig cells are compatible with parabolic subgroups. In a more precise sense, any left cell of a parabolic subgroup can be "induced" to obtain a union of left cells of the whole group. The main observation of this chapter is that the methods of [**15**] work in somewhat more general settings, so that we can "induce" from subsets which may not be parabolic subgroups. This leads to our "Generalized Induction Theorem".

In the final section, using this theorem, we show that, under specific technical conditions on the parameters, the cells of a certain finite parabolic subgroup are cells in the whole group.

In this section W denotes an arbitrary Coxeter group together with generating set S. Let L be a positive weight function on W and $\mathcal{H}$ be the Iwahori-Hecke algebra associated to W, L.

5.1. Main result

Consider a subset $U \subset W$ and a collection $\{X_u \mid u \in U\}$ of subsets of W satisfying the following conditions

I1. for all $u \in U$, we have $e \in X_u$,
I2. for all $u \in U$ and $x \in X_u$ we have $x.u$,
I3. for all $u, v \in U$ such that $u \neq v$ we have $X_u u \cap X_v v = \emptyset$,
I4. the submodule $\mathcal{M} := \langle T_x C_u |\ u \in U,\ x \in X_u \rangle_{\mathcal{A}}$ of $\mathcal{H}$ is a left ideal,
I5. for all $u \in U$, $x \in X_u$ and $u_1 < u$ we have

$$P_{u_1,u} T_x T_{u_1} \text{ is an } \mathcal{A}_{<0}\text{-linear combination of } T_z.$$

Let $u \in U$ and $x \in X_u$. We have

$$T_x C_u = T_{xu} + \text{a } \mathcal{A}\text{-linear combination of } T_z \text{ with } \ell(z) < \ell(xu).$$

Since the set $\{T_w|w \in W\}$ is a basis of $\mathcal{H}$, using **I3**, one can see that $\mathcal{B} = \{T_xC_u|u \in U, x \in X_u\}$ is a basis of $\mathcal{M}$.

Let $u \in U$ and $z \in W$. Using **I1** and **I4** and the fact that $\mathcal{B}$ is a basis of $\mathcal{M}$, we can write

$$T_zC_u = \sum_{u\in U, x\in X_u} a_{x,u}T_xC_u \quad \text{for some } a_{x,u} \in \mathcal{A}.$$

Let $\preceq$ be the relation on U defined as follows. Let $u, v \in U$. We write $v \preceq u$ if there exist $x \in W$ and $z \in X_v$ such that T_zC_v appears with a non-zero coefficient in the expression of T_xC_u in the basis $\mathcal{B}$. We still denote by $\preceq$ the pre-order induced by this relation (i.e the transitive closure). Since $C_u \in \mathcal{M}$, we have

$$\mathcal{H}C_u = \sum_{v\preceq u, z\in X_v} \mathcal{A}T_zC_v.$$

REMARK 5.1.1. If we choose $U = W$ and $X_w = \{e\}$ for all $w \in W$, the pre-order $\preceq$ is the left pre-order $\leq_L$ on W.

We are now ready to state the main result of this section.

THEOREM 5.1.2. *Let U be a subset of W and $\{X_u|u \in U\}$ be a collection of subsets of W satisfying conditions* **I1–I5**. *Let $\mathcal{U}$ be a subset of U such that the following holds:*

$$v \preceq u \in \mathcal{U} \Longrightarrow v \in \mathcal{U}.$$

Then, the set

$$\{x.u|u \in \mathcal{U},\ x \in X_u\}$$

is a left ideal of W.

The proof of this theorem will be given in the next section. First we discuss some consequences of this theorem.

COROLLARY 5.1.3. *Let $\mathcal{C}$ be an equivalence class on U with respect to $\preceq$. Then the subset $\{x.u|u \in \mathcal{C}, x \in X_u\}$ of W is a union of left cells.*

PROOF. Let $v \in \mathcal{C}$, $y \in X_v$ and $z \in W$ be such that $z \sim_L y.v$. Consider the set $\mathcal{U} = \{u \in U|u \preceq v\}$. Then $\mathcal{U}$ satisfies the requirement of Theorem 5.1.2, thus $\mathcal{M} := \{x.u|u \in \mathcal{U},\ x \in X_u\}$ is a left ideal of W. Since $z \leq_L y.v$ and $y.v \in \mathcal{M}$, there exist $u_z \in \mathcal{U}$ and $x \in X_{u_z}$ such that $z = x.u_z$ and $u_z \preceq v$.
We also have $y.v \leq_L x.u_z$. Applying the same argument as above to the set $\{u \in$

$U | u \preceq u_z\}$ yields that there exists $u_y \in \mathcal{U}$ and $w \in X_{u_y}$ such that $y = w.u_y$ and $u_y \preceq u_z$. By condition **I3**, we see that $u_y = v$. Thus $u_z \in \mathcal{C}$ and the result follows. $\square$

REMARK 5.1.4. In [**15**], Geck proved the following theorem, where (W, S) is an arbitrary Coxeter system.

THEOREM 5.1.5. *Let $W' \subset W$ be a parabolic subgroup of W and let X' be the set of all $w \in W$ such that w has minimal length in the coset wW'. Let $\mathcal{C}$ be a left cell of W'. Then $X'\mathcal{C}$ is a union of left cells of W.*

Let $U = W'$ and for all $w \in W'$ let $X_w = X'$. We claim that this theorem is a special case of Theorem 5.1.2 and Corollary 5.1.3. Indeed, conditions **I1**–**I3** and **I5** are clearly satisfied. Condition **I4** is a straightforward consequence of Deodhar's lemma; see [**15**, Lemma 2.2]. Hence, it is sufficient to show that the pre-order $\preceq$ on $U = W'$ coincides with the Kazhdan-Lusztig left pre-order defined with respect to W' (denoted $\leq'_L$) and the corresponding parabolic subalgebra $\mathcal{H}_{W'} := \langle T_w \mid w \in W' \rangle \subset \mathcal{H}$. In other words, we need to show the following

$$u \leq'_L v \quad \Longleftrightarrow \quad u \preceq v.$$

Let $u, v \in W'$ be such that $u \leq'_L v$. We may assume that there exists $s \in S'$ (where S' is the generating set of W') such that

$$T_s C_v = \sum_{w \in W'} a_{w,v} C_w \quad \text{where } a_{w,v} \in \mathcal{A} \text{ and } a_{u,v} \neq 0.$$

Since $C_w \in \mathcal{B}$ for all $w \in W'$, this is the expression of $T_s C_v$ in $\mathcal{B}$, which shows that $u \preceq v$.

Conversely, let $u, v \in W'$ be such that $u \preceq v$. We may assume that there exist $z \in W$ and $x \in X'$ such that

$$T_z C_v = \sum_{w \in W', y \in X'} a_{yw,zv} T_y C_w \quad \text{where } a_{yw,zv} \in \mathcal{A} \text{ and } a_{xu,zv} \neq 0.$$

We can write uniquely $z = z_1.z_0$ where $z_0 \in W'$ and $z_1 \in X'$. Then, we have

$$T_z C_v = T_{z_1}(T_{z_0} C_v) = T_{z_1}\big(\sum_{w \in W', w \leq'_L v} a_{w,v} C_w\big) = \sum_{w \in W', w \leq'_L v} a_{w,v} T_{z_1} C_w$$

and this is the expression of $T_z C_v$ in the basis $\mathcal{B}$. We assumed that $T_x C_u$ appears with a non-zero coefficient, thus $u \leq'_L v$ as desired.

5.2. Proof of Theorem 5.1.2

We keep the setting of the last section and we introduce the following relation. Let $u, v \in U$, $x \in X_u$ and $y \in X_v$. We write $xu \sqsubset yv$ if $xu < yv$ (Bruhat order) and $u \preceq v$. We write $xu \sqsubseteq yv$ if $xu \sqsubset yv$ or $x = y$ and $u = v$.

The main reference is the proof of [**15**, Theorem 1].

LEMMA 5.2.1. *Let $v \in U$, $y \in X_u$. We have*

$$T_{y^{-1}}^{-1} C_v = \sum_{u \in U,\ x \in X_u} \overline{r}_{xu,yv} T_x C_u$$

where $r_{yv,yv} = 1$ and $r_{xu,yv} = 0$ unless $xu \sqsubseteq yv$.

PROOF. Let $v \in U$ and $y \in X_v$. Recall that

$$T_{y^{-1}}^{-1} = T_y + \sum_{z<y} \overline{R}_{z,y} T_z$$

where $R_{z,y} \in \mathcal{A}$ are the usual R-polynomials as defined in Section 2.3. Thus

$$\begin{aligned} T_{y^{-1}}^{-1} C_v &= (T_y + \sum_{z<y} \overline{R}_{z,y} T_z) C_v \\ &= T_y C_v + \sum_{z<y} \overline{R}_{z,y} T_z C_v. \end{aligned}$$

Now we also have

$$T_z C_v = \ \mathcal{A}\text{-linear combination of } T_x C_u \text{ where } u \preceq v \text{ and } x \in X_u.$$

We still have to show that if $T_x C_u$ appears in this sum then $xu < yv$.
This comes from the fact that $T_z C_v$, expressed in the standard basis, is an $\mathcal{A}$-linear combination of terms of the form $T_{w_0.w_1}$ where $w_0 \leq z$ and $w_1 \leq v$. In particular, since $z < y$ we have $w_0 w_1 < yv$. Then, expressing the right hand side of the equality in the standard basis, one can see that we must have $xu < yv$ if $T_x C_u$ appears with a non-zero coefficient.
Finally, by definition of $\sqsubseteq$, we see that

$$T_{y^{-1}}^{-1} C_v = T_y C_v + \sum_{xu \sqsubset yv} \overline{r}_{xu,yv} T_x C_u.$$

The result follows. □

LEMMA 5.2.2. *Let $u, v \in U$, $x \in X_u$ and $y \in X_v$. Then*

$$\sum_{\substack{w \in U, z \in X_w \\ xu \sqsubseteq zw \sqsubseteq yv}} \overline{r}_{xu,zw} r_{zw,yv} = \delta_{x,y} \delta_{u,v}$$

PROOF. Since the map $h \mapsto \overline{h}$ is an involution and $C_v = \overline{C}_v$, we have

$$\begin{aligned}
T_y C_v &= \overline{T_{y^{-1}}^{-1} C_v} \\
&= \overline{\sum_{w \in U, z \in X_w} \overline{r}_{zw,yv} T_z C_w} \\
&= \sum_{w \in U, z \in X_w} r_{zw,yv} T_{z^{-1}}^{-1} C_w \\
&= \sum_{w \in U, z \in X_w} r_{zw,yv} \big(\sum_{u \in U, x \in X_u} \overline{r}_{xu,zw} T_x C_u \big) \\
&= \sum_{u \in U, x \in X_u} \big(\sum_{w \in U, z \in X_w} \overline{r}_{xu,zw} r_{zw,yv} \big) T_x C_u.
\end{aligned}$$

Since $\mathcal{B}$ is a basis of $\mathcal{M}$, using Lemma 5.2.1 and comparing the coefficients yields the desired result. □

LEMMA 5.2.3. *Let $u \in U$ and $x \in X_u$. We have*

$$T_x C_u \in T_{xu} + \bigoplus_{z < xu} \mathcal{A}_{<0} T_z.$$

PROOF. We have

$$\begin{aligned}
T_x C_u &= T_x \big(\sum_{u_1 \leq u} P_{u_1,u} T_{u_1} \big) \\
&= T_{xu} + \sum_{u_1 < u} P_{u_1,u} T_x T_{u_1}
\end{aligned}$$

and the result follows, using **I5**. □

PROPOSITION 5.2.4. *Let $v \in U$ and $y \in X_v$. We have*

$$C_{yv} = T_y C_v + \sum_{\substack{u \in U, x \in X_u \\ xu \sqsubset yv}} p^*_{xu,yv} T_x C_u \qquad \textit{where } p^*_{xu,yv} \in \mathcal{A}_{<0}.$$

PROOF. By Lemma 5.2.2, there exists a unique family $(p^*_{xu,yv})_{xu \sqsubset yv}$ of polynomials in $\mathcal{A}_{<0}$ such that

$$\tilde{C}_{yv} := T_y C_v + \sum_{\substack{u \in U, x \in X_u \\ xu \sqsubseteq yv}} p^*_{xu,yv} T_x C_u$$

is stable under the $^-$ involution; see [**12**], this contains a general setting to include the arguments in [**38**, Theorem 5.2], [**15**, Corollary 3.3]. Moreover, using the previous lemma, we see that $\tilde{C}_{yv}$ is an $\mathcal{A}_{<0}$-linear combination of T_z where $z < yv$. The result follows. □

Let $\mathcal{U} \subset U$ be as in Theorem 5.1.2. By definition of $\preceq$ one can see that

$$\mathcal{M}_{\mathcal{U}} := \langle T_y C_v \mid v \in \mathcal{U},\ y \in X_v \rangle_{\mathcal{A}}$$

is a left ideal of $\mathcal{H}$.

