APAL: 1636

pp. 1–23 (col. fig: NIL)

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Annals of Pure and Applied Logic xx (xxxx) xxx–xxx

ANNALS OF PURE AND APPLIED LOGIC

www.elsevier.com/locate/apal

On properties of theories which preclude the existence of universal models[☆]

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Received 20 August 2001; received in revised form 1 March 2003; accepted 1 June 2005

Communicated by T. Jech

Abstract

We introduce the oak property of first order theories, which is a syntactical condition that we show to be sufficient for a theory not to have universal models in cardinality λ when certain cardinal arithmetic assumptions about λ implying the failure of *GCH* (and close to the failure of *SCH*) hold. We give two examples of theories that have the oak property and show that none of these examples satisfy SOP_4, not even SOP_3. This is related to the question of the connection of the property SOP_4 to non-universality, as was raised by the earlier work of Shelah. One of our examples is the theory T^*_{feq} for which non-universality results similar to the ones we obtain are already known; hence we may view our results as an abstraction of the known results from a concrete theory to a class of theories.

We show that no theory with the oak property is simple.

MSC: 03C55; 03E04; 03C45

Keywords: Universal models; Oak property; Singular cardinals; pp

☆ This publication is numbered 710 in the list of publications of Saharon Shelah.

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0168-0072/$ - see front matter © 2005 Published by Elsevier B.V.
doi:10.1016/j.apal.2005.06.001

0. Introduction

Since the very early days of the mathematics of the infinite, the existence of a universal object in a category has been the object of continued interest to specialists in various disciplines of mathematics—even Cantor's work on the uniqueness of the rational numbers as the countable dense linear order with no endpoints is a result of this type. For some more recent examples see for instance [1,5]. We approach this problem from the point of view of model theory, more specifically, classification theory, and we concentrate on first order theories. In [10] the idea was to consider properties that can serve as good dividing lines between first order theories (in [10]; more general theories in other work). This is to be taken in the sense that useful information can be obtained both from the assumption that a theory satisfies the property, and the assumption that it does not, and in general we may expect several equivalent definitions for such properties. Preferably, there is an "outside property" and a "syntactical property" which end up being equivalent. The special outside property which was central in [10] was the number of pairwise non-isomorphic models, and it led to considering the notions of stability and superstability. It is natural to ask whether other divisions can be obtained using problems of similar nature. This is a matter of much investigation and some other properties have been looked at; see for example [6,21] and more generally [20]. One such property is universality, which is the main topic of this paper.

In a series of papers, e.g. Kojman–Shelah [8] (see there also for earlier references), [9], Kojman [7], Shelah [16,18], Džamonja–Shelah [3], the thesis claiming the connection between the complexity of a theory and its amenability to the existence of universal models has been pursued. Further research on the subject is in preparation in Shelah's [23]. It follows from the classical results in model theory (see [2]) that if *GCH* holds then every countable first order theory admits a universal model in every uncountable cardinal, so the question we need to ask is what happens when *GCH* fails. We may define the universality number of a theory T at a given cardinal λ as the smallest size of the family of models of T of size λ having the property that every model of T of size λ embeds into an element of the family. Hence, if *GCH* holds this number for uncountable λ and countable T is always at most 1. It is usually "easy" to force a situation in which such a universality number is as large as possible, namely 2^λ (by adding Cohen subsets, see [8]); however assuming that *GCH* fails and allowing ourselves a vague use of the words "many" and "often" for the moment, we can distinguish between those theories which for many cardinals have the largest possible universality number in that cardinal *whenever GCH* fails, and those for which it is possible to construct a model of set theory in which *GCH* fails, yet our theory has a small universality number at the cardinality under consideration. This division would suggest that the latter theories—let us call them for the sake of this introduction amenable—are of lower complexity than the former ones. The definition of amenability can be given in more precise terms. In the view of the preceding discussion involving the universality behaviour in models of *GCH*, it is not surprising that this definition is expressed in terms of forcing.

Definition 0.1. We say that a theory T is *amenable* iff whenever λ is an uncountable cardinal larger than the size of T and satisfying $\lambda^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$, while θ satisfies $\mathrm{cf}(\theta) > \lambda^+$, there is a λ^+-cc $(< \lambda)$-closed forcing notion that forces 2^λ to be θ and the universality number $\mathrm{univ}(T, \lambda^+)$ (see Definition 0.7) to be smaller than θ.

Localising this definition at a particular λ we define what is meant by theories that are *amenable at* λ.

Kojman and Shelah in [8] proved that the theory of a dense linear order exhibits high non-universality behaviour, making it a prototypical example of a non-amenable theory. That is, they proved (Section 3, proof of Theorem 3.10) that the theory of a dense linear order satisfies the property described in Definition 0.3, which we shall call high non-amenability. We shall indicate below that this name is well chosen, in the sense that high non-amenability implies the negation of amenability as introduced above. In order to define high non-amenability we shall need a somewhat technical definition of a tight (κ, μ, λ) club guessing sequence, but as this definition will be needed anyway in Section 2, we shall give the exact definition now rather than glancing over it for the sake of the introduction.

Definition 0.2. (1) Suppose that $\kappa < \lambda$ are regular cardinals and that $\kappa \leq \mu < \lambda$ while S is a stationary subset of λ consisting of points of cofinality κ. A sequence $\langle C_\delta : \delta \in S\rangle$ will be called *a tight [truly tight]* (κ, μ, λ) *club guessing sequence* iff

(i) for every $\delta \in S$ the set C_δ is a subset of δ with $\mathrm{otp}(C_\delta) = \mu$,
(ii) for every club E of λ there is $\delta \in S$ such that $C_\delta \subseteq E$, and
(iii) for every $\alpha \in \lambda$

$$|\{C_\delta \cap \alpha : \delta \in S \;\&\; \alpha \in (C_\delta \setminus \lim(C_\delta))\}| < \lambda.$$

[In addition to (i)–(iii) above,
(iv) $\sup(C_\delta) = \delta$.]

(2) Suppose that λ is a regular cardinal, $\mu < \lambda$ and $\langle C_\delta : \delta \in S\rangle$ satisfies (i)–(iii) from (1) with the possible exception of S not necessarily being a set of points of cofinality κ for any fixed κ. Then we say that $\langle C_\delta : \delta \in S\rangle$ is a *tight* (μ, λ) *club guessing sequence*.

Definition 0.3. A theory T is said to be *highly non-amenable* iff for every large enough regular cardinal λ and $\kappa < \lambda$ such that there is a truly tight $(\kappa, \kappa, \lambda)$ club guessing sequence $\langle C_\delta : \delta \in S\rangle$, the number $\mathrm{univ}(T, \lambda)$ is at least 2^κ.

Suppose that a theory T is both amenable and highly non-amenable, and let λ be a large enough regular cardinal while $V = L$ or simply $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(S^{\lambda^+}_\lambda)$ holds. Let P be the forcing exemplifying that T is amenable. Clearly there is a truly tight $(\lambda, \lambda, \lambda^+)$ club guessing sequence $\bar{C}$ in V, and since the forcing P is λ^+-cc, every club of λ^+ in V^P contains a club of λ^+ in V; hence $\bar{C}$ continues to be a truly tight $(\lambda, \lambda, \lambda^+)$ club guessing sequence in V^P. Then on the one hand we have that in V^P, $\mathrm{univ}(T, \lambda^+) \geq 2^\lambda$ by the high non-amenability, while $\mathrm{univ}(T, \lambda^+) < 2^\lambda$ by the choice of P, a contradiction.

In fact [8] proves that any theory with the strict order property is highly non-amenable. On the other hand Shelah proved in [18] that all simple theories are amenable at all successors of regular κ satisfying $\kappa^{<\kappa} = \kappa$. In that same paper Shelah introduced a hierarchy of complexity for first order theories, and showed that high non-amenability appears as soon as a certain level on that hierarchy is passed. The details of this hierarchy are described in the following Definition 0.8, but for the moment let us just mention the fact that the hierarchy describes a sequence SOP_n $(3 \leq n < \omega)$ of properties of increasing strength such that the theory of a dense linear order possesses all the properties, while on the other hand no simple theory can have the weakest among them, SOP_3. Shelah proved in [18] that the property SOP_4 of a theory T implies that T exhibits the same non-universality results as the theory of a dense linear order; in other words it is highly non-amenable. In the light of these results it might then be asked whether SOP_4 is a characterisation of high non-amenability, that is whether all highly non-amenable theories also have SOP_4.

The results available in the literature do not provide a counter-example, and the question in fact remains open after this investigation. However we provide a partial solution by continuing a result of Shelah about the theory T^*_{feq} of infinitely many indexed independent equivalence relations, [16]. It is proved there that this particular theory exhibits a non-amenability behaviour provided that some cardinal arithmetic assumptions close to the failure of the singular cardinal hypothesis *SCH* are satisfied (see Section 1 for details). This does not necessarily imply high non-amenability, as it was proved also in [16] that this theory is in fact amenable at any cardinal which is the successor of a cardinal κ satisfying $\kappa^{<\kappa} = \kappa$. Here we generalise the first of these two results by defining a property which implies such non-amenability results and is possessed by T^*_{feq}. This property is called the oak property, as its prototype is the model completion of $\mathrm{Th}(M_{\lambda,\kappa,f,g})$, a theory connected to that of the tree ${}^{\kappa\geq}\lambda$ (for details see Example 1.3). The oak property cannot be made a part of the SOP_n hierarchy, as we exhibit a theory which has oak, and is $NSOP_3$, while the model completion of the theory of triangle-free graphs is an example of a SOP_3 theory which does not satisfy the oak property. On the other hand we prove at the end of Section 1 that no oak theory is simple. We also exhibit a close connection between T^*_{feq} and $\mathrm{Th}(M_{\lambda,\kappa,f,g})$. These results indicate that in order to make the connection between the high non-amenability, amenability and the SOP_n hierarchy more exact one needs to consider the failure of *SCH* as a separate case. In addition the oak property not being compatible with the SOP_n hierarchy gives new evidence that this hierarchy does not exhaust the unstable theories that do not have the strict order property. Note that in [[18], 2.3(2)] there is an example of a first order theory that satisfies the strong order property but not the strict order property (and the strong order property implies all SOP_n, though it is not implied by their conjunction).

To finish this introduction, let us summarise the connection between the cardinal arithmetic and the universality number that is shown in this paper (a more detailed discussion of this can be found at the end of Section 2). Firstly, by classical model theory, if *GCH* holds then the universality number of any first order theory of size $< \lambda$, at any cardinal $\geq \lambda$, is 1—hence the situation is trivialised. Similarly, the results that we have here on sufficient conditions for non-amenability trivialise if the Strong Hypothesis *StH* of Shelah holds [15] because the conditions are never satisfied. *StH* says that $\mathrm{pp}(\mu) = \mu^+$ for every singular μ; hence $\mathrm{cf}([\mu]^{<\kappa}, \subseteq) \leq \mu^+$ for every $\kappa < \mu$, so *StH* implies the Singular Cardinal Hypothesis *SCH* (it is itself implied by $\neg 0^\sharp$). However, if *StH* fails, say κ, λ regulars satisfy that for some singular μ we have $\mathrm{cf}(\mu) = \kappa$ and $\mu^+ < \lambda$ while $\mathrm{pp}(\mu) > \lambda$, for all we know the results here hold and are not trivial, in the sense that not only do all known consistency proofs of the failure of *StH* show this, but it is not known whether it is consistent to have the failure of *StH* and at the same relevant cardinals a failure of our assumptions.

