Chain models, trees of singular cardinality and dynamic EF-games[*]

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Abstract

Let κ be a singular cardinal. Carol Karp's notion of a chain model of size κ is defined to be an ordinary model of size κ along with a decomposition of it into an increasing union of length $\mathrm{cf}(\kappa)$. With a notion of satisfaction and (chain)-isomorphism such models give an infinitary logic largely mimicking first order logic. In this paper we associate to this logic a notion of a dynamic EF-game which gauges when two chain models are chain-isomorphic. To this game is associated a tree which is a tree of size κ with no κ-branches (even no $\mathrm{cf}(\kappa)$-branches). The measure of how non-isomorphic the models are is reflected by a certain order on these trees, called reduction. We study the collection of trees of size κ with no κ-branches under this notion and prove that when $\mathrm{cf}(\kappa) = \omega$ this collection is rather regular; in particular it has universality number exactly κ^+. Such trees are then used to develop a descriptive set theory of the space $^{\mathrm{cf}(\kappa)}\kappa$.

The main result of the paper gives in the case of κ strong limit singular an exact connection between the descriptive set-theoretic complexity of the chain isomorphism orbit of a model, the reduction order on the trees and winning strategies in the corresponding dynamic EF games. In particular we obtain a neat analogue of the notion of Scott watershed from the Scott analysis of countable models.

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0 Introduction

We shall work with κ a singular cardinal of countable cofinality, and our interest is in rooted trees of height and cardinality κ. We call such trees κ-*Trees* (as opposed to the usual κ-*trees*, which are also required to have levels of size $< \kappa$). In particular, we are interested in κ-Trees which in addition do not have κ-branches. We call κ-Trees with this last property *bounded*, otherwise a κ-Tree is said to be *unbounded*. In analogy with the body of knowledge which is available about λ-Trees for λ regular (see [ToVä]) we study the class of bounded κ-Trees under the natural notion of reduction, which is simply a strict-order preserving function from one tree to another.

Let us now explain how these considerations have arisen in the context of model theory and descriptive set theory.

A relational structure of size λ can be considered as an element of 2^λ, which is an insight going back to 1930s, in particular to C. Kuratowski and A. Tarski [KuTa]. This insight was first exploited in the context of countable models. To understand the classification of such models up to isomorphism Ehrenfeucht [Eh], building on the work of Fraïssé [Fr1], [Fr2], introduced a game now known as the EF game. There is an even older method of classification, using the fact that for two fixed countable models the set of all isomorphisms between them is $F_{\delta\sigma}$, and hence the set of pairs of models that are non-isomorphic is co-analytic. It therefore can be covered by $\aleph_1$ many Borel sets, by the covering theorem of Lusin-Sierpiński [LuSi]. In fact, the main content of the Ehrenfeucht-Fraïssé Theorem (see e.g. [ChKe]) is that this set,

$$\{(A, B) :\ A, B \text{ countable models and } \exists f :\ A \cong B\}$$

is the same set as the set of pairs (A, B) of countable models for which II has a winning strategy in the EF game. By Dana Scott's analysis in the 1960s [Sc] the latter can be seen as the intersection of $\aleph_1$ Borel sets arising from ranking the EF game by the countable ordinals. This analysis made it possible to attach to each pair (A, B) of non-isomorphic countable models a rank, called Scott watershed $S(A, B)$, which in this case is an ordinal $\alpha < \omega_1$. The rank can be thought of as a clock in the following sense: during the EF game the Nonisomorphism player I has to at every stage go down this clock, starting at α itself, and a condition of winning for I is that he has not run out of time before the actual nonisomorphism has been exposed. This game is called the dynamic EF game of rank $\alpha + 1$ and denoted by $EFD_{\alpha+1}$ (the notation comes from [Vä2]). The fact that $S(A, B) = \alpha + 1$ means that II wins EFD_α (and hence EFD_β for any $\beta < \alpha$), while I wins $EFD_{\alpha+1}$ (and hence EFD_β for any $\beta > \alpha$).

For uncountable models, say of size $\aleph_1$, one can generalise the Ehrenfeucht-Fraïssé Theorem by considering games of length ω_1. One is then tempted to find the corresponding notion of the Scott watershed. It turns out that it is no longer enough to use the

ordinals, as the game is transfinite. The right notion this time is that of a tree of size $\aleph_1$ with no uncountable branches. The class $\mathcal{T}_{\aleph_1}$ of these trees is ordered by *reductions*, which are strict order-preserving functions. We denote the existence of a reduction from T to T' by $T \leq T'$. Furthermore, we write $T < T'$ if both $T \leq T'$ and $T' \not\leq T$. If $T \leq T'$ then it is easier for II to win EFD_T than $EFD_{T'}$. Respectively, if $T \leq T'$ then it is easier for I to win $EFD_{T'}$ than EFD_T. Finally, if II wins EFD_T and I wins $EFD_{T'}$, then one can prove $T < T'$. This approach was gradually developed in Helsinki in the 1980s, culminating in the paper [HyVä] where the general theory was established, and in particular it was shown that the analogue of the Scott watershed exists also in the uncountable case. Naturally, the importance of $(\mathcal{T}_{\aleph_1}, \leq)$ in this context led to a systematic study of its structural properties (e.g. [MeVä], [ToVä]). Descriptive set theory of the space 2^{ω_1} based on $\mathcal{T}_{\aleph_1}$ (see again [MeVä]) was subsequently developed. The theory of $(\mathcal{T}_\lambda, \leq)$ for λ successor of regular is also quite known, although it is not completely parallel to that of $(\mathcal{T}_{\aleph_1}, \leq)$-see [Vä2]. Juha Oikkonen [Oi] proved in 1991 a covering theorem for a certain infinite quantifier infinitary logic, using trees of singular cardinality. In this paper we develop the structure theory of $(\mathcal{T}_\kappa, \leq)$ for κ singular, concentrating especially on κ of countable cofinality.

When we move to such cardinals new possibilities open up. Let us look at the infinitary logic $L_{\kappa\kappa}$, assuming that κ is singular of cofinality ω. Starting with Carol Karp's Ph.D. thesis in 1959 and in later work (see [Ka]) Karp and others showed that one can develop model theory of $L_{\kappa\kappa}$ based on the concept of a *chain model*. This was further developed by Michael Makkai [Mak]. A chain model[1] is an ordinary model A equipped with a presentation of A as a union of a chain $A_0 \subseteq A_1 \ldots \subseteq A_n \subseteq \ldots$ for $n < \omega$. The chain is not assumed to be elementary. Let us denote such a system as (A_n). The point of chain models is the following modification of the truth definition of $L_{\kappa\kappa}$

$$(A_n) \models \exists \bar{x}\varphi(\bar{x}) \iff \text{ there are } n < \omega \text{ and } \bar{a} \in A_n \text{ with } A_n \models \varphi(\bar{a}), \tag{1}$$

where $\bar{x}$ is a sequence of length $< \kappa$. If we restrict to chain models, the model theory of $L_{\kappa\kappa}$ is very much like that of $L_{\omega_1\omega}$. The fundamental observation of Karp was that the proof of the Completeness Theorem of $L_{\omega_1\omega}$ using consistency properties works fine when we have infinite homogeneous strings of quantifiers if we modify truth definition as above in (1). Thus we can also prove undefinability of well order, Craig Interpolation Theorem, Beth Definability Theorem, etc. None of these theorems is true for the classical $L_{\kappa\kappa}$ logic, by the work of Gostanian and Hrbaček ([GoHr]) and Malitz ([Mal]). A discussion involving many of these developments can be found in [Vä1]. Our paper can be seen as an extension of Scott's analysis of countable models to chain models of size κ. This possibility was announced by C. Karp in [Ka]. Most likely this promising direction would have been continued by her had it not been for her premature death.

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[1]This is the definition used by C. Karp. We use a somewhat different language, see Definition 1.2.

There are several natural version of EF game for (chain) models of a singular cardinality κ. We show in Section 1 that the relevant clock trees of these games are trees of height κ without κ-branches. Thus to understand the Scott watershed phenomenon an investigation of the order of such trees under reduction is called for. This is what we set out to accomplish in this paper.

We shall show that κ-Trees for κ as above have properties that make them rather similar to ordinals. The reason for this is that there is a natural notion of rank. Using this notion we can for example show that the universality number of unbounded κ-Trees under reduction is just κ^+, and that within each rank in $[1, \kappa^+)$ it is ω. This is in sharp contrast with the situation of λ-Trees where λ is a regular cardinal. For example for $\lambda = \omega_1$ Mekler and Väänänen [MeVä] have established that the universality number for λ-Trees cannot be computed in ZFC.

Another insight of [MeVä] is the use of ω_1-Trees in the descriptive set theory in ${}^{\omega_1}\omega_1$. In analogy with that we consider Section 7 the descriptive set theory of ${}^{\mathrm{cf}(\kappa)}\kappa$ for κ singular strong limit. We present a covering theorem for Π^1_1 subsets of the topological space ${}^\theta\kappa$ where $\theta = \mathrm{cf}(\kappa)$ and κ is singular strong limit. This result was inspired by a theorem of Juha Oikkonen [Oi] from 1991 who proved a covering theorem for an infinitary logic. Our covering theorem and that one from [MeVä] are obtained by using a corresponding class of trees, in our case the κ-Trees which we studied in the rest of the paper.