Corollary 5.2.5.

$$\mathcal{M}_{\mathcal{U}} = \langle C_{yv} \mid v \in \mathcal{U},\ y \in X_v \rangle_{\mathcal{A}}.$$

Proof. Let $v \in \mathcal{U}$ and $y \in X_v$, using the previous proposition, we see that

$$C_{yv} = T_y C_v + \sum_{\substack{u \in \mathcal{U}, x \in X_u \\ xu \sqsubset yv}} \overline{p}^*_{xu,yv} T_x C_u.$$

Thus $C_{yv} \in \mathcal{M}_{\mathcal{U}}$. Now, a straightforward induction on the order relation $\sqsubseteq$ yields

$$T_y C_v = C_{yv} + \text{ an } \mathcal{A}\text{-linear combination of } C_{xu}$$

where $u \in \mathcal{U},\ x \in X_u$ and $xu \sqsubset yv$.
This yields the desired assertion. $\square$

We can now prove Theorem 5.1.2.
Let $\mathcal{U}$ be a subset of U such that

$$\{v \in U \mid v \preceq u \text{ for some } u \in \mathcal{U}\} \subset \mathcal{U}.$$

We know that $\mathcal{M}_{\mathcal{U}} = \langle T_z C_w \mid w \in \mathcal{U},\ z' \in X_w \rangle_{\mathcal{A}}$ is a left ideal of $\mathcal{H}$. We want to show that the set $\mathfrak{B} := \{y.v | v \in \mathcal{U},\ y \in X_v\}$ is a left ideal of W.
Let $v \in \mathcal{U}$, $y \in X_v$ and $z \in W$ be such that $z \leq_L y.v$. We may assume that there exists $s \in S$ such that C_z appears with a non-zero coefficient in the expression of $T_s C_{yv}$ in the Kazhdan-Lusztig basis. By Corollary 5.2.5, we have $C_{yv} \in \mathcal{M}_{\mathcal{U}}$. Since $\mathcal{M}_{\mathcal{U}}$ is a left ideal we have $T_s C_{yv} \in \mathcal{M}_{\mathcal{U}}$. Thus, using Corollary 5.2.5 once more, we have

$$T_s C_{yv} = \sum_{u \in \mathcal{U},\ x \in X_u} a_{xu,yv} C_{xu} \quad \text{where } a_{xu,yv} \in \mathcal{A}$$

and this is the expression of $T_s C_{yv}$ in the Kazhdan-Lusztig basis. Now, the fact that C_z appears with a non-zero coefficient in that expression implies that $z = xu$ for some $u \in \mathcal{U}$ and $x \in X_u$. Thus $z \in \mathfrak{B}$, as desired. $\square$.

5.3. On some finite cells

Recall that, for $J \subset S$, we denote by X_J the minimal left coset representatives with respect to the subgroup generated by J and by $\mathcal{R}^J$ the set $\{w \in W |\ J \subset \mathcal{R}(w)\}$.

Let $W' \subset W$ be a standard finite parabolic subgroup with generating set S' and longest element w_0. Let $t \in S - S'$ be such that $L(t) > L(w_0)$. We keep this setting in this section.

THEOREM 5.3.1. *The set*

$$\{w \in W |\ w = y.w',\ y \in \mathcal{R}^{\{t\}} \cap X_{S'}, w' \in W'\}$$

is a left ideal of W.

Once and for all, we fix $t \in S - S'$ such that $L(t) > L(w_0)$.
Let $U = tW'$. For $u \in U$ let

$$X_u = (\mathcal{R}^{\{t\}} \cap X_{S'})t.$$

We want to apply Theorem 5.1.2 to the set U. One can directly check that conditions **I1–I3** hold. In order to check conditions **I4–I5** we need some preliminary lemmas. We denote by $\mathcal{H}_{W'}$ the Hecke algebra associated to (W', S') and the weight function L (more precisely the restriction of L on S').

LEMMA 5.3.2. *Let* $w' \in W'$. *We have*

$$C_t C_{w'} = C_{tw'} \quad \text{and} \quad T_t C_{w'} = C_{tw'} - v^{-L(t)} C_{w'}$$

PROOF. We know that

$$\begin{aligned} C_t C_{w'} &= C_{tw'} + \sum_{tz<z<w'} M^t_{z,w'} C_z, \\ T_t C_{w'} &= C_{tw'} - v^{-L(t)} C_{w'} + \sum_{tz<z<w'} M^t_{z,w'} C_z. \end{aligned}$$

But $z < w'$ implies that $z \in W'$, thus we cannot have $tz < z$. The result follows. $\square$

REMARK 5.3.3. Let $s' \in S'$. Since $L(t) \neq L(s')$, the order of $s't$ has to be even or infinite (otherwise, s' and t would be conjugate and $L(s') = L(t)$).

LEMMA 5.3.4. *Let* $s' \in S'$ *and* $w \in W'$. *Let* $m \leq n$ *be such that* m *is less than or equal to the order of* $s't$. *We have*

$$T_{(s't)^m} C_w = \sum_{w' \in W'} \sum_{i=0}^{m-1} a_{w',i} T_{(s't)^i s'} C_{tw'} + h'_m$$

where $a_{w',i} \in \mathcal{A}$ *and* $h'_m \in \mathcal{H}_{W'}$, *and*

$$T_{(ts')^m} C_w = \sum_{w' \in W'} \sum_{i=0}^{m-1} b_{w',i} T_{(ts')^i} C_{tw'} + h''_m$$

where $b_{w',i} \in \mathcal{A}$ *and* $h''_m \in \mathcal{H}_{W'}$. *Furthermore,* $h'_m = h''_m$.

Proof. The first two equalities come from a straightforward induction. It is clear that $h_0 = h'_0 = C_w$. Even though it is not necessary, we do the case $m = 1$ to show how the multiplication process works. We have

$$T_{s'}C_w = \sum_{w' \in W'} a_{w'}C_{w'} \quad \text{for some } a_{w'} \in \mathcal{A}.$$

Thus we obtain (using the previous lemma)

$$T_{s't}C_{w'} = T_{s'}C_{tw'} - v^{-L(t)} \sum_{w' \in W'} a_{w'}C_{w'}$$

and

$$T_{ts'}C_{w'} = \sum_{w' \in W'} a_{w'}C_{tw'} - v^{-L(t)} \sum_{w' \in W'} a_{w'}C_{w'}.$$

It follows that

$$h'_1 = -v^{-L(t)} \sum_{w' \in W'} a_{w'}C_{w'} = h''_1.$$

Now, by induction, one can see that

$$h'_m = -v^{-L(t)}T_{s'}h'_{m-1} \in \mathcal{H}_{W'} \quad \text{and} \quad h''_m = -v^{L(t)}T_{s'}h''_{m-1} \in \mathcal{H}_{W'}.$$

The result follows. □

Proposition 5.3.5. *The submodule*

$$\mathcal{M} := \langle T_xC_u \mid u \in U,\ x \in X_u \rangle_{\mathcal{A}} \subset \mathcal{H}$$

is a left ideal of $\mathcal{H}$.

Proof. Let $z \in W$, $u \in U$ and $x \in X_u$. We need to show that $T_zT_xC_u \in \mathcal{M}$. Since T_zT_x is an $\mathcal{A}$-linear combination of T_y ($y \in W$), it is enough to show that $T_yC_u \in \mathcal{M}$ for all $y \in W$ and $u \in U$.
We proceed by induction on $\ell(y)$. If $\ell(y) = 0$, then the result is clear.
Assume that $\ell(y) > 0$. We may assume that $y \notin X_u$. Let $w' \in W'$ be such that $u = tw'$. Recall that $X_u = (\mathcal{R}^{\{t\}} \cap X_{S'})t$.
Suppose that $yt < y$ and let $y_0 = yt$. We have

$$T_yC_{tw'} = T_{y_0}T_tC_{tw'} = v^{L(t)}T_{y_0}C_{tw'} \in \mathcal{M}$$

by induction.
Suppose that $yt > y$. Since $yt \in \mathcal{R}^{\{t\}}$ and $yt \notin \mathcal{R}^{\{t\}} \cap X_{S'}$, there exists $s' \in S'$ such that $(yt)s' < yt$. Let $2n$ be the order of ts'. One can see that there exists y_0 (with $\ell(y_0) < \ell(y)$) such that $yt = y_0(ts')^n$.
Using Lemma 5.3.2 and the relation $C_t = T_t + v^{-L(t)}T_e$ we see that

$$C_{tw'} = C_t C_{w'} = T_t C_{w'} + v^{-L(t)} C_{w'}.$$

Since $s' \in S'$ and $w' \in W'$ we have

$$T_{s'} C_w = \sum_{w_i \in W'} a_{w_i} C_{w_i} \quad \text{for some } a_{w_i} \in \mathcal{A}.$$

Thus we get

$$\begin{aligned} T_y C_{tw'} &= T_{yt} C_{w'} + v^{-L(t)} T_y C_{w'} \\ &= T_{y_0} T_{(ts')^n} C_{w'} + v^{-L(t)} T_{y_0} T_{(s't)^{n-1}s'} C_{w'} \\ &= T_{y_0}\big(T_{(ts')^{n-1}} T_t \sum_{w_i \in W'} a_{w_i} C_{w_i} + v^{-L(t)} T_{(s't)^{n-1}} \sum_{w_i \in W'} a_{w_i} C_{w_i}\big) \\ &= \sum a_{w_i} T_{y_0.(ts')^{n-1}} C_{tw_i} + v^{-L(t)} T_{y_0} \sum a_{w_i}\big(T_{(s't)^{n-1}} C_{w_i} - T_{(ts')^{n-1}} C_{w_i}\big). \end{aligned}$$

By induction we see that

$$\sum a_{w_i} T_{y_0} T_{(ts')^{n-1}} C_{tw_i} \in \mathcal{M}.$$

Lemma 5.3.4 implies that

$$T_{(s't)^{n-1}} C_w - T_{(ts')^{n-1}} C_w$$

is an $\mathcal{A}$-linear combination of terms of the form $T_{(s't)^m s'} C_{tw'}$ and $T_{(ts')^m} C_{tw'}$, for some $tw' \in J$ and $m \leq n-2$ (it is 0 if $n = 1$). Thus it follows by induction that

$$T_{y_0} \sum a_{w_i}\big(T_{(s't)^{n-1}} C_{w_i} - T_{(ts')^{n-1}} C_{w_i}\big) \in \mathcal{M}$$

as required. □

Proposition 5.3.6. *For all $u \in U$, $u_1 < u$ and $y \in X_u$ we have*

$$P_{u_1,u} T_y T_{u_1} \quad \textit{is an } \mathcal{A}_{<0}\textit{-linear combination of } T_z.$$

Proof. Let $u = tw' \in U$, $u_1 < u$ and $y \in X_u$. One can see that we have either $u_1 \in W'$ (then $u_1 \leq w'$) or there exists $w \in W'$ such that $u_1 = t.w$ and $w < w'$.
Assume that $u_1 \in W'$. Then $tu_1 > u_1$ and we have (using ([**38**, Theorem 6.6])

$$P_{u_1,u} = P_{u_1,tw'} = v^{-L(t)} P_{tu_1,tw'} \in v^{-L(t)} \mathcal{A}_{\leq 0}.$$

Furthermore, the degree of the polynomials occurring in the decomposition of $T_yT_{u_1}$ in the standard basis is at most $L(u_1)$. Thus, since $L(t) > L(w_0) \geq L(u_1)$, we get the result in that case.
Assume that $u_1 = t.w$ ($w \in W'$). Then, since $y \in (\mathcal{R}^{\{t\}} \cap X_{S'})t$, we see that $y.u_1$ and $T_yT_{u_1} = T_{yu_1}$. The result follows. □

We are now ready to prove Theorem 5.3.1. Conditions **I4** and **I5** follow respectively from Proposition 5.3.5 and 5.3.6. Applying Theorem 5.1.2 yields that

$$\{x.u|\ u \in U, x \in X_u\} = \{w \in W|\ w = y.w',\ y \in \mathcal{R}^{\{t\}} \cap X_{S'}, w' \in W'\}$$

is a left ideal of W. We obtain the following corollary.

COROLLARY 5.3.7. *Let (W, S) be an arbitrary Coxeter system together with a weight function L. Let $W' \subset W$ be a finite standard parabolic subgroup with generating set S' and longest element w_0. If $L(t) > L(w_0)$ for all $t \in S - S'$ then the left cells (resp. two-sided cells) of W' with respect to the restriction of L to W' are left cells (resp. two-sided cells) of W.*

PROOF. For all $t \in S - S'$ we have $L(t) > L(w_0)$. Thus Theorem 5.3.1 yields that

$$\bigcup_{t \in S - S'} \{w \in W|\ w = y.w',\ y \in \mathcal{R}^{\{t\}} \cap X_{S'}, w' \in W'\} = W - W'$$

is a left ideal of W. Furthermore, since it is stable by taking the inverse, it's a two-sided ideal. Thus $W - W'$ is a union of cells and so is W'. Let $y, w \in W'$ be such that $y \leq_L w$ in W. Then using Theorem 5.1.5, one gets that $y \leq_L w$ in W'. Similarly, if $y \leq_R w$ in W then $y \leq_R w$ in W'. The result follows. □

EXAMPLE 5.3.8. Let W be of type $\tilde{G}_2$ with presentation as follows

$$W := \langle s_1, s_2, s_3 \mid (s_1s_2)^6 = 1, (s_2s_3)^3 = 1, (s_1s_3)^2 = 1 \rangle$$

and let L be a weight function on W. The longest element of the subgroup W' generated by s_2, s_3 is $w_0 = s_2s_3s_2$ and $L(w_0) = 3L(s_2)$. Thus if $L(s_1) > 3L(s_2)$, we can apply Theorem 5.3.1. We obtain that the following sets (which are the cells of W'):

$$\{e\} \cup \{s_2, s_3s_2\} \cup \{s_3, s_2s_3\} \cup \{w_0\} \qquad \text{(left cells)}$$
$$\{e\} \cup \{s_2, s_3, s_3s_2, s_2s_3\} \cup \{w_0\} \qquad \text{(two-sided cells).}$$

are left cells (resp. two-sided cells) of W.