Let us now commence the mathematical part of the paper by giving some background notions which will be used in the main sections of the paper, starting with some classical definitions of model theory.

Convention 0.4. A theory in this paper means a first order complete theory, unless otherwise stated. Such an object is usually denoted by T.

Notation 0.5. (1) Given a theory T, we let $\mathfrak{C} = \mathfrak{C}_T$ stand for "the monster model", i.e. a saturated enough model of T. As is usual, we assume without loss of generality that all our discussion takes place inside some such model, so all expressions to the extent "there is", "exists" and "$\models$" are to be relativised to this model, all models are $\prec \mathfrak{C}$, and all subsets of $\mathfrak{C}$ that we mention have size less than the saturation number of $\mathfrak{C}$. We let $\bar{\kappa} = \bar{\kappa}(\mathfrak{C}_T)$ be the size of $\mathfrak{C}$, so this cardinal is larger than any other cardinal mentioned in connection with T.

(2) For a formula $\varphi(\bar{x}; \bar{a})$ we let $\varphi(\mathfrak{C}; \bar{a})$ be the set of all tuples $\bar{b}$ such that $\varphi[\bar{b}; \bar{a}]$ holds in $\mathfrak{C}$.

Definition 0.6. (1) The tuple $\bar{b}$ is *defined by* $\varphi(\bar{x}; \bar{a})$ if $\varphi(\mathfrak{C}; \bar{a}) = \{\bar{b}\}$. It is defined by the type p if $\bar{b}$ is the unique tuple which realises p. It is definable over A if $\mathrm{tp}(\bar{b}, A)$ defines it.

(2) The formula $\varphi(\bar{x}; \bar{a})$ is *algebraic* if $\varphi(\mathfrak{C}; \bar{a})$ is finite. The type p is algebraic if it is realised by finitely many tuples only. The tuple $\bar{b}$ is algebraic over A if $\mathrm{tp}(\bar{b}, A)$ is.

(3) The *definable closure* of A is

$$\mathrm{dcl}(A) \stackrel{\text{def}}{=} \{b : b \text{ is definable over } A\}.$$

(4) The *algebraic closure* of A is

$$\mathrm{acl}(A) \stackrel{\text{def}}{=} \{b : b \text{ is algebraic over } A\}.$$

(5) If $A = \mathrm{acl}(A)$, we say that A is *algebraically closed*. When $\mathrm{dcl}(A)$ and $\mathrm{acl}(A)$ coincide, $\mathrm{cl}(A)$ denotes their common value.

Definition 0.7. (1) For a theory T and a cardinal λ, models $\{M_i : i < i^*\}$ of T, each of size λ, are *jointly universal* iff for every N a model of T of size λ there is an $i < i^*$ and an isomorphic embedding of N into M_i.

(2) For T and λ as above,

$$\mathrm{univ}(T, \lambda) \stackrel{\text{def}}{=} \min\{|\mathcal{M}| : \mathcal{M} \text{ is a family of jointly universal models of } T \text{ of size } \lambda\}.$$

To make Definition 0.7 more readable, note that $\mathrm{univ}(T, \lambda) = 1$ iff there is a universal model of T of size λ. The following is the main definition of Shelah's [18].

Definition 0.8 (*Shelah, [18]*). Let $n \geq 3$ be a natural number.

(1) A formula $\varphi(\bar{x}, \bar{y})$ is said to exemplify the *n-strong order property*, SOP_n if $lg(\bar{x}) = lg(\bar{y})$, and there are $\bar{a}_k$ for $k < \omega$, each of length $lg(\bar{x})$ such that
 (a) $\models \varphi[\bar{a}_k, \bar{a}_m]$ for $k < m < \omega$,
 (b) $\models \neg(\exists \bar{x}_0, \ldots, \bar{x}_{n-1})[\bigwedge\{\varphi(\bar{x}_\ell, \bar{x}_k) : \ell, k < n \text{ and } k = \ell + 1 \mod n\}]$.

 T has SOP_n if there is a formula $\varphi(\bar{x}, \bar{y})$ exemplifying this.

(2) A theory that does not possess SOP_n is said to have $NSOP_n$.

Note 0.9. Using a compactness argument and the Ramsey theorem, one can prove that if T is a theory with SOP_n and $\varphi(\bar{x}, \bar{y})$, and $\langle \bar{a}_n : n < \omega \rangle$ exemplify it, without loss of generality $\langle \bar{a}_n : n < \omega \rangle$ is an indiscernible sequence. See [10] or [6] for examples of such arguments.

Example 0.10. The model completion of the theory of triangle-free graphs is a prototypical example of a SOP_3 theory, with the formula $\varphi(x, y)$ just stating that x and y are connected. It can be shown that this theory is $NSOP_4$; see [18].

The following fact indicates that $SOP_n (3 \leq n < \omega)$ form a hierarchy, and the thesis is that this hierarchy is reflected in the complexity of the behaviour of the relevant theories under natural constructions in model theory.

Fact 0.11 (*Shelah, [18], Section 2*). *For $3 \leq n < \omega$ the property SOP_{n+1} of a theory implies the property SOP_n.*

1. The oak property

In this section we define a theory T^* that will serve as a prototype of a theory that possesses the oak property. Then we introduce the oak property and prove that the theory T^* has this property. We are interested in the connection between the oak property and the SOP hierarchy (see Definition 0.8). To this end we shall show that T^* satisfies $NSOP_3$ (so by Fact 0.11 it clearly does not satisfy SOP_4). As another example we shall show that the model completion of the theory of infinitely many indexed independent equivalence relations, $T^*_{\rm feq}$, also satisfies oak and $NSOP_3$. This theory is known not to be simple [18], but we shall in fact show that no theory with the oak property is simple.

We commence with some auxiliary theories which will allow us to define T^* (as the model completion of T_0^+).

Definition 1.1. (1) Let T_0 be the following theory in the language

$$\{Q_0, Q_1, Q_2, F_0, F_1, F_2, F_3\} :$$

(i) Q_0, Q_1, Q_2 are unary predicates which form a partition of the universe,
(ii) F_0 is a partial function from Q_1 to Q_0,
(iii) F_1 is a partial two-place function from $Q_0 \times Q_2$ to Q_1,
(iv) F_2 is a partial function from Q_0 to Q_2,
(v) F_3 is a partial function from Q_2 to Q_0,
(vi) the range of F_1 is included in the domain of F_0 and for all $(x, z) \in \mathrm{Dom}(F_1)$ we have $F_0(F_1(x, z)) = x$, and
(vii) the range of F_2 is included in the domain of F_3 and $F_3(F_2(x)) = x$ for all $x \in \mathrm{Dom}(F_2)$.

(2) Let T_0^+ be defined like T_0, but with the requirement that F_0, F_1, F_2 and F_3 are total functions.

Remark 1.2. It is to be noted that the above definition of T_0 uses partial rather than the more usual full function symbols. Using partial functions we have to be careful when we speak about submodels, where we have a choice of deciding whether statements of the form "$F_l(x)$ is undefined" are preserved in the larger model. We choose to request that the fact that F_l is undefined at a certain entry is not necessarily preserved in the larger model. Functions F_2 and F_3 are "dummies" whose sole purpose is to ensure that models of T_0^+ are non-trivial, while keeping T_0^+ a universal theory (which is useful when discussing the model completion). Also note that neither T_0 nor T_0^+ is complete, but every model M of T_0 in which $Q_0^M, Q_2^M \neq \emptyset$ and F_0 and F_3 are onto can be extended to a model of T_0^+ with the same universe (Claim 1.4(2)), and every model of T_0 is a submodel of a model of T_0^+ (Claim 1.4(4)). T_0^+ has a complete model completion (Claim 1.5). This model completion is the main theory we shall work with and, as we shall show, it has the oak property (Claim 1.11) and is *NSOP*$_4$ (Claim 1.7).

As we are only interested in the model completion T^* of T_0^+ we might have omitted the mention of T_0 altogether, but in the interest of possible future examples and also in order to make the proof of the existence of T^* easier, through Claim 1.4 we defined both T_0 and T_0^+ and then showed how to pass from models of one to models of the other.

Example 1.3. Suppose that κ and λ are infinite cardinals and f is any surjective function from ${}^{\kappa}\lambda$ to κ, while g is a function from κ to ${}^{\kappa}\lambda$ satisfying $g(f(\nu)) = \nu$ for all $\nu \in {}^{\kappa}\lambda$. Then we can construct a model $M = M_{\kappa,\lambda,f,g}$ as follows: let Q_0^M be κ, Q_1^M be ${}^{\kappa>}\lambda$, and $Q_2^M = {}^{\kappa}\lambda$. Further let $F_0^M(\eta)$ be the length of η for $\eta \in Q_1$, and let $F_1^M(\alpha, \nu) = \nu \restriction \alpha$. Let F_3^M be f and let F_2^M be g.

We consider such examples to be prototypical for models of T_0^+.

Claim 1.4. (1) *If M is a model of T_0^+, then Q_0^M, Q_1^M and Q_2^M are all non-empty, and F_0^M and F_3^M are onto.*

(2) *Every model M of T_0 in which $Q_0^M \neq \emptyset$ and $Q_2^M \neq \emptyset$, while F_0^M and F_3^M are onto, can be extended to a model of T_0^+ with the same universe (and every model of T_0^+ is a model of T_0).*

(3) *There are models M of T_0 with $Q_0^M \neq \emptyset$ and $Q_2^M \neq \emptyset$ and F_3^M onto which cannot be extended to a model of T_0^+ with the same universe.*

(4) *Every model of T_0 is a submodel of a model of T_0^+.*

(5) *T_0^+ has the amalgamation property and the joint embedding property JEP.*

(6) *If $M \models T_0$ and $A \subseteq M$ is finite, then the closure B of A under F_0^M, F_1^M, F_2^M and F_3^M is finite (in fact $|B| \leq |A|^2 + 2|A|$); moreover:*

(a) $B \cap Q_2^M = (A \cap Q_2^M) \cup \{F_2^M(a) : a \in A \cap Q_0^M\}$,

(b) $B \cap Q_0^M = (A \cap Q_0^M) \cup \{F_0^M(b) : b \in A \cap Q_1^M\} \cup \{F_3^M(c) : c \in A \cap Q_2^M\}$ *and*

(c) $B \cap Q_1^M = (A \cap Q_1^M) \cup \{F_1^M(a, c) : a \in B \cap Q_0^M \ \&\ c \in B \cap Q_2^M\}$.

In this case, $B \models T_0$ and if $M \models T_0^+$, then $B \models T_0^+$.

To declutter the notation we shall from now on whenever possible in discussing T_0, T_0^+ (and its model completion T^* which will be introduced later) omit the superscript M from the function symbols.

Proof. (1) As M is a model we have that $M \neq \emptyset$, so at least one among Q_0^M, Q_1^M, Q_2^M is not empty.

If $Q_0^M \neq \emptyset$, then F_2 guarantees that $Q_2^M \neq \emptyset$, so $Q_1^M \neq \emptyset$ because of F_1. If $Q_1^M \neq \emptyset$, then $Q_0^M \neq \emptyset$ because of F_0. Finally, if $Q_2^M \neq \emptyset$, then $Q_0^M \neq \emptyset$ because of F_3, and we can again argue as above.

