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Precalibre pairs of measure algebras

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Abstract

We consider Radon measures μ and pairs (κ, λ) of cardinals such that among every κ many positive measure sets there are λ many whose intersection is nonempty. Such families are connected with the cardinal invariants of the ideal of μ-null sets and have found applications in various subjects of topological measure theory. We survey many of such connections and applications and give some new ones. In particular we show that it is consistent to have a Corson compact space carrying a Radon measure of type $\mathfrak{c} > \aleph_1$ and we partially answer a question of Haydon about measure precalibres.
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0. Introduction

Combinatorial properties of families of sets and their intersections are a well studied subject in set theory and topology, starting from the Delta-System Lemma to numerous chain conditions of topological spaces. The general ilk of such investigations is that one is given a large family of sets with a certain common property, for example, a family of κ many sets of some fixed size, and one looks for a large subfamily with strong intersection properties: being centred, independent, et cetera. In addition to its intrinsic

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combinatorial interest, the notion has become very central to independence proofs because of its applications to chain conditions in forcing.

Calibres and precalibres form a fruitful area of interest in general topology. The monograph by Comfort and Negrepontis [9] is a very general reference, of particular relevance to the present paper is its Chapter 6; Todorčević [41] gives an excellent recent survey; see also Turzański [42] and Juhász and Szentmiklóssy [27].

The present paper studies precalibres of measure algebras or, equivalently, calibres of Radon measures on topological spaces. The exact notions we work with are defined in Section 2, but for the sake of this introduction the reader may concentrate on the situation in which one is given a family of κ many positive elements in some measure algebra and faced with the question of the existence of a subfamily of λ many whose all finite intersections are nonzero. Given the relevance of chain conditions in mathematics it is not at all surprising that this and similar notions have found their way into a number of applications regarding measure algebras and topological measure theory. We give some of them in the references and explain some in the paper, whilst including some new applications in Section 5.

In contrast with the general theory in the context of pure sets or the one of topological spaces, where extensive literature exists, there seems to be a lack of the similarly general treatment of the concept of precalibres in measure algebras. We hope that this paper will narrow that gap. We of course hasten to add that many authors have already considered precalibres of measure algebras within various contexts and we include their results here; in particular the list includes Cichon et al. [6], Cichoń and Pawlikowski [8], Cichoń [7], Fremlin [16,17]. In the fifth volume of his extensive monograph on Measure Theory (in preparation as [24]), D.H. Fremlin surveys several cardinal invariants related to measures. In particular, Chapter 524 of [24] contains many of the facts we discuss here.

Our intention is to present a unified treatment of the subject including some of the results mentioned in the references above and some new results, while avoiding as much as possible an unnecessary repetition of what is already available in the literature. Striking the right balance has not always been easy and we apologise in advance to the authors of the many related theorems that have not been mentioned for the lack of space. Among new results presented here there are two results on cardinal numbers $\kappa > \mathfrak{c}$ which are precalibres of measure algebras; see Section 4. Theorem 4.3 partially answers a question of Haydon about measure precalibres; Theorem 4.7 was inspired by Shelah's result from [40] on independent families in measure algebras. It turned out that the methods developed in the proof of Theorem 4.7 could be used to give a somewhat easier proof of Shelah's theorem which also has slightly weaker assumptions than the original; see Section 6. In Section 5 we prove that it is consistent to have a Corson compact space carrying a Radon measure of type $\mathfrak{c} > \aleph_1$.

The paper is organised as follows: Section 1 gives all the necessary background and is divided into the following subsections: Radon measures, measure algebras, ideals of null sets and combinatorics. Section 2 introduces the main notions, those of calibres and precalibres and shows that for our purposes they are more or less equivalent. Section 3 studies the connections between precalibres and the ideals of null sets, mostly concentrating on the situation below and at the continuum. The situation above $\mathfrak{c}$ is studied separately in Section 4. In Section 5 we give some applications. Section 6 is devoted to the

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independent families in measure algebras and in particular to Shelah's theorem mentioned above. Finally, Section 7 gives some open questions.

1. Background and the notation

In the interest of clarity we include a section giving our notation and some basic facts that will be used later.

Notation 1.1.

(1) Greek letters κ, λ and θ always stand for infinite cardinals.
(2) χ_A denotes the characteristic function of the set A. For a set A contained in some universal set B which is clear from the context, we write A^1 for A and A^0 for the complement of A.
(3) For a set X of ordinals 2^X denotes the set ${}^X 2$ endowed with the product topology. The subbasic clopen sets here are

$$C^l_{\alpha,X} \overset{\text{def}}{=} \{f \in {}^X 2\colon\ f(\alpha) = l\} \quad \text{for } l \in \{0,1\}.$$

If X is clear from the context then we write C^l_α for $C^l_{\alpha,X}$. We also write (following (2)) C_α for C^1_α.
(4) For $Z \subseteq X$ we denote by $\pi_Z : 2^X \to 2^Z$ the coordinatewise projection.