Our notation is standard, in particular, $h_T(t)$ denotes the height of an element t in a tree T. The order on T is denoted by $\leq_T$. If T is clear from the context we may omit it. A tree is said to be *normal* if for any $t \neq t'$ of a limit height, we have $\{s : s <_T t\} \neq \{s : s <_T t'\}$. Level α of T is denoted by $\mathrm{lev}_\alpha(T)$. If $t \in T$ then $T[\geq t]$ is the set $\{s \in T : s \geq t\}$ equipped with the order inherited from T. The following is a crucial definition:

Definition 0.1 *A reduction from a tree T into a tree T' is a function $f : T \to T'$ such that whenever $x <_T y$ in T, then $f(x)' <_T f(y)$. If there is such a reduction we write $T \leq T'$.*

We shall also make use of the following operator σ^2 defined on the class of trees:

Definition 0.2 *If T is a tree then σT is the tree whose elements are strictly $<_T$-increasing sequences of the elements of T, ordered by end extension.*

The importance of this operator is that it is easily checked that there is no reduction from σT into T.

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[2]The definition of σ comes from Kurepa's work on Souslin's hypothesis, see [ToVä] for more.

Most of the trees we consider in this paper will have size and height κ for some fixed cardinal κ, with no restriction on the size of the levels of the tree. We call such trees *κ-Trees*, to be distinguished from the usual κ-trees whose levels are also required to have size $< \kappa$. A κ-Tree is said to be *bounded* if it does not have a branch of length κ. A κ-Tree that has no branches of length μ is called a *(κ, μ)-Tree*.

The paper is organised as follows. In section 1 we show how κ-Trees and games involving them arise naturally in the context of C. Karp's chain models. In section 2 we introduce a rank operation and show that it characterizes bounded κ-Trees, and then we introduce a game characterisation of ranks. In section 3 we introduce various operations on trees which will allow us to study the structure of the class of κ-Trees. In §4 we consider how one can relate the known results about universality of trees at uncountable regular θ, say at $\theta = \aleph_1$, to singular cardinals of that cofinality. This shows that the ZFC results we obtain later in §5 for cofinality ω must be special to that cofinality. In §5 we obtain these ZFC results, allowing us the exact calculation of the universality number of the class of κ-Trees, even within any given rank. In section 6 we consider the problem of the existence of incomparable trees. Finally, in section 7 we give the promised descriptive set-theory results for the space $^{\mathrm{cf}(\kappa)}\kappa$ for κ singular strong limit, and we finish by giving a theorem which in the case of such cardinals connects all of the mentioned notions, giving an exact connection between chain isomorphisms, descriptive set-theoretic complexity and the generalised EF games and trees.

1 Chain models and κ-Trees

To show how κ-Trees and games involving them arise naturally in the context chain models, we first review the case of ordinary models. Recall the notion of the Scott watershed from the Introduction.

Let κ be an infinite cardinal. The original EF game, denoted $EF_\kappa(A, B)$, for models A and B of size κ is as follows: players play elements of $A \cup B$ for κ rounds, and at each round II plays in the model in which I did not play at that round. II wins this game if the end result is a partial isomorphism between A and B. Obviously, $A \cong B$ iff II has a winning strategy in $EF_\kappa(A, B)$. Suppose that in addition to the models A and B we are given a tree T. The dynamical version $EFD_T(A, B)$ of $EF_\kappa(A, B)$ is defined by requiring the player I to have the extra task of moving up the tree T at every move, and if he runs out of the space to move, he loses.

Let τ be the strategy of I in which he systematically lists elements of $A \cup B$. Let S_1 be the tree of plays of II against τ where she has not lost yet. Let K_1 be the tree of winning strategies of II in shorter versions of $EF_\kappa(A, B)$ where we limit the length of the game to be an ordinal $< \kappa$. The ordering of the strategies of K_1 is canonical: a

strategy ρ precedes another strategy ρ' if ρ is the restriction of ρ' to the length of the game in ρ.

Lemma 1.1 *Assume that κ is a cardinal and A, B are models. Then $K_1 \leq S_1$. If $A \not\cong B$ then the trees S_1 and K_1 do not have κ-branches, and $K_1 \leq S_1$. Moreover, in this case II wins $EFD_{K_1}(A, B)$ and I wins $EFD_{\sigma S_1}(A, B)$.*

Proof. If $\rho \in K_1$, define $f(\rho)$ to be the result of applying ρ to τ- i.e. both players play their strategies. The result is in S_1. Clearly this function is increasing. Thus $K_1 \leq S_1$. It is an immediate consequence of $A \not\cong B$ that S_1 and K_1 do not have κ-branches. The reason why II wins $EFD_{K_1}(A, B)$ is that the moves of I in K_1 reveal longer and longer strategies that II can use against I. The reason why I wins $EFD_{\sigma S_1}(A, B)$ is that he can submit the sequence of previous moves of II as his moves in the tree σS_1. ★

So we have found upper (σS_1) and lower (K_1) bounds for a Scott watershed. Our cardinal number κ can be also singular. However, the trees S_1 and K_1 need not be of cardinality κ. All we can say is that $|S_1| \leq \kappa^{<\kappa}$ and $|K_1| \leq \kappa^{\kappa^{<\kappa}}$, and so in the case of singular cardinal κ no cardinal arithmetic assumption may assure us that these trees have size κ.

We shall now modify the above EF game to take into account further information about the models A and B. The further information is that the models are built up from a chain of smaller models.

Definition 1.2 *A chain model (of length μ) consists of a model A and a decomposition $\langle A_\alpha : \alpha < \mu \rangle$. It is assumed that $A = \bigcup_{\alpha < \mu} A_\alpha$, and that $\langle A_\alpha : \alpha < \mu \rangle$ is an increasing sequence of models satisfying $|A_\alpha| < |A|$. A chain isomorphism between A and B is an isomorphism $f : A \to B$ such that for all α the image of A_α is contained in some B_β and, conversely, the preimage of any B_α is contained in some A_β. If there is a chain isomorphism between A and B we write $A \cong_{chain} A$.*

The original language used by Karp to study chain models did not involve trees as clocks, however we can now show that chain models lend themselves to an analysis using exactly the κ-Trees as considered in this paper.

The most obvious EF game for *chain* models A and B of size κ is the following: The players play elements of $A \cup B$ for κ rounds. At any given moment all the played elements have to be contained in some $A_\alpha \cup B_\alpha$. This game is denoted by $EF^c_\kappa(A, B)$, and the respective dynamical version by $EFD^c_T(A, B)$. II wins this game if the end result is a partial chain isomorphism between A and B. Obviously, $A \cong_{\text{chain}} B$ iff II has a winning strategy in $EF^c_\kappa(A, B)$. Let τ be the strategy of I in which he systematically lists elements of $A_\alpha \cup B_\alpha$ for each α, smaller values of α before the larger values. Similarly

to the way S_1 was defined, let S_2 be the tree of plays of II against τ where she has not lost yet. Also, let K_2 be again the tree of winning strategies of II in shorter versions of $EF_\kappa^c(A, B)$ where we limit the length of the game to be an ordinal $< \kappa$. It is again easy to see, as in Lemma 1.1, that I wins $EFD_{\sigma S_2}^c(A, B)$, II wins the game $EFD_{K_2}^c(A, B)$, and $K_2 \leq S_2$. The trees S_2 and K_2 do not have κ-branches, if we assume $A \not\cong_{\text{chain}} B$. The difference to the previous game is that now, if we assume that κ is a strong limit, $|\sigma S_2| \leq \kappa$ and $|K_2| \leq \kappa$, so these are κ-Trees. Such trees are the topic of Sections 2-6.

The EF game for models A and B of size κ, needed to prove $L_{\infty\kappa}$-elementary equivalence in the classical sense is the following: players play sequences of elements of $A \cup B$ for ω steps. The sequences have to be shorter than κ. This game is denoted by $EF_\omega^{<\kappa}(A, B)$, and the respective dynamic version by $EFD_T^{<\kappa}(A, B)$. II wins this game if the end result is a partial isomorphism between A and B. Player II has a winning strategy in $EF_\omega^{<\kappa}(A, B)$ if and only if the models A and B satisfy (in the classical sense) the same $L_{\infty\kappa}$-sentences, in symbols $A \equiv_{\infty\kappa} B$. Note that II having a winning strategy in $EF_\omega^{<\kappa}(A, B)$ is not a guarantee for the isomorphism of A and B (see [NaSt]). In this respect the games $EF_\kappa(A, B)$ and $EF_\kappa^c(A, B)$ are "better". Assuming $A \not\equiv_{\infty\kappa} B$, there is a minimal $\alpha < (\kappa^{<\kappa})^+$ such that I wins $EFD_{\sigma T}^{<\kappa}(A, B)$, where T is the tree of descending sequences of elements of α. By determinacy then II wins $EFD_T^{<\kappa}(A, B)$. In this case $|T| \leq \kappa^{<\kappa}$, so T is a $(\kappa^{<\kappa}, \omega)$-Tree.

Finally, the EF game for chain models A and B of size κ and length μ, needed to prove their $L_{\infty\kappa}$-elementary equivalence as chain models (in the sense of (1)) is the following: players play sequences of length $< \kappa$ of elements of A or B for μ steps. The sequences have to be contained in some $A_\alpha \cup B_\alpha$. This game is denoted by $EF_\mu^{c, <\kappa}(A, B)$ and the respective dynamic version by $EFD_T^{c, <\kappa}(A, B)$. II wins this game if the end result is a partial chain isomorphism between A and B. Player II has a winning strategy in $EF_\mu^{c, <\kappa}(A, B)$ if and only if A and B are chain isomorphic. Moreover, if $\mu = \omega$, then both are equivalent to the models A and B satisfying (in the sense of (1)) the same $L_{\infty\kappa}$-sentences. Note the difference to the game $EF_\omega^{<\kappa}(A, B)$ in the case $\mu = \omega$. This difference is the whole point of chain models.