CHAPTER 6

The lowest two-sided cell of an affine Weyl group

Bremke and Xi ([**9, 46**]) determined the lowest two-sided cell for an irreducible affine Weyl group with unequal parameters. In [**9**], it is shown that it consists of at most $|W_0|$ left cells where W_0 is the associated Weyl group. We prove that this bound is exact. Previously, this was known in the equal parameter case ([**41, 42**]) and when the parameters are coming from a graph automorphism ([**9**]). Our argument works uniformly for any choice of parameters.

Consider the set

$$c_0 := \{w \in W \mid w = z.w_\lambda.z',\ w, w' \in W, \lambda \in T\}.$$

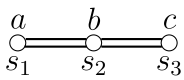
We will show that it is a two-sided cell, that it is the lowest (with respect to $\leq_{LR}$) and we will determine the left cells lying in c_0.

In this chapter, (W, S) denotes an irreducible affine Weyl group together with a weight function L.

6.1. Presentation of the lowest two-sided cell

We begin this section by giving an example.

EXAMPLE 6.1.1. Let W be an affine Weyl group of type $\tilde{C}_2$, with parameters as follows



By symmetry of the graph we may assume that $a \geq c$. In Example 3.3.6, we have determined the set of special points in the case $a > c$ and $a = c$. In the next two figures, the gray sets describe the lowest two-sided cell in those two cases.

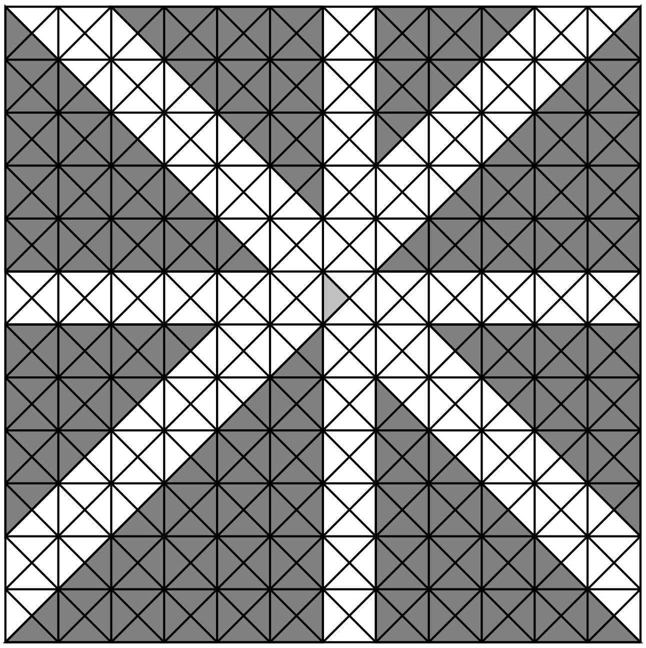


FIGURE 1. Lowest two-sided cell of $\tilde{C}_2$ in the case $a > c$.

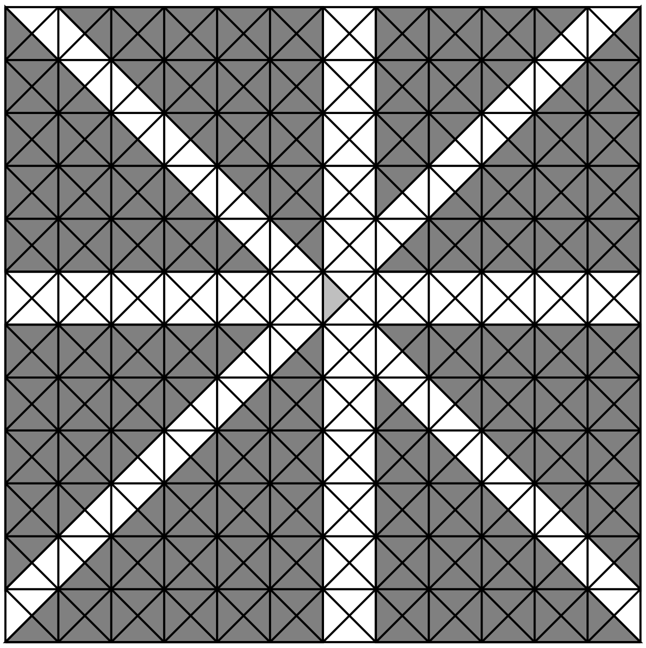


FIGURE 2. Lowest two-sided cell of $\tilde{C}_2$ in the case $a = c$.

REMARK 6.1.2. In fact, using the classification of special points (see 3.3), we see that if W is not of type $\tilde{C}_n$ $(n \geq 2)$ or $\tilde{A}_1$, then the lowest two-sided cell is the same for any choice of parameters.

We can describe c_0 in terms of strips (see [**3, 9**]).

THEOREM 6.1.3. *Let $\mathcal{U}$ be the set which consists of all maximal strips (of any direction) which contain A_0. We have*

$$c_0 := \{w \in W \mid wA_0 \not\subset \mathcal{U}\}.$$

REMARK 6.1.4. Compare the previous example and Example 3.4.4.

Finally, we have the following description; see [**9**, Theorem 6.13].

THEOREM 6.1.5. *We have*

$$c_0 := \{w \in W \mid \mathbf{a}(w) = \tilde{\nu}\}$$

where $\tilde{\nu} = L(w_\lambda)$ for any special point λ.

6.2. The lowest two-sided cell

We recall some notation, definitions and facts from Chapter 3. Let T be the set of special points and let $\lambda \in T$. We denote by

(1) W_λ the stabilizer in W of the set of alcoves containing λ in their closure;
(2) w_λ the longest element of W_λ;
(3) S_λ the subset of S defined by $S_\lambda := S \cap W_\lambda$.

Note that we have

$$sw_\lambda < w_\lambda \text{ for any } s \in S_\lambda.$$

Let $\lambda \in T$ and $z \in W$ be such that $w_\lambda.z$. We set

$$N_{\lambda,z} = \{w \in W \mid w = z'.w_\lambda.z,\ z' \in W\}.$$

PROPOSITION 6.2.1. *Let $\lambda \in T$ and $z \in W$ be such that $w_\lambda.z$. Then the set $N_{\lambda,z}$ is included in a left cell.*

PROOF. We only sketch the proof here and refer to [**9**] for details.

Let $y = z'.w_\lambda.z \in N_{\lambda,z}$ for some $z' \in W$. Let $x = z^{-1}.w_\lambda.z \in N_{\lambda,z}$. Note that $x^{-1} = x$. Using Proposition 2.2.2 we know that

$$\tau(T_xT_{y^{-1}}T_y) = \tau(T_{y^{-1}}T_yT_x) = \tau(T_yT_xT_{y^{-1}}).$$

Furthermore, one can check that

$$\tau(T_{y^{-1}}T_yT_x) = f_{y^{-1},y,x^{-1}} = f_{y^{-1},y,x} \quad \text{and} \quad \tau(T_yT_xT_{y^{-1}}) = f_{y,x,y}.$$

It can be shown that $v^{-\tilde{\nu}}\tau(T_xT_{y^{-1}}T_y)$ has a non-zero constant term, thus the degree of $f_{y^{-1},y,x}$ and $f_{y,x,y}$ in v is equal to $\tilde{\nu}$. We have seen that (see Proposition 2.7.1)

(1) $f'_{x,y,z} = f_{x,y,z} + \sum_{z',z<z'} Q_{z,z'} f_{x,y,z'}$
(2) $h_{x,y,z} = h_{x,y,z} + \sum_{x'<x,y'<y} P_{x',x}P_{y',y}f'_{x',y',z}$

where $Q_{z,z'} \in \mathcal{A}_{<0}$. Since $\tilde{\nu}$ is a bound for W, all the powers of v which appear in the "big" sum above are stricly less than $\tilde{\nu}$. It follows that $v^{\tilde{\nu}}$ appears with a non-zero coefficient in $h_{y,x,y}$ and $h_{y^{-1},y,x}$. Hence $y \leq_L x$ and $x \leq_L y$. In other words, any $y \in N_{\lambda,z}$ is $\sim_L$ equivalent to x and $N_{\lambda,z}$ is included in a left cell. □

Let

$$M_\lambda = \{z \in W \mid w_\lambda.z,\ sw_\lambda z \notin c_0 \text{ for all } s \in S_\lambda\}.$$

Following [**42**], we choose a set of representatives for the Ω-orbits on T and denote it by R. Then

$$c_0 = \bigcup_{\lambda\in R,\ z\in M_\lambda} N_{\lambda,z}$$

where the union is disjoint and is over $|W_0|$ elements.

We are now ready to state the main results of this chapter.

THEOREM 6.2.2. *Let $\lambda \in R$ and $z \in M_\lambda$. The set $N_{\lambda,z}$ is a left ideal of W. In particular it is a union of left cells.*

The proof of this theorem will be given in the next section. We first discuss a number of consequences of this result.

COROLLARY 6.2.3. *Let $\lambda \in T$ and $z \in M_\lambda$. The set $N_{\lambda,z}$ is a left cell.*

PROOF. The set $N_{\lambda,z}$ is a union of left cells (Theorem 6.2.2) which is included in a left cell (Proposition 6.2.1). Hence it is a left cell. □

The next step is to prove the following.

PROPOSITION 6.2.4. *The set c_0 is included in a two-sided cell.*

PROOF. Recall that $R = \{\lambda_1, ..., \lambda_n\}$ is a set of representatives for the Ω-orbits on T (see Example 3.3.6). Set

$$c_{\lambda_i} = \{w \in W |\ w = z'.w_{\lambda_i}.z,\ z, z' \in W\}.$$

One can see that

$$c_0 = \bigcup_{i=1}^{i=n} c_{\lambda_i}$$

and for $1 \leq i \leq j \leq n$ we have $c_{\lambda_i} \cap c_{\lambda_j} \neq \emptyset$. Therefore to prove the proposition, it is enough to show that each of the sets c_{λ_i} is included in a two-sided cell.

Fix $1 \leq i \leq n$. Let $z'.w_{\lambda_i}.z \in c_{\lambda_i}$ and $y'.w_{\lambda_i}.y \in c_{\lambda_i}$. Using Proposition 6.2.1, together with its version for right cells, we obtain

$$z'w_{\lambda_i}z \;\sim_L\; w_{\lambda_i}.z \;\sim_R\; w_{\lambda_i}y \;\sim_R\; y'w_{\lambda_i}y.$$

The result follows. □

Finally, combining the previous results of Shi, Xi and Bremke with Theorem 6.2.2, we obtain the following description of the lowest two-sided cell in complete generality.

THEOREM 6.2.5. *Let W be an irreducible affine Weyl group with associated Weyl group W_0. Let*

$$c_0 = \{w \in W \mid w = z'.w_\lambda.z,\; z, z' \in W, \lambda \in T\}$$

where T is the set of special points. We have:

(1) *c_0 is a two-sided cell.*
(2) *c_0 is the lowest two-sided cell with respect to the partial order on the two-sided cell induced by the pre-order $\leq_{LR}$.*
(3) *c_0 contains exactly $|W_0|$ left cells.*
(4) *The decomposition of c_0 into left cells is as follows*

$$c_0 = \bigcup_{\lambda \in R,\; z \in M_\lambda} N_{\lambda,z}.$$

PROOF. We have seen that c_0 is included in a two-sided cell. Let $w \in c_0$ and $y \in W$ be such that $y \sim_{LR} w$. In particular we have $y \leq_{LR} w$. We may assume that $y \leq_L w$ or $y \leq_R w$. We know that

$$c_0 = \bigcup_{\lambda \in R,\; z \in M_\lambda} N_{\lambda,z}.$$

Thus $w \in N_{\lambda,z}$ for some $\lambda \in R$ and $z \in M_\lambda$. If $y \leq_L w$ then, using Theorem 6.2.2, we see that $y \in N_{\lambda,z}$ and thus $y \in c_0$. If $y \leq_R w$ then using [**38**, §8.1], we have $y^{-1} \leq_L w^{-1}$. But c_0 is stable by taking the inverse, so as before, we see that $y^{-1} \in c_0$ and $y \in c_0$. This implies that c_0 is a two-sided cell and that it is the lowest one with respect to $\leq_{LR}$.

By [**42**], we know that

$$c_0 = \bigcup_{\lambda \in R,\; z \in M_\lambda} N_{\lambda,z}$$

is a disjoint union over $|W_0|$ terms. By Corollary 6.2.3, (3) and (4) follow. □

6.3. Proof of Theorem 6.2.1

Once and for all we fix $\lambda \in R$ and $z \in M_\lambda$. We want to apply Theorem 5.1.2. We set $U = \{w_\lambda.z\} = \{u\}$ and $X_u = X_{S_\lambda}$ (the minimal coset representatives with respect to W_λ). Conditions **I1**–**I3** are clearly satisfied. We have

LEMMA 6.3.1. *The set*

$$\mathcal{M} := \langle T_y C_u \mid y \in X_u \rangle$$

is a left ideal of $\mathcal{H}$*.*

PROOF. Since $\mathcal{H}$ is generated by T_s for $s \in S$, it is enough to check that $T_s T_x C_v \in \mathcal{M}$ for $x \in X_u$. According to Deodhar's lemma, there are three cases to consider

(1) $sx \in X_u$ and $\ell(sx) > \ell(x)$. Then $T_s T_x C_v = T_{sx} C_v \in \mathcal{M}$ as required.
(2) $sx \in X_u$ and $\ell(sx) < \ell(x)$. Then $T_s T_x C_v = T_{sx} C_v + (v_s - v_s^{-1}) T_x C_v \in \mathcal{M}$ as required.
(3) $t := x^{-1} sx \in S_\lambda$. Then $\ell(sx) = \ell(x) + 1 = \ell(tx)$. Now, since $tv < v$, we have

$$T_t C_v = v^{L(t)} C_v.$$

Thus, we see that

$$T_s T_x C_v = T_{sx} C_v = T_{xt} C_v = T_x T_t C_v = v^{L(t)} T_x C_v,$$

which is in $\mathcal{M}$ as required.

