Representation theorems for connected compact Hausdorff spaces

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Abstract

We present two theorems which can be used to represent compact connected Hausdorff spaces in an algebraic context, using a Stone-like representation. The first theorem stems from the work of Wallman and shows that every distributive disjunctive normal lattice is the lattice of closed sets in a unique up to homeomorphism connected compact Hausdorff space. The second theorem stems from the work of Jung and Sünderhauf. Introducing the notion of strong proximity involution lattices, it shows that every such lattice can be uniquely represented as the lattice of pairs of compact and open sets of connected compact Hausdorff space. As a consequence we easily obtain a somewhat surprising theorem birepresenting distributive disjunctive normal lattices and strong proximity involution lattices.

1 Introduction

Arguably one of the most important theorems about Boolean algebras is the theorem by M. Stone [10] which states that every Boolean algebra $\mathfrak{B}$ is isomorphic to a field of sets, namely a subalgebra of the

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algebra of all subsets of a certain (totally disconnected) 0-dimensional compact Hausdorff topological space. This space is called the Stone space of $\mathfrak{B}$ and denoted by $\mathrm{St}(\mathfrak{B})$. Stone considered this connection important because among other thing it "is a precise analogue of the theorem that every abstract group is represented by an isomorphic group of permutations". Conversely, Stone also proved that to every compact 0-dimensional Hausdorff topological space X there corresponds a unique up to isomorphism Boolean algebra $\mathfrak{B}$ such that $X = \mathrm{St}(\mathfrak{B})$. Even though the motivation of the Stone representation theorem was the forward direction, now it is more often the latter direction of the theorem that gets used in applications, when one wishes to construct a topological space with certain properties and instead one constructs the Boolean algebra whose Stone space is the desired space (see [7], [5], [2] for examples). The advantage of this approach from the point of view of logic is that Boolean algebras are first order objects, while topological spaces are second order, and therefore it is much easier to control the properties of Boolean algebras, be it in direct or in forcing constructions. However, the approach necessarily runs in difficulties when one needs to construct a connected space because Stone spaces are totally disconnected. Examples of such a construction arose most recently in [8], [6] where there are constructions of connected compact Hausdorff spaces K having the property that the space of continuous functions $C(K)$ is not isomorphic to $C(L)$ for any 0-dimensional space L and solving an important question in the isomorphic theory of Banach spaces. In order to approach such a construction through a representation theorem one needs a Stone-style representation theorem for compact Hausdorff spaces, *not* the 0-dimensional such spaces. The 0-dimensional aspect of the Stone representation theorem stems from the existence of complements in Boolean algebras, therefore one needs to work with structures in which there is no complement. Birkhoff in [1] points out the necessity of complementation in Stone's theorem and gives a representation theorem for general distributive lattices, given however in terms of families of functions without topological considerations.

Wallman announced in [14] and gave detailed proofs in [13] of a topological representation theorem for disjunctive distributive lattices in which to each such lattice one associates a compact T_1 space. He noted that the space is Hausdorff iff the lattice is normal. Wallman's motivation was that if one starts with the lattice of closed sets of a given topological space X then one obtains through his representation a compact space in which X is embedded as a dense subspace and which has the same homology and dimension as X. He was not concerned with the connectedness of the space, but nevertheless, with small changes, his original theorem can be used to obtain a representation of connected compact spaces. In §2 we give such a representation, largely based on an appendix in [8]. Wallman's research was continued and generalised by others later, notably by Šhanin in [12] as well as in a number of more recent papers, however connectedness does not seem to have been an issue.

Another approach is motivated by questions in logic, and later computer science. Namely in [11] Stone considers Heyting algebras, which are a generalisation of Boolean algebra in which there is only a certain pseudocomplementation in place of complementation. Heyting algebras are used as models of intuitionistic logic, where there is no law of excluded middle. Another important example of a Heyting algebra is the collection of open sets in any topological space. Stone in [11] gives a representation theorem for such algebras. This line of research was taken up again by Priestley in [9], where she provided what is now known as Priestley duality. It associates to distributive lattices a compact Hausdorff space endowed with an order. This has led to a large body of research. Such dualities are of special interest in theoretical computer science, and in particular Jung and Sünderhauf in [4] introduce a notion of strong proximity lattices that is used to represent the so called stably compact spaces. The spaces to which this type of representation are applied are in general not Hausdorff and the interests in them stems from the fact that stably compact spaces capture by topological means most semantic domains in the mathematical theory of computation. Research of [4] is continued in recent work of Jung and Moshier in [3], where they provide a bitopological setting for Stone duality. In this line of generalisation of Stone's representation theorem the concern seems to have been on the non-Hausdorff case. Like in the case of Wallman-like representation, the situation of compact Hausdorff connected spaces, with which we are concerned here, does not seem to have been directly considered. This is not to say that there was no awareness of the possibility, and as we shall see in §3 only one additional twist is needed in Jung's and Sünderhauf's work to obtain a representation of connected Hausdorff spaces using strong proximity lattices. The main ideas of the representation of compact Hausdorff spaces were laid to us by Jung in a conversation in 2003. Connectedness was not discussed at the time and does not seem to appear as an issue in published work.

Wallman's paper [14] and Stone's [11] appeared the same year, yet it is not clear if the authors were aware of each other's work and the connection between them. This seems to have continued to be the case between those who continued to study representations from the set-theoretic topology point of view and those who have studied them from the computing or logical point of view. A purpose of this note is to bring the two representation theorems on compact Hausdorff spaces together. This allows us to easily obtain the unexpected result stating that distributive disjunctive normal lattices and strong involution proximity lattices are representable by each other. This connection does not seem to have been noticed before.