Let τ be a strategy of I in $EFD_\mu^{c, <\kappa}(A, B)$ in which he lists during his move number $\alpha = \nu + 2n < \mu$, $\nu = \cup\nu$, all elements of $A_{\nu+n} \setminus \bigcup_{\beta < \nu+n} A_\beta$, and during move number $\alpha = \nu + 2n + 1 < \mu$, $\nu = \cup\nu$, all elements of $B_{\nu+n} \setminus \bigcup_{\beta < \nu+n} B_\beta$. Let S_4 be again the tree of plays of II against τ where she has not lost yet. Let K_4 be again the tree of winning strategies of II in shorter versions of $EF_\mu^{c, <\kappa}(A, B)$ where we limit the length of the game to be an ordinal $< \mu$. It is again easy to see that I wins $EFD_{\sigma S_4}^{c, <\kappa}(A, B)$, II wins the game $EFD_{K_4}^{c, <\kappa}(A, B)$, and $K_4 \leq S_4$. The trees S_4 and K_4 do not have branches of length μ if $A \not\cong_{\text{chain}} B$. If, moreover, κ is a strong limit, then $|\sigma S_4| \leq \kappa$ and $|K_4| \leq \kappa$. So these are κ-Trees, but in fact more: they do not even have branches of length μ. So they are (κ, μ)-Trees. Such trees are the topic of Section 7.

In conclusion, the investigation of Scott watersheds for models of cardinality κ has led us to study the order of trees without κ-branches. In the case of chain models of cardinality κ singular strong limit of cofinality ω, the relevant trees are moreover κ-Trees.

For the rest of the paper, unless stated otherwise, let κ be a singular cardinal of cofinality ω and T a κ-Tree. We also fix an increasing sequence $\langle \kappa_n : n < \omega \rangle$ of regular cardinals converging to κ.

2 Ranks and Games

In this section we introduce a rank operation and show that it characterizes bounded κ-Trees.

Definition 2.1 *For $t \in T$ let $\rho_T(t) = \rho(t)$ be defined by recursion on α as follows:*
$\rho(t) \geq 0$ for any $t \in T$;
$\rho(t) \geq \alpha$ iff for all $\beta < \alpha$ and $n < \omega$ there is $t' \geq t$ such that $h(t') \geq \kappa_n$ and $\rho(t') \geq \beta$.

Example 2.2 *Note that $\rho(t) \geq 1$ iff t has extensions of unbounded height in κ. Let F be the "fan", namely a tree with a single root with ω maximal branches where each two only intersect at the root, and where there is one branch of length κ_n for each n. Then F is a κ-Tree, and one can verify that $\rho(F) = 1$.*

Some basic properties of the rank function are given by the following

Claim 2.3   *1. If $t \leq t'$ and $\rho(t') \geq \alpha$, then $\rho(t) \geq \alpha$.*

   2. If δ is a limit ordinal and $\rho(t) \geq \alpha$ for all $\alpha < \delta$, then $\rho(t) \geq \delta$.

   3. Suppose that $f : T \to T'$ is a reduction between κ-Trees, and $\rho_T(t) \geq \alpha$. Then $\rho_{T'}(f(t)) \geq \alpha$.

   4. The truth of '$\rho(t) \geq \alpha$' does not depend on the choice of the sequence $\bar{\kappa} = \langle \kappa_n : n < \omega \rangle$.

Proof. (4) Suppose that $\bar{\kappa}' = \langle \kappa_n' : n < \omega \rangle$ is another sequence with supremum κ, and define ρ' in the same way as ρ but using $\bar{\kappa}'$ in place of $\bar{\kappa}$. We show by induction on α that for any $t \in T$, $\rho(t) \geq \alpha$ iff $\rho'(t) \geq \alpha$.

Suppose that $\rho(t) \geq \alpha$ and let $n < \omega$ and $\beta < \alpha$. Let $m < \omega$ such that $\kappa_m \geq \kappa_n'$. Hence there is $t' \geq t$ with $h(t') \geq \kappa_m$ and such that $\rho(t') \geq \beta$. By the inductive assumption we also have $\rho'(t') \geq \beta$. This shows that $\rho'(t) \geq \alpha$, and the other direction is symmetric. $\bigstar_{2.3}$

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Definition 2.4 *For any $t \in T$ if there is α such that $\rho(t) \geq \alpha$ but not $\rho(t) \geq \alpha + 1$, we say that $\rho(t) = \alpha$. We let $\rho(T) = \rho(t^*)$. It follows from Claim 2.3(1) that if $t \leq t'$ then $\rho(t) \geq \rho(t')$.*

The point of the rank operation is that it characterises bounded trees, in the following sense:

Theorem 2.5 *T is bounded iff $\rho(T) < \kappa^+$.*

Proof. In the forward direction, we use the following

Claim 2.6 *Suppose that $t \in T$ and $\rho(t) \geq \kappa^+$, while $n < \omega$. Then there is $t' \geq t$ such that $\rho(t') \geq \kappa^+$ and $h(t') \geq \kappa_n$.*

Proof. Suppose otherwise, so for every $t' \geq t$ of height $\geq \kappa_n$ we have $\rho(t') < \kappa^+$. Therefore $\rho(t) \leq \sup_{t' \geq t, h(t) \geq \kappa_n} \rho(t) < \kappa^+$, which is a contradiction. ★$_{2.6}$

So if $\rho(t^*) \geq \kappa^+$, we could use Claim 2.6 to build a branch $t^* = t_0 \leq t_1 \leq t_2 \ldots$ such that for each n we have $\rho(t_{n+1}) \geq \kappa^+$ and $h(t_{n+1}) \geq \kappa_n$. Any maximal branch containing this branch must be a κ-branch in T, which is a contradiction.

In the other direction, suppose that we had a κ-branch in T and show by induction on $\alpha \leq \kappa^+$ that $\rho(t^*) \geq \alpha$. Let $t_0 \leq t_1 \leq t_2 \ldots$ be such that $h(t_n) \geq \kappa_n$- such a sequence exists by the assumption. The nontrivial case is that $\alpha = \beta + 1$. The inductive hypothesis is that for all n, $\rho(t_n) \geq \beta$. The choice of t_n's shows that in fact for all n we have $\rho(t_n) \geq \alpha$, so $\rho(t^*) \geq \alpha$. ★$_{2.5}$

Corollary 2.7 *Suppose that T, T' are bounded κ-Trees and $T \leq T'$. Then $\rho(T) \leq \rho(T')$.*

Proof. By Theorem 2.5 and Claim 2.3(3). ★$_{2.7}$

It follows from Lemma 2.5 that in a bounded κ-Tree there is for every t the first n such that t does not have an extension $t' \geq t$ of height $\geq \kappa_n$ and satisfying $\rho(t') \geq \rho(t)$. We denote this n by $n_T(t)$ or $n(t)$ if T is clear from the context .

Claim 2.8 *Suppose that T is a bounded κ-Tree and $t \leq_T t'$ satisfy $\rho(t) = \rho(t')$. Then $n(t) \geq n(t')$.*

Proof. t does not have an extension t'' of height $\geq \kappa_{n(t)}$ satisfying $\rho(t'') \geq \rho(t)$, so certainly t' does not have such an extension. Therefore $n(t') \leq n(t)$. ★$_{2.8}$

We introduce a game that can be used to characterise the rank of bounded trees.

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Definition 2.9 *Let T be a κ-Tree and let $t_0 \in T$. Game $\Gamma_T(t_0)$ is defined as follows. Two players I and II play and I commences by $n_0 < \omega$ and $\beta_0 < \kappa^+$, II replies by $t \geq t_0$ of height at least κ_{n_0}. I chooses $n_1 \geq n_0$ and $\beta_1 < \beta_0$, II replies by $t_1 \geq t$ of height at least κ_{n_1} etc. The game continues until either of the players can no longer play according to these rules. The winner is the player who played the last move.*

We define $\Gamma_T(\alpha, t_0)$ for $\alpha < \kappa^+$ similarly but we require $\beta_0 < \alpha$.

Note that the above games are determined, because they only involve finitely many moves.

Claim 2.10 *If T is a κ-Tree with root t^* then $\rho(T) \geq \alpha$ iff II has a winning strategy in $\Gamma_T(\alpha) \stackrel{\text{def}}{=} \Gamma_T(\alpha, t^*)$.*

Proof. In the forward direction, let II respond to any (n_m, β_m) by an extension of t_m of height at least κ_{n_m} and rank at least β_m- which is possible by the definition of the rank.

For the other direction, suppose that $\beta < \alpha$ and n are given and that there is no $t \in T$ with $\rho(t) \geq \beta$ and $h(t) \geq \kappa_n$. Let β_0 be the first for which there is such n, and let $n_0 = n_T(t)$, i.e. it is the first n with this property for $\beta = \beta_0$. Start a play of $\Gamma_T(\alpha)$ by letting I move with (n_0, β_0). Because II has a winning strategy, this strategy must respond with $t \in T$ with $h(t) \geq \kappa_{n_0}$. However, it must be that $\rho_T(t) < \beta_0$, so let $\beta_1 < \beta_0$ be the first for which there is no $t_1 \geq t$ of rank $\geq \beta_1$ and let $n_1 = n_T(t)$. Therefore $n_1 \geq n_0$. Let I play (n_1, β_1). Since II is playing according to the winning strategy, the play can never stop at a move for II. However, the choice of (n_k, β_k) guarantees that I can respond to each move of II, therefore obtaining an infinite decreasing sequence $\langle \beta_k : k < \omega \rangle$ of ordinals. Contradiction. Therefore $\rho(T) \geq \alpha$. $\bigstar_{2.10}$

Corollary 2.11 *If $T \leq T'$ are κ-Trees and $\alpha < \kappa^+$ then II wins $\Gamma_{T'}(\alpha) \implies$ II wins $\Gamma_T(\alpha)$.*

Proof. By Corollary 2.7 and Claim 2.10. $\bigstar_{2.11}$

Theorem 2.12 *If T is a κ-Tree with root t^* then it has a κ-branch iff II can win all $\Gamma_T(\alpha, t^*)$ for $\alpha < \kappa^+$.*

Proof. In the forward direction, any κ-Tree with a κ-branch satisfies $\rho(t^*) \geq \kappa^+$, and therefore the conclusion follows from Claim 2.10.