If $a \in Q_0^M$, let $b \in Q_2^M$ be arbitrary. Then $F_1(a, b) \in Q_1^M$ and $F_0(F_1(a, b)) = a$. Hence, F_0 is onto. Also, $F_3(F_2(a)) = a$, so F_3^M is onto.

(2) Let $M \models T_0$ and $Q_0^M, Q_2^M \neq \emptyset$. For $x \in Q_0^M$ and $z \in Q_2^M$ such that $(x, z) \notin \mathrm{Dom}(F_1^M)$, let $F_1(x, z) = y$ for any $y \in Q_1^M$ such that $F_0(y) = x$, which exists as F_0^M is already onto. For $x \in Q_0^M$ for which $F_2(x)$ is not already defined, let $F_2(x) = z$ for any z such that $F_3(z) = x$, which exists as F_3^M is onto. Finally, extend F_0 and F_3 to be total. The model described is a model of T_0^+ with the same universe as M.

(3) Let $\kappa_1 < \kappa_2 < \lambda$ and let $Q_0^M = \kappa_2$, $Q_1^M = {}^{\kappa_1 >}\lambda$, while $Q_2^M = {}^{\kappa_1}\lambda$. For $\alpha < \kappa_2$ let $F_2(\alpha)$ be the function in ${}^{\kappa_1}\lambda$ which is constantly α, and for $\nu \in {}^{\kappa_1}\lambda$ let $F_3(\nu) = \min(\mathrm{Rang}(\nu))$ if this value is $< \kappa_2$, and 0 otherwise. Also, let $F_0(\eta) = lg(\eta)$ and $F_1(\alpha, \nu) = \nu \restriction \alpha$ be defined for $\nu \in {}^{\kappa_1}\lambda$ and $\alpha < \kappa_1$.

This is a model of T_0, but not of T_0^+ because F_1 is not total. If this model were to be extended to a model of T_0^+ with the same universe, we would have that for every $\nu \in {}^{\kappa_1}\lambda$

$$F_0(F_1(\kappa_1, \nu)) = \kappa_1 \ \& \ F_1(\kappa_1, \nu) = \eta$$

for some $\eta \in {}^{\kappa_1 >}\lambda$. As $F_0(\eta)$ is already defined, $F_0(\eta) = lg(\eta) < \kappa_1$, which is a contradiction.

(4) Given a model M of T_0. First ensure that $Q_0^M, Q_1^M, Q_2^M \neq \emptyset$ by adding new elements if necessary. Then make sure that F_0 and F_3 are total and onto, which might require adding new elements to M (and hence redefining Q_0^M, Q_1^M, Q_2^M if needed). Now for each $x \in Q_0^M$ choose $y(x) \in Q_1^M$ such that $F_0(y(x)) = x$, which is possible since F_0 is onto, and then define for every $(x, z) \in Q_0^M \times Q_2^M$ the value of $F_1(x, z)$ to be $y(x)$, unless $F_1(x, z)$ has already been defined to start with, in which case we leave it at that value. Finally declare for $x \in Q_0^M$ for which $F_2(x)$ has not already been defined that $F_2(x) = z$ for any z such that $F_3(z) = x$, which can be done since F_3 is onto.

(5) We first prove the amalgamation property. Suppose that M_0, M_1 and M_2 are models of T_0^+ with $|M_1| \cap |M_2| = |M_0|$, and $M_0 \subseteq M_1, M_2$. We define M_3 as follows. Let $|M_3| = |M_1| \bigcup |M_2|$, and for $m \in \{0, 2, 3\}$ let $F_m^{M_3}(x) = F_m^{M_l}(x)$ if $x \in M_l$ for some l. This is well defined, because M_1 and M_2 agree on M_0. Also, the identity $F_3(F_2(x)) = x$ is satisfied in M_3. Now we let $F_1^{M_3} = F_1^{M_1} \cup F_1^{M_2}$. This does not necessarily give us a total function, but we still have a model of T_0 with universe $|M_1| \cup |M_2|$ and so to obtain the desired amalgam (which has the same universe) we apply part (2) of this claim. From this definition it follows that both M_1 and M_2 are submodels of M_3 and equal to its restriction to their respective universes.

To see that JEP holds, suppose that we are given two models M_1, M_2 of T_0^+. Define M by letting its universe be the disjoint union of M_1 and M_2, and define the functions F_m for $m \in \{0, 1, 2, 3\}$ by $F_m^M = F_m^{M_1} \cup F_m^{M_2}$. Then M is a model of T_0, but like in the proof of amalgamation, the function F_1 might happen to be only partial, in which case we extend M to a model of T_0^+ by applying part (2) of this claim. Then it can easily be checked that M embeds both M_1 and M_2.

(6) Suppose that A and M are as in the assumptions. Then items (a)–(c) of the statement uniquely define a subset of M, which we shall call B. The proof will be complete if we can prove that B is of the required size and is the closure of A.

Clearly B is contained in the closure of A and the size of B is as claimed. That is, letting for $l \in \{0, 1, 2\}$ the size of $A \cap Q_l^M$ be n_l and $n = \Sigma_{l<3} n_l$, we have first that $|B \cap Q_2^M| \leq n_2 + n_0$, then $|B \cap Q_0^M| \leq n_0 + n_1 + n_2 \leq n$, and so $|B \cap Q_1^M| \leq n_1 + n^2$. It can be checked directly that B is closed, using the equations for T_0, and it also easily follows that B is a model of T_0, or of T_0^+ if M is. $\square$

Claim 1.5. *T_0^+ has a complete model completion T^* which admits elimination of quantifiers, and is $\aleph_0$-categorical. In this theory the closure and the algebraic closure coincide.*

Proof. We can construct T^* directly. T^* admits elimination of quantifiers because T_0^+ has the amalgamation property and is universal ([2] 3.5.19). It can be seen from the construction of T^* that it is complete, or alternatively, it can be seen that T^* has JEP and so by [2] 3.5.11, it is complete. To see that the theory is $\aleph_0$-categorical, observe that Claim 1.4(6) implies that for every n there are only finitely many T_0-types in n-variables. Then by the Characterisation of Complete $\aleph_0$-categorical Theories ([2] 2.3.13), T^* is $\aleph_0$-categorical. Using the elimination of quantifiers and the fact that all relational symbols of the language of T^* have infinite domains in every model of T^*, we can see that the algebraic closure and the definable closure coincide in T^*. $\square$

Observation 1.6. If $A, B \subseteq \mathfrak{C}_{T^*}$ are closed and $c \in \mathrm{cl}(A \cup B) \setminus A \setminus B$, <u>then</u> $c \in Q_1^{\mathfrak{C}_{T^*}}$.

Proof. Notice that

$$\mathrm{cl}(A \cup B) = A \cup B \cup \{F_1(a, c) : a \in (A \cup B) \cap Q_0 \ \& \ c \in (A \cup B) \cap Q_2 \\ \& \ \{a, c\} \nsubseteq A \ \& \ \{a, c\} \nsubseteq B\}$$

by Claim 1.4(6). $\square$

Claim 1.7. *T^* is NSOP$_3$, consequently NSOP$_4$.*

Proof. Suppose that T^* is SOP_3 and let $\varphi(\bar{x}, \bar{y})$, and $\langle \bar{a}_n : n < \omega \rangle$ exemplify this in a model M (see Definition 0.8(1)). Without loss of generality, by redefining φ if necessary, each $\bar{a}_n$ is without repetition and is closed (recall Claim 1.4(6)). By the Ramsey theorem and compactness, we can assume that the given sequence is a part of an indiscernible sequence $\langle \bar{a}_k : k \in \mathbb{Z} \rangle$; hence $\bar{a}_k$'s form a Δ-system. Let for $k \in \mathbb{Z}$

$$X_k^< \stackrel{\mathrm{def}}{=} \bigcap_{m<k} \mathrm{cl}(\bar{a}_m \hat{\ } \bar{a}_k), \qquad X_k^> \stackrel{\mathrm{def}}{=} \bigcap_{m>k} \mathrm{cl}(\bar{a}_m \hat{\ } \bar{a}_k), \qquad X_k = \mathrm{cl}(X_k^< \cup X_k^>).$$

Hence $\mathrm{Rang}(\bar{a}_k) \subseteq X_k$, and X_k is closed. By Claim 1.4(6), there is an a priori finite bound on the size of X_k; hence by indiscernibility, we have that $|X_k| = n^*$ for some fixed n^* not depending on k. Let $\bar{a}_k^+$ list X_k with no repetition. By Observation 1.6, Claim 1.4(6), indiscernibility and the fact that each $\bar{a}_k$ is closed, we have that for $l \in \{0, 2\}$

$$\mathrm{cl}(\bar{a}_m \hat{\ } \bar{a}_k) \cap Q_l^{\mathfrak{C}} = (\mathrm{Rang}(\bar{a}_m) \cup \mathrm{Rang}(\bar{a}_k)) \cap Q_l^{\mathfrak{C}}$$

and

$$X_k \cap Q_0^{\mathfrak{C}} \subseteq \mathrm{Rang}(\bar{a}_k) \cap Q_0^{\mathfrak{C}} \text{ and } X_k \cap Q_2^{\mathfrak{C}} \subseteq \mathrm{Rang}(\bar{a}_k) \cap Q_2^{\mathfrak{C}}.$$

Applying the Ramsey theorem again, without loss of generality we have that $\langle \bar{a}_k^+ : k \in \mathbb{Z} \rangle$ are indiscernible. Let

$$w_0^* \stackrel{\text{def}}{=} \{l : \bar{a}_{k_1}^+(l) = \bar{a}_{k_2}^+(l) \text{ for some (equivalently all) } k_1 \neq k_2\}.$$

If $\bar{a}_{k_1}^+(l_1) = \bar{a}_{k_2}^+(l_2)$ for some $k_1 \neq k_2$, without loss of generality $k_1 < k_2$, by indiscernibility and symmetry. By transitivity and the fact that each $\bar{a}_k^+$ is without repetition, using $k_1 < k_2 < k_3$ we get $l_1 = l_2 \in w_0^*$. Let $w_1^* \stackrel{\text{def}}{=} n^* \setminus w_0^*$, and let $\bar{a} = \bar{a}_k^+ \restriction w_0^*$ and $\bar{a}'_k = \bar{a}_k^+ \restriction w_1^*$. Hence, $\langle \bar{a}\hat{\ }\bar{a}'_k : k \in \mathbb{Z} \rangle$ is an indiscernible sequence, and $\mathrm{Rang}(\bar{a}) \cap \mathrm{Rang}(\bar{a}'_k) = \emptyset$ for all k. In addition, for $k_1 \neq k_2$ we have $\mathrm{Rang}(\bar{a}'_{k_1}) \cap \mathrm{Rang}(\bar{a}'_{k_2}) = \emptyset$ and $\mathrm{Rang}(\bar{a}\hat{\ }\bar{a}'_k) = X_k$.