A Wallman-style representation is given §2 and representation using strong proximity involution lattice is given in S3. The origins of these results are explained in the above. The birepresentation of distributive disjunctive normal lattices and strong involution proximity lattices is given in §4.

All lattices we mention will be *bounded*, which means that they will have the smallest element 0 and the largest element 1. Therefore by saying 'distributive lattice' we mean a distributive bounded lattice. We shall only be concerned with Hausdorff spaces.

2 Wallman representation

Let $\mathfrak{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a distributive lattice. The notions of a filter, prime filter and an ultrafilter of such a lattice are introduced similarly to the analogous notions in a Boolean algebra.

Definition 2.1 *An* $\mathfrak{L}$-filter *is a family* $\mathcal{F} \subseteq \mathfrak{L}$ *closed under* $\wedge$ *and satisfying that* $1 \in \mathcal{F}, 0 \notin \mathcal{F}$*, whilst for any* $a \in \mathcal{F}, b \in \mathfrak{L}$*, if* $a \leq b$ *then* $b \in \mathcal{F}$*.*

An $\mathfrak{L}$*-filter* $\mathcal{F}$ *is* prime *if whenever* $a \vee b \in \mathcal{F}$ *then* $a \in \mathcal{F}$ *or* $b \in \mathcal{F}$*. An* $\mathfrak{L}$-ultrafilter *is an* $\mathfrak{L}$*-filter which is maximal under* $\subseteq$*.*

In the case of Boolean algebras all prime filters are ultrafilters but this is not the case in distributive lattices in general. Using Zorn's Lemma and basic lattice manipulations one can still prove the following facts:

Lemma 2.2 *(i) Every subset of* $\mathfrak{L}$ *satisfying that the meet of any of its finite subset is non-zero, is contained in an* $\mathfrak{L}$*–ultrafilter.*

(ii) *If $\mathcal{F}$ is an $\mathfrak{L}$-ultrafilter and $b \in \mathfrak{L}$ has the property that $b \wedge a \neq 0$ for every $a \in \mathcal{F}$, then $b \in \mathcal{F}$.*

(iii) *Every $\mathfrak{L}$-ultrafilter is prime.*

Let $\mathrm{ULT}(\mathfrak{L})$ be the set of all $\mathfrak{L}$-ultrafilters. For $a \in \mathfrak{L}$ we put

$$V(a) = \{\mathcal{F} \in \mathrm{ULT}(\mathfrak{L}) : a \notin \mathcal{F}\}$$

and we let $F(a) = \mathrm{ULT}(\mathfrak{L}) \setminus V(a)$. We shall show below that these sets can be interpreted as the basic open and closed sets in a topology on $\mathrm{ULT}(\mathfrak{L})$. In the interesting cases this topology will have a nice connection with $\mathfrak{L}$, for which we need an additional property of $\mathfrak{L}$:

Definition 2.3 *A lattice $\mathfrak{L}$ is said to be* disjunctive *if for any $a \neq 1$ there is $b \neq 0$ such that $a \wedge b = 0$.*

Then the following can easily be checked:

Lemma 2.4 *Let $\mathfrak{L}$ be a distributive disjunctive lattice. Then:*

(i) *$V(a) \cap V(b) = V(a \vee b)$ and $V(a) \cup V(b) = V(a \wedge b)$ for any $a, b \in \mathfrak{L}$.*

(ii) *$V(a) = \emptyset$ if and only if $a = 1$.*

(iii) *$V(a) = \mathrm{ULT}(\mathfrak{L})$ if and only if $a = 0$.*

Proof. We only prove (ii) to see how the assumption of disjunctivity is used.

If $a = 1$ then clearly for every $\mathcal{F} \in \mathrm{ULT}(\mathfrak{L})$ we have $a \in \mathcal{F}$, so $V(a) = \emptyset$. On the other hand, if $a \neq 1$ then by disjunctivity there is $b \neq 0$ such that $a \wedge b = 0$. Let $\mathcal{F} \in \mathrm{ULT}(\mathfrak{L})$ be such that $b \in \mathcal{F}$. Then $a \notin \mathcal{F}$, so $\mathcal{F} \in V(a)$, showing that $V(a) \neq \emptyset$. $\bigstar_{2.4}$

Now we shall show using Lemma 2.4 that sets $V(a)$ form a topology on $\mathrm{ULT}(\mathfrak{L})$, and that under additional assumptions on $\mathfrak{L}$ this topology is connected.