In the backwards direction, again by Claim 2.10 we conclude $\rho(t^*) \geq \kappa^+$ and hence T cannot be bounded. $\bigstar_{2.12}$

3 Operations on κ-Trees

In this section we introduce various operations on trees which will allow us to study bounded trees of a given rank. A reader interested directly in structure results about the class of κ-Trees may skip this section and use it as a reference.

Notation 3.1 *If T is a tree and $t \in T$ we let $T[\geq t]$ stand for $\{s \in T : s \geq t\}$ with the order induced by T.*

3.1 Topping

This operation is used to construct from a given tree a tree of the same rank with a long stem. It will be useful when considering the universality problem within a rank.

Definition 3.2 *Suppose that T is a bounded κ-Tree and $\gamma < \kappa$. Let $T_{\lfloor \gamma \rfloor}$ be the tree obtained by taking a stem of length γ and putting a disjoint copy of T on top of it. Formally, $T_{\lfloor \gamma \rfloor}$ is $(\{0\} \times \gamma) \cup (\{1\} \times T)$ ordered lexicographically.*

Definition 3.3 *A κ-Tree T with root t^* is said to be* unfoldable *if for no $t \in T \setminus \{t^*\}$ do we have $T \leq T[\geq t]$.*

Claim 3.4 *Suppose that T is a bounded κ-Tree and $\beta \leq \gamma < \kappa$.*

    1. $T_{\lfloor \beta \rfloor} \leq T_{\lfloor \gamma \rfloor}$.

    2. If T is unfoldable then $T_{\lfloor \gamma \rfloor} \not\leq T_{\lfloor \beta \rfloor}$.

    3. $\rho(T_{\lfloor \gamma \rfloor}) = \rho(T)$.

Proof. (1) The identity function is a reduction from $T_{\lfloor \beta \rfloor}$ to $T_{\lfloor \gamma \rfloor}$.

(2) Suppose f is a reduction from $T_{\lfloor \gamma \rfloor}$ to $T_{\lfloor \beta \rfloor}$. Then $f((1, t^*)) = (1, t)$ for some $t \neq t^*$. Hence f gives rise to a reduction from T to $T[\geq t]$, a contradiction.

(3) Since T is a copy of $T_{\lfloor 0 \rfloor}$ we have by (1) that $T \leq T_{\lfloor \gamma \rfloor}$ and therefore $\rho(T) \leq \rho(T_{\lfloor \gamma \rfloor})$, by Corollary 2.7.

Let $\rho(T) = \alpha$. We denote by t^* the root of T and by t^{**} the root of $T_{\lfloor \gamma \rfloor}$. Clearly, $\rho_{T_{\lfloor \gamma \rfloor}}(1, t^*) = \alpha$. By the definition of the rank there is $n > 0$ such that t^* does not have an extension in T of rank $\geq \alpha$ and height $\geq \kappa_n$. Note that for every $(1, s) \in T_{\lfloor \gamma \rfloor}$ we have $\rho_{T_{\lfloor \gamma \rfloor}}((1, s)) = \rho_T(s)$. Let m be the first such that $\gamma + \kappa_{n-1} < \kappa_m$. Suppose that $x \in T_{\lfloor \gamma \rfloor}$ is an extension of $t^{*,*}$ of rank $\geq \alpha$ and height $\leq \kappa_m$. Then $x = (1, s)$ for some $s \in T$, Therefore $\rho_T(s) \geq \alpha$ and by the choice of m, also $h_T(s) \geq \kappa_n$, a contradiction. In conclusion, $\rho(T_{\lfloor \gamma \rfloor}) \leq \rho(T)$. $\bigstar_{3.4}$

3.2 Bunching

The idea of the bunching operation is to produce bounded κ-Trees of arbitrarily large rank $< \kappa^+$. Let F be the fan tree described above.

Definition 3.5 *Suppose that $\alpha^* < \kappa^+$ and $\bar{T} = \langle T_i : i < i^* \rangle$ are bounded κ-Trees. Then the* bunch $\mathrm{Bn}(\bar{T})$ *is the κ-Tree obtained by putting at the end of every maximal branch of F a node followed by a disjoint family of copies of T_is.*
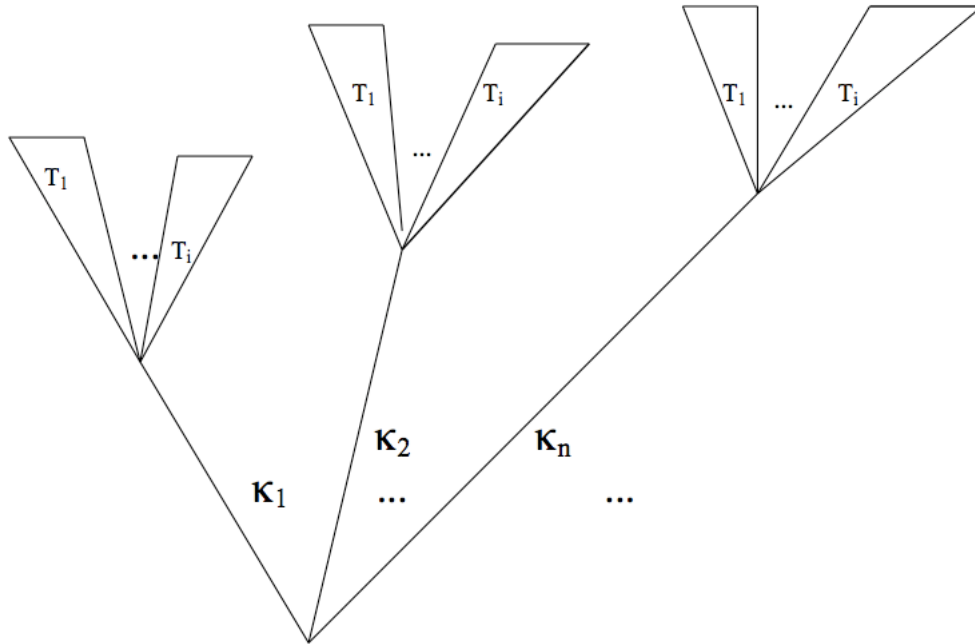

Figure 1: A Bunch $Bn(\langle T_i : i < i^* \rangle)$

Claim 3.6 *(1) Suppose that $\bar{T} = \langle T_i : i < i^* \rangle$ are bounded κ-Trees and let $T^* = \mathrm{Bn}(\bar{T})$. Then T^* is bounded and $\rho(T^*) \geq \sup\{i < i^* : \rho(T_i) + 1\}$.*
(2) If all T_i are the same then $\rho(T^*) = \sup\{i < i^* : \rho(T_i) + 1\}$

Proof. (1) Clearly, T^* is bounded. Let t^* be the root of T^*. First suppose that there is j such that
$$\alpha = \rho(T_j) = \max\{\rho(T_i) : i < i^*\}.$$

Let $n < \omega$ and let t be the root of the copy of T_j which is on top of the branch of F whose length is κ_n. Therefore $\rho_{T^*}(t) \geq \alpha$, as follows from Claim 2.3(3) by taking the

reduction from T_j into T^* given by mapping T_j identically into this particular copy of itself. Therefore $\rho_{T^*}(t^*) \geq \alpha + 1$.

If there is no such j then we note $\sup\{i < i^* : \rho(T_i) + 1\} = \sup\{i < i^* : \rho(T_i)\}$ and repeat the above argument with an arbitrary $\beta < \alpha$ in the place of α above.

(2) Let $T_i = T$ for all i. Note that then we are in the situation of the first case of the proof of (1), so we can simply follow that argument. ★$_{3.6}$

Corollary 3.7 *There is a family $\{T_\alpha : \alpha < \kappa^+\}$ of bounded κ-Trees with*

$$\sup\{\rho(T_\alpha) : \alpha < \kappa^+\} = \kappa^+.$$

Proof. By induction on α we construct a bounded tree T_α satisfying $\rho(T_\alpha) \geq \alpha$. For $\alpha = 1$ let $T_1 = F$. For $\alpha + 1$ let $\bar{T}$ consist of a unique element T_α, and then let $T_{\alpha+1} = \mathrm{Bn}(\bar{T})$. For α limit let $T_\alpha = \mathrm{Bn}(\langle T_\beta : \beta < \alpha\rangle)$. ★$_{3.7}$

In fact, Corollary 3.7 can be improved to obtain a better control on the trees obtained as well as on the ranks. We use the notion of unfoldable trees from Definition 3.3. We isolate a specific property of the trees obtained by the bunching operation which will interest us.

Claim 3.8   *1. F is unfoldable.*

2. Suppose that $\bar{T} = \langle T_i : i < i^*\rangle$ is a sequence of unfoldable κ-Trees. Then $\mathrm{Bn}(\bar{T})$ is unfoldable.

Proof. (1) has a proof that can easily be extracted from the proof of (2), so let us prove T. Let $T^* = \mathrm{Bn}(\bar{T})$ and let t^* be the root of T^*. Suppose that T^* is not unfoldable, so let $t \neq t^*$ be such that $T^* \leq T^*[\geq t]$ and let f be a reduction witnessing this.

First suppose that t is one of the nodes of the bottom copy of F or the nodes added as an immediate successor of such a branch. Therefore it is a node on a branch of length $\kappa_n + 1$ for some n. Since $T^* \leq T^*[\geq t]$, in particular the image of the branch B_{n+1} of F of length κ_{n+1} is contained in $T^*[\geq t]$, and it has to be a branch of length at least κ_{n+1}. Therefore an end segment of this branch is completely within a copy of exactly one T_i. Let s be any element of that end segment which is not the root of T_i. Then in particular we have that f maps the copy of T_i on top of B_{n+1} into $T^*[\geq s]$, which is isomorphic to $T_i[\geq s']$ for some s' not the root of T_i. This is a contradiction.