Now we define a model N. Its universe is $\cup_{0 \leq l < 3}\{\mathrm{cl}_M(\bar{a}\hat{\ }\bar{a}'_l\hat{\ }\bar{a}'_{l+1})\}$, and $Q_i^N = Q_i^M \cap N$, $F_j^N = \cup\{F_{j,l} : l < 3\}$, where $F_{j,l} = F_j^M \restriction \mathrm{cl}_M(\bar{a}\hat{\ }\bar{a}'_l\hat{\ }\bar{a}'_{l+1})$, or $F_{j,l} = F_j^M \restriction (\mathrm{cl}_M(\bar{a}\hat{\ }\bar{a}'_l\hat{\ }\bar{a}'_{l+1}))^2$, as appropriate. Note that N is well defined, and that it is a model of T_0. N is not necessarily a model of T_0^+, as the function F_1 may be only partial. Notice that $X_l \subseteq N$ for $l \in [0, 3]$. We wish to define N' like N, but identifying $\bar{a}_0^+$ and $\bar{a}_3^+$ coordinatewise. We shall now check that this will give a well defined model of T_0. Note that by the proof of Observation 1.6 we have

$$\begin{aligned} N' = \bigcup_{0\leq l<3} X_l \cup \bigcup_{0\leq l<3} \{F_1^N(c,d) : c, d \in X_l \cup X_{l+1} \\ \& \{c,d\} \nsubseteq X_l \ \& \ \{c,d\} \nsubseteq X_{l+1} \ \& \ F_1^N(c,d) \notin X_l \cup X_{l+1}\}. \end{aligned}$$

The possible problem is that $F_i^{N'}$ might not be well defined, i.e. there could perhaps be a case defined in two distinct ways. We verify that this does not happen, by discussing various possibilities.

Case 1. For some $b \in \mathrm{Rang}(\bar{a}_0^+)$, say $b = \bar{a}_0^+(t)$, $b' = \bar{a}_3^+(t)$ and $j \in \{0, 2, 3\}$, we have $F_j(b) \neq F_j(b')$ after the identification of $\bar{a}_0^+$ with $\bar{a}_3^+$. As $\bar{a}_k^+$'s are closed, we have $F_j(b) = \bar{a}_0^+(s)$ and $F_j(b') = \bar{a}_3^+(s')$ for some s, s'. By indiscernibility, we have $s = s'$, hence the identification will make $F_j(b) = F_j(b')$.

Case 2. For some s, t we have that $F_1(\bar{a}_0^+(s), \bar{a}_0^+(t))$ and $F_1(\bar{a}_3^+(s), \bar{a}_3^+(t))$ are well defined, but not the same after the identification of $\bar{a}_0^+$ and $\bar{a}_3^+$. This case cannot happen, as can be seen similarly to in Case 1.

Case 3. For some $\tau(x, y) \in \{F_1(x, y), F_1(y, x)\}$ and $d_1 = \bar{a}_0^+(s), d_2 = \bar{a}_3^+(s)$ and some $e \in N$ we have that $\tau^N(e, d_1), \tau^N(e, d_2)$ are well defined but do not get identified when N' is defined.

By Case 2, we have that $e \notin \bar{a}$ and $s \notin w_0^*$. As $\tau(e, d_1)$ is well defined and $d_1 \in X_0 \setminus \bar{a}$, necessarily $e \in \mathrm{cl}_M(X_0 \cup X_1)$. Similarly, as $\tau(e, d_2)$ is well defined and $d_2 \in X_3 \setminus \bar{a}$, we have $e \in \mathrm{cl}_M(X_2 \cup X_3)$. But, as $F_1(e, d_l)$ is well defined, we have $e \in Q_2 \cup Q_0$. Hence $e \in \mathrm{cl}_M(X_0 \cup X_1) \setminus Q_1 \subseteq X_0 \cup X_1$ and similarly $e \in X_2 \cup X_3$. This implies $e \in \bar{a}$, a contradiction.

As M is a model of T_0, F_0^M is onto (Claim 1.4(1)). Suppose $y \in Q_0^N$; then for some $l \in [0,3)$ we have that $y \in \mathrm{cl}_M(X_l \cup X_{l+1})$, so by Observation 1.6, we have $y \in X_l \cup X_{l+1}$. As each X_l is closed in M, by Claim 1.4(6) each X_l is a model of T_0^+, so $y \in \mathrm{Rang}(F_0^M \restriction X_l)$; hence $y \in \mathrm{Rang}(F_0^N)$ and $y \in \mathrm{Rang}(F_0^{N'})$. We can similarly prove that $F_3^{N'}$ is onto, and as each X_l is a model of T_0^+ we have by Claim 1.4(1) that $Q_0^{N'}$, $Q_1^{N'}$ and $Q_2^{N'}$ are all non-empty. By Claim 1.4(2), N' can be extended to a model of T_0^+.

By the choice of φ and the fact that T^* is complete we have that

$$T^* \models (\forall \bar{x}_0, \bar{x}_1, \bar{x}_2) \neg[\varphi(\bar{x}_0, \bar{x}_1) \wedge \varphi(\bar{x}_1, \bar{x}_2) \wedge \varphi(\bar{x}_2, \bar{x}_0)].$$

As T^* is the model completion of T_0^+, in particular T^* and T_0^+ are cotheories, so we have that

$$T_0^+ \models (\forall \bar{x}_0, \bar{x}_1, \bar{x}_2) \neg[\varphi(\bar{x}_0, \bar{x}_1) \wedge \varphi(\bar{x}_1, \bar{x}_2) \wedge \varphi(\bar{x}_2, \bar{x}_0)],$$

yet in N' we have

$$N' \models \varphi(\bar{a}_0, \bar{a}_1) \wedge \varphi(\bar{a}_1, \bar{a}_2) \wedge \varphi(\bar{a}_2, \bar{a}_0),$$

by the identification of $\bar{a}_0$ and $\bar{a}_3$. This is a contradiction. □

Definition 1.8. (1) A theory T is said to *satisfy* the *oak property as exhibited by a formula* $\varphi(\bar{x}, \bar{y}, \bar{z})$ iff for any infinite λ, κ there are $\bar{b}_\eta (\eta \in {}^{\kappa >}\lambda)$ and $\bar{c}_\nu (\nu \in {}^{\kappa}\lambda)$ and $\bar{a}_i (i < \kappa)$ such that

(a) $[\eta \triangleleft \nu \ \& \ \nu \in {}^{\kappa}\lambda] \implies \varphi[\bar{a}_{\lg(\eta)}, \bar{b}_\eta, \bar{c}_\nu]$,

(b) If $\eta \in {}^{\kappa >}\lambda$ and $\eta\hat{\ }\langle\alpha\rangle \triangleleft \nu_1 \in {}^{\kappa}\lambda$ and $\eta\hat{\ }\langle\beta\rangle \triangleleft \nu_2 \in {}^{\kappa}\lambda$, while $\alpha \neq \beta$ and $i > lg(\eta)$, <u>then</u> $\neg\exists \bar{y}\, [\varphi(\bar{a}_i, \bar{y}, \bar{c}_{\nu_1}) \wedge \varphi(\bar{a}_i, \bar{y}, \bar{c}_{\nu_2})]$,
and in addition φ satisfies

(c) $\varphi(\bar{x}, \bar{y}_1, \bar{z}) \wedge \varphi(\bar{x}, \bar{y}_2, \bar{z}) \implies \bar{y}_1 = \bar{y}_2$.

We allow for the replacement of $\mathfrak{C}_T$ by $\mathfrak{C}_T^{\mathrm{eq}}$ (i.e. allow $\bar{y}$ to be a definable equivalence class).

(2) We say that oak holds for T if this is true for some φ.

Observation 1.9. If some infinite λ, κ exemplify that oak(φ) holds, then so do all infinite λ, κ. (This holds by the compactness theorem.)

Remark 1.10. We shall not need to use this, but let us remark that witnesses $\bar{a}, \bar{b}, \bar{c}$ to oak(φ) can be chosen to be indiscernible along an appropriate index set (a tree). This can be proved using the technique of [10], Chapter VII, which employs the compactness argument and an appropriate partition theorem.

Claim 1.11. *T^* has oak.*

Proof. Let

$$\varphi(x, y, z) \stackrel{\text{def}}{=} Q_0(x) \wedge Q_1(y) \wedge Q_2(z) \wedge F_0(y) = x \wedge F_1(x, z) = y.$$

Clearly, (c) of Definition 1.8(1) is satisfied. Given λ, κ, we shall define a model $N = N_{\lambda,\kappa}$ of T_0^+. This will be a submodel of $\mathfrak{C} = \mathfrak{C}_{T^*}$ such that its universe consists of $Q_0^N \stackrel{\text{def}}{=}$

$\{a_i : i < \kappa\}$ with no repetitions, $Q_1^N \stackrel{\text{def}}{=} \{b_\eta : \eta \in {}^{\kappa>}\lambda\}$ with no repetitions and $Q_2^N \stackrel{\text{def}}{=} \{c_\nu : \nu \in {}^{\kappa}\lambda\}$ with no repetitions, while Q_0, Q_1, Q_2 are pairwise disjoint. We also require that the following are satisfied in $\mathfrak{C} = \mathfrak{C}_{T^*}$:

$$F_0(b_\eta) = a_{\lg(\eta)},\, F_1(a_i, c_\nu) = b_{\nu \restriction i}$$

and that N is closed under F_2 and F_3. That such a choice is possible can be seen by writing the corresponding type and using the saturativity of $\mathfrak{C}$.

We can check that $N \models T_0^+$, and that N is a submodel of $\mathfrak{C}$ when understood as a model of T_0^+. Clearly, (a) from Definition 1.8(1) is satisfied for φ and a_i, b_η, c_ν in place of $\bar{a}_i, \bar{b}_\eta, \bar{c}_\nu$ respectively. To see (b), suppose that $\eta, \alpha, \beta, \nu_1, \nu_2$ and i are as there, but d is such that $\varphi(a_i, d, c_{\nu_1}) \wedge \varphi(a_i, d, c_{\nu_2})$. Hence $F_1(a_i, c_{\nu_1}) = F_1(a_i, c_{\nu_2})$, so $\nu_1 \restriction i = \nu_2 \restriction i$, a contradiction. This shows that φ is a witness for T^* having oak. $\square$

A similar argument can be used to show that T^* is not simple, but in fact we shall prove that no theory with the oak property is simple (this in particular answers a question of A. Dolich raised in a private communication).

Claim 1.12. *No theory with the oak property is simple.*

Proof. Let T be a theory with the oak property and let κ, λ be cardinals such that $\kappa > |T|$, $2^\kappa < \lambda$ and $\lambda = \lambda^{<\kappa} < \lambda^\kappa$ (such cardinals always exist). By Observation 1.9 we may assume that the oak property of T is exemplified by a formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ and sequences $\langle \bar{a}_i : i < \kappa\rangle$, $\langle \bar{b}_\eta : \eta \in {}^{\kappa>}\lambda\rangle$ and $\langle \bar{c}_\nu : \nu \in {}^{\kappa}\lambda\rangle$. For $\nu \in {}^{\kappa}\lambda$ let $p_\nu = p_\nu(\bar{z}) \stackrel{\text{def}}{=} \{\varphi(\bar{a}_i, \bar{b}_{\nu\restriction i}, \bar{z}) : i < \kappa\}$. Hence each p_ν is a type of cardinality κ and the set $\{p_\nu : \nu \in {}^{\kappa}\lambda\}$ consists of pairwise incompatible types. The set of parameters used in $\bigcup\{p_\nu : \nu \in {}^{\kappa}\lambda\}$ has size $\leq \kappa \cdot \lambda^{<\kappa} = \lambda$. By [[10], III, 7.7, pg. 141] this implies that T is not simple. $\square$

We now pass to another example of a theory with oak that satisfies $NSOP_3$, which is the theory T^*_{feq} of infinitely many indexed independent equivalence relations. This example also shows why it is that this research continues [16]. The readers uninterested in T^*_{feq} can skip to the next section without loss of continuity. We use the notation for T^*_{feq} which was used in [4], while the fact that this is equivalent to the notation in [16] was explained in [4]. The existence of the required model completion is explained in [4].