Definition 2.5 *A lattice $\mathfrak{L}$ is* normal *if whenever $a, b \in \mathfrak{L}$ satisfy $a \wedge b = 0$ then there are $u, v \in \mathfrak{L}$ such that $u \vee v = 1$ while $u \wedge b = v \wedge a = 0$.*

An element a of $\mathfrak{L}$ is complemented *if there is $b \in \mathfrak{L}$ such that $a \wedge b = 0$ and $a \vee b = 1$. Then such a b is unique and is denoted by a^c.*

Theorem 2.6 *For any normal lattice* $\mathfrak{L}$ *the space* $K = \mathrm{ULT}(\mathfrak{L})$ *is compact and Hausdorff. If the set of complemented elements of* $\mathfrak{L}$ *is* $\{0,1\}$ *and the lattice is disjunctive then* K *is connected.*

Proof. Let $\mathcal{F},\mathcal{G}$ be two distinct $\mathfrak{L}$–ultrafilters. Then $\mathcal{F}$ is not contained in $\mathcal{G}$ so we may take $a \in \mathcal{F}\setminus\mathcal{G}$. By Lemma 2.2 (ii) there is $b \in \mathcal{G}$ such that $a \wedge b = 0$. By the normality of $\mathfrak{L}$ there are $u, v \in \mathfrak{L}$ such that $u \vee v = 1$ while $u \wedge b = v \wedge a = 0$. By the choice of u, v we cannot have $v \in \mathcal{F}$ or $u \in \mathcal{G}$. Therefore $\mathcal{F} \in V(v)$ and $\mathcal{G} \in V(u)$. Moreover, by Lemma 2.2 (iii) we have $V(u) \cap V(v) = V(u \vee v) = V(1) = \emptyset$. This shows that K is Hausdorff.

To check compactness consider a cover of K of the form $V(a_t)$, $t \in T$ and suppose for contradiction that it has no finite subcover. Then using Lemma 2.2 and 2.4 we obtain that for any finite $I \subseteq T$

$$V(\bigwedge_{t\in I} a_t) = \bigcup_{t\in I} V(a_t) \neq K, \text{ so } \bigwedge_{t\in I} a_t \neq 0.$$

Hence a_t are centered and there is an ultrafilter $\mathcal{F}$ containing them all. It follows that $\mathcal{F} \notin V(a_t)$ for any $t \in T$, a contradiction.

Suppose now that the only complemented elements in $\mathfrak{L}$ are 0 and 1, $\mathfrak{L}$ is disjunctive and that $M \subseteq K$ is a clopen set. Then by compactness and Lemma 2.4 $M = V(a)$ and $K \setminus M = V(b)$ for some $a, b \in \mathfrak{L}$. We have $K = V(a) \cup V(b) = V(a \wedge b)$ so by 2.4(iii) $a \wedge b = 0$; similarly $\emptyset = V(a) \cap V(b) = V(a \vee b)$ so $a \vee b = 1$. It follows that $a = b^c$ so $a = 0$ or $a = 1$ and $M = K$ or $M = \emptyset$. $\bigstar_{2.6}$

Claim 2.7 *Suppose that* $\mathfrak{L}$ *is a distributive lattice. Then the mapping* $a \mapsto F(a)$ *is a lattice isomorphism between* $\mathfrak{L}$ *and the family of closed subsets of* $\mathrm{ULT}(\mathfrak{L})$.

If K *is a compact Hausdorff space then the family of its closed subsets forms a normal distributive lattice. If* K *is connected then this lattice is disjunctive.*

Proof. The mapping preserves lattice operations by Lemma 2.4 (i). For $a, b \in \mathfrak{L}$, if $a \neq b$ then $a \Delta b \neq 0$. Say $a \setminus b \neq 0$. Therefore there is a filter $\mathcal{F}$ such that $a \setminus b \in \mathcal{F}$, implying that $a \in \mathcal{F}$ and $b \notin \mathcal{F}$. Then $\mathcal{F} \in F(a) \setminus F(b)$, showing that the mapping is injective. Finally, by compactness and Lemma 2.4 (i), any closed set in $\mathrm{ULT}(\mathfrak{L})$ is of the form $F(a)$ for some $a \in \mathfrak{L}$.

It is clear that the family of closed sets of a connected compact Hausdorff space forms a normal disjunctive distributive lattice. $\bigstar_{2.7}$

3 Spils

Definition 3.1 *A* strong proximity involution lattice (spil) *is given by a structure* $\langle B, \vee, \wedge, ', 0, 1, \prec\rangle$ *where* $\langle B, \vee, \wedge, 0, 1\rangle$ *is a distributive lattice and the following additional axioms hold:*

(i) $\prec$ *is a transitive binary relation which is also* interpolating*: for all* $a, b, c \in B$ *if* $a \prec c$ *then there is some* b *such that* $a \prec b$ & $b \prec c$*.*

(ii) for all finite $M \subseteq B$ *and* $a \in B$

$$(\forall m \in M) m \prec a \iff \bigvee M \prec a,$$

$$(\forall m \in M) a \prec m \iff a \prec \bigwedge M$$

(iii) Involution ' is a unary operation satisfying that

(a) $x'' = x$ *for all* x *(we say the involution is* proper*);*

(b) for all x, y *and* z *we have* $x \wedge y \prec z$ *iff* $x \prec z \vee y'$ *and*

(c) (De Morgan laws) $(x \vee y)' = x' \wedge y'$ *and its dual* $(x \wedge y)' = x' \vee y'$ *hold;*

(iv) $x \prec y \wedge y' \implies x \prec 0$.

It is convenient to use the notation $M \prec a$ for $(\forall m \in M)\, m \prec a$ and similarly for $a \prec M$.

The idea of a spil is that it is a substitute for a Boolean algebra, where the involution plays the role of the complement and $\prec$ the role of the order $\leq$ induced by the Boolean operations. As in the classical case of the Boolean algebras there is a duality in the axioms, as seen in (ii) and (iii).

Basic properties of strong proximity lattices are given by the following Lemma, which is Lemma 7 in [4]. For the sake of completeness we give the proof.