1.1. Radon measures

We remind the reader of some basic concepts from topological measure theory and fix the notation concerning product measures on Cantor cubes.

Definition 1.2. We say that μ is *a Radon measure* on a (Hausdorff topological) space T when μ is a complete finite measure defined on some σ-algebra Σ of subsets of T, and

(i) every open subset of T is in Σ (so that Σ contains the Borel σ-algebra of T);
(ii) $\mu(A) = \sup\{\mu(K)\colon\ K \subseteq A,\ K \text{ compact}\}$ for every $A \in \Sigma$.

Such a measure is called *a Radon probability measure* if $\mu(T) = 1$.

Notation 1.3. For an arbitrary set X, by *the measure* on 2^X we mean the completed product measure on 2^X induced by giving each subbasic clopen set measure $1/2$. It will be denoted by μ_X, and its domain by Σ_X.

We shall now recall some basic properties of μ_X; more facts on measures μ_X can be found in Fremlin [16, 1.15–1.16]; see also Fremlin [22, 254]; [23, 416]. The following definition is crucial in understanding product measures.

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Definition 1.4. A set $A \subseteq 2^X$ depends only on the coordinates in or is determined by the coordinates in $Z \subseteq X$ iff $A = \pi_Z^{-1}[\pi_Z[A]]$.

In other words, if $A \subseteq 2^X$ is determined by the coordinates in $Z \subseteq X$ then $x \in A$ and $y_{|Z} = x_{|Z}$ imply $y \in A$ for $y \in 2^X$. Clearly every clopen subset of 2^X is determined by the coordinates in some finite set.

Fact 1.5. *Let X be an infinite set and let us write $\Sigma = \Sigma_X$ and $\mu = \mu_X$ for simplicity.*

(1) *Every compact G_δ set in 2^X is the intersection of countably many basic clopen sets and hence is determined by the coordinates in a countable subset of X.*
(2) *For every $A \in \Sigma$ we have*

$$\mu(A) = \sup\bigl\{\mu(K):\ K \subseteq A,\ K \text{ is a compact } G_\delta\bigr\}. \tag{$*$}$$

(3) *Every open subset of 2^X is in Σ, so μ is a Radon probability measure on 2^X.*
(4) *For every subset A of 2^X of positive measure there is a compact G_δ set F which is determined by countably many coordinates and satisfies $F \subseteq A$ and $\mu(F) > 0$.*
(5) *For every $A \in \Sigma$ there is $B \in \Sigma$ such that B is determined by countably many coordinates, $B \subseteq A$ and $\mu(A \setminus B) = 0$.*
(6) *For every $A \in \Sigma$ and $\delta > 0$ there is a clopen set C such that $\mu(A \triangle C) < \delta$.*

Proof. (1) Let C be a compact G_δ-set such that $C = \bigcap_{n<\omega} O_n$ where each O_n is open. By compactness we can find for each n a basic clopen set C_n such that $C \subseteq C_n \subseteq O_n$. Hence $C = \bigcap_{n<\omega} C_n$.

(2) Let $\mathcal{F}$ be the family of those $A \in \Sigma$ for which $(*)$ holds. Then $\mathcal{F}$ contains all clopen sets and $\mathcal{F}$ is a monotone class (i.e., is closed under increasing unions and countable decreasing intersections). So $\mathcal{F}$ contains the smallest monotone class generated by the clopen sets; i.e., $\mathcal{F}$ contains the product σ-algebra, and hence its (measure-theoretic) completion Σ.

(3) This follows from the fact that the measure is completion regular, which is a well-known theorem of Kakutani from [28].

(4) and (5) follow immediately from (1), (2). To check (6) first find a compact $K \subseteq A$ such that $\mu(A \setminus K) < \delta/2$; next find a clopen set $C \supseteq K$ with $\mu(C \setminus K) < \delta/2$. Then C is as required. $\square$

Fact 1.5(4) will be in frequent use, which is why we state it explicitly above. Actually we do not use Kakutani's theorem anywhere—we may think of μ_X as the usual product measure, but it seems to be worth recalling that μ_X is really Radon.

1.2. Measure algebras

Concerning measure algebras we generally follow Fremlin [16] but again we tacitly assume that all measures are finite, so by *a measure algebra* we mean a σ-complete Boolean algebra equipped with a finite strictly positive and countably additive functional.

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Throughout this subsection assume that μ is a (finite) complete (i.e., all subsets of any set of measure 0 are measurable) measure with domain Σ and $\mathfrak{A}$ is its measure algebra. For $A \in \Sigma$ we denote by $A^{\bullet}$ the corresponding element of $\mathfrak{A}$. Recall that a *lifting* of μ is a Boolean homomorphism $\varphi : \mathfrak{A} \to \Sigma$ such that $\varphi(0) = \emptyset$ and $\varphi(a)^{\bullet} = a$ for every $a \in \mathfrak{A}$. Part (2) of the following Fact is one of the most useful properties of liftings.