Definition 1.13. (1) T^+_{feq} is the following theory in $\{Q, P, E, R, F\}$:

(a) Predicates P and Q are unary and disjoint, and $(\forall x)\,[P(x) \vee Q(x)]$.
(b) E is an equivalence relation on Q.
(c) R is a binary relation on $Q \times P$ such that

$$[x\,R\,z \;\&\; y\,R\,z \;\&\; x\,E\,y] \implies x = y.$$

(Explanation: so R picks for each $z \in Q$ (at most) one representative of any E-equivalence class.)

(d) F is a (total) binary function from $Q \times P$ to Q, which satisfies

$$F(x, z) \in Q \;\&\; (F(x, z)\,R\,z) \;\&\; (x\,E\,F(x, z)).$$

(Explanation: so for $x \in Q$ and $z \in P$, the function F picks the representative of the E-equivalence class of x which is in the relation R with z.)

(2) T^*_{feq} is the model completion of T^+_{feq}.

Remark 1.14. After renaming, $\mathfrak{C}^{\mathrm{eq}}_{T^*_{\mathrm{feq}}}$ is a reduct of $\mathfrak{C}^{\mathrm{eq}}_{T^*}$; formally T^*_{feq} is interpretable in T^*. Given a model M of T^*, we define $N = N_1[M]$ by letting its universe be $Q_1^M \bigcup Q_2^M$ and $P^N = Q_2^M$, while $Q^N = Q_1^M$. We let

$$y\,E\,z \text{ iff } F_0^M(y) = F_0^M(z) \text{ and } F^N(x,z) = F_1(F_0(x), z).$$

We also let $x\,R\,z \iff F^N(x,z) = x$. It is easily seen that $N \models T^+_{\mathrm{feq}}$, and moreover, $N \models T^*_{\mathrm{feq}}$.

Using the above Remark and the fact that oak and $NSOP_3$ are preserved up to isomorphism of $\mathfrak{C}^{\mathrm{eq}}$, we obtain:

Corollary 1.15. (1) *T^*_{feq} has oak.*
(2) *T^*_{feq} has $NSOP_3$.*[1]

Proof. (1) Use the formula $\varphi(x,y,z) \equiv F(x,z) = y$.
(2) Follows by Remark 1.14. □

Part (2) of Corollary 1.15 was stated without proof in [18]. The results here suggest the following questions.

Question 1.16. (1) Does T^* satisfy SOP_2 or SOP_1?
(2) Are there any nontrivial examples of oak theories that have SOP_3?

Properties SOP_2 or SOP_1 were introduced in [4] where it was shown that $SOP_3 \implies SOP_2 \implies SOP_2 \implies$ not simple, but it was left open to decide whether any of these implications is reversible. These properties are studied further in [24] where it is proved that T^*_{feq} has $NSOP_1$. This makes it reasonable to conjecture that the answer to both parts of 1.16 is positive.

We finish the section by quoting a result of Shelah from [16], which can be compared with our non-universality results from Section 2. The notation is explained in Section 2.

Theorem 1.17 (*Shelah*). *Suppose that κ, μ and λ are cardinals satisfying*

(1) $\kappa = \mathrm{cf}(\mu) < \mu$, $\lambda = \mathrm{cf}(\lambda)$,
(2) $\mu^+ < \lambda$,
(3) *there is a family*

$$\{(a_i, b_i) : i < i^*, a_i \in [\lambda]^{<\mu}, b_i \in [\lambda]^{\kappa}\}$$

such that $|\{b_i : i < i^*\}| \le \lambda$ and satisfying that for every $f : \lambda \to \lambda$ there is i such that $f(b_i) \subseteq a_i$; and
(4) $\mathrm{pp}_{\Gamma(\kappa)}(\mu) > \lambda + |i^*|$.

Then $\mathrm{univ}(T^+_{\mathrm{feq}}, \lambda) \ge \mathrm{pp}_{\Gamma(\kappa)}(\mu)$.

[1] It has subsequently been proved by Shelah and Usvyatsov in [24] that T^*_{feq} has a stronger property $NSOP_1$.

2. Non-universality results

In this section we present two general theorems showing that under certain cardinal arithmetic assumptions oak theories do not admit universal models. Let us start by introducing some common abbreviations that we shall use in the statements and the proofs in this section.

Notation 2.1. (1) Let $\kappa \le \lambda$ be cardinals. We let

$$[\lambda]^\kappa \stackrel{\text{def}}{=} \{A \subseteq \lambda : |A| = \kappa\}.$$

If κ is regular we let

$$S^\lambda_\kappa \stackrel{\text{def}}{=} \{\alpha < \lambda : \mathrm{cf}(\alpha) = \kappa\}.$$

(2) For a set A of ordinals we let the set of *accumulation points* of A be $\mathrm{acc}(A) \stackrel{\text{def}}{=} \{\alpha \in A : \alpha = \sup(A \cap \alpha)\}$ and the set of *non-accumulation points* be $\mathrm{nacc}(A) \stackrel{\text{def}}{=} A \setminus \mathrm{acc}(A)$.

Before proceeding to the non-universality theorems recall from the Introduction the definition of a tight club guessing sequence (Definition 0.2). Note that the definition does not require sets C_δ to be either closed or unbounded in δ. It can be deduced from the existing literature on club guessing sequences that tight and truly tight club guessing sequences exist for many triples (κ, μ, λ). We shall indicate in Claim 2.10 how this deduction can be made, but let us leave this for the discussion on the consistency of the assumptions of the non-universality theorems, which will be given after their proofs. We shall now give two non-universality theorems. These theorems have set-theoretic and model-theoretic assumptions. The model-theoretic assumption is the same in both cases: that we are dealing with an oak theory of size $< \lambda$, with the desired conclusion being that the universality number $\mathrm{univ}(T, \lambda)$ is larger than λ. The set-theoretic assumptions, which are different for the two theorems, will be phrased in the form of certain combinatorial statements that are needed for the proofs of the theorem. As with tight club guessing sequences, it might not be immediately clear to the reader that these assumptions are consistent. However, after we prove the theorems we shall give some sufficient conditions for these assumptions to be satisfied and as a corollary get some non-universality results whose set-theoretic assumptions are phrased in the form of cardinal arithmetic and known to be consistent.

Theorems 2.2 and 2.4 have similar proofs, as we explain below, so we shall first state both theorems and then give the proofs simultaneously.

Theorem 2.2. *Assume that κ, μ, σ and λ are cardinals satisfying*

(1) $\mathrm{cf}(\kappa) = \kappa < \mu < \lambda = \mathrm{cf}(\lambda)$ *and there is a tight (μ, λ) club guessing sequence,*
(2) $\lambda < \mu^\kappa$,
(3) $\kappa \le \sigma \le \lambda$,
(4) *there are families $\mathcal{P}_1 \subseteq [\lambda]^\kappa$ and $\mathcal{P}_2 \subseteq [\sigma]^\kappa$ such that*
  (i) *for every injective $g : \sigma \to \lambda$ there is $X \in \mathcal{P}_2$ with $\{g(i) : i \in X\} \in \mathcal{P}_1$,*
  (ii) $|\mathcal{P}_1| < \mu^\kappa$, $|\mathcal{P}_2| \le \lambda$,
(5) *T is a theory of size $< \lambda$ which has the oak property.*

Then

$$\mathrm{univ}(T,\lambda) \geq \mu^\kappa.$$

Definition 2.3. For cardinals $\kappa \leq \mu$ we define

$$\mathcal{U}_{J_\kappa^{\mathrm{bd}}}(\mu) \stackrel{\mathrm{def}}{=} \min\{|\mathcal{P}| : \mathcal{P} \subseteq [\mu]^\kappa \ \&\ (\forall b \in [\mu]^\kappa)(\exists a \in \mathcal{P})(|a \cap b| = \kappa\}.$$

More on $\mathcal{U}_{J_\kappa^{\mathrm{bd}}}(\mu)$ can be found in [22].

Theorem 2.4. *Assume that κ, μ, σ and λ are cardinals satisfying*

(1) $\mathrm{cf}(\kappa) = \kappa < \mu < \lambda = \mathrm{cf}(\lambda)$ *and there is a tight (μ, λ) club guessing sequence,*
(2) $\lambda < \mathcal{U}_{J_\kappa^{\mathrm{bd}}}(\mu)$,
(3) $\kappa \leq \sigma \leq \lambda$,
(4) *there are families $\mathcal{P}_1 \subseteq [\lambda]^\kappa$ and $\mathcal{P}_2 \subseteq [\sigma]^\kappa$ such that*
  (i) *for every injective $g : \sigma \to \lambda$ there is $X \in \mathcal{P}_2$ such that for some $Y \in \mathcal{P}_1$*
  $$|\{g(i) : i \in X\} \cap Y| = \kappa,$$
  (ii) $|\mathcal{P}_1| < \mathcal{U}_{J_\kappa^{\mathrm{bd}}}(\mu)$, $|\mathcal{P}_2| \leq \lambda$,
(5) *T has the oak property.*

Then

$$\mathrm{univ}(T,\lambda) \geq \mathcal{U}_{J_\kappa^{\mathrm{bd}}}(\mu).$$

Before we start the proof let us give an introduction to the methods that appear within it. When proving that the universality number of a certain category with given morphisms (so not just in the context of first order model theory) is high it is often the case that one can associate with each object in the category a certain construct, an invariant, which is to some extent preserved by morphisms. For example such an invariant might be an ordinal number and then one can prove that such an invariant may only increase after an embedding. The proof then proceeds by contradiction by showing that any candidate for the universal would have to satisfy too many invariants. A trivial example would be to show that there is no countable well-ordering that is universal under order preserving embeddings: the order type of the ordering is an invariant that satisfies that if $f : P \to Q$ is an order preserving embedding, then the order type of Q is at least as large as that of P. Any Q that would be universal would have to have a countable well-order type that is larger than that of all countable ordinals, a contradiction. As trivial as it is, this example points out two stages of a non-universality proof: *construction* which associates an object with every invariant prescribed by a certain set (e.g. the uncountable set of all countable ordinals) and *preservation* that shows that some essential features of the invariant are preserved (e.g. the order type does not decrease) under embeddings. In our proofs we shall use the same method, except that the invariants will be defined as certain λ-sequences of subsets of μ, unique modulo the club filter on λ, and that the preservation and the resulting contradiction will be dependent on a certain club guessing sequence. Using such invariants is a technique that was first used by Kojman and Shelah in [8] and has appeared in a number of papers since. The main point tends to be the right definition of an invariant and the use of a right kind of club guessing.

Proof. We shall use the same proof for both Theorems 2.2 and 2.4. The two main Lemmas are the same for the two theorems, and we shall indicate the differences which occur toward the end of the proof. Suppose that $\varphi(\bar{x}, \bar{y}, \bar{z})$ shows that T has the oak property and let a_i $(i < \kappa)$, b_η $(\eta \in {}^{\kappa>}\lambda)$ and $c_\nu (\nu \in {}^{\kappa}\lambda)$ exemplify the oak property of $\varphi(\bar{x}, \bar{y}, \bar{z})$ for λ and κ. For notational simplicity, let us assume that $lg(\bar{x}) = lg(\bar{y}) = lg(\bar{z}) = 1$.