Lemma 3.2 *Suppose that* B *is a spil. Then for all* $a, b, c, d \in B$ *we have*

(1) $0 \prec a \prec 1$,

(2) $a \prec b \implies a \prec b \vee c$,

(3) $a \prec b \implies a \wedge c \prec b$,

(4) $a \prec b$ & $c \prec d \implies a \vee b \prec c \vee d$,

(5) $a \prec b$ & $c \prec d \implies a \wedge b \prec c \wedge d$.

Proof. (1) We have $\emptyset \prec a$ trivially so $0 = \bigvee \emptyset \prec a$ by axiom (ii). Similarly $a \prec \bigwedge \emptyset = 1$. For (2) write $b = b \wedge (b \vee c)$ and use (ii). (3) is proved similarly. For (4) first use (2) to get $\{a, b\} \prec c \vee d$, and then use (ii). (5) is proved similarly. $\bigstar_{3.2}$

The next Lemma gives further basic properties, this time involving the involution.

Lemma 3.3 *Suppose that B is a spil. Then B satisfies:*
(1) for all x, y and z we have $x \wedge y' \prec z$ iff $x \prec z \vee y$, and
(2) for all x and y, $y \vee y' \prec x \implies 1 \prec x$.

Proof. (1) Suppose that $x \wedge y' \prec z$, so by (iii)(b) we have $x \prec z \vee y'' = z \vee y$. The other direction is proved similarly.
(2) Suppose that $y \vee y' \prec x$. We have by the properness of the involution that $y \vee y' = y'' \vee y'$ which is by De Morgan laws equal to $(y' \wedge y)'$. Hence $1 \wedge (y' \wedge y)' = (y' \wedge y)' \prec x$. By (iii)(b) we have $1 \prec x \vee (y' \wedge y)$. Therefore $1 \prec (x \vee y') \wedge (x \vee y)$, giving us by (ii) that $1 \prec x \vee y'$ and $1 \prec x \vee y$. From $1 \prec x \vee y'$ we obtain by (iii)(b) that $1 \wedge y \prec x$, so $y \prec x$. Using that $x \prec 1$ from 3.2(1), we conclude that $x \vee y \prec x$ by 3.2(4). Then $1 \prec x$ by transitivity. $\bigstar_{3.3}$

We now proceed to associate to every spil a compact Hausdorff space, in a manner similar to the classical Stone representation theorem. The main difference is that filters are defined in connection with the $\prec$ relation rather than the Boolean-algebraic order $\leq$ and that there are no complements.

Definition 3.4 *Suppose that B is a spil.*
(1) For $A \subseteq B$ we define $\uparrow A \stackrel{\text{def}}{=} \{x \in B : (\exists a \in A)\, a \prec x\}$.
(2) A $\prec$-filter *F on B is a non-empty subset of B which is closed under (finite) meets and satisfies $F = \uparrow F$.*
(3) A $\prec$-filter F on B is called prime *iff for every finite $M \subseteq B$ with $\bigvee M \in F$ we have that $a \in F$ for some $a \in M$.*
(4) $\mathbf{spec}(B)$ *is the set of all prime $\prec$-filters with the topology generated by the sets*

$$O_x \stackrel{\text{def}}{=} \{F \in \mathbf{spec}(B) : x \in F\}$$

for $x \in B$. (We shall prove below that these sets really form a basis).

Note that a prime $\prec$-filter is not necessarily an ultrafilter in the sense of containing every set or its complement, as there is no complement to speak of– the involution does not necessarily satisfy $x \wedge x' = 0$ for all x. It is also not necessarily a $\subseteq$-maximal filter. That is why $\mathbf{spec}(B)$ is not necessarily isomorphic to a subspace of 2^B and in fact it is not necessarily zero-dimensional. Some basic properties of prime filters are given by the following

Lemma 3.5 *Let B be a spil. Then:*

(1) if F is a prime $\prec$-filter on B then $0 \notin F$, and $1 \in F$,

(2) if $a, b \in B$ then $O_{a\wedge b} = O_a \cap O_b$ and $O_{a\vee b} = O_a \cup O_b$,

(3) if F is a prime $\prec$-filter on B and $a \in F$ then $a' \notin F$,

(4) if F is a prime $\prec$-filter on B, $a, b \in B$ and for some x we have $x \prec a$ and $x' \prec b$, then $a \in F$ or $b \in F$,

(5) if $F \neq G$ are two prime $\prec$-filters on B, there is a such that $a \in F$ and $a' \in G$ or $a' \in F$ and $a \in G$.