Therefore suppose that t is in a copy of some T_i on top of some branch of length $\kappa_n + 1$. Then argue as above with t in place of s to obtain a contradiction. ★$_{3.8}$

Theorem 3.11 below is an improvement of Corollary 3.7. Its proof will feature two variants on the bunching operation which we isolate in the following definition. In fact,

the operation of a disjoint bunch is sufficient for the purposes of Theorem 3.11, but we describe the mixed bunch as well, as it seems like an interesting variant.

Definition 3.9 *(1) Suppose that $\bar{T} = \langle T_n : n < \omega \rangle$ are κ-Trees. Then the* mixed bunch *of $\bar{T}$ is the tree $\mathrm{MBn}(\bar{T})$ obtained by putting a copy of T_n on top of the branch of length κ_n of F.*
(2) Suppose that $\bar{T} = \langle T_i : i < i^* \rangle$ are κ-Trees. Then the disjoint bunch *of of $\bar{T}$ is the tree $\mathrm{DBn}(\bar{T})$ obtained by putting a copy of each T_i on top of a common root.*

Claim 3.10 *(1) Suppose that $\bar{T} = \langle T_n : n < \omega \rangle$ are unfoldable κ-Trees with strictly increasing ranks. Then $T^* = \mathrm{MBn}(\bar{T})$ is unfoldable and satisfies $\rho(T^*) = \sup_{n<\omega} \rho(T_n)$.*
(2) Suppose that $\bar{T} = \langle T_i : i < i^* \rangle$ are unfoldable κ-Trees of increasing rank and $i^* < \kappa^+$. Then $T^* = \mathrm{DBn}(\bar{T})$ is an unfoldable κ-Tree and satisfies $\rho(T^*) = \sup_{i<i^*} \rho(T_i)$.

Proof. The argument is similar to the ones in the proof of Claim 3.6 and Claim 3.8.
(1) Suppose that $t \in T^* \setminus \{t^*\}$ is such that $T \leq T[\geq t]$. Let n be such that t is on the branch of F of length κ_n or in the tree T_n above that branch. Then the reduction witnessing $T \leq T[\geq t]$ in particular gives rise to a reduction from T_{n+1} into $T_{n\lfloor\kappa_n\rfloor}$, which is a contradiction because $\rho(T_{n\lfloor\kappa_n\rfloor}) = \rho(T_n) < \rho(T_{n+1})$. Therefore T^* is unfoldable.

Clearly $T_n \leq T^*$ for all n, and hence $\rho(T^*) \geq \alpha \overset{\mathrm{def}}{=} \sup_{n<\omega} \rho(T_n)$. Suppose $\rho(T^*) \geq \alpha+1$ and let $t \neq t^*$ be such that $\rho(t) \geq \alpha$. Let n be such that t is on the branch of F of length κ_n or in the tree T_n above that branch. Let S_n denote the branch and the tree T_n on top of it, so S_n is isomorphic to $T_{n\lfloor\kappa_n\rfloor}$ whose rank is $< \alpha$, a contradiction.
(2) Because $i^* < \kappa^+$ we have $|T^*| \leq \kappa$. Suppose that $t \in T^* \setminus \{t^*\}$ is such that $T \leq T[\geq t]$ and let i be such that t is in the copy of T_i used to construct T^*. The reduction witnessing $T \leq T[\geq t]$ in particular gives rise to a reduction from T_{i+1} into T_i, a contradiction.

As in (1), since $T_i \leq T^*$ for all i we have $\rho(T^*) \geq \alpha \overset{\mathrm{def}}{=} \sup_{i<i^*} \rho(T_i)$. If $\rho(T^*) \geq \alpha+1$ then there is $t \in T^*$ such that $\rho(t) \geq \alpha$ and t is not the root of T^*. Therefore t is in a copy of some T_i and we obtain a contradiction as in (1). ★$_{3.10}$

Theorem 3.11 *For every rank $1 \leq \alpha < \kappa^+$ there is an unfoldable κ-Tree T^α of rank α.*

Proof. The proof is by induction on α. For $\alpha = 1$ let $T^1 = F$. For $\alpha + 1$ let $\bar{T} = \langle T^\alpha \rangle$ and let $T^{\alpha+1} = \mathrm{Bn}(\bar{T})$, so $T^{\alpha+1}$ is unfoldable by Claim 3.8(2) and has rank $\alpha + 1$ by Claim 3.6(2). For α limit we use Claim 3.10(2) and the induction hypothesis. ★$_{3.11}$

3.3 Cutting branches

The idea of the following operation is to associate to every κ-Tree a bounded tree of height κ (but possibly of size κ^+).

Definition 3.12 *(1) Let T be a κ-Tree. Let*

$$R(T) = \{r^*(T)\} \cup \{(\alpha, t, n) : \alpha < \kappa^+, t \in T, h(t) < \kappa_n\},$$

ordered by letting $(\alpha, t, n) < (\alpha', t', n')$ if $\alpha > \alpha'$ or $\alpha' = \alpha, t < t'$ and $n \geq n'$, and letting every $(\alpha, t, n) \geq r^*(T)$.
(2) For T as above and $\alpha^* < \kappa^+$ let $R_{\alpha^*}(T) = \{r^*(T)\} \cup \{(\alpha, t, n) \in R(T) : \alpha \leq \alpha^*\}$, with the order inherited from $R(T)$.

Lemma 3.13 *Let T be a κ-Tree.*

1. *$R(T)$ is a tree of height at most κ with no κ branches, and so is every $R_{\alpha^*}(T)$.*

2. *T is bounded iff $T \leq R(T)$. If T is bounded then $T \leq R_{\rho(T)}(T)$.*

Proof. (1) If $(\alpha, t, n) \in R(T)$ then the predecessors of (α', t', n') are clearly well ordered in $R(T)$. Suppose that $\langle (\alpha_i, t_i, n_i) : i < \kappa \rangle$ is increasing in $R(T)$. Then there must be $i^* < \kappa$ such that all α_i for $i \geq i^*$ are a fixed α. Therefore $\langle t_i : i^* \leq i < \kappa \rangle$ is increasing in T, and all t_i have height $< \kappa_n$, a contradiction. We argue similarly for $R_{\alpha^*}(T)$.

(2) In the forward direction, let $f(t) = (\rho_T(t), t, n_T(t))$. This is well defined because T is bounded, as by Theorem 2.5. If $t < t'$ we have $\rho(t) \geq \rho(t')$ so if $\rho(t) > \rho(t')$ then $f(t) < f(t')$ and if $\rho(t) = \rho(t')$ then $t < t'$ and $n(t) \geq n(t')$ (by Claim 2.8). Therefore f is a reduction. In the other direction, since $R(T)$ has no branches of size κ and $T \leq R(T)$, T cannot have any branches of size κ.

The last sentence follows from the proof of the forward direction above. ★$_{3.13}$

4 Twins and other cofinalities

We consider two functors which make a connection between κ-Trees and trees of size κ and height ω. It is convenient to assume that $\kappa_0 = 0$.

Definition 4.1 *Suppose that T is a κ-Tree. Then the small twin $\mathrm{tw}(T)$ of T is the tree T' of height ω such that $\mathrm{lev}_n(T') = \mathrm{lev}_{\kappa_n}(T')$ with the order induced from T. If T is a tree of size κ and height ω then the big twin $\mathrm{Tw}(T)$ is a normal κ-Tree T such that $\mathrm{lev}_{\kappa_n}(T) = \mathrm{lev}_n(T')$ and for every n and $t \in T'$ of height n, the set of immediate successors of t is in a bijective correspondence f_t with $\{t' \in T : h(t') = n + 1 \ \& \ t \leq t'\}$.*

For any $\alpha \in [\kappa_n + 1, \kappa_{n+1})$, every point of height α in T' has exactly one immediate successor, and finally we have that for t of height κ_n and t' of height κ_{n+1}, $t \leq'_T t'$ iff $t \leq_T t'$. We also have that $s \leq f_t(s)$ whenever s is an immediate successor of t.

Basic properties of the twinning functors are the following:

Claim 4.2   *1. For any κ-Tree T we have $\operatorname{tw}(T)$ has an infinite branch iff T is unbounded. If T' is a tree of height ω and size κ, then $\operatorname{Tw}(T')$ is unbounded iff T has an infinite branch.*

    2. $\operatorname{tw}(\operatorname{Tw})(T) = T$, for any relevant T.

    3. If $T \leq T'$ are κ-Trees then $T \leq T' \implies \operatorname{tw}(T) \leq \operatorname{tw}(T')$, by the same witnessing function.

    4. Suppose that $T \leq T'$ are trees of height ω and size κ. Then $\operatorname{Tw}(T) \leq \operatorname{Tw}(T')$.

Proof. (4) Let $f : T \to T'$ be a reduction. The Twinning operation gives an embedding of T into $\operatorname{Tw}(T)$ which maps level n of T into the level κ_n of $\operatorname{Tw}(T)$, and similarly for T'. Let us denote both of these mappings as g. Now we define $f^* : \operatorname{Tw}(T) \to \operatorname{Tw}(T')$ so that $f^*(g(t)) = g(f(t))$ for all $t \in T$. Notice that f^* is a partial reduction and if $h_{\operatorname{Tw}(T)}(t) = \kappa_n$ then $h_{\operatorname{Tw}(T')}(f^*)(t) = \kappa_m$ for some $m \geq n$. Also note that $\operatorname{Tw}(T)$ cannot have a maximal branch that terminates at a level $\alpha \notin \{\kappa_n : n < \omega\}$. Therefore we can extend f^* to a full reduction from $\operatorname{Tw}(T)$ to $\operatorname{Tw}(T')$. ★$_{4.2}$

5   Universals

The *universality* number of a class of trees is the smallest cardinal number θ such that there is a family $\mathcal{U}$ of θ many members of the class, satisfying that for every T in the class there is $T' \in \mathcal{U}$ such that $T \leq T'$. A family $\mathcal{U}$ satisfying the latter property is called a *universal* family.