Let $\langle C_\delta : \delta \in S\rangle$ be a tight (μ, λ) club guessing sequence. For each δ, let $\langle \alpha(\delta, \zeta) : \zeta < \mu\rangle$ be the increasing enumeration of C_δ. Let $\mathfrak{C}^+$ be a (saturated enough) expansion of $\mathfrak{C}_T$ by the Skolem functions for $\mathfrak{C}_T$.

Definition 2.5. (1) For $\bar{N} = \langle N_\gamma : \gamma < \lambda\rangle$ an $\prec$-increasing continuous sequence of models of T of size $< \lambda$, and for $a, c \in N_\lambda \overset{\text{def}}{=} \bigcup_{\gamma<\lambda} N_\gamma$, and $\delta \in S$, we let

$$\mathrm{inv}_{\bar{N}}(c, C_\delta, a) \overset{\text{def}}{=} \{\zeta < \mu : (\exists b \in N_{\alpha(\delta,\zeta+1)} \setminus N_{\alpha(\delta,\zeta)})(N_\lambda \models \varphi[a, b, c])\}.$$

(2) For a set A and δ, $\bar{N}$ as above, let

$$\mathrm{inv}^A_{\bar{N}}(c, C_\delta) \overset{\text{def}}{=} \bigcup\{\mathrm{inv}_{\bar{N}}(c, C_\delta, a) : a \in A\}.$$

Note 2.6. Following the notation of Definition 2.5, notice that $\mathrm{inv}_{\bar{N}}(c, C_\delta, a)$ is always a singleton or empty, since if there is $b \in N_\lambda$ such that $\varphi[a, b, c]$ holds then such b is unique (by part (c) of Definition 1.8). Consequently $\mathrm{inv}^A_{\bar{N}}(c, C_\delta) \in [\mu]^{\leq|A|}$.

Construction Lemma 2.7. *For every $A^* \in [\mu]^\kappa$ of order type κ, there is an $\prec$-increasing continuous sequence $\bar{N}^{A^*} = \langle N^{A^*}_\gamma : \gamma < \lambda\rangle$ of models of T of size $< \lambda$ and a set $\{\hat{a}_i : i < \sigma\}$ of elements of $N_{A^*} \overset{\text{def}}{=} \bigcup_{\gamma<\lambda} N^{A^*}_\gamma$ such that for some club E^* of λ, for every $X \in \mathcal{P}_2$, for some $\alpha_X < \lambda$, for every $\delta \in S$ satisfying $\min(C_\delta) > \alpha_X$, there is $c \in N_{A^*}$ such that $\mathrm{inv}^{\{\hat{a}_i : i \in X\}}_{\bar{N}_{A^*}}(c, C_\delta) = A^*$.*

In addition, the universe of N_{A^*} is λ.

Proof of the Lemma. Let $\mathcal{P}_2 = \{X_\alpha : \alpha < \alpha^* \leq \lambda\}$. Without loss of generality $\sigma \subseteq \bigcup_{\alpha<\alpha^*} X_\alpha$.

Given A^*. Let $f = f_{A^*}$ be an increasing function from the successor ordinals $< \kappa$ into μ such that $\mathrm{Rang}(f) = A^*$. For $\delta \in S$ let ν_δ be the function from κ into λ such that $\nu_\delta(\zeta) = \alpha(\delta, f(\zeta))$ for all $\zeta < \kappa$. Note that ν_δ is increasing. Hence c_{ν_δ} is well defined, as is b_η for $\eta \lhd \nu_\delta$. For $X \in \mathcal{P}_2$, let ρ_X be a bijection between the ordinals $< \kappa$ that have the form $\beta + 2$ for some β and X. For $\eta \in {}^{\kappa>}\lambda$ let us say that η is *good* iff the domain of η is of the form $\beta + 2$ for some $\beta < \kappa$.

By a compactness argument, we can see that there are $\langle \hat{a}_i : i < \sigma\rangle$ and for $X \in \mathcal{P}_2$, sequences $\langle c^X_{\nu_\delta} : \delta \in S\rangle$, $\langle b^X_\eta : \eta \lhd \nu_\delta \ \&\ \eta \text{ good } \ \&\ \delta \in S\rangle$ such that for η good and $\delta \in S$

$$\eta \lhd \nu_\delta \implies \models \varphi[\hat{a}_{\rho_X(\lg(\eta))}, b^X_\eta, c^X_{\nu_\delta}]$$

and the appropriate translation of (b) from Definition 1.8 holds. By taking an isomorphic copy of $\mathfrak{C}^+$ if necessary, we can assume that the Skolem hull in $\mathfrak{C}^+$ of

$$\{\hat{a}_i : i < \sigma\} \cup \{b^X_\eta : X \in \mathcal{P}_2 \ \&\ (\exists\delta \in S)\eta \lhd \nu_\delta\} \cup \{c^X_{\nu_\delta} : X \in \mathcal{P}_2 \ \&\ \delta \in S\}$$

is contained in λ. Let for $\gamma < \lambda$ the model $N^{A^*}_\gamma$ be the reduction to $\mathcal{L}(T)$ of the Skolem hull in $\mathfrak{C}^+$ of

$$\gamma \cup \{\hat{a}_i : i \in \cup_{\alpha < \min\{\alpha^*,\gamma\}} X_\alpha\} \cup$$
$$\cup \bigcup_{\alpha < \min\{\alpha^*,\gamma\}} \{c^{X_\alpha}_{\nu_\delta} : \delta \in S \cap \gamma \ \&\ \sup(\mathrm{Rang}(\nu_\delta)) < \gamma\} \cup$$
$$\cup \bigcup_{\alpha < \min\{\alpha^*,\gamma\}} \{b^{X_\alpha}_\eta : \eta \lhd \nu_\delta \text{ for some } \delta \in S \ \&\ \eta \text{ good } \&\ \sup(\mathrm{Rang}(\eta)) < \gamma\}.$$

Hence $\bar{N}^{A^*} = \langle N^{A^*}_\gamma : \gamma < \lambda\rangle$ is $\prec$-increasing continuous, and it also follows that the universe of $N^{A^*} \stackrel{\text{def}}{=} \bigcup_{\gamma<\lambda} N^{A^*}_\gamma$ is λ. We observe also that for $\gamma < \lambda$ we have $|N^{A^*}_\gamma| < \lambda$ because λ is regular, T has size $< \lambda$ and the Skolem hull needed to obtain $N^{A^*}_\gamma$ is taken over a set of size $< \lambda$. That this set has size $< \lambda$ might not be immediate, since in the last clause of its definition we allow δ to range over the entire set S, whose size is λ. However, for every η appearing in this part of the definition, η is increasing (as an initial segment of some ν_δ) and it satisfies $\sup(\mathrm{Rang}(\eta)) < \gamma$. Since the domain of η is of the form $\beta + 2$ for some β, this means $\eta(\beta + 1) < \gamma$. For any $\delta \in S$ such that $\eta \lhd \nu_\delta$ we have that $\eta(\beta + 1) \in C_\delta$, so either $\eta(\beta + 1) \in \mathrm{nacc}(C_\delta)$ or for some $\gamma' \in \mathrm{nacc}(C_\delta)$ we have that $\eta(\beta) < \gamma' < \eta(\beta + 1)$. At any rate, $\mathrm{Rang}(\eta)$ is a subset of size $< \kappa$ of a set of the form $C_\delta \cap \xi \cup \{o\}$ for some $\xi \in \mathrm{nacc}(C_\delta)$ and ξ, o are both $< \gamma$. As part of the choice of $\bar{C}$ we obtain that for any $\xi < \gamma$

$$|\{C_\delta \cap \xi : \delta \in S,\ \xi \in \mathrm{nacc}(C_\delta)\}| < \lambda.$$

For $\delta \in S$ and $\xi \in \mathrm{nacc}(C_\delta)$ let $\zeta^*(\delta, \xi) \stackrel{\text{def}}{=} \min\{\zeta : \alpha(\delta, f(\zeta)) \geq \xi\}$, if this is well defined, and let $\zeta^*(\delta, \xi) = \kappa$ otherwise. Now notice that if $C_\delta \cap \xi = C_{\delta'} \cap \xi$ then we have $\zeta^*(\delta, \xi) = \zeta^*(\delta', \xi)$ and that $\nu_\delta \restriction \zeta^*(\delta, \xi) = \nu_{\delta'} \restriction \zeta^*(\delta', \xi)$. Our analysis shows that any η relevant to the third clause of the definition of $N^{A^*}_\gamma$ and having domain $\beta + 2$ satisfies that $\eta \restriction (\beta+1) = (\nu_\delta \restriction \zeta^*(\delta, \xi)) \restriction (\beta+1)$ for some $\delta \in S$ and $\xi < \gamma$ and hence that there are $< \lambda$ choices for $b^{X_\alpha}_\eta$. Let E^* be a club of λ such that for every $\delta \in E^*$ and good η we have

$$b^{X_\beta}_\eta \in N^{A^*}_\delta \text{ iff } \beta < \delta \ \&\ (\exists \delta' \in S \cap \delta)[\eta \lhd \nu_{\delta'}].$$

Given $\alpha < \alpha^*$, $X = X_\alpha$ and $\delta \in S$ with $\min(C_\delta) \geq \alpha + 1$ and $C_\delta \subseteq E^*$, we shall show that with

$$I \stackrel{\text{def}}{=} \mathrm{inv}^{\{\hat{a}_i : i \in X\}}_{\bar{N}^{A^*}}(c^X_{\nu_\delta}, C_\delta)$$

we have $I = A^*$. Notice that $\varepsilon < \kappa \implies \alpha(\delta, f(\varepsilon)) > \alpha$ trivially since $\min(C_\delta) > \alpha$. Let $i \in X$, $\beta + 2 = \rho^{-1}_X(i)$ and let $\eta = \langle \alpha(\delta, f(\varepsilon)) : \varepsilon \leq \beta + 1\rangle$. We have that $\eta \lhd \nu_\delta$ and $i = \rho_X(lg(\eta))$. Hence $\varphi[\hat{a}_i, b^X_\eta, c^X_{\nu_\delta}]$ holds. Let $\zeta = f(\beta + 1)$. We then have that $b^X_\eta \in N^{A^*}_{\alpha(\delta,\zeta)+1} \subseteq N^{A^*}_{\alpha(\delta,\zeta+1)}$ (as $\alpha(\delta, \zeta) + 1$ is strictly larger than $\sup(\mathrm{Rang}(\eta)) = \alpha(\delta, \zeta)$ and $\alpha < \alpha(\delta, \zeta) + 1$), but $b^X_\eta \notin N^{A^*}_{\alpha(\delta,\zeta)}$ by the choice of E^*. Hence $\zeta = f(\beta + 1) \in I$. So $A^* \subseteq I$ because every element of A^* is $f(\beta + 1)$ for some β as above.

In the other direction, suppose $\zeta \in I$ and let $i \in X$ be such that ζ is in $\mathrm{inv}_{\bar{N}^{A^*}}(c^X_{\nu_\delta}, C_\delta, \hat{a}_i)$. Hence for some $b \in N^{A^*}_{\alpha(\delta,\zeta+1)} \setminus N^{A^*}_{\alpha(\delta,\zeta)}$ we have $\models \varphi[\hat{a}_i, b, c^X_{\nu_\delta}]$.