Fact 1.6.

(1) *Every* (*finite*) *complete measure admits a lifting.*
(2) *If $\varphi : \mathfrak{A} \to \Sigma$ is a lifting then for every family $\{a_\xi \colon \xi < \kappa\} \subseteq \mathfrak{A}$ the union $\bigcup_{\xi<\kappa} \varphi(a_\xi)$ is measurable, and in fact there is a countable $J \subseteq \kappa$ such that the measure of $\bigcup_{\xi<\kappa} \varphi(a_\xi)$ is the same as that of $\bigcup_{\xi \in J} \varphi(a_\xi)$.*

Proof. For (1), which is a celebrated result with a long proof and a long history see [16, Theorem 4.4].

To check (2) let

$$Z = \bigcup_{\xi<\kappa} \varphi(a_\xi), \qquad r = \sup\left\{\mu\left(\bigcup_{\xi \in I} \varphi(a_\xi)\right) \colon I \in [\kappa]^{\aleph_0}\right\}.$$

Then there is a set $J \in [\kappa]^{\aleph_0}$ such that writing $A = \bigcup_{\xi \in J} \varphi(a_\xi) \in \Sigma$ we have $\mu(A) = r$. Therefore for every $\xi < \kappa$ the set $\varphi(a_\xi) \setminus A$ is null. This implies that $a_\xi = \varphi(a_\xi)^{\bullet} \leqslant A^{\bullet}$, and $\varphi(a_\xi) \subseteq \varphi(A^{\bullet})$. Hence $A \subseteq Z \subseteq \varphi(A^{\bullet})$; as $\varphi(A^{\bullet}) \setminus A$ is of measure zero this gives that $Z \in \Sigma$ and that J is as required. $\square$

The Maharam type $\tau(\mathfrak{A})$ of $\mathfrak{A}$ (or of a measure μ itself) can be defined as the density of the metric space $(\mathfrak{A}, \rho)$, where $\rho(a, b) = \mu(a \triangle b)$. In other words

$$\tau(\mathfrak{A}) = \min\{|\mathcal{C}| \colon \mathcal{C} \subseteq \Sigma,\ \mathcal{C} \text{ is } \triangle\text{-dense in } \Sigma\},$$

where $\mathcal{C}$ is said to be $\triangle$-dense in Σ if for every $E \in \Sigma$ and every $\varepsilon > 0$ there is $C \in \mathcal{C}$ such that $\mu(E \triangle C) < \varepsilon$.

A measure μ is *Maharam homogeneous* or just *homogeneous* if it has the same type on every $E \subseteq \Sigma$ with $\mu(E) > 0$, and then we also say that its measure algebra is homogeneous.

Notation 1.7. For every κ we denote by $\mathfrak{A}_\kappa$ the measure algebra of μ_κ. The set of positive elements of a Boolean algebra $\mathfrak{A}$ endowed with the induced operations is denoted by $\mathfrak{A}^+$.

Recall that for every κ, $\mathfrak{A}_\kappa$ is a homogeneous measure algebra of type κ. The essence of the Maharam theorem (see [16, p. 908, Paragraph 1]) states that if μ is a homogeneous probability measure of type κ then its measure algebra $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_\kappa$. Recall also the following (see [16, Corollary 3.12]):

Fact 1.8. *If $(\mathfrak{A}, \mu)$ is a probability measure algebra of type κ then there is a measure preserving homomorphism $f : \mathfrak{A} \to \mathfrak{A}_\kappa$ (so $\mu_\kappa[f(a)] = \mu(a)$ for every $a \in \mathfrak{A}$ and f is necessarily injective).*

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1.3. Ideals of null sets

Let $\mathcal{N}$ be a proper ideal of subsets of a space X with $\bigcup\mathcal{N} = X$. Recall that the cardinal numbers $\operatorname{add}(\mathcal{N})$, $\operatorname{cov}(\mathcal{N})$ and $\operatorname{non}(\mathcal{N})$ of $\mathcal{N}$ are defined as follows

$$\begin{aligned}
&\operatorname{add}(\mathcal{N}) = \min\Bigl\{|\mathcal{E}|\colon\ \mathcal{E}\subseteq\mathcal{N}, \bigcup\mathcal{E}\notin\mathcal{N}\Bigr\},\\
&\operatorname{cov}(\mathcal{N}) = \min\Bigl\{|\mathcal{E}|\colon\ \mathcal{E}\subseteq\mathcal{N}, \bigcup\mathcal{E} = X\Bigr\},\\
&\operatorname{non}(\mathcal{N}) = \min\bigl\{|Y|\colon\ Y\notin\mathcal{N}\bigr\}.
\end{aligned}$$