□

Thus condition **I4** is satisfied.
We now have a look at condition **I5**.

LEMMA 6.3.2. *Let $y \in X_u$ and $u_1 < u = w_\lambda.z$. Then, $P_{u_1,u} T_y T_{u_1}$ is an $\mathcal{A}$-linear combination of T_z with coefficients in $\mathcal{A}_{<0}$.*

PROOF. We can write $u_1 = w.u'$, where $w \in W_\lambda$ and $u'^{-1} \in X_{S_\lambda}$. First, assume that $w = w_\lambda$. In that case, we have $y.u_1$ and $T_y T_{u_1} = T_{yu_1}$. Since $P_{u_1,u} \in \mathcal{A}_{<0}$ the result follows.

Next, assume that $w < w_\lambda$. Let $w_{u_1} \in W$ be such that $w_{u_1}.w = w_\lambda$. We know that the Kazhdan-Lusztig polynomials satisfy the following relation

$$P_{x,z} = v^{-L(s)} P_{sx,z}, \text{ where } x < sx \text{ and } sw < w\ .$$

Therefore, one can see that

(1) $P_{u_1,u} \in v^{-L(w_{u_1})}\mathcal{A}_{<0}$ if $w_\lambda.u' < u$,
(2) $P_{u_1,u} = v^{-L(w_{u_1})}$ if $w_\lambda.u' = u$.

Thus, to prove the lemma, it is sufficient to show that the polynomials occuring in $T_yT_{u_1}$ are of degree less than or equal to $L(w_{u_1})$ in the first case and $L(w_{u_1}) - 1$ in the second case.

Using Theorem 3.5.1, we know that the degree of these polynomials is less than or equal to c_{y,u_1} (for the definition of c_{y,u_1}, see Section 3.5).

Let $w_{u_1} = s_n...s_{m+1}$ and $w = s_m...s_1$ be reduced expressions, and let H_i be the unique hyperplane which separates $s_{i-1}...s_1u'A_0$ and $s_i...s_1u'A_0$. Note that $c_{H_i} = L(s_i)$. Let λ' be the unique special point contained in the closure of $u'A_0$ and $w_\lambda u'A_0$. One can see that yu_1A_0 lies in the quarter $\mathcal{C}$ with vertex λ' which contains u_1A_0.

Let $1 \leq i \leq m$. Let α_i and $k \in \mathbb{Z}$ be such that $H_i = H_{\alpha_i,k}$. Assume that $k > 0$ (the case $k \leq 0$ is similar). We have $u_1A_0 \in V^+_{H_i}$. Now, since λ' lies in the closure of u_1A_0 and $\lambda' \in H_i$, one can see that

$$k < \langle x, \check{\alpha}_i \rangle < k+1 \text{ for all } x \in u_1A_0.$$

Moreover, $yu_1A_0 \in \mathcal{C}$ implies that

$$k < \langle x, \check{\alpha}_i \rangle \text{ for all } x \in yu_1A_0.$$

From there, we conclude that, if $l \leq k$ then $H_{\alpha_i,l} \notin H(u_1A_0, yu_1A_0)$ and that none of the hyperplanes $H_{\alpha_i,l}$ with $l > k$ lie in $H(A_0, u_1A_0)$. Thus $\overline{H_i} \notin I_{u_1,y}$ and we have

$$I_{y,u_1} \subset \{\overline{H_{m+1}}, ..., \overline{H_n}\},$$

which implies

$$c_{y,u_1} \leq \sum_{i=m+1}^{i=n} c_{y,u_1}(\overline{H_i}).$$

Now, if W is not of type $\tilde{C}_r$ or $\tilde{A}_1$ then any two parallel hyperplanes have same weight and we have

$$c_{y,u_1}(\overline{H_i}) = \begin{cases} 0 & \text{if } i \notin I_{y,u_1}, \\ L(s_i) & \text{otherwise.} \end{cases}$$

Thus

$$c_{y,u_1} \leq \sum_{i=m+1}^{i=n} L(s_i) = L(w_{u_1}),$$

as required in the first case.

Assume that W is of type $\tilde{C}_r$ or $\tilde{A}_1$. Then, one can see that, since λ' is a special point, we have for all $1 \leq i \leq n$, $c_{H_i} = c_{\overline{H_i}} = L(s_i)$. Thus we can argue as above.

Assume that we are in case 2. We have $u_1 = w.u' < w_\lambda u' = u$ and $u = w_\lambda.z$ (where $z \in M_\lambda$). Recall that

$$M_\lambda = \{z \in W \mid w_\lambda.z,\ sw_\lambda z \notin c_0 \text{ for all } s \in S_\lambda\}$$

thus $u_1 = w.u' \notin c_0$. The elements of c_0 are characterized by the fact that they don't lie in any strip of maximal weight which contains the identity. Thus, since $u_1 \notin c_0$, u_1A_0 lies in a strip of maximal weight which contains A_0. Let $1 \leq i \leq m$. By definition, H_i separates A_0 and u_1A_0 and since λ' is a special point, $c_{H_i} = c_{\overline{H_i}}$. Thus, the maximal strip which contains A_0 and u_1A_0 has to be a strip of direction $\overline{H_k}$ with $k > m$.

If W is not of type $\tilde{C}_r$ or $\tilde{A}_1$, then our strips and the strip as defined in [**9**] are the same. Therefore, since A_0 and u_1A_0 lie in the same strip of direction $\overline{H_k}$, we have $\overline{H_k} \notin I_{u_1,y}$ and

$$c_{y,u_1} \leq \sum_{\substack{i=m+1 \\ i\neq k}}^{i=n} c_{y,u_1}(\overline{H_i}) \leq \sum_{\substack{i=m+1 \\ i\neq k}}^{i=n} L(s_i) < L(w_{u_1}),$$

as required.

Assume that W is of type $\tilde{C}_r$ or $\tilde{A}_1$. First, if all the hyperplanes with direction $\overline{H_k}$ have same weight then we have $\overline{H_k} \notin I_{y,u_1}$ and we can conclude as before. Assume not, then we must have $c_{H_k} = c_{\overline{H_k}}$ (since $\lambda' \in H_k$) and there is no hyperplane with direction $\overline{H_k}$ and maximal weight which separates A_0 and u_1A_0. Therefore

$$c_{y,u_1} \leq \sum_{i=m+1}^{i=n} c_{y,u_1}(\overline{H_i}) < \sum_{\substack{i=m+1 \\ i\neq k}}^{i=n} c_{y,u_1}(\overline{H_i}) + c_{\overline{H_k}} \leq \sum_{i=m+1}^{i=n} L(s_i) = L(w_{u_1}),$$

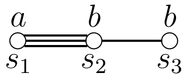
as required. $\square$

We have checked that conditions **I1**–**I5** are satisfied. Thus Theorem 6.2.2 is a consequence of the generalized induction theorem.

CHAPTER 7

Decomposition of $\tilde{G}_2$

Let W be an affine Weyl group of type $\tilde{G}_2$ with diagram and weight function given by



where a, b are positive integers.

The main aim of this chapter is to find the decomposition of W into left cells and two-sided cells for any weight function L such that $a/b > 4$. Furthermore we will determine the partial left (resp. two-sided) order on the left (resp. two-sided) cells (see Section 7.2.4). In the final section, we introduce a conjecture of Bonnafé concerning the behaviour of Kazhdan-Lusztig cells when the parameters are varying. We then discuss the decomposition of $\tilde{G}_2$ for other values of the parameters and show that it agrees with the conjecture.

In Figure 1, we present a partition of W using the geometric realization as defined in Chapter 3, where the pieces are formed by the alcoves lying in the same connected component after removing the thick lines. Using the same methods as in Chapter 4, Section 4.4, one can show that each of these pieces are included in a left cell for any weight function L such that $a/b > 4$. Thus in order to show that this is the actual decomposition of W into left cells, it is enough to show that each of these pieces are included in a union of left cells.

REMARK 7.0.3. Note that we have changed the notation of the sets A_i, B_i and C_i (see Chapter 4). The notation here are more convenient to describe the partial left order on the left cells.

We now consider the union of all subsets of W whose name contains a fixed capital letter; we denote this union by that capital letter. For instance

$$A = (\overset{6}{\underset{i=1}{\cup}} A_i) \bigcup (\overset{6}{\underset{i=1}{\cup}} A'_i).$$

The decomposition of W into two-sided cells is as follows

$$W = A \cup B \cup C \cup D \cup E \cup F \cup \{e\}.$$

REMARK 7.0.4. In this section we need to compute some Kazhdan-Lusztig polynomials $P_{x,y}$ $(x, y \in W)$ for a whole class of weight functions. This is done using Proposition 4.1.6. In particular, this involved some computation with GAP ([**39**]).

For any subset $J \subseteq \{1, 2, 3\}$, let

(1) $\mathcal{R}^J := \{w \in W \mid \{s_j | j \in J\} \subseteq \mathcal{R}(w)\}$;
(2) W_J be the subgroup of W generated by $\{s_j; j \in J\}$;
(3) $X_J := \{w \in W |\ w$ has minimal length in $wW_J\}$.

For details of the computations, see [**20**].

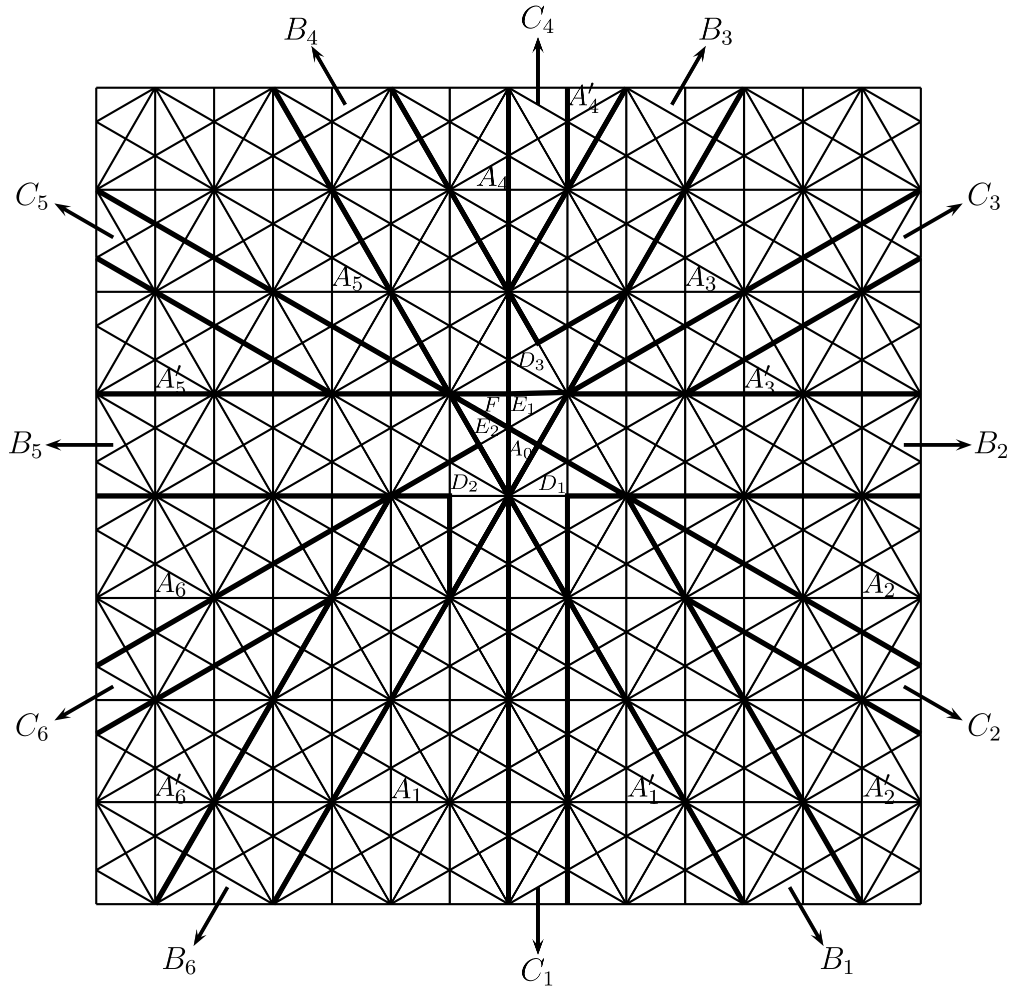


FIGURE 1. Decomposition of $\tilde{G}_2$ into left cells in the case $a > 4b$

7.1. Preliminaries

In this section (W, S) denotes an arbitrary Coxeter system and L a positive weight function on W. We give a number of lemmas which will be needed later on.

LEMMA 7.1.1. *Let $S' \subseteq S$ be such that*

(1) *for all $s'_1, s'_2 \in S'$, we have $L(s'_1) = L(s'_2)$,*
(2) *for all $t \in S - S'$ and $s' \in S'$ we have $L(t) > L(s')$.*

Let $y, w \in W$ and $s' \in S'$ be such that $s'y < y < w < s'w$. Then if $M^{s'}_{y,w} \neq 0$, we have either $\mathcal{L}(w) \subseteq \mathcal{L}(y)$ or there exists $s \in S'$ such that $w = sy$, in which case $M^{s'}_{y,w} = 1$.