Constructing η as in the previous paragraph we have that $\models \varphi[\hat{a}_i, b^X_\eta, c^X_{\nu_\delta}]$ holds. Using the uniqueness property from (c) of Definition 1.8 we see that $b = b^X_\eta$ so $\zeta = f(\beta + 1)$ for some β. So $A^* = I$. $\square$

Note 2.8. With the notation of Lemma 2.7, for any $i \in \bigcup_{\alpha < \min\{\alpha^*, \delta\}} X_\alpha$ we have $\mathrm{inv}_{\bar{N}^{A^*}}(c^X_\delta, C_\delta, \hat{a}_i) \neq \emptyset$, as follows from the forward direction of the proof that $A^* = I$.

Preservation Lemma 2.9. *Suppose that N and N^* are models of T both with universe λ, and $f : N \to N^*$ is an elementary embedding, while $\langle N_\gamma : \gamma < \lambda \rangle$ and $\langle N^*_\gamma : \gamma < \lambda \rangle$ are continuous increasing sequences of models of T of cardinality $< \lambda$ with $\bigcup_{\gamma<\lambda} N_\gamma = N$ and $\bigcup_{\gamma<\lambda} N^*_\gamma = N^*$. Further suppose that $\{\hat{a}_\alpha : \alpha < \kappa\} \subseteq N$ is given. Let*

$$E \stackrel{\text{def}}{=} \left\{ \gamma : \begin{array}{r} (N, N^*, f) \restriction \gamma \prec (N, N^*, f) \;\&\; \sup(\{a_\alpha : \alpha < \kappa\}) < \gamma \;\& \\ \text{the universes of } N_\gamma \text{ and } N^*_\gamma \text{ are both the set } \gamma \end{array} \right\}.$$

<u>Then</u> for every $c \in N$ and δ with $C_\delta \subseteq E$, and for every $\alpha < \kappa$ we have

$$\mathrm{inv}_{\bar{N}}(c, C_\delta, \hat{a}_\alpha) = \mathrm{inv}_{\bar{N}^*}(f(c), C_\delta, f(\hat{a}_\alpha)).$$

Proof of the Lemma. Note that E is a club of λ. Fix $c \in N$ and $\delta \in S$ as required, and let $a = a_\alpha$ for some $\alpha < \kappa$. We shall see that $\mathrm{inv}_{\bar{N}}(c, C_\delta, a) = \mathrm{inv}_{\bar{N}^*}(f(c), C_\delta, f(a))$.

Suppose $\zeta < \mu$ is an element of $\mathrm{inv}_{\bar{N}}(c, C_\delta, a)$, so there is $b \in N_{\alpha(\delta,\zeta+1)}$ with $N \models \varphi[a, b, c]$, while there is no such $b \in N_{\alpha(\delta,\zeta)}$ (we are using the uniqueness property from (c) of Definition 1.8). We have that N^* satisfies $\varphi[f(a), f(b), f(c)]$. As $C_\delta \subseteq E$ we have that $\alpha(\delta, \zeta + 1) \in E$, and as $b \in N_{\alpha(\delta,\zeta+1)}$, clearly $f(b) \in N^*_{\alpha(\delta,\zeta+1)}$. Similarly, by the definition of E again and the fact that f is injective we have $f(b) \notin N^*_{\alpha(\delta,\zeta)}$. By the assumptions on φ we have

$$N^* \models \text{“}(\forall y)[\varphi(f(a), y, f(c)) \implies y = f(b)]\text{”},$$

so $\zeta \in \mathrm{inv}_{\bar{N}^*}(f(c), C_\delta, f(a))$.

In the other direction, suppose $\zeta < \mu$ is an element of $\mathrm{inv}_{\bar{N}^*}(f(c), C_\delta, f(a))$, so there is $b^* \in N^*_{\alpha(\delta,\zeta+1)}$ with $N^* \models \varphi[f(a), b^*, f(c)]$, while there is no such $b^* \in N^*_{\alpha(\delta,\zeta)}$. Hence $N^* \models \exists y\,(\varphi[f(a), y, f(c)])$, so $N \models \exists y\,(\varphi[a, y, c])$. Let $b \in N$ be such that $N \models \varphi[a, b, c]$. Hence $N^* \models \varphi[f(a), f(b), f(c)]$. Again by (c) of Definition 1.8, we have $f(b) = b^*$, so $b \in N_{\alpha(\delta,\zeta+1)} \setminus N_{\alpha(\delta,\zeta)}$ because $\{\alpha(\delta, \zeta), \alpha(\delta, \zeta + 1)\} \subseteq E$, so by the choice of E we have that for $\gamma \in \{\alpha(\delta, \zeta), \alpha(\delta, \zeta + 1)\}$, $(N, N^*, f) \restriction \gamma$ is an elementary submodel of (N, N^*, f). As this b is unique (by (c) of Definition 1.8) we have that ζ belongs to $\mathrm{inv}_{\bar{N}}(c, C_\delta, a)$. $\square$

Proof of the Theorems continued (*Theorem* 2.2 (*Theorem* 2.4)). To conclude the proof of the theorems, given $\theta < \mu^\kappa$ [$\theta < \mathcal{U}_{J^{\mathrm{bd}}_\kappa}(\mu)$], we shall see that $\mathrm{univ}(T, \lambda) > \theta$. Without loss of generality, we can assume that $\theta \geq \lambda + |\mathcal{P}_1|$. Given $\langle N^*_j : j < \theta \rangle$ a sequence of models of T each of size λ, we shall show that these models are not jointly universal. So suppose they were. Without loss of generality, the universe of each N^*_j is λ. Let $\bar{N}^*_j = \langle N^*_{\gamma,j} : \gamma < \lambda \rangle$ be an increasing continuous sequence of models of T of size $< \lambda$ such that $N^*_j = \bigcup_{\gamma<\lambda} N^*_{\gamma,j}$, for $j < \theta$. For each $A \in \mathcal{P}_1$ (so $A \in [\lambda]^\kappa$), $\delta \in S$,

$j < \theta$ and $d \in N^*_j$, we compute $\mathrm{inv}^{A}_{\bar{N}^*_j}(d, C_\delta)$, each time obtaining an element of $[\mu]^{\leq\kappa}$. The number of elements of $[\mu]^{\leq\kappa}$ obtained in this way is

$$\leq |\mathcal{P}_1| \cdot |S| \cdot \theta \cdot \lambda \leq \theta.$$

By the choice of θ [and the definition of $\mathcal{U}_{J^{\mathrm{bd}}_\kappa}(\mu)$], we can choose $A^* \in [\mu]^\kappa$ such that A^* is not equal to any of these sets [is almost disjoint (i.e. has intersection of size $< \kappa$) to any one of these sets]. Let $N \stackrel{\mathrm{def}}{=} N_{A^*}$ be as guaranteed to exist by the Construction Lemma, and let $\{\hat{a}_i : i < \sigma\}$, $\bar{N}^{A^*} \stackrel{\mathrm{def}}{=} \langle N^{A^*}_\gamma : \gamma < \lambda\rangle$ and E^* be as in that Lemma. In particular, the universe of N is λ. Suppose that $j < \theta$ and $f : N \to N^*_j$ is an elementary embedding, and let

$$E^{**} \stackrel{\mathrm{def}}{=} \left\{ \begin{array}{c} \delta \in E^* : (N, N^*_j, f) \restriction \delta \prec (N, N^*_j, f) \ \& \\ \text{the universe of each } N^*_{\delta,j}, N^{A^*}_\delta \text{ is } \delta \end{array} \right\}.$$

Let $g : \sigma \to \lambda$ be given by $g(i) = f(\hat{a}_i)$. Note that g is injective because f is an isomorphic embedding. By assumption (4)(i) of Theorem 2.2 [2.4], there is $X = X_\alpha \in \mathcal{P}_2$ such that $\{f(\hat{a}_i) : i \in X\} \in \mathcal{P}_1$ [for some $Y \in \mathcal{P}_1$ we have

$$|\{f(\hat{a}_i) : i \in X\} \cap Y| = \kappa].$$

Let $\alpha_X < \lambda$ be as provided by the Construction Lemma, and let

$$E \stackrel{\mathrm{def}}{=} (E^{**} \setminus \alpha_X) \cap \{\delta : \{\hat{a}_i : i \in X\} \subseteq \delta\}.$$

Since we have that the universe of N is λ we have $\{\hat{a}_i : i < \sigma\} \subseteq \lambda$, so as X is a set of size $\kappa < \lambda$ we can conclude that E is a club of λ. We now choose $\delta \in S$ such that $C_\delta \subseteq E$, so in particular $C_\delta \subseteq E^*$ and $\min(C_\delta) > \alpha_X$.

The Construction Lemma guarantees that there is $c \in N$ such that $\mathrm{inv}^{\{\hat{a}_i : i \in X\}}_{\bar{N}}(c, C_\delta) = A^*$. By the Preservation Lemma we have

$$\mathrm{inv}^{\{f(\hat{a}_i) : i \in X\}}_{\bar{N}^*_j}(f(c), C_\delta) = A^*$$

$[\mathrm{inv}^{\{f(\hat{a}_i) : i \in X\}}_{\bar{N}^*_j}(f(c), C_\delta) \cap A^*$ includes $\mathrm{inv}^{\{f(\hat{a}_i) : i \in X\} \cap Y}_{\bar{N}^*_j}(f(c), C_\delta)]$.

In the case of Theorem 2.2 we have a contradiction with the choice of A^* and we are done. We are almost done also in the case of Theorem 2.4, but we need to know that $\mathrm{inv}^{\{f(\hat{a}_i) : i \in X\} \cap Y}_{\bar{N}^*_j}(f(c), C_\delta)$ has size κ. We know that $\{f(\hat{a}_i) : i \in X\} \cap Y$ has size κ, but it is a priori possible that for some $i \in X$ we have $\mathrm{inv}_{\bar{N}^*_j}(f(c), C_\delta, f(\hat{a}_i)) = \emptyset$. However, by Note 2.8 and the choice of E we have that $\mathrm{inv}_{\bar{N}}(c, C_\delta, \hat{a}_i) \neq \emptyset$ for all i, and then by the Preservation Lemma $\mathrm{inv}_{\bar{N}^*_j}(f(c), C_\delta, f(\hat{a}_i)) \neq \emptyset$. This finishes the proof of Theorem 2.4. □ □

Let us now pass to the promised discussion of the consistency of our assumptions. The following is a claim about the existence of tight club guessing sequences. If we were to concentrate on truly tight club guessing sequences then we could quote further results, for example a theorem of Shelah from [15], so in this sense Claim 2.10 is not optimal.

However for what we need in the main theorems tight club guessing sequences suffice; hence the claim is formulated in a form that is not optimal but is sufficient, with a gain of simplicity in presentation.

Claim 2.10. *Suppose that $\kappa < \lambda$ are regular.*

(1) *If $\kappa^+ < \lambda$ then there is a truly tight $(\kappa, \kappa, \lambda)$ club guessing sequence.*
(2) *If $\kappa = \mathrm{cf}(\mu) \le \mu$ and $\mu^+ < \lambda$ then there is a tight (μ, λ) guessing sequence.*

Proof. (1) This is proved in [[22], 1.3(a)]. An alternative proof is to deduce the statement from Claim 1.6. of [15] (for uncountable κ) by letting $\mathcal{P}_\delta = \{C_\delta\}$ for $\delta \in S$.
(2) If $\mu^{++} < \lambda$ we simply find a truly tight (μ^+, μ^+, λ) sequence $\langle E_\delta : \delta \in S \rangle$, which exists by (1), and then let C_δ be the first μ elements of E_δ. If $\lambda = \mu^{++}$, the statement is proved in [[22], 1.3(b)]. Alternatively, this follows from the partial square for successors of regulars proved in [[14], Section 4]. $\square$

Remark 2.11. A problematic but natural case for (2) in Claim 2.10 would be when $\kappa = \mathrm{cf}(\mu) < \mu$ and $\lambda = \mu^+$. The conclusion still "usually" holds (i.e. it holds in most natural models of set theory).