It is clear that $\operatorname{add}(\mathcal{N}) \leqslant \operatorname{cov}(\mathcal{N})$, and $\operatorname{add}(\mathcal{N}) \leqslant \operatorname{non}(\mathcal{N})$. The ordering of $\operatorname{cov}(\mathcal{N})$ and $\operatorname{non}(\mathcal{N})$ depends on the model. See, e.g., the proof in [4] that Mathias forcing increases $\operatorname{non}(\mathcal{N})$ and leaves intact $\operatorname{cov}(\mathcal{N})$ where $\mathcal{N}$ is the ideal of Lebesgue null sets, while [4] also gives a model (Model 7.5.5, pg. 384) in which $\operatorname{non}(\mathcal{N}) < \operatorname{cov}(\mathcal{N})$. In fact a fundamental example of such a model is provided by Solovay's random real model. If $V \models GCH$ and $V[G]$ is the extension obtained by adding κ random reals for $\kappa > \aleph_1$ regular, then in $V[G]$ there is a Sierpiński set of size $\aleph_1$ and 2^ω is not a union of fewer than κ null sets. So $\aleph_1 = \operatorname{add}(\mathcal{N}) = \operatorname{non}(\mathcal{N}) < \operatorname{cov}(\mathcal{N}) = \kappa$. This may be found in Kunen's exposition [30], including Theorem 3.18 where one takes $\mathcal{N}$ for $\mathcal{S}$, and Theorem 3.19 where the notation $\mathrm{BAIRE}(\mathcal{N})$ is used to say that $\operatorname{cov}(\mathcal{N}) = \kappa$; see also Remark 1.10(6) below. We shall consider these cardinal functions on the ideals of μ_κ-null sets.

Notation 1.9. For every κ we denote by $\mathcal{N}_\kappa$ the σ-ideal $\{N\subseteq 2^\kappa\colon\ \mu_\kappa(N) = 0\}$.

Basic facts concerning ideals $\mathcal{N}_\kappa$ and their cardinal functions, as well as further references, may be found, e.g., in Fremlin [16]; Vaughan [43] surveys many other cardinal functions related to combinatorics, measure and category; Kraszewski [29] offers a detailed discussion on cardinal functions on a larger class of σ-ideals in Cantor cubes.

A useful fact is that if μ is a Radon measure then the cardinal functions of the ideal of μ-null sets can be expressed in terms of the measure algebra of μ, see Fremlin [16], Section 6 (in particular, Theorem 6.13). This implies that if two Radon measures have isomorphic measure algebras, then the cardinal invariants agree on their corresponding ideals of null sets.

Remark 1.10.

(1) If $\mathcal{N}$ is a σ-ideal, in particular if $\mathcal{N}$ is the ideal of null sets for a non-trivial measure, then $\operatorname{add}(\mathcal{N}) > \aleph_0$ (hence $\operatorname{cov}(\mathcal{N}), \operatorname{non}(\mathcal{N}) > \aleph_0$ as well).
(2) The function $\kappa\mapsto\operatorname{cov}(\mathcal{N}_\kappa)$ is nonincreasing; in particular $\operatorname{cov}(\mathcal{N}_{\aleph_0}) \geqslant \operatorname{cov}(\mathcal{N}_{\aleph_1})$ and the equality need not hold (adding $\aleph_\omega$ random reals over a model of GCH produces a model of this; see [29, Remark after Theorem 5.5]).
(3) The function $\kappa\mapsto\operatorname{non}(\mathcal{N}_\kappa)$ is nondecreasing; however,

$$\operatorname{non}(\mathcal{N}_{\aleph_0}) = \operatorname{non}(\mathcal{N}_{\aleph_1}) = \operatorname{non}(\mathcal{N}_{\aleph_2}),$$

where the first equality is standard while the latter is a striking result due to Kraszewski (see [29, Corollary 3.11]).

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(4) $\mathrm{non}(\mathcal{N}_{\aleph_1}) < \mathrm{cov}(\mathcal{N}_{\aleph_1})$ is relatively consistent (adding $\aleph_\omega$ random reals over a model of GCH produces a model of this; see [29, Remark after Theorem 5.5]).
(5) The existence of an atomlessly measurable cardinal implies $\aleph_1 = \mathrm{non}(\mathcal{N}_{\aleph_0}) < \mathrm{cov}(\mathcal{N}_{\aleph_1})$ (see [17, 6G and 6L]).
(6) Bartoszyński et al. [5] (see also [4, Theorem 3.2.57]) construct a model V of set theory such that adding a random real over it produces a model $V[G]$ that satisfies $\mathrm{add}(\mathcal{N}_{\aleph_0}) < \mathrm{cov}(\mathcal{N}_{\aleph_0})$.

1.4. Combinatorics

When dealing with calibres and precalibres one often encounters the combinatorial Δ-System Lemma. We quote the instances of it that we need. The complete references, proofs and a historical discussion can be found in [9]. We note only that Theorem 1.12 has a much simpler proof than 1.13 and was proved about thirty years earlier (1940s versus 1970s).