PROOF. We proceed by induction on $\ell(w)-\ell(y)$. Assume first that $\ell(w)-\ell(y) = 1$. Since $s'y < y$ and $s'w > w$ one can see that there exist $s \in S$ such that $s \neq s'$ and $w = sy$. In that case we have (see [**28**, Proposition 5])

$$M^{s'}_{y,w} = \begin{cases} 0, & \text{if } L(s) > L(s'), \\ 1, & \text{if } L(s) = L(s'). \end{cases}$$

Thus if $M^{s'}_{z,w} \neq 0$ we must have $s \in S'$.
Assume that $\ell(w) - \ell(y) > 1$ and that $\mathcal{L}(w) \not\subseteq \mathcal{L}(y)$. Let $s \in S$ be such that $s \in \mathcal{L}(w)$ and $s \notin \mathcal{L}(y)$. We have

$$M^{s'}_{y,w} + \sum_{z;y<z<w,s'z<z} P_{y,z}M^{s'}_{z,w} - v_{s'}P_{y,w} \in \mathcal{A}_{<0}.$$

Thus in order to show that $M^{s'}_{y,w} = 0$ it is enough to show that

$$\sum_{z;y<z<w,s'z<z} P_{y,z}M^{s'}_{z,w} - v_{s'}P_{y,w} \in \mathcal{A}_{<0}.$$

Let $z \in W$ be such that $M^{s'}_{z,w} \neq 0$. By induction we have either $M^{s'}_{z,w} = 1$ or $\mathcal{L}(w) \subseteq \mathcal{L}(z)$. In the first case we have $P_{y,z}M^{s'}_{z,w} \in \mathcal{A}_{<0}$. Assume that we are in the second case (then $s \in \mathcal{L}(z)$). By ([**38**, proof of Theorem 6.6]) we know that

$$P_{y,z} = v_s^{-1}P_{sy,z} \in \mathcal{A}_{\leq 0}.$$

Furthermore the degree in v of $M^{s'}_{z,w}$ is at most $L(s')-1$ ([**38**, Proposition 6.4]). Since $s' \in S'$ we have $L(s) \geq L(s')$ and

$$P_{y,z}M^{s'}_{z,w} \in \mathcal{A}_{<0}.$$

Similarly $v_{s'}P_{y,w} \in \mathcal{A}_{<0}$ (since $\ell(w) - \ell(y) > 1$). Thus if $\mathcal{L}(w) \not\subseteq \mathcal{L}(y)$ we must have $M^{s'}_{y,w} = 0$, as required.

$\square$

LEMMA 7.1.2. *Let $\mathfrak{B} \subseteq W$ be a left ideal of W. Let $s \in S$ and $\mathfrak{B}'_s$ (resp. $\mathfrak{B}_s$) be the subset of $\mathfrak{B}$ which consists of all $w \in \mathfrak{B}$ such that $ws > w$ (resp. $ws < w$). Assume that there exists a left ideal $\mathfrak{A}$ of W such that, for all $w' \in \mathfrak{B}'_s$, we have*

$$C_{w'}C_s = C_{w's} + \sum_{z \in \mathfrak{A}} \mathcal{A}C_z.$$

Then $\mathfrak{A} \cup \mathfrak{B}_s \cup \mathfrak{B}'_s.s$ is a left ideal of W. Furthermore, $\mathfrak{B}'_s.s$ is a union of left cells.

PROOF. Let $w \in \mathfrak{A} \cup \mathfrak{B}_s \cup \mathfrak{B}'_s.s$. Let $y \in W$ be such that $y \leq_L w$. We need to show that $y \in \mathfrak{A} \cup \mathfrak{B}_s \cup \mathfrak{B}'_s.s$.
If $w \in \mathfrak{A}$ then $y \in \mathfrak{A}$, since $\mathfrak{A}$ is a left ideal.
If $w \in \mathfrak{B}_s$ then $y \in \mathfrak{B}$. Note that since

$$y \leq_L w \Longrightarrow \mathcal{R}(w) \subseteq \mathcal{R}(y)$$

we have $s \in \mathcal{R}(y)$ and $y \in \mathfrak{B}_s$. This shows that $\mathfrak{B}_s$ is a left ideal.
Finally, assume that $w \in \mathfrak{B}'_s.s$ and let $w' \in \mathfrak{B}'_s$ be such that $w's = w$. We may assume that there exists $t \in S$ such that C_y appears with a non-zero coefficient in the expression of C_tC_w in the Kazhdan-Lusztig basis. We have

$$\begin{aligned} C_tC_w &= C_tC_{w's} \\ &= C_t\big(C_{w'}C_s + \sum_{z \in \mathfrak{A}} \mathcal{A}C_z\big) \\ &= \big(\sum_{z \in \mathfrak{B}} \mathcal{A}C_z\big)C_s + \sum_{z \in \mathfrak{A}} \mathcal{A}C_z \\ &= \sum_{z \in \mathfrak{B}'_s s} \mathcal{A}C_z + \sum_{z \in \mathfrak{B}_s} \mathcal{A}C_z + \sum_{z \in \mathfrak{A}} \mathcal{A}C_z. \end{aligned}$$

Thus we see that $y \in \mathfrak{A} \cup \mathfrak{B}_s \cup \mathfrak{B}'_s.s$ as desired. Now, since $\mathfrak{A}$ and $\mathfrak{B}_s$ are unions of left cells, we obtain that $\mathfrak{B}'_s s$ is a union of left cells. $\square$

LEMMA 7.1.3. *Let T be a union of left cells which is stable by taking the inverse. Let $T = \cup\, T_i$ $(1 \leq i \leq N)$ be the decomposition of T into left cells. Assume that for all $i, j \in \{1, ..., N\}$ we have*

$$T_i^{-1} \cap T_j \neq \emptyset \tag{$*$}$$

Then T is included in a two-sided cell.

PROOF. Let $y, w \in T$ and $i, j \in \{1, ..., N\}$ be such that $y \in T_i$ and $w \in T_j$. Using $(*)$, there exist $y_1, y_2 \in T_i$ such that $y_1^{-1} \in T_i$ and $y_2^{-1} \in T_j$. We have

$$y \sim_L y_1 \sim_L y_2 \quad \Longrightarrow \quad y \sim_L y_1^{-1} \sim_R y_2^{-1} \sim_L w$$

as required. $\square$

7.2. Decomposition of $\tilde{G}_2$ in the "asymptotic case"

Let W be an affine Weyl group of type $\tilde{G}_2$. Let L be a weight function on W such that $a/b > 4$. We keep the notation of Figure 1 (i.e. B_i, C_i...).

7.2.1. The sets C_i. In this section we want to prove that C_i is a left cell (for all $1 \leq i \leq 6$) and that $C = \cup C_i$ is a two-sided cell.

For $1 \leq i \leq 6$, let

(1) $u_i \in C_i$ be the element of minimal length in C_i;
(2) $v_i \in A_i$ be the element of minimal length in A_i;
(3) $v'_i \in A'_i$ be the element of minimal length in A'_i.

For instance, we have

$$u_1 = s_1s_2s_1s_2s_1;$$
$$v_1 = s_1s_2s_1s_2s_1s_2;$$
$$v'_1 = s_2s_1s_2s_1s_2s_3s_1s_2s_1s_2s_1.$$

We set $U := \{u_i, v_i, v'_i \mid 1 \leq i \leq 6\}$, $X_{v_i} = X_{v'_i} = X_{1,2}$ for all $1 \leq i \leq 6$ and

$$X_{u_i} = \{z \in W \mid z.u_i \in C_i\}.$$

We want to apply Corollary 5.1.3. One can check that conditions **I1**–**I3** of Theorem 5.1.2 hold. We now have a look at condition **I4**.

LEMMA 7.2.1. *The set*
$$\mathcal{M} := \langle T_xC_u \mid u \in U, x \in X_u\rangle$$
is a left ideal of $\mathcal{H}$.

PROOF. We know that, for all $1 \leq i \leq 6$, the sets

$$\langle T_xC_{v_i} \mid x \in X_{1,2}\rangle \quad \text{and} \quad \langle T_xC_{v'_i} \mid x \in X_{1,2}\rangle$$

are left ideals of $\mathcal{H}$; see Lemma 6.3.1. Thus, in order to show that $\mathcal{M}$ is a left ideal of $\mathcal{H}$, it is enough to prove that $T_xC_{u_i} \in \mathcal{M}$ for all $1 \leq i \leq 6$ and all $x \in W$. We proceed by induction on $\ell(x)$. If $\ell(x) = 0$ it's clear. Assume that $\ell(x) > 0$. We may assume that $x \notin X_{u_i}$. Then, one can see that we have either $x = x_0.s_2$ or $x = x_1.s_2s_1s_2s_1s_2s_3$. Now, doing explicit computations, one can show that $T_{s_2}C_{u_i}$ is an $\mathcal{A}$-linear combination of C_u with $u \in U$. For example, we have

$$T_{s_2}C_{u_1} = C_{v_1} - v^{-L(s_2)}C_{u_1}$$

and

$$T_{s_2}C_{u_5} = C_{v_5} - v^{-L(s_2)}C_{u_5} + C_{v_1}.$$

Thus, by induction, $T_x C_{u_i} = T_{x_0}T_{s_2}C_{u_i} \in \mathcal{M}$ as required.

Similarly, one can show that $T_{s_2s_1s_2s_1s_2s_3}C_{u_i}$ is an $\mathcal{A}$-linear combination of terms of the form T_zC_u where $u \in U$, $z \in X_u$ and $\ell(z) < \ell(s_2s_1s_2s_1s_2s_3)$. For example we have

$$\begin{aligned} T_{s_2s_1s_2s_1s_2s_3}C_{u_1} = C_{v_1'} &+ \mathcal{A}T_{s_1s_2s_1s_2s_3}C_{u_1} + \mathcal{A}T_{s_2s_1s_2s_3}C_{u_1} + \mathcal{A}T_{s_1s_2s_3}C_{u_1} \\ &+ \mathcal{A}T_{s_2s_3}C_{u_1} + \mathcal{A}T_{s_3}C_{u_1} + \mathcal{A}C_{u_1} + \mathcal{A}C_{v_1}. \end{aligned}$$

Thus by induction, we obtain $T_xC_{u_i} \in \mathcal{M}$ as required. □

We now have a look at condition **I5**. Let $u \in U$, $u' < u$ and $y \in X_u$. We need to show that

$$P_{u',u}T_yT_{u'} \text{ is an } \mathcal{A}_{<0}\text{-linear combination of } T_z.$$

For $u = v_i$ or $u = v_i'$, it is proved in Lemma 6.3.2. In order to prove it for $u = u_i$ we proceed as follows. We determine an upper bound for the degree of the polynomials occurring in the expression of $T_yT_{u'}$ (where $y \in C_i$, $u' < u_i$) in the standard basis using either Theorem 3.5.1 or explicit computations. Then we compute the polynomials $P_{u',u}$ and we can check that the condition is satisfied.

We can now apply Corollary 5.1.3. We need to find the equivalence classes on U with respect to $\preceq$. Using the fact that $\langle T_xC_{v_i} \mid x \in X'\rangle$ and $\langle T_xC_{v_i'} \mid x \in X'\rangle$ are left ideals of $\mathcal{H}$ for all $1 \leq i \leq 6$ and the relations computed in the previous proof, one can check that

$$\{\{u_i\}\{v_i\}, \{v_i'\} \mid 1 \leq i \leq 6\}$$

is the decomposition of U into equivalence classes. Hence by Corollary 5.1.3, the set $X_{u_i}.u_i = C_i$ is a union of left cells for all $1 \leq i \leq 6$. Since C_i is included in a left cell, we obtain that C_i is a left cell, for all $1 \leq i \leq 6$.
More precisely, the following sets are left ideals of W

$$C_i \cup A_i \cup A_i' \quad \text{for } i = 1, 2, 3, 6$$

$$C_4 \cup A_4 \cup A_4' \cup A_2,$$

$$C_5 \cup A_5 \cup A_5' \cup A_1.$$

PROPOSITION 7.2.2. *The set C is a two-sided cell.*

PROOF. Applying Theorem 5.1.2 to the set U yields that $A \cup C$ is a left ideal of W. One can check that $A \cup C$ is stable by taking the inverse, thus it is a two-sided ideal and $A \cup C$ is a union of two-sided cells. Since A is a two-sided cell, we see that C is a union of two-sided cells. Now one can check that $C = \cup C_i$ satisfies the requirement of Lemma 7.1.3 thus C is included in a two-sided cell. It follows that C is a two-sided cell. $\square$

7.2.2. The sets B_i. We want to prove that B_i (for all $1 \leq i \leq 6$) is a left cell and that the set B is a two-sided cell.

CLAIM 7.2.3. The set B_4 is a left cell.

PROOF. The set $\mathcal{R}^{2,3}$ is a left ideal of W (see Example 2.5.6). Furthermore, we have

$$\mathcal{R}^{2,3} = \{s_2s_3s_2\} \cup B_4 \cup A_4 \cup A_5.$$

Since A_3, A_6 and $\{s_2s_3s_2\}$ are left cells (for $\{s_2s_3s_2\}$ see Example 5.3.8), it follows that B_4 is a left cell. $\square$

REMARK 7.2.4. We have seen in Example 5.3.8 that $W - W_{2,3}$ is a left ideal of W. Thus

$$\mathcal{R}^{2,3} \cap (W - W_{2,3}) = B_4 \cup A_4 \cup A_5$$

is a left ideal of W.

CLAIM 7.2.5. B_2 is a left cell.

PROOF. The set $\mathcal{R}^{1,3}$ is a left ideal of W. Since we have

$$\mathcal{R}^{1,3} = B_2 \cup A_3 \cup A_3' \cup A_2 \cup C_3$$

it follows that B_2 is a left cell. $\square$

CLAIM 7.2.6. B_5 is a left cell.