Proof. (1) If $0 \in F$ then $\bigvee \emptyset \in F$ so $F \cap \emptyset \neq \emptyset$ by primeness, a contradiction. Since $\emptyset \subseteq F$ we have $\bigwedge \emptyset \in F$ so $1 \in F$.
(2) If F is a $\prec$-filter containing both a, b then it also contains $a \wedge b$ by the closure under meets. If F is a $\prec$-filter containing $a \wedge b$ then by $F = \uparrow F$ we get that for some $x \in F$ the relation $x \prec a \wedge b$ holds. Then $x \prec a$ and $x \prec b$ by the axioms of a spil, and hence $a, b \in F$. This shows the first equality. For the second equality, if $F \in \mathbf{spec}(B)$ and $a \vee b \in F$ then by the primeness of F we have $a, b \in F$; hence $O_{a\vee b} \subseteq O_a \cup O_b$. If $F \in O_a$ then $a \in F = \uparrow F$, so for some $c \in F$ we have $c \prec a$. By Lemma 3.2(2) we have $c \prec a \vee b$ and hence $a \vee b \in \uparrow F = F$. This shows $O_a \subseteq O_{a\vee b}$ and similarly $O_b \subseteq O_{a\vee b}$.
(3) Suppose otherwise and let $a, a' \in F$, hence $a \wedge a' \in F = \uparrow F$. By axiom (iv)(b) we have $a \wedge a' \prec 0$ so $0 \in F$, contradicting (1).
(4) By Lemma 3.2(4) we have $x \vee x' \prec a \vee b$. By Lemma 3.3(2) we have $1 \prec a \prec b$ then by (1) above and $F = \uparrow F$ we get $a \vee b \in F$, and hence $a \in F$ or $b \in F$.
(5) Suppose $F \neq G$ and say $a \in F \setminus G$ (if there is no such a, then there is $a \in G \setminus F$ and that case is handled by symmetry). Since $a \in F = \uparrow F$ there is $b \in F$ with $b \prec a$, and for the same reason there is $c \in F$ with $c \prec b$. By transitivity we have $c \prec a$. By Lemma 3.2(4) it follows that $c \prec a \vee b$, so by axiom (iii)(b) of a spil we have $c \wedge b' \prec a$. On the other hand, by Lemma 3.2(5) we have $c \wedge b \prec a$. Putting these two

conclusions together and using Lemma 3.2(4) we have $c \wedge (b \vee b') \prec a$. Using axiom (iii)(b) we have $b \vee b' \prec a \vee c'$ and then by Lemma 3.3(2). this implies $1 \prec a \vee c'$. By (1) of this Lemma we have $a \vee c' \in G$ so by the primeness of G we have $a \in G$ or $c' \in G$. Since $a \notin G$ we have $c' \in G$. $\bigstar_{3.5}$

To prove Theorem 3.9 below we need to assure Hausdorffness and compactness of the resulting space. The former will follow by Lemma 3.5 and for the latter we shall need the following lemmas:

Lemma 3.6 *Suppose that B is a spil and $A \subseteq B$. Then:*
(1) $\uparrow (\uparrow A) = \uparrow A$ and,
(2) if A is closed under meets then so is $\uparrow A$.

Proof. (1) If $c \in \uparrow A$ then there is $a \in A$ with $a \prec c$, so by axiom (i)(b) of spils there is some b such that $a \prec b$ and $b \prec c$. Then $b \in \uparrow A$, so $c \in \uparrow (\uparrow A)$.

If $c \in \uparrow (\uparrow A)$ then there is $b \in \uparrow A$ such that $b \prec c$, hence $a \in A$ such that $a \prec b$ and $b \prec c$. Since $\prec$ is transitive we have that $c \in \uparrow A$.
(2) Let $b, d \in \uparrow A$, hence there are $a, c \in A$ such that $a \prec b$ and $c \prec d$. Then by Lemma 3.2(5) we have $a \wedge b \prec c \wedge d$ and since $a \wedge b \in A$ we conclude $c \wedge d \in \uparrow A$. $\bigstar_{3.6}$

Lemma 3.7 *Suppose that B is a spil and $A \subseteq B$ is closed under meets and satisfies that for no $x \in A$ do we have $x \prec 0$. Then there is a prime filter F containing A as a subset.*

Proof. Let $\mathcal{F}$ be given by

$$\mathcal{F} = \{F \subseteq B : A \subseteq F, 0 \notin F \text{ and } F \text{ is a filter}\}.$$

By the choice of A we have $0 \notin \uparrow A$ and by Lemma 3.6(2) we have $\uparrow A$ is closed under meets. By Lemma 3.6(1) we have $\uparrow (\uparrow A) = \uparrow A$, so $A \in \mathcal{F}$. Consequently $\mathcal{F} \neq \emptyset$. Now we observe the following

Claim 3.8 *If $F \in \mathcal{F}$ then $\uparrow (F \cup \{1\}) \in \mathcal{F}$.*

Proof of the Claim. By Lemma 3.6 it suffices to check that $F \cup \{1\}$ is closed under meets and does not contain 0, which follows by the choice of F. $\bigstar_{3.8}$

It is easily seen that $\mathcal{F}$ is closed under $\subseteq$-increasing unions so by Zorn's lemma there is a maximal element F of $\mathcal{F}$. We claim that F is prime. By Claim 3.8 and maximality we have that $1 \in F$. Now we shall show that for all $p \in B$ either p or p' are in F (not both as then $0 \in F$). So suppose that $p \in B$ is such that $p, p' \notin F$. The family $X = \uparrow (F \cup \{p \wedge q : q \in F\})$ is clearly a set satisfying $X = \uparrow X$ that is closed under meets and is proper a superset of F because it includes p. By maximality of F we have that $0 \in F$ so for some $q \in F$ the relation $p \wedge q \prec 0$ holds. Similarly we can find $r \in F$ such that $p' \wedge r \prec 0$ holds. Applying axiom (iii)(b) of a spil we obtain that $q \prec p'$ and $r \prec p''$, so $p \wedge q \prec p' \wedge p''$ by Lemma 3.2(5), and hence by axiom (iv) of a spil, $q \wedge r \prec 0$, which is a contradiction with the choice of F.