Definition 1.11. We say that κ is $\aleph_1$-*inaccessible* and write $\aleph_1 \ll \kappa$ iff for every $\tau < \kappa$ also $\tau^{\aleph_0} < \kappa$.

In particular for $\aleph_1$-inaccessible κ we have $\aleph_1 \leqslant \mathfrak{c} = 2^{\aleph_0} < \kappa$.

Theorem 1.12. *If κ is regular and $\aleph_1 \ll \kappa$ then for every family $\{J_\xi\colon \xi < \kappa\}$ of countable sets there is $X \in [\kappa]^\kappa$ such that the family $\{J_\xi\colon \xi \in X\}$ forms a Δ-system with some root J, meaning that for every $\xi \neq \eta \in X$ we have $J_\xi \cap J_\eta = J$.*

Theorem 1.13. *Suppose that θ is a singular cardinal satisfying $\aleph_1 \ll \theta$. Then for every family $\{J_\alpha\colon \alpha < \theta\}$ of countable sets and for any increasing sequence of regular $\aleph_1$-inaccessible cardinals $\langle \theta_i\colon i < \mathrm{cf}(\theta)\rangle$, converging to θ, there are $\langle I_j\colon j < \mathrm{cf}(\theta)\rangle$ and $\langle R_j\colon j < \mathrm{cf}(\theta)\rangle$ such that*

(i) *$I_j \in [\theta]^{\theta_j}$ are pairwise disjoint*;
(ii) *$J_\alpha \cap J_\beta = R_j$ for $\alpha \neq \beta \in I_j$; and*
(iii) *$J_\alpha \cap J_\beta \subseteq R_{j'}$ for $\alpha \in I_j$, $\beta \in I_{j'}$ and $j < j'$.*

Another fact about $\aleph_1$-inaccessible cardinals that will be useful to us is contained in the following simple Lemma, which we give with a proof.

Lemma 1.14. *Let κ be an $\aleph_1$-inaccessible cardinal of countable cofinality. Then there is an increasing sequence $\langle \kappa_n\colon n < \omega\rangle$ of regular $\aleph_1$-inaccessible cardinals with limit κ.*

Proof. Let $\langle \rho_n\colon n < \omega\rangle$ be any sequence of cardinals increasing to κ. By induction on n define τ_n, κ_n as follows.

Let $\tau_0 = \aleph_0$. For any n, assuming that $\tau_n < \kappa$ let $\kappa_n \stackrel{\mathrm{def}}{=} (\tau_n^{\aleph_0})^+$. Then $\kappa_n < \kappa$ is regular, and if $\tau < \kappa_n$ then $\tau \leqslant \tau_n^{\aleph_0}$ so $\tau^{\aleph_0} \leqslant \tau_n^{\aleph_0} < \kappa_n$. We define $\tau_{n+1} \stackrel{\mathrm{def}}{=} \max\{\rho_n, \kappa_n\}$. $\square$

We shall also use the following Theorem of Engelking and Karłowicz from [15].

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Theorem 1.15. *Suppose that $\theta = \theta^{\aleph_0}$. Then there is a family of functions $\{f_\gamma\colon \gamma < 2^\theta\}$ in ${}^\theta\theta$ such that for all sequences $\langle \gamma_n\colon n < \omega\rangle$ in 2^θ and $\langle \zeta_n\colon n < \omega\rangle$ in θ, there is $\zeta < \theta$ such that $f_{\gamma_n}(\zeta) = \zeta_n$ holds for all n.*

2. Calibres and precalibres

In this section we introduce the definition of the precalibre of a measure algebra and note some elementary properties. With only a few exceptions, the facts given below are either basic, from the literature or belong to the mathematical folklore.

Definition 2.1. If $\kappa \geqslant \lambda$ are cardinal numbers and $\mathfrak{A}$ is a Boolean algebra we say that (κ, λ) is a *precalibre of* $\mathfrak{A}$ iff for every family $\{a_\xi\colon \xi < \kappa\}$ of (not necessarily distinct) elements of $\mathfrak{A}^+$, there is $X \in [\kappa]^\lambda$ such that $\{a_\xi\colon \xi \in X\}$ is centred, i.e., $\bigwedge_{\xi\in J} a_\xi \neq 0$ for any finite $J \subseteq X$. In the case $\kappa = \lambda$ we simply say that κ is a precalibre of $\mathfrak{A}$.

We shall consider this concept mainly for measure algebras. Note also that there is interesting combinatorics involving calibre (κ, κ, n) for measure algebras, see 6.12–6.17 of [9] but we shall not go into it for reasons of space. It will be convenient to use the following notation.