Let us now comment on the assumptions (3) and (4) used in Theorems 2.2 and 2.4. An impatient reader might have accused us at this point of unnecessary generalisation and introduction of too many cardinals into the theorem, only to obscure the real issues. Why not set $\kappa = \mu = \sigma$? The reason is that in this case (2) would prevent us from fulfilling (4). For example, suppose that $\kappa^{<\kappa} = \kappa$ and we are considering the requirements of Theorem 2.2. We can let $\mathcal{P}$ of size $\theta \stackrel{\text{def}}{=} \kappa^\kappa$ be a family of almost disjoint elements of $[\kappa]^\kappa$. Let $\langle g_j : j < \theta \rangle$ be some sequence enumerating all increasing enumerations of the elements of $\mathcal{P}$. Hence for $j \ne j'$ the set $\{\gamma : g_j(\gamma) = g_{j'}(\gamma)\}$ has size $< \kappa$. Suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ exemplify that (3) and (4) hold with $\sigma = \kappa$, and assume also that (1) and (2) hold with $\mu = \kappa$. Let $\mathcal{P}_2 = \{X_\alpha : \alpha < \alpha^* \le \lambda\}$. For every $j < \theta$ there is $\alpha(j) < \alpha^*$ such that $\{g_j(i) : i \in X_{\alpha(j)}\} \in \mathcal{P}_1$. Since $|\mathcal{P}_1|, \lambda < \theta$, there is $A \in \mathcal{P}_1$ such that $B_A \stackrel{\text{def}}{=} \{j < \theta : \{g_j(i) : i \in X_{\alpha(j)}\} = A\}$ has size at least λ^+. Since $|\mathcal{P}_2| \le \lambda$, there is β such that

$$|\{j : \alpha(j) = \beta \ \&\ \{g_j(i) : i \in X_{\alpha(j)}\} = A\}| \ge \lambda^+.$$

This is a contradiction to the fact that the elements of $\mathcal{P}$ are almost disjoint.

In fact the situation that is natural for us to consider is when μ is a strong limit singular, because of the following Claim, which follows from the "generalised *GCH*" theorem of Shelah proved in [17] (Theorem 0.1).

Claim 2.12. *Suppose that θ is a strong limit singular cardinal (for example $\theta = \beth_\omega$) and that $\kappa = \mathrm{cf}(\kappa)$ and λ satisfy $\theta \in (\kappa, \lambda]$. Then for every large enough regular $\sigma \in (\kappa, \theta)$, there are $\mathcal{P}_1, \mathcal{P}_2$ satisfying parts (4) of the assumptions of Theorem 2.2 and $|\mathcal{P}_1|, |\mathcal{P}_2| \le \lambda$.*

Proof. By Theorem 0.1 of [17] for every large enough regular $\sigma \in (\kappa, \theta)$ there is a family $\mathcal{P} = \mathcal{P}(\sigma)$ of elements of $[\lambda]^\sigma$ whose size is λ and such that any element of $[\lambda]^\sigma$ can be covered by the union of $< \sigma$ members of $\mathcal{P}$ (in the notation of [17], $\lambda^{[\sigma]} = \lambda$). Let us fix such a σ and let $\mathcal{P} = \mathcal{P}(\sigma)$. Let $\mathcal{P}_2 = [\sigma]^\kappa$, so since θ is a strong limit we have $|\mathcal{P}_2| < \theta \leq \lambda$. Let $\mathcal{P}_1$ be the family of all subsets of size κ of the elements of $\mathcal{P}$, so $|\mathcal{P}_1| \leq \lambda \cdot \sigma^\kappa \leq \lambda$.

Suppose now that $g : \sigma \to \lambda$ is injective; hence the range of g is an element of $[\lambda]^\sigma$. By the choice of $\mathcal{P}$ and the regularity of σ there is $Z \in \mathcal{P}$ such that $\mathrm{Rang}(g) \cap Z$ has size σ. Let Y be any subset of Z of size κ, so $Y \in \mathcal{P}_1$. Letting X be such that $\{g(i) : i \in X\} = Y$ we have that $X \in \mathcal{P}_2$ since g is injective. $\square$

Putting together Claims 2.10 and 2.12 we can see that our non-universality results apply in a large number of set-theoretic situations that are known to be consistent, and moreover follow just from the assumptions on the cardinal arithmetic:

Corollary 2.13. *Suppose that θ is a strong limit singular cardinal and that κ, μ and λ satisfy*

(1) $\mathrm{cf}(\mu) = \kappa < \theta \leq \mu < \mu^+ < \lambda = \mathrm{cf}(\lambda)$,
(2) $\lambda < \mu^\kappa$.

Then for any theory T of size $< \lambda$ satisfying the oak property, we have $\mathrm{univ}(T, \lambda) \geq \mu^\kappa$.

Proof. The assumptions in (1) specifically say that $\lambda > \mu^+$. By Claim 2.10, assumption (1) of Theorem 2.2 is satisfied. By Claim 2.12, assumption (4) of Theorem 2.2 is satisfied for all large enough regular $\sigma \in (\kappa, \theta)$. The conclusion follows by Theorem 2.2. $\square$

We shall now show that a conclusion similar to the one obtained in Corollary 2.13 can be obtained from an assumption whose *negation* is not known to be consistent (i.e. for all we know this assumption is true just in ZFC).

Claim 2.14. *Suppose that κ and λ are regular and $\lambda \geq \kappa^{+\omega+1}$. Further suppose that*

$$\text{for some } n, \mathrm{cov}(\lambda, \kappa^{+n+1}, \kappa^{+n+1}, \kappa^{+n}) = \lambda. \qquad (*_{\lambda,\kappa})$$

Then for any n showing that $(*_{\lambda,\kappa})$ holds, letting $\sigma = \kappa^{+n}$ we have that clause (4) of the assumptions of Theorem 2.4 holds with some $\mathcal{P}_1, \mathcal{P}_2$ satisfying $|\mathcal{P}_1|, |\mathcal{P}_2| \leq \lambda$.

Here we use the familiar pcf notation:

Notation 2.15. For cardinals $\lambda \geq \mu \geq \theta \geq \sigma$ we let $\mathrm{cov}(\lambda, \mu, \theta, \sigma)$ be the smallest possible size of a family $\mathcal{P}$ of elements of $[\lambda]^{<\mu}$ such that every element of $[\lambda]^{<\theta}$ is covered by the union of $< \sigma$ elements of $\mathcal{P}$.

Proof. By the choice of n there is $\mathcal{P}_0 \subseteq [\lambda]^{\kappa^{+n}}$ with $|\mathcal{P}_0| \leq \lambda$ and such that for every $A \in [\lambda]^{\kappa^{+n}}$ there are $\alpha < \kappa^{+n}$ and $A_i \in \mathcal{P}_0$ for $i < \alpha$ such that $A \subseteq \cup_{i<\alpha} A_i$. As κ is regular, $\mathrm{cf}([\kappa^{+n}]^\kappa, \subseteq) \leq \kappa^{+n+1}$. Let $\mathcal{P}_2 \subseteq [\sigma]^\kappa$ exemplify this. For $A \in \mathcal{P}_0$ let h_A be a one-to-one function from σ onto A, and let $\mathcal{P}_1 = \{h_A\text{“}B : A \in \mathcal{P}_0, B \in \mathcal{P}_2\}$. We have that $|\mathcal{P}_1|, |\mathcal{P}_2| \leq \lambda$ and that $\mathcal{P}_1 \subseteq [\lambda]^\kappa$.

As for the clause (i) of (4), let an injective $g : \sigma \to \lambda$ be given. By the choice of $\mathcal{P}_1$, there are $\alpha < \sigma$ and $A_i \in \mathcal{P}_0$ for $i < \alpha$ such that $\mathrm{Rang}(g) \subseteq \cup_{i<\alpha} A_i$. Hence for some $i < \alpha$ we have $|\mathrm{Rang}(g) \cap A_i| = \sigma$. Let $B = \{\zeta < \sigma : h_{A_i}(\zeta) \in \mathrm{Rang}(g)\}$, so $B \in [\sigma]^\sigma$.

Hence for some $B' \in \mathcal{P}_2$ we have $|B \cap B'| = \kappa$. Let $Y = h_{A_i}``B'$, so $Y \in \mathcal{P}_1$. Now choose $X \in \mathcal{P}_2$ that includes $\{\varepsilon < \sigma : g(\varepsilon) \in Y\}$, so clearly $|\{g(i) : i \in X\} \cap Y| = \kappa$. $\square$

Remark 2.16. In the notation of Claim 2.14, the failure of $(*_{\lambda,\kappa})$ is not known to be consistent for any λ, κ as above. For example, consider the hypothesis (F) of [15] Section 6, which states:

for every λ the set of singular cardinals $\chi < \lambda$ whose cofinality is uncountable and that satisfy $\mathrm{pp}_{\Gamma(\mathrm{cf}(\chi))}(\chi) \geq \lambda$ is finite,

and the consistency of whose negation is not known. By the "cov versus pp" theorem of [12], II 5.4, we have that for every $n \geq 1$,

$$\mathrm{cov}(\lambda, \kappa^{+n+1}, \kappa^{+n+1}, \kappa^{+n}) = \sup\{\mathrm{pp}_{\Gamma(\kappa^{+n})}(\chi) : \chi \in [\kappa^{+n+1}, \lambda], \mathrm{cf}(\chi) = \kappa^{+n}\},$$

so Hypothesis (F) implies $(*_{\lambda,\kappa})$. One can see from the proof of Claim 2.14 that for our purposes even weaker statements suffice.

Corollary 2.17. *Suppose that*

(1) $\mathrm{cf}(\mu) = \kappa < \mu < \mu^+ < \lambda$,
(2) $(*_{\lambda,\kappa})$, *and*
(3) $\lambda < \mathcal{U}_{J^{\mathrm{bd}}_\kappa}(\mu)$.

Then for every theory T of size $< \lambda$ satisfying the oak property we have $\mathrm{univ}(T, \lambda) \geq \mathcal{U}_{J^{\mathrm{bd}}_\kappa}(\mu)$.

Proof. The conclusion follows by Claim 2.10, 2.14 and Theorem 2.4. $\square$

Let us also comment on the connection between the assumptions of Theorems 2.2 and 2.4. If $\aleph_0 < \kappa = \mathrm{cf}(\mu) < \mu$ and for all $\theta < \mu$ we have $\theta^\kappa < \mu$, then

$$\mathrm{pp}_{J^{\mathrm{bd}}_\kappa}(\mu) = \mu^\kappa = \mathcal{U}_{J^{\mathrm{bd}}_\kappa}(\mu)$$

(by [12], Chapter VII, Section 1).

Uncited references

[11,13,19].

Acknowledgements

The authors thank the United States–Israel Binational Science Foundation and NSF for their support during the preparation of this paper, and Mirna Džamonja thanks the Academic Study Group for their support during the summer of 1999 and Leverhulme Trust for their grant number F/00204B.

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