Now suppose that $M \subseteq B$ is finite such that $m = \bigvee M \in F$ but no $p \in M$ is in F. Hence for all $p \in M$ we have $p' \in F$ and so $\bigwedge \{p' : p \in M\} = m' \in F$. But then $m \wedge m' \in F$, which contradicts axiom (iv) and the fact that $0 \notin F$. We have shown that F is as required. $\bigstar_{3.7}$

Theorem 3.9 *Let* $\mathbf{spec}(B)$ *be as defined in Definition 3.5. Then* $\mathbf{spec}(B)$ *is a compact Hausdorff space with* $\{O_x : x \in B\}$ *a base.*

Proof. Clearly every element of $\mathbf{spec}(B)$ is contained in some O_a. It follows by Lemma 3.5(2) that the family $\{O_a : a \in B\}$ indeed forms a base for a topology on $\mathbf{spec}(B)$. Now we show that the topology is Hausdorff.

Suppose that $F \neq G$ are prime $\prec$-filters. By Lemma 3.5(5) there is a such that $a \in F$ and $a' \in G$, or vice versa. Let us say that $a \in F$. Then $F \in O_a$ and $G \in O_{a'}$ and by Lemma 3.5(3), the sets O_a and $O_{a'}$ are disjoint.

Finally we need to show that $\mathbf{spec}(B)$ is compact. So suppose that $\{O_p : p \in A\}$ covers $\mathbf{spec}(B)$ but no finite subfamily does. By Lemma 3.5(2) we may assume that A is closed under finite joins. By the choice of A for all finite $M \subseteq A$ there is $F \in \mathbf{spec}(B)$ with $\bigvee M \notin F$. Fix such M, F and let $q = \bigvee M$. If for some $p \in F$ we have that $p \wedge q' \prec 0$ then $p \prec 0 \vee q'' = q$, so $q \in F$ as F is a filter, a contradiction. So for no $p \in F$ do we have $p \wedge q' \prec 0$ and in particular we cannot have $q' \prec 0$ by Lemma 3.2(3). This means that the family $\{p' : p \in A\}$ is closed under meets (as A is closed under joins) and none of its elements is $\prec 0$. By Lemma 3.7 there is a prime filter F that contains this family as a subset. By the choice of A there is $p \in A$ such that $F \in O_p$. But then $p, p' \in F$ which contradicts Lemma 3.5(3). $\bigstar_{3.9}$

We are in particular interested in the situation when $\mathbf{spec}(B)$ is connected. Characterising this situation will become easier once we prove the whole representation theorem.

The idea behind the direction from the space to a spil in the representation theorem is that the pairs of the form (O, K) where O is open and $K \supseteq O$ compact will replace the clopen sets in the Stone representation. The relation $\prec$ will be a replacement for $\subseteq$ (so $\leq$ in the Ba representation), so we shall have $(O_0, K_0) \prec (O_1, K_1)$ iff $K_0 \subseteq O_1$.

Theorem 3.10 *Suppose that X is a compact Hausdorff space. We define*

- $B \stackrel{\text{def}}{=} \{(O, K) : O \textit{ is open } \subseteq X, K \textit{ is compact } \subseteq X, O \subseteq K\}$,
- $(O_0, K_0) \vee (O_1, K_1) \stackrel{\text{def}}{=} (O \cup O_1, K_0 \cup K_1)$,
- $(O_0, K_0) \wedge (O_1, K_1) \stackrel{\text{def}}{=} (O_0 \cap O_1, K_0 \cap K_1)$,
- $0 \stackrel{\text{def}}{=} (\emptyset, \emptyset), 1 \stackrel{\text{def}}{=} (X, X)$,
- $(O_0, K_0) \prec (O_1, K_1) \iff K_0 \subseteq O_1$,
- $(O, K)' \stackrel{\text{def}}{=} (X \setminus K, X \setminus O)$.

Then $\langle B, \vee, \wedge, 0, 1, \prec, ' \rangle$ is a spil such that $\mathbf{spec}(B)$ is homeomorphic to X.

Proof. It is clear that $\langle B, \vee, \wedge, 0, 1\rangle$ is a distributive (bounded) lattice, as well as that $\prec$ is transitive. Since X is compact Hausdorff it is normal so the operation $\prec$ is indeed interpolating. The second axiom from the list in Definition 3.1 is easily seen to hold by the definition of $\wedge$ and $\vee$. Let us consider axiom (iii).

The involution is clearly proper. For part (b) suppose that $(O_0, K_0) \wedge (O_1, K_1) \prec (O_2, K_2)$, so $K_0 \cap K_1 \subseteq O_2$. We have $(O_1, K_1)' = (X \setminus K_1, X \setminus O_1)$ so $(O_2, K_2) \vee (O_1, K_1)' = (O_2 \cup (X \setminus K_1), K_2 \cup (X \setminus O_1))$. Since $K_0 \subseteq O_2 \cup (X \setminus K_1)$ we obtain that $(O_0, K_0) \prec (O_2, K_2) \vee (O_1, K_1)'$, as required. The remaining direction of the axiom is proved similarly. De Morgan laws clearly hold.

For axiom (iv), if $(O, K) \prec (U, H) \wedge (X \setminus H, X \setminus U)$ then since $U \subseteq H$ we have $X \setminus U \supseteq X \setminus H$ and hence $U \cap (X \setminus H) = \emptyset$ (as a side note observe that it does not necessarily follow that $H \cap (X \setminus U) = \emptyset$). Since $(O, K) \prec (U, H)$ we have $K = \emptyset$, so $O = \emptyset$ and clearly $(O, K) \prec (\emptyset, \emptyset)$.