Notation 2.2. We write $\mathrm{pc}_\theta(\kappa, \lambda)$ to say that (κ, λ) is a precalibre of $\mathfrak{A}_\theta$ (i.e., the measure algebra of the usual product measure μ_θ on 2^θ). Let $\mathrm{pc}(\kappa, \lambda)$ mean that $\mathrm{pc}_\theta(\kappa, \lambda)$ holds for every cardinal number θ.

We shall use some obvious conventions in the case $\lambda = \kappa$. In particular, we say that κ is a precalibre of $\mathfrak{A}_\theta$ iff $\mathrm{pc}_\theta(\kappa, \kappa)$ holds.

Notice that if $\mathfrak{A}$ is any nonatomic Boolean algebra then $\mathfrak{A}$ contains a sequence of pairwise disjoint nonzero elements, so $\aleph_0$ is trivially not a precalibre of $\mathfrak{A}$. Hence $\aleph_0$ is not a precalibre of any nonatomic measure algebra. One can similarly check that $\mathrm{pc}(\kappa, \kappa)$ does not hold for any κ with countable cofinality. The following version of the notion of a precalibre enables us to avoid such trivialities when dealing with κ with $\mathrm{cf}(\kappa) = \aleph_0$. It was suggested by R. Haydon.

Definition 2.3. If κ and λ are cardinal numbers and $(\mathfrak{A}, \mu)$ is a measure algebra we say that (κ, λ) is a *measure precalibre of* $\mathfrak{A}$ iff for every $\{a_\xi\colon \xi < \kappa\} \subseteq \mathfrak{A}$ satisfying $\inf_{\xi<\kappa} \mu(a_\xi) > 0$ (and again not necessarily consisting of distinct elements), there is $X \in [\kappa]^\lambda$ such that $\{a_\xi\colon \xi \in X\}$ is centred.

Note that $(\aleph_0, \aleph_0)$ is a measure precalibre of every measure algebra (see the remark after the proof of Lemma 2.5), and also that as opposed to the notion of precalibres which has a well-known analogue in the theory of compact ccc spaces, the notion of a measure precalibre seems to be restricted to the context of measures.

Our notation for measure precalibres follows the one we use for precalibres, so we write $\mathrm{mpc}_\theta(\kappa, \lambda)$ to say that (κ, λ) is a measure precalibre of $\mathfrak{A}_\theta$, and $\mathrm{mpc}(\kappa, \lambda)$ means that

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$\mathrm{mpc}_\theta(\kappa,\lambda)$ holds for every cardinal number θ. In a similar manner we define when κ itself is a measure precalibre.

It is often convenient to use the language of measure spaces rather than that of measure algebras.

Definition 2.4. If κ and λ are cardinal numbers and (T,Σ,μ) is a finite measure space we say that (κ,λ) is a *calibre of* μ iff for every subfamily $\{A_\xi\colon \xi<\kappa\}\subseteq\Sigma$ of (not necessarily distinct) sets of positive measure there is $X\in[\kappa]^\lambda$ such that $\bigcap\{A_\xi\colon \xi\in X\}\neq\emptyset$. The definition of a *measure calibre of* μ is similar, but the sets $\{A_\xi\colon \xi<\kappa\}\subseteq\Sigma$ we start with are required to have measure bounded away from 0.

In our context it turns out that precalibres and calibres express the same property in slightly different languages:

Lemma 2.5. *Let $\mathfrak{A}$ be the measure algebra of a measure space (T,Σ,μ). Then the following are equivalent*

(i) *(κ,λ) is a precalibre of $\mathfrak{A}$;*
(ii) *for every family $\{E_\xi\colon \xi<\kappa\}\subseteq\Sigma$ of not necessarily distinct sets of positive measure, there is $X\in[\kappa]^\lambda$ such that the family $\{E_\xi\colon \xi\in X\}$ is centred.*

Consequently, if μ is a Radon measure then (κ,λ) is a precalibre of $\mathfrak{A}$ if and only if (κ,λ) is a calibre of μ. A similar statement holds for measure precalibres and measure calibres.

Proof. The implication from (i) to (ii) follows immediately from the fact that if $\{E_\xi^\bullet\colon \xi\in X\}\subseteq\mathfrak{A}$ is a centred family then so is $\{E_\xi\colon \xi\in X\}\subseteq\Sigma$.

To prove the reverse implication, notice first that without loss of generality we can assume that (T,Σ,μ) is a complete measure space. Let $\varphi:\mathfrak{A}\to\Sigma$, be a lifting (so $\varphi(a)^\bullet=a$ for every $a\in\mathfrak{A}$; see Fact 1.6(1)). Now if $\{a_\xi\colon \xi<\kappa\}$ is any family in $\mathfrak{A}^+$ then $\{\varphi(a_\xi)\colon \xi<\kappa\}$ is a family of sets of positive measure so there is $X\in[\kappa]^\lambda$ such that $\{\varphi(a_\xi)\colon \xi\in X\}$ is centred. This implies that the family $\{a_\xi\colon \xi\in X\}$ is centred (as φ is a homomorphism and $\varphi(0)=\emptyset$).