PROOF. Let $w \in \mathcal{R}^{1,3}$ and $w' \in W$ be such that $w = w'.s_1s_3$. We have $ws_2 > w$ and

$$C_wC_{s_2} = C_{ws_2} + \sum_{z \in W, zs_2 < z} \mu^{s_2,r}_{z,w} C_z.$$

Applying Lemma 7.1.1 (in its right version), if $M^{s_2,r}_{z,w} \neq 0$ we have either $\{s_1, s_2, s_3\} \subseteq \mathcal{R}(z)$ which is impossible or there exists $w'' \in W$ such that

$$w = w''.s_2s_3 \quad \text{and} \quad z = w''s_2.$$

Since $w = w''.s_2s_3 = w's_1s_3$ we must have $w \in A_3$, which, in turn, implies that $z \in A_1$ (recall that A_1 is a left ideal). Thus applying Lemma 7.1.2 to $\mathfrak{A} = A_1$ and $\mathfrak{B} = \mathcal{R}^{1,3}$ yields that

$$\mathcal{R}^{1,3}.s_2 \cup A_1 = A_1 \cup A_5 \cup A'_5 \cup A_6 \cup C_5 \cup B_5$$

is a left ideal of W. In particular B_5 is a left cell. □

CLAIM 7.2.7. The set B_1 is a left cell.

PROOF. Set $u = s_1s_3s_2s_1$ and

$$X_{u_1} = \{z \in W \mid z.s_1s_3s_2s_1 \in B_1\}.$$

Recall that

$$\begin{aligned}
u_1 &= s_1s_2s_1s_2s_1,\\
v_1 &= s_1s_2s_1s_2s_1s_2,\\
v'_1 &= s_1s_2s_1s_2s_1s_2s_3s_2s_1s_2s_1,\\
u_2 &= s_1s_2s_1s_2s_1s_3s_2s_1,\\
v_2 &= s_2s_1s_2s_1s_2s_1s_3s_2s_1,\\
v'_2 &= s_2s_1s_2s_1s_2s_3s_1s_2s_1s_2s_1s_3s_2s_1\\
v_3 &= s_2s_1s_2s_1s_2s_1s_3,
\end{aligned}$$

and

$$\begin{aligned}
X_{u_i} &= \{z \in W \mid z.u_i \in C_2\},\\
X_{v_i} &= X_{v'_i} = X_{1,2}
\end{aligned}$$

for $1 \leq i \leq 6$. Using similar arguments as in Lemma 7.2.1 and the results in Section 7.2.1, one can check that we can apply Theorem 5.1.2 to $U := \{u, u_1, v_1, v'_1, u_2, v_2, v'_2, v_3\}$. We obtain that

$$\{x.u_i \mid u_i \in J, x \in X_{u_i}\} = A_2 \cup A'_2 \cup C_2 \cup B_1 \cup A_1 \cup A'_1 \cup C_1 \cup A_3$$

is a left ideal. In particular, B_1 is a left cell. □

CLAIM 7.2.8. The set B_6 is a left cell.

PROOF. Applying Lemma 7.1.2 (in a similar way as in Claim 7.2.6) to

$$\mathfrak{B} = A_2 \cup A'_2 \cup C_2 \cup B_1 \cup A_1 \cup A'_1 \cup C_1 \cup A_3$$

and $\mathfrak{A} = A_1$ we obtain that

$$A_1 \cup A'_1 \cup C_1 \cup A_6 \cup A'_6 \cup C_6 \cup A_5 \cup B_6$$

is a left ideal. Thus B_6 is a left cell. In fact since $\mathcal{R}^1$ is a left ideal and $(C_1 \cup A'_1) \cap \mathcal{R}^1 = \emptyset$, we see that

$$A_1 \cup A_6 \cup A'_6 \cup C_6 \cup A_5 \cup B_6$$

is a left ideal of W. $\square$

Claim 7.2.9. The set B_3 is a left cell.

Proof. One could use Lemma 7.1.2 to show that B_3 is a left cell. However, later on we will need a more precise result in order to determine the left order on the left cells. To this end, we need to go through the proof of Theorem 5.1.2.
We use the notation of the previous section (u_i, X_{u_i} etc...). Let $v = s_1s_3s_2s_1s_2s_3$ and

$$X_v := \{z \in W | z.v \in B_3\} \qquad Y_v := \{y \in X_v | y = y_0.s_2s_1s_2\}.$$

We want to apply Theorem 5.1.2 to the set $U = \{v, u_4, v_4, v'_4, v_3, v_2, v_5\}$ and the corresponding X_u. Arguing as before, one can show that conditions **I1**–**I4** hold. However, condition **I5** does not hold if (and only if) $v_1 = s_1s_2s_1s_2s_3 < v$ and $y \in Y_v$. Indeed, in this case we have $P_{v_1,v} = v^{-L(s_3)}$ and

$$T_{y_0}T_{s_2s_1s_2}T_{v_1} = T_{y_0}\big(T_{s_1s_2s_1s_2s_1s_3} + (v^{L(s_2)} - v^{-L(s_2)})T_{s_1s_2s_1s_2s_1s_2s_3}\big).$$

Note that $y_0.s_1s_2s_1s_2s_1s_3$ and $y_0.s_1s_2s_1s_2s_1s_2s_3$. However, we can certainly construct the elements $\tilde{C}_{x.u}$ such that

$$\tilde{C}_{x.u} = \overline{\tilde{C}_{x.u}} \quad \text{for all } u \in U \text{ and } x \in X_u.$$

Now one can check that

(1) $\tilde{C}_{x.u} = C_{x.u}$ for all $u \in U - \{v\}$ and $x \in X_u$.
(2) $\tilde{C}_{y.v} = C_{y.v}$ if $y \in X_v - Y_v$.

Let $y \in Y_v$. We have

$$\begin{aligned}
\tilde{C}_{yv} &= T_yC_v + \sum_{\substack{u \in U, x \in X_u \\ xu \sqsubset yv}} p^*_{xu,yv}T_xC_u \\
&= T_yC_v + \sum_{x<y} p^*_{xv,yv}T_xC_v + \sum_{\substack{u \in U, x \in X_u \\ u \neq v}} p^*_{xu,yv}T_xC_u \\
&= T_yC_v + \sum_{x<y} p^*_{xv,yv}T_xC_v & \mod \mathcal{H}_{<0} \\
&= T_yC_v & \mod \mathcal{H}_{<0} \\
&= T_yT_v + T_y(P_{v_1,v}T_{v_1}) & \mod \mathcal{H}_{<0} \\
&= T_yT_v + T_{y_0}T_{s_1s_2s_1s_2s_1s_2s_3} & \mod \mathcal{H}_{<0}
\end{aligned}$$

Thus since $\tilde{C_{yv}}$ is stable under the involution $^{-}$, it follows that

$$\tilde{C_{yv}} = C_{yv} + C_{y_0.s_1s_2s_1s_2s_1s_2s_3}.$$

Furthermore, since $y_0.s_1s_2s_1s_2s_1s_2s_3 \in A_3$, we obtain that

$$\mathcal{M}_U = \langle T_xC_u | u \in U, x \in X_u\rangle = \langle C_{x.u} | u \in U, x \in X_u\rangle$$

is a left ideal of $\mathcal{H}$. It follows that

$$B_3 \cup C_4 \cup A_4 \cup A'_4 \cup A_3 \cup A_2 \cup A_5$$

is a left ideal of W. □

PROPOSITION 7.2.10. *The set $B = \cup B_i$ is a two-sided cell.*

PROOF. By the previous proofs, we see that $A \cup C \cup B$ is a left ideal of W. Arguing as in the proof of Proposition 7.2.2, we obtain that B is a two-sided cell. □

7.2.3. Finite cells. We know that E_1, E_2 and F are left cells and that $E_1 \cup E_2$ and F are two-sided cells (see Example 5.3.8). Thus

$$W - A \cup B \cup C \cup F = D_1 \cup D_2 \cup D_3 \cup \{e\}$$

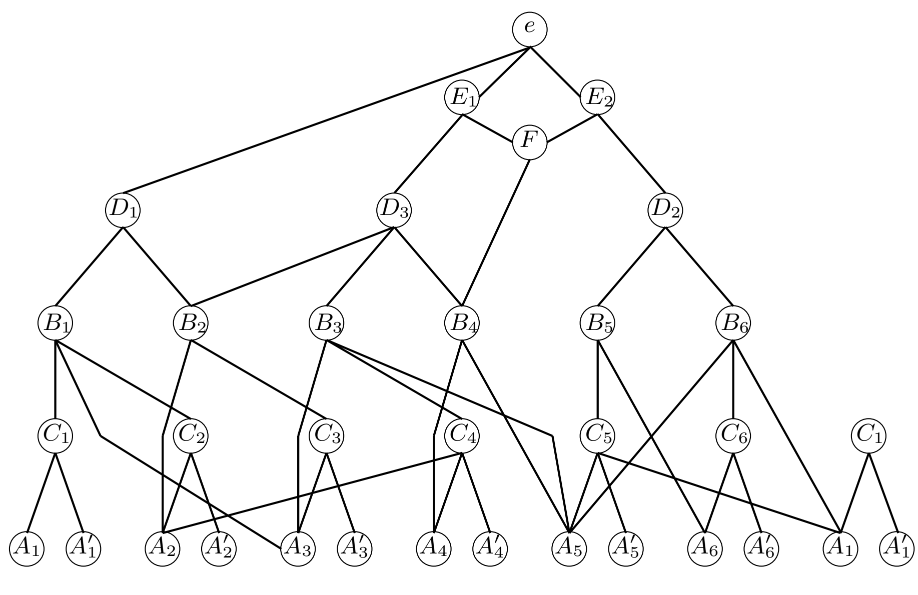
is a union of left and two-sided cells. The set $\{e\}$ is clearly a left cell and a two-sided cell. Now if $w \in D_i$ then $\mathcal{R}(w) = \{s_i\}$, thus each of the D_i's is a left cell. Now one can easily see that

$$D_1 \cup D_2 \cup D_3$$

is a two-sided cell.

7.2.4. Left and two-sided order.

THEOREM 7.2.11. *The partial order induced by $\leq_L$ on the left cells can be described by the following Hasse diagram*



PROOF. Most of the relations can be deduced using the fact that for $s \in S$ and $w \in W$, if $sw > w$ then $sw \leq_L w$. For instance, for all $1 \leq i \leq 6$ we have $A_i \leq_L C_i$ and $A'_i \leq_L C_i$.
Some of the relations require some explicit computations, we refer to [**20**] for details. The fact that there is no other links comes from the last two sections, where we have determined many left ideals of W.

□

THEOREM 7.2.12. *Let $T = D$ or $T = F$. The partial order induced by $\leq_{LR}$ on the two-sided cell is as follows*

$$A \leq C \leq B \leq T \leq E \leq \{e\}$$

and D and F are not comparable.

PROOF. This is easily checked. □

Using the explicit decomposition of $\tilde{G}_2$ in our case, we can check some of Lusztig conjectures ([**38**, Chap14]). For instance

P14. For any $z \in W$, we have $z \sim_{LR} z^{-1}$

is certainly true. The following statement can be deduced from **P4** and **P9**

$$x \leq_L y \quad \text{and} \quad x \sim_{LR} y \quad \Longrightarrow \quad x \sim_L y.$$

In our case, it follows from the partial left order on the left cells. Indeed, there is no relation between two left cells lying in the same two-sided cell.

7.3. Other parameters

In the next two sections we work in the following setting. Let (W, S) be a Coxeter group. Let I, J be two non-empty subsets of S such that

- $S = I \cup J$ (disjoint union);
- if $s \in I$ and $t \in J$ then s and t are not conjugate.

7.3.1. Parameters equal to 0**.** Let L be a weight function on W such that $L(s) = 0$ for all $s \in I$. Set

$$\tilde{J} := \{wtw^{-1} | w \in W_I, t \in J\}.$$

Let $\tilde{W}$ be the subgroup of W generated by $\tilde{J}$. Then it can be shown that $(\tilde{W}, \tilde{J})$ is a Coxeter group (see [**6**, Theorem 1] and the references there) and

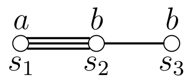
$$W = W_I \ltimes \tilde{W}.$$

Let $\tilde{t} \in \tilde{J}$. We denote by $\nu(\tilde{t})$ the unique element of J which is conjugate to $\tilde{t}$. Then the function $\tilde{L} : \tilde{J} \to \mathbb{N}$ defined by $\tilde{L}(\tilde{t}) = L(\nu(\tilde{t}))$ is a weight function on $\tilde{W}$; see [**6**]. Let $\mathcal{H}_{\tilde{W}}$ be the Hecke algebra associated to the Coxeter group $\tilde{W}$ and the weight function $\tilde{L}$. The group W_I acts on $\tilde{W}$ and stabilize J, thus it acts naturally on $\mathcal{H}_{\tilde{W}}$ and we can form the semi direct product

$$W_I \ltimes \mathcal{H}_{\tilde{W}}.$$

It turns out (see [**5**, Corollary 5.14]) that the left cells of W with respect to the weight function L are of the form $W_I.C$ where C is a left cell of $\tilde{W}$ with respect to $\tilde{L}$. In particular W_I is a left cell of W.

Example 7.3.1. Let W be an affine Weyl group of type $\tilde{G}_2$ with diagram and weight function given by



where $a, b \in \mathbb{N}$. Assume that $b = 0$ and $a > 0$. Then we have

$$W = W_{2,3} \ltimes \tilde{W}$$

where $\tilde{W}$ is of type $\tilde{A}_2$ and is generated by $\tilde{J} = \{s_1, s_2s_1s_2, s_3s_2s_1s_2s_3\}$. Since we know the decomposition of $\tilde{A}_2$ into left cells (see [**30**]) one can easily find the decomposition of W into left cells in that case (see Figure 2).

Assume that $a = 0$ and $b > 0$. Then we have

$$W = W_1 \ltimes \tilde{W}$$

where $\tilde{W}$ is of type $\tilde{A}_2$ and is generated by $\tilde{J} = \{s_2, s_3, s_1s_2s_1\}$. As before, one can easily find the decomposition of W into left cells in that case (see Figure 10).