This shows that B is a spil and we have to verify that X is homeomorphic to $\mathbf{spec}(B)$. To this end let us define for $x \in X$ the set $F_x = \{(O, K) \in B : x \in O\}$.

Claim 3.11 *Each F_x is an element of* $\mathbf{spec}(B)$.

Proof of the Claim. Let $x \in X$. Since $(X, X) \in F_x$ we have that $F_x \neq \emptyset$. It is clear that F_x is closed under meets, so F_x is a filter. Suppose that $(\bigcup_{i<n} O_i, \bigcup_{i<n} K_i) \in F_x$, where each $(O_i, K_i) \in B$. Hence $x \in \bigcup_{i<n} O_i$ so there is some $i < n$ such that $x \in O_i$ and so $(O_i, K_i) \in F_x$. $\bigstar_{3.11}$

Let g be the function associating F_x to x. We claim that g is a homeomorphism between X and $\mathbf{spec}(B)$. If $x \neq y$ then there is O open containing x and not containing y. Hence $(O, X) \in F_x \setminus F_y$ and hence $F_x \neq F_y$. So g is 1-1.

Suppose that $F \in \mathbf{spec}(B)$ and let $\mathcal{K} = \{K : (\exists O)(O, K) \in F\}$. Since this is a centred family of compact sets its intersection is nonempty, so let $x \in \bigcap \mathcal{K}$. We claim that $F = F_x$. If not, then there is $a = (O, K) \in F_x$ such that $a' \in F$ (by Lemma 3.5(5) and the fact that the involution is proper in B). But then $x \in O$ and hence $x \notin X \setminus O$, contradicting the assumption that $a' = (X \setminus K, X \setminus O) \in F$. Hence g is bijective.

Suppose that U is basic open in $\mathbf{spec}(B)$ so $U = O_a$ for some $a = (O, K)$. Then

$$g^{-1}(O_a) = \{x : F_x \in O_a\} = \{x : a \in F_x\} = \{x : x \in O\} = O,$$

so open in X. Hence g is continuous.

Finally, if O is open in X then $g``O = \{g(x) : x \in O\} = \{F_x : x \in O\}$. If U is open $\subseteq O$ and K is a compact superset of U then if $F = O_{(U,K)}$, $F = F_x$ for some $x \in U$, as follows from the argument showing the surjectivity of g. Hence $O_{(U,K)} \in \{F_x : x \in O\}$, which shows that $= \{F_x : x \in O\}$ contains $\bigcup\{O_{(U,K)} : U \text{ open} \subseteq O, K \text{ compact} \supseteq U\}$. In fact we claim that these two sets are equal, which shows that g is an open mapping and hence a homeomorphism. So let $x \in O$ and $(U, K) \in F_x$. Hence $(O \cap U, K) \in F_x$ and so $F_x \in O_{(U,K)}$. $\bigstar_{3.10}$

Now we are able to state

Theorem 3.12 $\mathbf{spec}(B)$ *is connected iff for no $x \in B \setminus \{0, 1\}$ do we have $x \wedge x' = 0$.*

Proof. By Theorem 3.10 we may assume that B is given in the form stated in that theorem. Then $X = \mathbf{spec}(B)$ is connected iff there are no open disjoint sets $O, V \neq X, \emptyset$ such that $X = O \cup V$.

Suppose that X is connected and let $x = (O, K) \in B$, therefore $(O, K)' = (X \setminus K, X \setminus O)$. Suppose $x \wedge x' = 0$. Then $O \cap (X \setminus K) = K \cap (X \setminus O) = \emptyset$. This means that $O = K$ and that letting $V = X \setminus O$ we obtain $O \cup V = X$, and V is open. Hence $O \in \{\emptyset, X\}$ and therefore $x \in \{0, 1\}$.

In the other direction, suppose for contradiction that X is not connected and let O, V exemplify that. Hence both O, V are compact and letting $x = (O, O)$ we obtain $x \wedge x' = 0$, in contradiction with $x \notin \{0, 1\}$. $\bigstar_{3.12}$

We finish this section by explaining the use of the word "strong" in the name for a spil. In the terminology of [4], proximity lattices are structures that satisfy the axioms of a spil but without the involution, and such structures are called strong if they in addition satisfy the following axioms

(A) for all $a, x, y \in B$

$$x \wedge y \prec a \implies (\exists x^+, y^+ \in B)\, x \prec x^+, y \prec y^+ \;\&\; x^+ \wedge y^+ \prec a;$$

(B) for all $a, x, y \in B$

$$a \prec x \vee y \implies (\exists x^+, y^+ \in B)\, x^+ \prec x, y^+ \prec y \;\&\; a \prec x^+ \vee y^+;$$

Note that $\prec$ is not necessarily reflexive in a spil hence axioms (A) and (B) are not trivially met. We shall however demonstrate that every spil satisfies them.

Claim 3.13 *Suppose B is a spil. Then axioms (A) and (B) above are satisfied.*

Proof of the Claim. Let us first show (A), so suppose that $x \wedge y \prec a$. Then by the interpolating property of $\prec$ there is b such that $x \wedge y \prec b \prec a$. By axiom (iii)(b) of a spil this gives $x \prec b \vee y'$. Similarly we obtain $y \prec b \vee x'$. Letting $x^+ = b \vee y'$ and $y^+ = b \vee x'$ we have $x^+ \wedge y^+ = b \wedge (x' \vee y')$. Since $b \prec a$, by Lemma 3.2(3) we have $b \wedge (x' \vee y') \prec a$, hence x^+ and y^+ are as required.