If μ is a Radon measure and $\{E_\xi\colon \xi<\kappa\}$ is a family of sets of positive measure then by 1.2(ii) we may assume that each E_ξ is compact, and hence every centred subfamily has nonempty intersection. $\square$

As one can notice from the above, the fact that in the definition of calibres and precalibres the family we start with does not necessarily consist of distinct elements appears rather often, so we shall take it for granted in every such instance. To continue, it is a classical fact from measure theory that $\aleph_0$ is a measure calibre of every finite measure (T,Σ,μ). Recall the proof: writing for a given sequence $\langle E_n\colon n<\omega\rangle$ of sets whose measures are bounded away from 0 by ε

$$E=\bigcap_{n<\omega}\bigcup_{k\geqslant n}E_k,$$

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we have $\mu(E) \geqslant \varepsilon$, so E is nonempty. Any $s \in E$ is in infinitely many sets E_n so we are done. It is also easy to verify the following.

Observation 2.6. Suppose that $\mathrm{cf}(\kappa) > \aleph_0$ and let $\mathfrak{A}$ be any measure algebra.

(a) For every $\lambda \leqslant \kappa$ of uncountable cofinality, (κ, λ) is a measure precalibre of $\mathfrak{A}$ iff (κ, λ) is a precalibre of $\mathfrak{A}$.
(b) $(\kappa, \aleph_0)$ is a precalibre of $\mathfrak{A}$.

We now collect some implications about various calibre pairs and note some cases when basic cardinal arithmetic of κ and λ leads to a conclusion about the calibre pair (κ, λ).

Lemma 2.7. *For infinite cardinal numbers κ, λ, θ the following are satisfied*:

(i) *if* $\mathrm{pc}_\theta(\kappa, \lambda)$ *then* $\mathrm{pc}_\theta(\kappa', \lambda')$ *whenever* $\kappa' \geqslant \kappa$ *and* $\lambda' \leqslant \lambda$;
(ii) *if* $\mathrm{pc}_\theta(\kappa, \lambda)$ *then* $\mathrm{pc}_{\theta'}(\kappa, \lambda)$ *whenever* $\theta' \leqslant \theta$;
(iii) *if* $\kappa > \theta^{\aleph_0}$ *then* $\mathrm{pc}_\theta(\kappa, \kappa)$.

Proof. (i) is obvious; (ii) follows from the fact that $\mathfrak{A}_{\theta'}$ is embeddable as a subalgebra of $\mathfrak{A}_\theta$ when $\theta' \leqslant \theta$. Part (iii) follows from Fact 1.5 (4), because there are only $\theta^{\aleph_0}$ compact G_δ sets in 2^θ (see Fact 1.5(1)). □

The following fact is very useful; it has been noted by D.H. Fremlin (unpublished).

Theorem 2.8. *If $\kappa \geqslant \lambda \geqslant \aleph_0$ then the following are equivalent*:

(i) (κ, λ) *is a precalibre of every measure algebra*;
(ii) $\mathrm{pc}(\kappa, \lambda)$;
(iii) $\mathrm{pc}_\kappa(\kappa, \lambda)$.

The analogous equivalence holds when we replace 'precalibre' by 'measure precalibre'.

Proof. Trivially, (i) implies (ii), and (ii) implies (iii).

Assume now $\mathrm{pc}_\kappa(\kappa, \lambda)$ and suppose that $\{a_\xi\colon \xi < \kappa\}$ is a family of nonzero elements in some measure algebra $\mathfrak{A}$. Let $\mathfrak{B}$ be the complete subalgebra of $\mathfrak{A}$ generated by the family $\{a_\xi\colon \xi < \kappa\}$. Then $\mathfrak{B}$ is a measure algebra of Maharam type $\leqslant \kappa$, and there is a homomorphic measure preserving embedding $\phi : \mathfrak{B} \to \mathfrak{A}_\kappa$ (see Fact 1.8). Since $\mathrm{pc}_\kappa(\kappa, \lambda)$ holds, there is $X \in [\kappa]^\lambda$ such that $\{\phi(a_\xi)\colon \xi \in X\}$ is a centred family. Then $\{a_\xi\colon \xi \in X\}$ is centred too. The same argument can be applied to measure precalibres, as ϕ preserves measure. □

Finally we note an obvious connection with topological calibres, which follows immediately from the Stone representation theorem.

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Remark 2.9. Assume that (κ,λ) is a calibre of all ccc compact spaces (i.e., whenever we have κ many nonempty open sets in a compact ccc space then we can choose λ of them having a nonempty intersection). Then (κ,λ) is a precalibre of all measure algebras.

3. Precalibres and ideals of null sets

In this section we analyse calibre-like properties in terms of suitable properties of ideals of null sets. This enables us to discuss when small uncountable cardinals are precalibres of measure algebras. The discussion is based on Cichoń [7] and Fremlin [17].