7.3.2. Semicontinuity. Recently, Bonnafé has conjectured that the Kazhdan-Lusztig cells should satisfy some "semicontinuity" properties (see [**5**]) when the parameters are varying. We describe briefly this conjecture in the two parameter case and we refer to [**5**] for a more general setting.

Let $L_{a,b}$ $(a, b \in \mathbb{N})$ be the weight function which takes the value a on I and b on J. Let $r = a/b$ (we set $r = 0$ if $a = 0$ and $b > 0$ and $r = \infty$ if $a > 0$ and $b = 0$). The decomposition of W into cells only depends on r. We denote by $\mathcal{L}_r(W)$ the decomposition into left cells associated to r.

Bonnafé has conjectured the following.

CONJECTURE 7.3.2. *There exists an integer m and some rational numbers $0 < r_1 < ... < r_m$ (we set $r_0 = 0$ and $r_{m+1} = \infty$) which depends on W such that for all $\theta, \theta' \in \mathbb{Q}$ we have*

(1) *if $r_i < \theta, \theta' < r_{i+1}$ for some $0 \leq i \leq m$, then $\mathcal{L}_\theta(W) = \mathcal{L}_{\theta'}(W')$;*
(2) *if $r_{i-1} < \theta < r_i < \theta' < r_{i+1}$ for some $1 \leq i \leq m$, then the left cells of $\mathcal{L}_{r_i}(W)$ are the smallest subsets of W which are at the same time unions of left cells of $\mathcal{L}_\theta(W)$ and unions of left cells of $\mathcal{L}_{\theta'}(W)$.*
(3) *if $0 < \theta < r_1$ then the left cells of $\mathcal{L}_0(W)$ are the smallest subsets of W which are at the same time unions of left cells of $\mathcal{L}_\theta(W)$ and stable by multiplication on the left by W_I.*
(4) *if $r_m < \theta$ then the left cells of $\mathcal{L}_\infty(W)$ are the smallest subsets of W which are at the same time unions of left cells of $\mathcal{L}_\theta(W)$ and stable by multiplication on the left by W_J.*

In the finite case, the existence of the rational numbers $0 < r_1 < ... < r_m$ is clear. In the case, $\tilde{G}_2$ it is proved in Chapter 4.

REMARK 7.3.3. We have seen that W_I is a left cell of $\mathcal{L}_0(W)$. Thus the conjecture implies that, for θ small enough, W_I should be a union of left cells of $\mathcal{L}_\theta(W)$. In the case where W_I is finite, it has been proved in Corollary 5.3.7.

REMARK 7.3.4. One can easily state similar conjectures for right and two-sided cells.

7.3.3. Semicontinuity in $\tilde{G}_2$. Let W be an affine Weyl group of type $\tilde{G}_2$ with diagram and weight function as defined in Example 7.3.1. We denote by r the ratio a/b (we set $r = 0$ if $a = 0$ and $b > 0$ and $r = \infty$ if $a > 0$ and $b = 0$). The following figures present some conjectural decompositions of W into left cells for different values of r. In each case, using our GAP3 program, we can show that the decomposition is included in the left cell decomposition. One can check that these computations agree with the "semicontinuity conjecture".

REMARK 7.3.5. The decomposition in the case $r = \infty$ and $r = 0$ are the actual left cell decomposition (see Example 7.3.1). The decomposition in the equal parameter case ($r = 1$) has been proved by Lusztig in **[30]**.