We can obtain some results about universality of κ-Trees as a consequence of the observations made in the previous section.

Claim 5.1 *The universality number of the class of all κ-Trees is larger or equal to the universality number of the class of all trees of size κ and height ω. The same is true if we restrict the former class to bounded members, and the latter to members with no infinite branch.*

Proof. Suppose that $\mathcal{U}$ is a universal family for the class of all κ-Trees. Let

$$\mathcal{U}' = \{\text{tw}(T) : T \in U\}.$$

Suppose that T is any tree of size κ and height ω. Therefore there is $T^* \in \mathcal{U}$ such that $\text{Tw}(T) \leq T^*$. Hence $T = \text{Tw}(\text{tw}(T)) \leq \text{twT}^*$, by Claim 4.2(2) and (3) above. Therefore $\mathcal{U}'$ is a universal family for the class of all trees of size κ and height ω. The latter statement is proved in the same way. $\bigstar_{5.1}$

The following gives an idea about the universality number of κ-Trees and bounded κ-Trees.

Claim 5.2 *(1) The universality number of the class of all κ-Trees is 1.*
(2) The universality number of the class of all bounded κ-Trees is at least κ^+.

Proof. (1) Let T be any κ-Trees and view κ as a κ-Tree with one unique branch. We define $f : T \to \kappa$ by letting $f(t) = \alpha$ if $h(t) = \alpha$. This is clearly a reduction.
(2) Suppose this universality number is 1, as demonstrated by a T^*. Let $\alpha = \rho(T)$. The bunching operation $\bar{T} = \langle T^* \rangle$ gives rise to a bounded κ-Tree with rank T at least $\alpha + 1$. However $T \leq T^*$, in contradiction with Corollary 2.7.

Suppose now that the universality number is some $\lambda \in [2, \kappa]$, as exemplified by trees $\{T_i : i < \lambda\}$. Then we can construct a bounded κ-Tree T^* by defining a root and λ many successors of it, on top of i-th of which we add a copy of T_i. This is a κ-Tree which reduces all T_i. Therefore the universality number is not in $[2, \kappa]$. $\bigstar_{5.2}$

We now have

Theorem 5.3 *The universality number of the class of all bounded κ-Trees is exactly κ^+.*

Proof. We exhibit a universal family of size κ^+. This suffices by Claim 5.2(2). We consider the ordinal κ as a κ-Tree. Because this tree has a branch of length κ, every bounded κ-Tree T $\leq$-embeds into it, simply by mapping every element of T into its height. Let now $T_\alpha = R_\alpha(\kappa)$, for $\alpha < \kappa^+$ (recall the branch cutting operation from Subsection 3.3). Therefore $\{T_\alpha : \alpha < \kappa^+\}$ is a family of bounded κ-Trees, and we now claim that it is universal.

Let T be any bounded κ-Tree and let $f : T \to \kappa$ be a reduction. Let $\alpha = \rho(T) + 1$, therefore $\alpha < \kappa^+$. Define $g : T \to T_\alpha$ by letting $g(t) = (\rho_T(t), f(t), n_T(t))$. If $t < t'$ in T then either $\rho(t) > \rho(t)'$, or $\rho(t) = \rho(t)'$ and $f(t) < f'(t)$. At any rate, $g(t) < g(t')$. $\bigstar_{5.3}$

Given the above definite results above universality within the class of bounded κ-Trees we may wonder if there are universality results about the bounded κ-Trees within the same rank. We have the following:

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Theorem 5.4 *(1) For every $1 \leq \alpha < \kappa^+$ the universality number of the class of κ-Trees of rank α is exactly ω. Moreover, for every such α there is a single unfoldable κ-Tree T^α of rank α such that $\{T_n^\alpha \overset{\text{def}}{=} T_{\lfloor \kappa_n \rfloor}^\alpha\}$ is a minimal universal family for κ-Trees of rank α.*

(2) For every $1 \leq \alpha < \kappa^+$ there is a single unfoldable κ-Tree T_α^* of rank $\alpha + 1$ which is universal for κ-Trees of rank α.

Proof. We commence with a simple lemma.

Lemma 5.5 *Suppose that $\alpha < \kappa^+$ and that $\mathcal{F}$ is a family of κ-Trees of rank α universal for κ-Trees of rank α. Then for every $\beta < \alpha$ and κ-Tree T of rank β, T embeds into some member of $\mathcal{F}$.*

Proof. For such β, T let T^* be the disjoint bunch of T and any tree of rank α. Clearly T^* has rank α, and therefore it embeds into some member of $\mathcal{F}$. The restriction of the witnessing reduction to T demonstrates the lemma. $\bigstar_{5.5}$

We now prove the theorem by induction on α, proving (1) and (2) simultaneously.

(1) We distinguish several cases of the induction.

For $\alpha = 1$ let $T^1 = F$, and so $T_n^1 = F_{\lfloor \kappa_n \rfloor}$. By Claim 3.8(1) trees F is unfoldable and by Claim 3.4(3) T_n^1 are of rank 1. By Claim 3.4(2) we have $T_{n+1} \not\leq T_n$, so there is no proper subset of $\{T_n^1 : n < \omega\}$ which can be universal. We now have to prove that every κ-Tree T of rank 1 embeds into some T_n. Let T be such a tree.

By the definition of the rank, there is $n = n_T$ such that for every $t \in T$ of height $\geq n$ we have $\rho(t) < 1$. We define $f : T \to T_n^1$ by letting for $s \in T$, if $h(s) < \kappa_n$, $f(s)$ be the node on the stem of T_n^1 of height $h(s)$. The choice of n implies that for every $t \in T$ with $h(t) \geq n$ we have that the whole tree $T[\geq t]$ has height $< \kappa$. Therefore for every such t we can choose m with $h(t) < \kappa_m$ and define f on $T[\geq t]$ to carry all points s to the branch of F of length κ_m, assigning to each s the unique point $f(s)$ such that $h_F(f(s)) = h_{T[\geq t]}(s)$.

Let $1 < \alpha < \kappa^+$.

<u>Case 1</u>. $\alpha = \beta + 1$ for some β. We claim that letting $T^\alpha = T_\beta^*$ we obtain the desired conclusion. By (2) at β we have that T^α is unfoldable and has rank $\alpha + 1$. Therefore by Claim 3.4(2) we have $T_n^\alpha \not\leq T_{n+1}^\alpha$. Like in the case $\alpha = 1$, for a given tree T of rank α we let $n = n_T$ and define a reduction $f : T \to T_{\alpha \lfloor \kappa_n \rfloor}^*$ by mapping all s of height $< \kappa_n$ to the stem. On the other hand, for any $t \in T$ of height $\geq \kappa_n$ we have that the rank of $T[\geq t]$ is at most β, and so by Lemma 5.5, $T[\geq t]$ embeds into T_α^*.

<u>Case 2</u>. α is a limit ordinal. Let $T^\alpha = \text{DBn}(\langle T^\beta : \beta < \alpha \rangle)$, so by Claim 3.10(2), T^α is unfoldable and has rank α. The rest of the proof follows closely the proof of Case 1.

(2) If we are given T^α let T_α^* be $\text{DBn}(\langle T_n^\alpha : n < \omega \rangle)$. It is easily seen that T_α^* has rank $\alpha + 1$, it is unfoldable and satisfies $T_n^\alpha \leq T_\alpha^*$ for all n. Therefore it is universal for trees of rank α. $\bigstar_{5.4}$

6 Incomparable κ-Trees

In this section we are concerned with the antichains of κ-Trees. We commence by showing that consistently there can be large antichains of κ-Trees in every rank ≥ 2.

We shall use the following useful and well known claim.

Claim 6.1 *Suppose that T and T' and $T \leq T'$. Then there is a reduction $f : T \to T'$ satisfying $h(t) = h(f(t))$ for all $t \in T$.*

Proof. Let $g : T \to T'$ be any reduction and define $f(t) = g \upharpoonright h(t)$. Note that the fact that g is a reduction implies that $h(g(t)) \geq g(t)$ for any $t \in T$, so f is well defined. If $s <_T t$ then $g(s) <_{T'} g(t)$, and hence $g(s) \upharpoonright h(s) = g(t) \upharpoonright h(s) < g(t)$, but since $h(s) < h(t)$ we also obtain $g(s) \upharpoonright h(s) < g(t) \upharpoonright h(s)$. Therefore f is a reduction. $\bigstar_{6.1}$

We call the reductions satisfying the requirement of Claim 6.1 *level preserving* reductions.

Theorem 6.2 *Suppose λ, θ are cardinals such that $\mu \overset{\text{def}}{=} 2^{\lambda^{<\theta}} \leq \kappa$, and $\rho < \kappa^+$. Then there is a family of μ many pairwise incomparable κ-Trees of rank ρ.*

Proof. Let $\langle S_\alpha : \alpha < \mu \rangle$ be a family of copies of the tree $^{\theta>}\lambda$, therefore each S_α is a tree of size κ-Tree (here we have used $\lambda^{<\theta} \leq \kappa$). Let $\{f_i : i < \mu\}$ list all level preserving reductions from S_α into S_β for $\alpha \neq \beta < \mu$. We shall extend each S_α into a tree T_α by selecting certain cofinal branches of S_α and attaching a copy of a κ-Tree of rank ρ on top of it, as described below. The purpose of these extensions is to assure that no f_i extends into a reduction between T_α and T_β for $\alpha \neq \beta$. Since we are using up to μ cofinal branches, we are again using $\mu \leq \kappa$ to assure that T_α's are κ-Trees.