(B) is shown similarly. $\bigstar_{3.13}$

4 Lattices

In the previous sections we have given two theorems which both can be viewed as algebraic representation theorems for connected compact Hausdorff spaces. A corollary of this is that the algebraic notions used are birepresentable. In particular, every distributive disjunctive normal lattice can be adjoined an order and a convolution operator to make it into a spil.

Suppose that $\mathfrak{L}$ is a distributive disjunctive normal lattice and let $X = ULT(\mathfrak{L})$ be the connected compact Hausdorff space constructed in §2. Therefore $\mathfrak{L}$ is isomorphic to the lattice of closed sets of X endowed with $\cap, \cup$ and this space is unique up to homeomorphism. Let $\mathfrak{A}$ be the Boolean algebra generated by the closed and open sets in X. We consider $\mathfrak{L}$ as a sublattice of $\mathfrak{A}$ and therefore the family of open subsets of X is the set of complements in $\mathfrak{A}$ of the elements of $\mathfrak{L}$. For $a \in \mathfrak{L}$ we denote by a^c the complent of a in $\mathfrak{A}$, which agrees with the previous definition in the case that $a^c \in \mathfrak{L}$.

Definition 4.1 *Let $\mathfrak{L}, X$ be as above. We define* the spil *induced by $\mathfrak{L}$ by letting*

$$B = \{(u, k) : u \in \mathfrak{L}^c, k \in \mathfrak{L} \text{ and } u \subseteq k\},$$

endowing it with the following operations:

- $(u, k) \wedge (v, h) = (u \cup v, k \cap h)$,
- $(u, k) \vee (v, h) = (u \cup v, k \cup h)$,
- $(u, k)' = (k^c, u^c)$

and the relation $(u, k) \prec (v, h)$ iff $k \subseteq v$. We let $1 = (X, X)$ and $0 = (\emptyset, \emptyset)$.

Theorem 4.2 *Suppose that $\mathfrak{L}$ is as in Definition 4.1. Then*
(1) the spil B induced by $\mathfrak{L}$ is a spil and
(2) the space $\mathbf{spec}(B)$ is homeomorphic to X and its lattice of closed subsets is isomorphic to $\mathfrak{L}$.

Proof. (1) Clearly B is a distributive lattice with the 0 and 1 as specified. We check the rest of the axioms of Definition 3.1.

It is clear that $\prec$ is transitive. Checking that the relation $\prec$ is interpolating uses the normality of $\mathfrak{L}$. Suppose that $(u, k) \prec (v, h)$ holds, hence $k \subseteq v$ and hence k, v^c are disjoint elements of $\mathfrak{L}$. Let

$w, z \in \mathfrak{L}$ be such that $w \cup z = X$ while $w \cap v^c = \emptyset$ and $z \cap k = \emptyset$. From $w \cup z = X$ we conclude that $w \supseteq z^c$ and from $z \cap k = \emptyset$ we have $k \subseteq z^c$. Hence $k \supseteq w$. From $w \cup z = X$ and $w \cap v^c = \emptyset$ we conclude $v^c \subseteq z$, and hence $v \supseteq z^c$. Therefore $(w, z^c) \in B$ satisfies $(u, k) \prec (w, z^c) \prec (v, h)$.

Axiom (ii) of a spil follows by the corresponding properties of the Boolean algebra $\mathfrak{A}$. Similarly for axioms (iii) and (iv).
(2) Let B^* be the spil consisting of pairs (O, K) of pairs of open and compact subsets of $\mathbf{spec}(B)$ such that $O \subseteq K$. By the representation theorem in §3 we have that $\mathbf{spec}(B^*)$ and $\mathbf{spec}(B)$ are homeomorphic and this induces an isomorphism between B and B^*. Hence $\mathfrak{L}$ is isomorphic to the lattice of closed, equivalently, compact, subsets of $\mathbf{spec}(B)$. This implies that $\mathbf{spec}(B)$ is homeomorphic to $ULT(\mathfrak{L})$. $\bigstar_{4.2}$

Corollary 4.3 *Every distributive normal disjunctive lattice induces a unique spil B satisfying that $\mathfrak{L}$ is isomorphic to the lattice of closed subsets of* $\mathbf{spec}(B)$.

Running the proof of Theorem 4.2 backwards will naturally show how each spil induces a distributive disjunctive normal lattice.

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Sadržaj

Predstavljena su dva teorema koja se mogu koristiti da predstave kompaktne povezane Hausdorff-ove prostore na algebarski način, putem reprezentacije u stilu Stone-a. Prvi teorem proizilazi iz Wallman-ovog rada i pokazuje da svaka distributivna disjunktna normalna latica jeste latica zatvorenih skupova u kompaktnom povezanom Hausdorff-ovom prostoru koji je do homeomorfizama, jedinstveno odredjen. Drugi teorem proizilazi iz rada Jung-a i Sünderhauf-a. Definisan je pojam snažno bliske latice sa involucijom i pokazano je da se svaka takva latica može jednoznačno interpretirati kao latica parova kompaktnih i otvorenih skupova u kompaktnom povezanom Hausdorff-ovom prostoru. Kao posljedicu ova dva teorema smo dobili neočekivan rezultat koji pokazuje da je moguće interpretirati distributivne disjunktne normalne latice i snažno bliske latice sa involucijom jedne putem drugih.