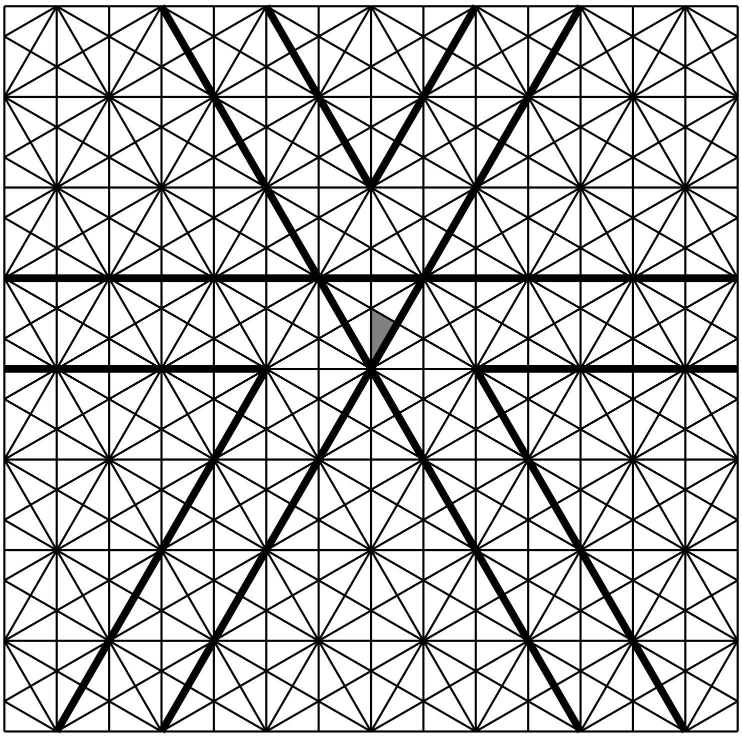


FIGURE 2. $r = \infty$

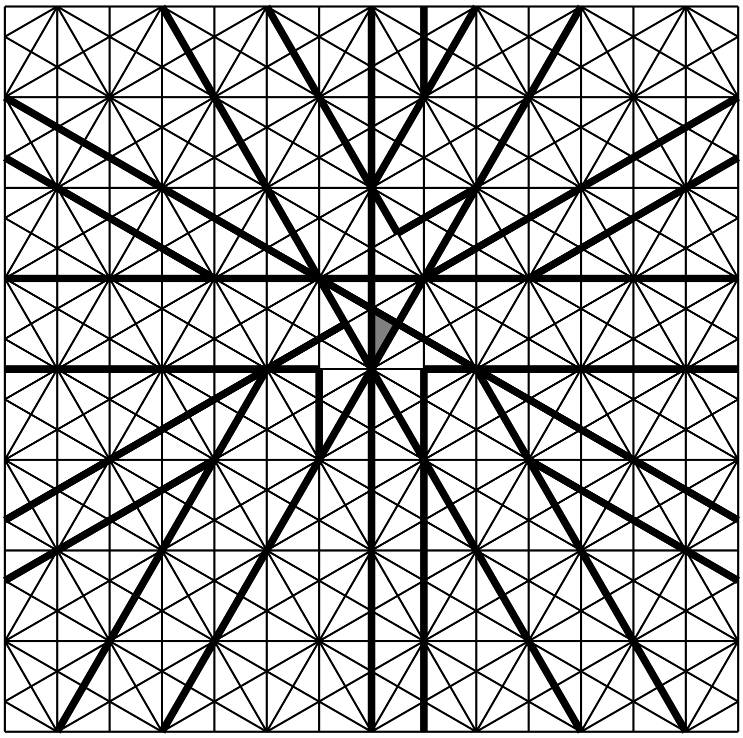


FIGURE 3. $r > 2$

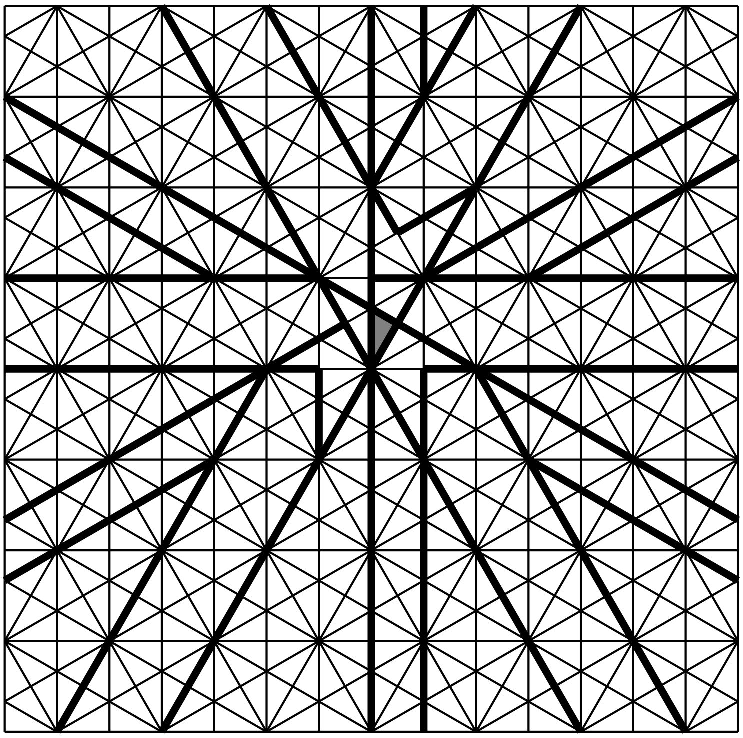


FIGURE 4. $r = 2$

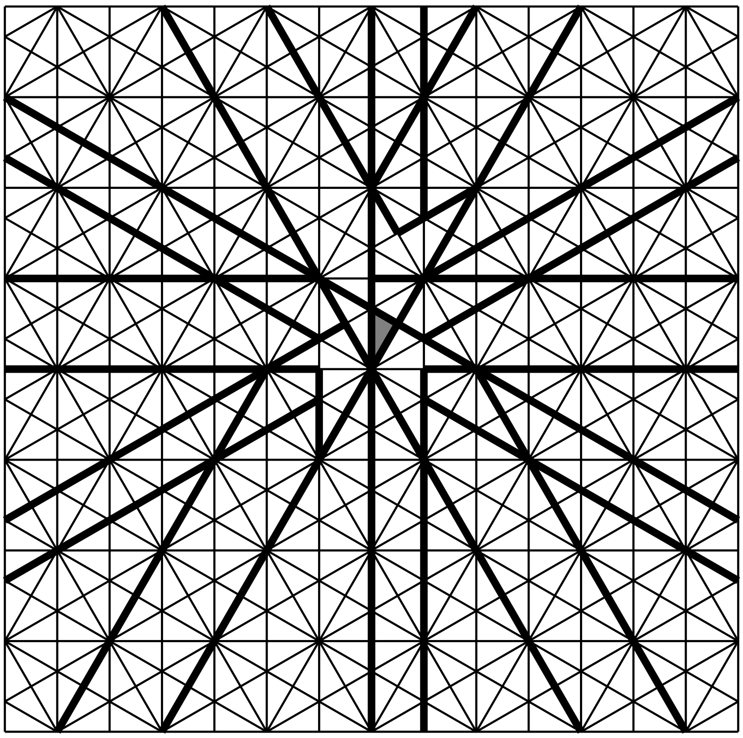


FIGURE 5. $2 > r > 3/2$

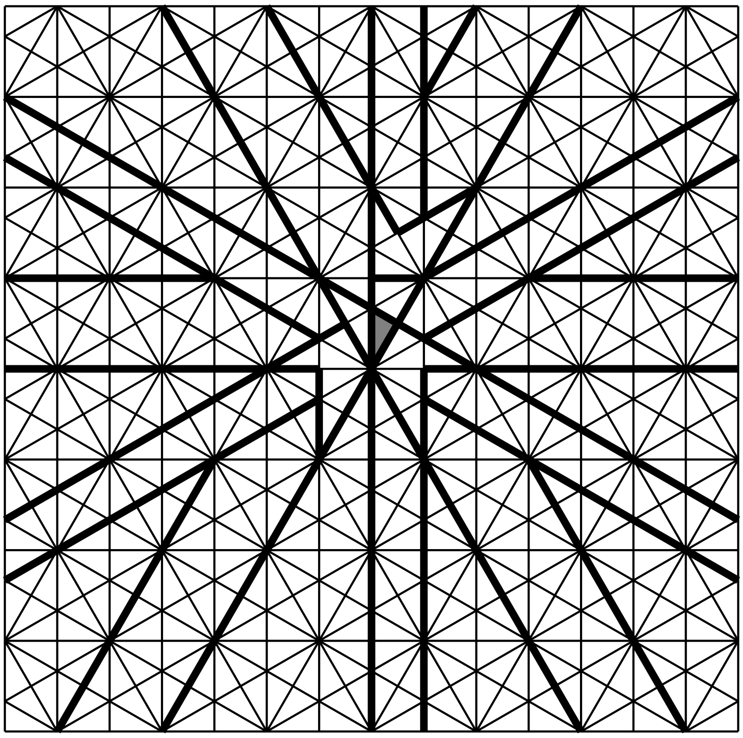


FIGURE 6. $r = 3/2$

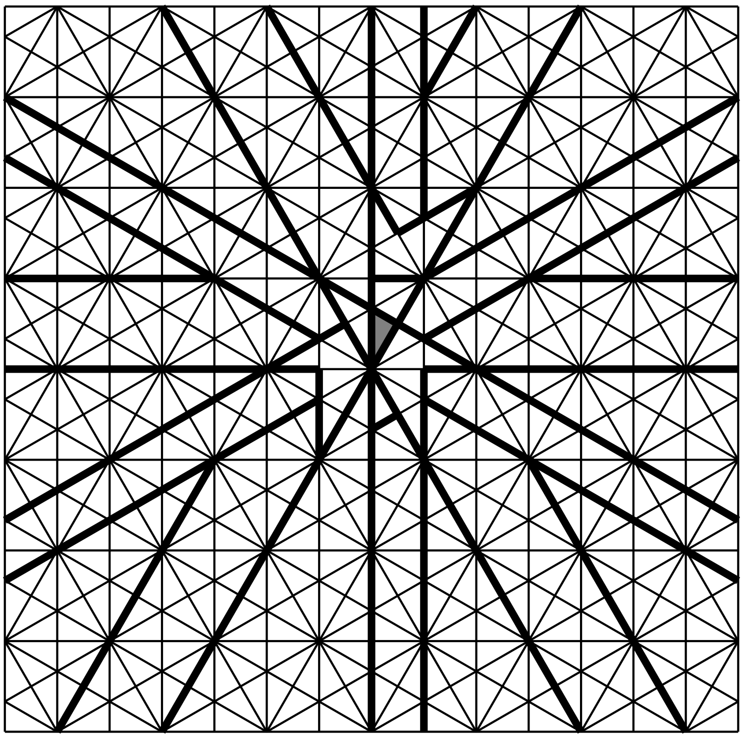


FIGURE 7. $3/2 > r > 1$

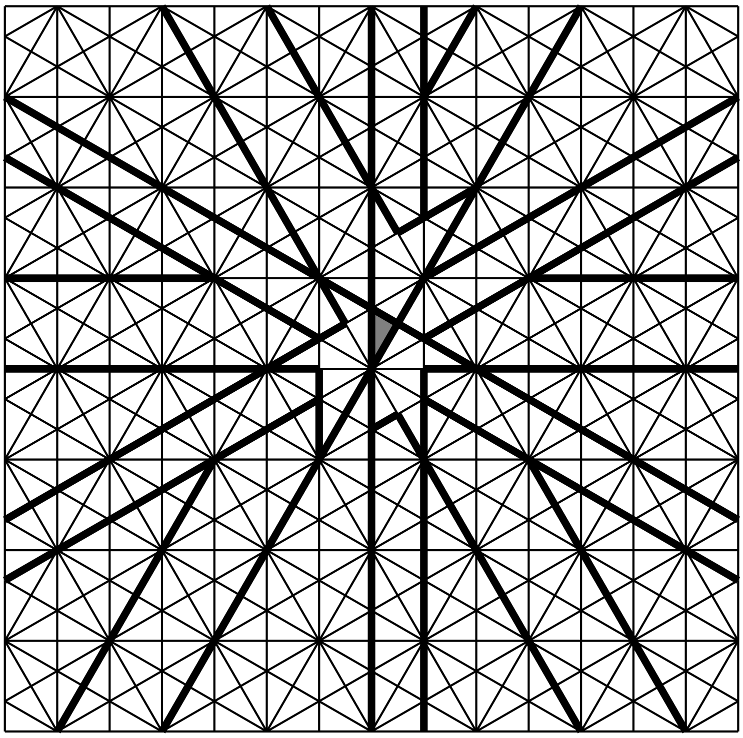


FIGURE 8. $r = 1$

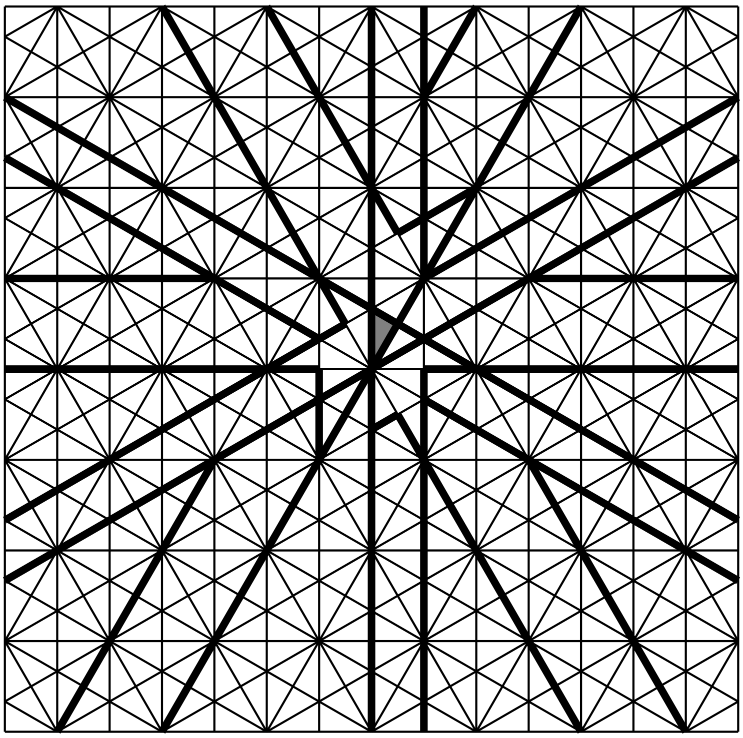


FIGURE 9. $r < 1$

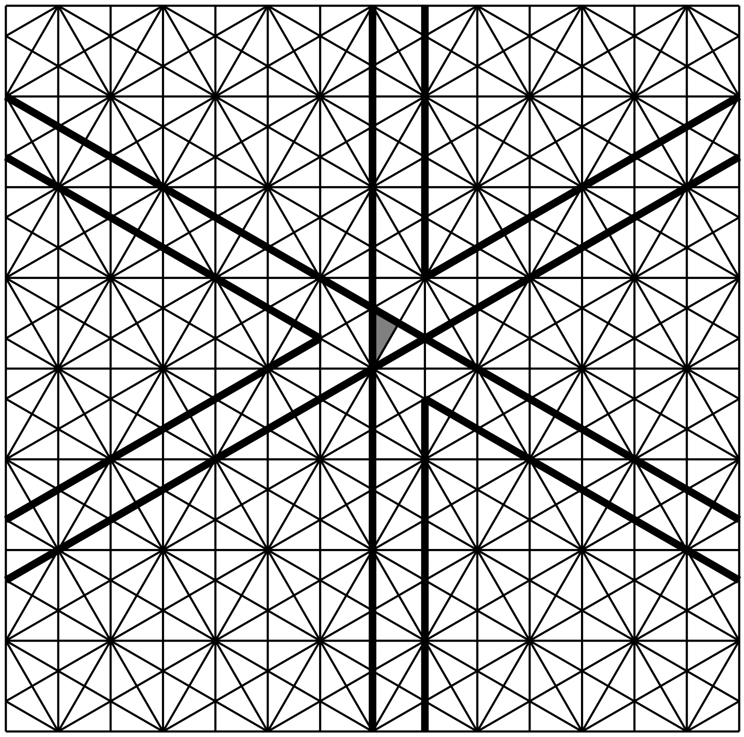


FIGURE 10. $r = 0$

REMARK 7.3.6. Note that not all the cases that we have described above arise "in nature"; we have seen in the introduction that the only parameters on $\tilde{G}_2$ that arise in the framework of representation of reductive groups over p-adic field are

$$(9,1,1),\ (3,1,1),\ (1,1,1),\ (1,3,3).$$

Bibliography

[1] D. L. Alvis, The left cells of the Coxeter group of type H_4, *J. Algebra* **107**, *160–168*, 1987.

[2] R. Bedard, Cells for two Coxeter groups, *Comm. Algebra* **14**, *1253–1286*, 1986.

[3] R. Bedard, The lowest two-sided cell for an affine Weyl group, *Comm. Algebra* **16**, *1113–1132*, 1988.

[4] C. Bonnafé, Two-sided cells in type B (asymptotic case), *J. Algebra* **304**, *216–236*, 2006.

[5] C. Bonnafé, Semi-continuité des cellules de Kazhdan-Lusztig, *Preprint available at http://arxiv.org/abs/0805.3038*, 2008.

[6] C. Bonnafé and M. Dyer, Semidirect product decomposition of Coxeter groups, *Preprint available at http://arxiv.org/abs/0805.4100*, 2008.

[7] C. Bonnafé and L. Iancu, Left cells in type B_n with unequal parameters, *Represent. Theory* **7**, *587–609*, 2003.

[8] N. Bourbaki, *Groupes et algèbres de Lie, Chap 4–6*, Hermann, Paris, 1968; Masson, Paris, 1981.

[9] K. Bremke, On generalized cells in affine Weyl groups, *J. Algebra* **191**, *149–173*, 1997.

[10] C. Chen, The decomposition into left cells of the affine Weyl group of type $\tilde{D}_4$, *J. Algebra* **163**, *692–728*, 1994.

[11] F. Du Cloux, Coxeter, version 3, *available at http://math.univ-lyon1.fr/∼ducloux/coxeter/coxeter3/english/coxeter3_ e.html.*

[12] F. Du Cloux, An abstract model for Bruhat intervals, *European J. Combin.* **21**, *197–222*, 2000.

[13] F. Du Cloux, Positivity results for the Hecke algebras of noncrystallographic finite Coxeter groups, *J. Algebra* **303**, *731–741*, 2006.

[14] J. Du, The decomposition into cells of the affine Weyl group of type $\tilde{B}_3$, *Comm. Algebra* **16**, *1383–1409*, 1988.

[15] M. Geck. On the induction of Kazhdan-Lusztig cells *Bull. London Math. Soc.* **35**, *608–614*, 2003.

[16] M. Geck, Computing Kazhdan-Lusztig cells for unequal parameters, *J. Algebra* **281**, *342–365*, 2004.

[17] M. Geck, Hecke algebras of finite type are cellular, *Invent. Math.* **169**, *501–517*, 2007.

[18] M. Geck, Remarks on Iwahori-Hecke algebras with unequal parameters, *Preprint available at http://arxiv.org/pdf/0711.2522*, 2007.

[19] J. J. Graham and G. I. Lehrer, Cellular algebras, *Invent. Math.* **123**, *1–34.*, 1996.

[20] http://math.univ-lyon1.fr/∼guilhot/Computation.pdf

[21] R. B. Howlett and Y. Yin, Inducing W-graphs, *Math. Z.* **244**, *415–431*, 2003.

[22] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advance Mathematics **29**, Cambridge University Press, 1990.

[23] N. Iwahori, On the structure of a Hecke ring of a Chevalley group over a finite field, *J. Fac. Sci. Univ. Tokyo Sect. I,* **10**, *215–236*, 1964.

[24] D. A. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras *Invent. Math* **53**, *165–184*, 1979.

[25] D. A. Kazhdan and G. Lusztig, Schubert varieties and Poincaré duality, *Proc. Sympos. Pure Math.* **36**, *Amer. Math. Soc., 185–203*, 1980.

[26] G. Lusztig, Representations of finite Chevalley groups, *CBMS Regional Conf. Ser. in Math., vol. 39, Amer. Math. Soc., Providence, RI*, 1977.

[27] G. Lusztig, Hecke algebras and Jantzen's generic decomposition patterns, *Adv. in Math.* **37**, *121–164*, 1980.

[28] G. Lusztig, Left cells in Weyl groups, *Lie Group Representations, I (eds R. L. R. Herb and J. Rosenberg), Lecture Notes in Math., Springer, Berlin, vol. 1024, 99–111*, 1983.

[29] G. Lusztig, *Characters of reductive groups over a finite field*, Ann. of Math. Stud., vol 107, Princeton University Press, Princeton, NJ, 1984.

[30] G. Lusztig, Cells in affine Weyl groups, *Advanced Studies in Pure Math.,* **6**, *Algebraic groups and related topics, Kinoku-niya and North-Holland, 255–287*, 1985.

[31] G. Lusztig, The two-sided cells of the affine Weyl group of type $\tilde{A}_n$, *Infinite dimensional groups with applications, Math. Sci. Res. Inst. Publ.,* **4**, *Springer, New York, 275–283*, 1985.

[32] G. Lusztig, Sur les cellules gauches des groupes de Weyl, *C. R. Acad. Sci. Paris Sér. I Math.* **302**, *5–8*, 1986.

[33] G. Lusztig, Cells in affine Weyl groups II, *J. Algebra* **109**, *536–548*, 1987.

[34] G. Lusztig, Cells in affine Weyl groups III, *J. Fac. Sci. Univ. Tokyo* **34**, *223–243*, 1987.

[35] G. Lusztig, Cells in affine Weyl groups IV, *J. Fac. Sci. Univ. Tokyo* **36** *297–328*, 1989.

[36] G. Lusztig. Intersection cohomology methods in representation theory *Proc. Int. Congr. Math. Kyoto., Springer Verlag, 155–174*, 1991.

[37] G. Lusztig, Classification of unipotent representations of simple p-adic groups, *Internat. Math. Res. Notices, 517–589*, 1995.

[38] G. Lusztig, Hecke algebras with unequal parameters, *CRM Monograph Series* **18**, *Amer. Math. Soc., Providence, RI*, 2003.

[39] Martin Schoenert et al. *GAP – Groups, Algorithms, and Programming – version 3 release 4 patchlevel 4*. Lehrstuhl D fur Mathematik, Rheinisch Westfalische Technische Hochschule, Aachen, Germany, 1997.

[40] J.-Y. Shi, The Kazhdan-Lusztig cells in certain affine Weyl groups, *Lecture Notes in Math., Springer-Verlag, Berlin, vol. 1179*, 1986.

[41] J.-Y. Shi, A two-sided cell in an affine Weyl group, *J. London Math. Soc.* **36**, *407–420*, 1987.

[42] J.-Y. Shi, A two-sided cell in an affine Weyl group II, *J. London Math. Soc.* **37**, *253–264*, 1988.

[43] J.-Y. Shi, Left cells in affine Weyl groups, *Tokohu Math. J.* **46**, *105–124*, 1994.

[44] J.-Y. Shi, Left cells in the affine Weyl group $W_a(\tilde{D}_4)$, *Osaka J. Math.* **31**, *27–50*, 1994.

[45] T. A. Springer, Quelques applications de la cohomologie d'intersection, *Séminaire Bourbaki (1981/82), exp. 589, Astérisque* **92–93**, 1982.

[46] N. Xi, Representations of affine Hecke algebras, *Lecture Notes in Math., Springer-Verlag, Berlin, vol. 1587*, 1994.