Let $\{T_j^* : j < \omega\}$ list a universal family of trees of rank ρ, which exists by Theorem 5.4. By induction on $i < \mu$ we consider f_i. The inductive hypothesis is that for each $\alpha < \mu$ and $j < i$ we have a tree S_α^j such that $S_\alpha^0 = S_\alpha$ and $\langle S_\alpha^j : j < i \rangle$ forms a sequence of trees that end-extend each other, while in the union $S_\alpha^{0,i}$ the tree S_α has been extended by putting terminal nodes and a copy of some κ-Tree of rank $\leq \rho$ on top of $< \mu$ cofinal branches of the original S_α. We assume that each cofinal branch of S_α has been extended at most once.

At stage i of the induction, there are unique α, β such that $f_i : S_\alpha \to S_\beta$. For $\gamma \notin \{\alpha, \beta\}$ let $S_\gamma^i = S_\gamma^{0,i}$. Next we choose a cofinal branch b_α^i in S_α which still does not have a terminal node in $S_\alpha^{0,i}$ (this is possible since there are μ many cofinal branches in S_α, of which we only used $< \mu$). Since f_i is a level preserving reduction, the image $b_\beta^i = f \text{``} b_\alpha^i$ is a cofinal branch of S_β.

The first possibility is that b_β^i still does not have a terminal node in $S_\beta^{0,i}$. Then extend $S_\alpha^{0,i}$ to S_α^i by putting a terminal node and a copy of T_0^* on top of b_α^i if $\rho > 1$, or $F_{\lfloor 1 \rfloor}$ if

$\rho = 1$. Extend $S^{0,i}_\beta$ to S^i_β by putting a terminal node and a copy of the fan F on top of b^i_β.

Note that f_i cannot be extended to a level preserving reduction from S^i_α to S^i_β because any such reduction would have to move a κ-Tree of rank ρ into F, or $F_{\lfloor 1 \rfloor}$ if $\rho = 1$ into F, which is in contradiction with Claim 2.3. The second possibility is that there is already a κ-Tree T of rank $\delta \le \rho$ on top of b^i_β. By the choice of the family $\{T^*_j : j < \omega\}$ there must be j such that T^*_j does not embed into T. We choose some such j and place T^*_j on top of b^i_α.

At the end let T_α be the union of all S^i_α. It is clear that $\rho(T_\alpha) \ge \rho$. Since κ is of cofinality ω we cannot have $\mu = \kappa$, by König's Lemma. Therefore $\mu < \kappa$ If there is a reduction from T_α to T_β for some $\alpha \neq \beta$ then there is a level preserving reduction f, by Claim 6.1. However, f would have to extend some f_i, which was made impossible by the step i of the induction. $\bigstar_{6.2}$

7 A covering theorem for $\lambda^{\mathrm{cf}(\lambda)}$

In this section we prove some results about (λ, κ)-Trees, i.e. trees of singular cardinality λ without branches of length $\kappa = \mathrm{cf}(\lambda)$. Such trees arise from consideration of the game $EF^{c,<\lambda}_\kappa(A, B)$ on chain models A and B of length κ. In particular, we prove a covering theorem which can be seen as a generalization of the covering theorem of the Baire space ω^ω.

We fix λ strong limit singular of cofinality κ, along with a continuous cofinal sequence $\langle \lambda_\xi : \xi < \kappa \rangle$ of cardinals in λ such that $2^{\lambda_\xi} \le \lambda_{\xi+1}$. We shall throughout make the usual identification between $^\kappa\lambda$ and λ^κ.

We equip the set λ^κ with the topology generated by the neighbourhoods

$$N(f, \beta) = \{g : \kappa \to \lambda : g \restriction \beta = f \restriction \beta\},$$

for $f \in {}^\kappa\lambda$ and $\beta < \lambda$. Note that in this topology there is a dense subset of λ^κ of size $|\bigcup_{\xi<\kappa} \lambda^{\lambda_\xi}| = \lambda$, the topology is 0-dimensional and each open set is the union of λ closed sets. We make an expected definition:

Definition 7.1 *A set $A \subseteq \lambda^\kappa$ is Π^1_1 if there is an open set $B \subseteq \lambda^\kappa \times \lambda^\kappa$ (in the product topology where each copy of λ^κ has the topology described above) such that for every $f \in {}^\kappa\lambda$*

$$f \in A \iff \forall g\,((f,g)) \in B). \tag{2}$$

A set is Σ^1_1 if its complement is Π^1_1 and it is Δ^1_1 if it is both Π^1_1 and Σ^1_1.

Now we introduce some notation:

Let $\mathcal{T}$ denote the class of trees of cardinality $\leq \lambda$ and height $\leq \kappa$ with no branches of length κ. Let $pair : \lambda \times \lambda \to \lambda$ be the standard bijection. If $f, g \in {}^{\kappa}\lambda$, let $\langle f, g \rangle \in {}^{\kappa}\lambda$ be a function such that $\langle f, g \rangle(pair(\alpha, \beta)) = pair(f(\alpha), g(\beta))$ for $\alpha, \beta \in \lambda$. Let $\sigma : {}^{\kappa >}\lambda \to \lambda$ be a bijection.

Now we shall fix a bijection H between ${}^{\kappa}(\mathcal{P}(\lambda))$ and ${}^{\kappa}\lambda$ which has the following properties. Namely, identifying $\mathcal{P}(\lambda)$ with ${}^{1}\mathcal{P}(\lambda)$ the function H induces an injection F from $\mathcal{P}(\lambda)$ to ${}^{\kappa}\lambda$. We require this injection F to have the property that there is a sequence $\langle F_{\xi} : \xi < \kappa \rangle$ of injections $F_{\xi} : \mathcal{P}(\lambda_{\xi+1} \setminus \lambda_{\xi}) \to \lambda_{\xi+2} \setminus \lambda_{\xi+1}$ such that for any $A \subseteq \lambda$ the value $F(A) \in {}^{\kappa}\lambda$ is a function such that $F(A)(\lambda_{\xi}) = F_{\xi}(A \cap (\lambda_{\xi+1} \setminus \lambda_{\xi}))$. Such an H is easy to find, by constructing first the functions F_{ξ}. For most of the this sections we shall only work with the function F, but the whole of function H will be needed once we start talking about chain models in Theorem 7.8.

If $f : \kappa \to \lambda$, we denote the initial segment $\langle f(0), \ldots, f(\beta), \ldots \rangle_{\beta < \alpha}$ by $\bar{f}(\alpha)$. By a *tree on* $\lambda^{<\kappa}$ we mean a subset of $\lambda^{<\kappa}$ which is closed under initial segments. Respectively, a *tree on* $\lambda^{<\kappa} \times \lambda^{<\kappa}$ we mean a set T of pairs $(\bar{f}(\alpha), \bar{g}(\alpha))$ such that if $(\bar{f}(\alpha), \bar{g}(\alpha))$ and $\beta < \alpha$, then $(\bar{f}(\beta), \bar{g}(\beta))$.

Example 7.2 *Let* $\mathrm{Tr} = \{F(\sigma``T) : T$ *is a tree on* $\lambda^{<\kappa}\}$. *Therefore* $\mathrm{Tr} \subseteq \lambda^{\kappa}$ *and we claim that it is closed. Suppose that f is in the closure of* Tr*, hence for every* $\beta < \lambda$ *there is* $g_{\beta} \in N(f, \beta)$ *and some tree* $T_{g\beta}$ *such that* $F(\sigma``T_{g\beta}) = g_{\beta}$. *If* $\zeta < \xi_1 < \xi_2$ *then* $g_{\lambda_{\xi_1}}(\lambda_{\zeta}) = g_{\lambda_{\xi_1}}(\lambda_{\zeta}) = f(\lambda_{\zeta})$, *therefore* $F_{\zeta}(\sigma``T_{g_{\lambda_{\xi_1}}} \cap (\lambda_{\zeta+1} \setminus \lambda_{\zeta})) = F_{\zeta}(\sigma``T_{g_{\lambda_{\xi_2}}} \cap (\lambda_{\zeta+1} \setminus \lambda_{\zeta}))$. *Let* $A = \bigcup_{\zeta < \kappa} F_{\zeta}^{-1}(\sigma``T_{g_{\lambda_{\xi}}} \cap (\lambda_{\zeta+1} \setminus \lambda_{\zeta}))$, *where* $\xi \in (\zeta, \kappa)$ *is arbitrary. The discussion above shows that A is well defined. Let* $T^* = \sigma^{-1}(A)$, *so* $T^* \subseteq \lambda^{<\kappa}$. *Now if* $g \in T^*$ *and h is an initial segment of g we have* $\sigma(g) \in A$ *and therefore for some* $\zeta < \xi < \kappa$ *we have* $\sigma(g) \in F_{\zeta}^{-1}(\sigma``T_{g_{\lambda_{\xi}}} \cap (\lambda_{\zeta+1} \setminus \lambda_{\zeta}))$. *Hence* $\sigma(g) \in \sigma``T_{g_{\lambda_{\xi}}}$ *and so* $g \in T_{g_{\lambda_{\xi}}}$. *Therefore* $h \in T_{g_{\lambda_{\xi}}}$ *and we can show that* $h \in T^*$ *by reversing the implications. Therefore T^* is a tree and by its definition* $F(T^*) = f$, *showing that* $f \in \mathrm{Tr}$.

If $f = F(\sigma``T) \in \mathrm{Tr}$ *we denote the tree T by* T_f. *Let*

$$\mathcal{TO} = \{f : T_f \text{ has no branches of length } \kappa\}.$$

It can be checked that this set is Π_1^1.

Proposition 7.3 (Boundedness Theorem) *Suppose* $B \subseteq \mathcal{TO}$ *is* Σ_1^1. *Then there is* $g \in \mathcal{TO}$ *such that* $T_f \leq T_g$ *for all* $f \in B$.

Proof. If T is a tree on $\lambda^{<\kappa} \times \lambda^{<\kappa}$ and $f \in {}^{\kappa}\lambda$, let

$$T(f) = \{\bar{g}(\alpha) : (\bar{f}(\alpha), \bar{g}(\alpha)) \in T\}.$$

Note that $|T(f)| \leq \lambda$. Let $S = \{(\bar{f}(\alpha), \bar{g}(\alpha)) : f \in \text{Tr}, \bar{g}(\alpha) \in T_f\}$. Now $S(f) = T_f$ for all $f \in \text{Tr}$ and

$$f \in \mathcal{TO} \iff S(f) \text{ has no branches of length } \kappa.$$

Let T be a tree such that

$$f \in B \iff T(f) \text{ has a branch of length } \kappa.$$

Let T' be the tree of triples $(\bar{f}(\alpha), \bar{g}(\alpha), \bar{h}(\alpha))$ such that $\bar{g}(\alpha) \in S(f)$ and $\bar{h}(\alpha) \in T(f)$. If T' has a branch of length κ then we get $f \in B \setminus \mathcal{TO}$. To prove $T_f \leq T'$ for all $f \in B$, let $f \in B$ be arbitrary. Let $(\bar{h}(\alpha))_{\alpha < \kappa}$ be a branch of length κ in $T(f)$. For $\bar{g}(\alpha) \in S(f)$ let

$$\phi(\bar{g}(\alpha)) = (\bar{f}(\alpha), \bar{g}(\alpha), \bar{h}(\alpha)).$$

Now $\phi : S(f) \to T'$ is order-preserving.

To prove the corresponding result for all Π_1^1-sets, we present the basic *tree-representation* of Π_1^1-sets. This emerges in the same natural way as in the space ${}^\omega\omega$.

Lemma 7.4 *A set $A \subseteq {}^\kappa\lambda$ is Π_1^1 if and only if there is a relation $R \subseteq \lambda^{<\kappa} \times \lambda^{<\kappa}$ such that*

$$f \in A \iff \forall g \exists \alpha R(\bar{f}(\alpha), \bar{g}(\alpha)). \tag{3}$$

Proof. If A and B satisfy (2), then

$$R = \{(\bar{f}(\alpha), \bar{g}(\alpha)) : N((f, g), \alpha) \subseteq B\}$$

satisfies (3). Conversely, if A and B satisfy (3), then

$$B = \bigcup \{N((f, g), \alpha) : R(\bar{f}(\alpha), \bar{g}(\alpha))\}$$

satisfies (2).

Proposition 7.5 *Let $A \subseteq {}^\kappa\lambda$. A is Π_1^1 if and only if there is a tree T on $\lambda^{<\kappa} \times \lambda^{<\kappa}$ such that*

$$f \in A \iff T(f) \text{ has no branches of length } \kappa. \tag{4}$$

Proof. Suppose A is Π_1^1 and R is as in (3). Let

$$T = \{(\bar{f}(\alpha), \bar{g}(\alpha)) : \forall \beta \leq \alpha \neg R(\bar{f}(\alpha), \bar{g}(\alpha))\}.$$

Hence $f \in A$ if and only if $T(f)$ has no branches of length κ. For the converse, suppose such a T exists. Let

$$R(\bar{f}(\alpha), \bar{g}(\alpha)) \iff \exists \beta \leq \alpha (\bar{f}(\beta), \bar{g}(\beta)) \notin T.$$

Now $f \in A$ if and only if $\forall g \exists \alpha R(\bar{f}(\alpha), \bar{g}(\alpha))$. Thus A is Π_1^1.

Suppose A is Π_1^1 and T is a tree such that (4) holds. If $g \in \mathcal{TO}$, let

$$A^{T,g} = \{f \in {}^\kappa \lambda : T(f) \leq T_g\}.$$

Then clearly, $A^{T,g} \subseteq A$ for $g \in \mathcal{TO}$, and $\lambda^{<\kappa} = \lambda$ implies $\bigcup_{g \in \mathcal{TO}} A^{T,g} = A$. Using the functions σ and F it is not hard to see that the sets $A^{T,g}$ are Σ_1^1 subsets of A.

Proposition 7.6 (Covering Theorem) *Suppose A is Π_1^1 and T is a tree such that*

$$f \in A \iff T(f) \text{ has no branches of length } \kappa.$$

Suppose $B \subseteq A$ is Σ_1^1. Then there is $g \in \mathcal{TO}$ such that $B \subseteq A^{T,g}$.

Proof. Let S be a tree such that $f \in B$ if and only if $S(f)$ has a branch of length κ. Let T' be the tree of triples $(\bar{f}(\alpha), \bar{g}(\alpha), \bar{h}(\alpha))$ such that $\bar{g}(\alpha) \in T(f)$ and $\bar{h}(\alpha) \in S(f)$. If T' has a branch of length κ, then $f \in B \setminus A$. Let $T_e \cong T'$. To prove $B \subseteq A^{T,e}$, let $f \in B$. Let $(\bar{h}(\alpha))_{\alpha < \kappa}$ be a branch of length κ in $S(f)$. For $\bar{g}(\alpha) \in T(f)$ let $\phi(\bar{g}(\alpha)) = (\bar{f}(\alpha), \bar{g}(\alpha), \bar{h}(\alpha))$. Now $\phi : T(f) \to T'$ is order-preserving, whence $f \in A^{T,e}$.

Corollary 7.7 (Boundedness Theorem) *Suppose A is Π_1^1 and T is a tree such that*

$$f \in A \iff T(f) \text{ has no branches of length } \kappa.$$

Then A is Δ_1^1 if and only if there is $g \in \mathcal{TO}$ such that $\forall f \in A(T(f) \leq T_g)$.

We finish the section by showing how these ideas and the concept of chain isomorphism join to give us at strong limit singular cardinals an analogue of the cornerstone of the analysis of countable models using descriptive set theory. Recall that we have fixed a cofinal κ-sequence $\vec{\lambda} = \langle \lambda_\xi : \xi < \kappa \rangle$ in λ. We shall only consider chain models with λ as the universe and a decomposition $\langle A_\alpha : \alpha < \kappa \rangle$ such that every A_α is a subset of some λ_β, and vice verse, every λ_β is a subset of some A_α. Let us call these $\vec{\lambda}$-*chain models*. Recall that we have fixed a bijection $H : {}^\kappa \mathcal{P}(\lambda) \to {}^\kappa \lambda$. If A is a $\vec{\lambda}$-chain model with decomposition $\bar{A} = \langle A_\alpha : \alpha < \kappa \rangle$, then and $\langle A \rangle \frown \bar{A}$ is in the domain of H and it is mapped to some $f \in {}^\kappa \lambda$. We say that A is *coded by* f. The *orbit* of A is the set of $g \in {}^\kappa \lambda$ which code some B (and a decomposition of it) chain-isomorphic to A. The orbit depends on the function H.

Theorem 7.8 *Suppose that λ is a singular strong limit cardinal of cofinality κ. Then the orbit of a $\vec{\lambda}$-chain model A of cardinality λ and length κ is a Σ_1^1-set in the space ${}^\kappa \lambda$ equipped with the above topology. The orbit is Δ_1^1 iff there is a (λ, κ)-Tree T such that for any chain model B of cardinality λ and length κ, I has a winning strategy in $EFD_T^{c, <\lambda}(A, B)$ if and only if $A \ncong_{chain} B$.*

Proof. Suppose A is a $\vec{\lambda}$-chain model with decomposition $\langle A_\alpha : \alpha < \kappa \rangle$. Let τ be a strategy of I in $EF_\kappa^{c,<\lambda}(A, B)$ in which the αth move is for $\alpha = \nu + 2n$, $\nu = \cup\nu$, a sequence that enumerates $\lambda_{\nu+n} \setminus \bigcup_{\beta<\nu+n} \lambda_\beta$ in A, and for $\alpha = \nu + 2n + 1$, $\nu = \cup\nu$, a sequence that enumerates $\lambda_{\nu+n} \setminus \bigcup_{\beta<\nu+n} \lambda_\beta$ in B. Note that the strategy is independent of B. For any $\vec{\lambda}$-chain model B with some decomposition $\langle B_\alpha : \alpha < \kappa \rangle$ let $S(B)$ be the tree of plays in which I has used τ and II has played but has not lost yet. Then for all B,

$$A \ncong_{chain} B \iff S(B) \text{ has no branches of length } \kappa.$$

This set if Π_1^1 by Proposition 7.5, so the orbit of A, which is the complement of this set, is Σ_1^1.

If the set $\{B : A \cong_{chain} B\}$ is Δ_1^1, then we can apply Boundedness Theorem to its complement. Therefore there is a tree T of cardinality λ without branches of length κ such that

$$A \ncong_{chain} B \text{ implies } S(B) \leq T.$$

We can now show that

$$A \ncong_{chain} B \text{ iff player I has a winning strategy in } EFD_{\sigma T}^{c,<\lambda}(A, B):$$

Suppose $A \ncong_{chain} B$. Then $S(B) \leq T$. Hence I wins $EFD_{\sigma T}^{c,<\lambda}(A, B)$ with the strategy τ. On the other hand, if $A \cong_{chain} B$, then II wins $EF_\kappa^{c,<\lambda}(A, B)$, so I cannot win $EFD_{\sigma T}^{c,<\lambda}(A, B)$. $\bigstar_{7.8}$

If $\lambda = \kappa = \omega$, then every orbit is Δ^1_1 and the Scott height of A is - essentially - the tree T of the above Theorem. So Theorem 7.8 generalizes the Scott analysis of countable models to chain models of singular strong limit cardinality.

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