Definition 3.1. Suppose that $\mathcal{N}$ is a σ-ideal of subsets of T. A family $\mathcal{R}=\{N_\xi\colon \xi<\kappa\}\subseteq\mathcal{N}$ is a (κ,λ)-*Rothberger family for* $\mathcal{N}$ if for every $X\in[\kappa]^\lambda$ we have $\bigcup_{\xi\in X}N_\xi=T$.

The following theorem combines Theorem 7.1 from Cichoń [7] and Lemma A2U from Fremlin [17].

Theorem 3.2. *Suppose that (T,Σ,μ) is a finite complete measure space, $\mathcal{N}$ is its ideal of null sets and $\mathfrak{A}$ is the corresponding measure algebra.*

(i) *If $\kappa\geqslant\lambda$, $\mathrm{cf}(\kappa)>\aleph_0$ and (κ,λ) is not a precalibre of $\mathfrak{A}$ then there is a set $A\in\Sigma$ of positive measure and a (κ,λ)-Rothberger family for the ideal $\mathcal{N}_A=\{N\in\mathcal{N}\colon N\subseteq A\}$ of subsets of A.*
(ii) *If κ is regular uncountable and is not a precalibre of $\mathfrak{A}$ then there is an increasing sequence $\langle N_\xi\colon \xi<\kappa\rangle$ of elements of $\mathcal{N}$ such that $\bigcup_{\xi<\kappa}N_\xi\in\Sigma\setminus\mathcal{N}$.*

Proof. (i) Take a family $\{E_\xi\colon \xi<\kappa\}\subseteq\Sigma$ witnessing that (κ,λ) is not a precalibre of $\mathfrak{A}$. We define inductively a sequence $\langle I_\alpha\colon \alpha<\kappa\rangle$ of pairwise disjoint countable subsets of κ such that for every α

$$\bigvee_{\xi\in I_\alpha}E_\xi^{\bullet}=\bigvee_{\xi\in R_\alpha}E_\xi^{\bullet}\quad\text{where } R_\alpha=\kappa\setminus\bigcup_{\beta<\alpha}I_\beta.$$

Since $\mathrm{cf}(\kappa)>\aleph_0$, there is $\alpha_0<\kappa$ and $a\in\mathfrak{A}^+$ such that

$$\bigvee_{\xi\in R_\alpha}E_\xi^{\bullet}=a\quad\text{for every } \alpha\geqslant\alpha_0.$$

Now we take $A\in\Sigma$ with $A^{\bullet}=a$ and for every $\alpha<\kappa$ put

$$N_\alpha=A\setminus\bigcup_{\xi\in I_\alpha}E_\xi.$$

Then we claim that $\{N_\alpha\colon \alpha<\kappa\}$ is a (κ,λ)-Rothberger family for $\mathcal{N}_A$.

Indeed, it is clear that $N_\alpha\in\mathcal{N}_A$ for every $\alpha<\kappa$; suppose that $\bigcup_{\alpha\in X}N_\alpha\neq A$ for some $X\in[\kappa]^\lambda$. Taking $t\in A\setminus\bigcup_{\alpha\in X}N_\alpha$, we have $t\in\bigcup_{\xi\in I_\alpha}E_\xi$ for every $\alpha\in X$, hence t is in λ many sets E_ξ, a contradiction.

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(ii) Take a family $\{a_\xi\colon \xi < \kappa\} \subseteq \mathfrak{A}$ witnessing that κ is not a precalibre of $\mathfrak{A}$. Let $\varphi : \mathfrak{A} \to \Sigma$ be a lifting. For every $\xi < \kappa$ we put

$$F_\xi = \bigcup_{\xi \leqslant \eta < \kappa} \varphi(a_\eta).$$

Then $F_\xi \in \Sigma$ by Fact 1.6(2). Since $\mathrm{cf}(\kappa) > \aleph_0$ there is η_0 such that $\mu(F_\eta) = \mu(F_{\eta_0})$ whenever $\eta_0 \leqslant \eta < \kappa$.

It is clear that the sets $N_\eta = F_{\eta_0} \setminus F_\eta$ form an increasing family of null sets. We claim that $\bigcup_{\eta<\kappa} N_\eta = F_{\eta_0}$. Otherwise, there is a point $t \in F_{\eta_0}$ such that $t \in \bigcap_{\eta<\kappa} F_\eta$. Then the set $X = \{\xi\colon t \in \varphi(a_\xi)\}$ is cofinal in κ, so $|X| = \kappa$ as κ is regular. But then $\{a_\xi\colon \xi \in X\}$ is centred, a contradiction. □

Lemma 3.3. *If (T, Σ, μ) is a nontrivial Radon measure space and there is a (κ, λ)-Rothberger family for the ideal $\mathcal{N}$ of μ-null sets, then (κ, λ) is not a calibre of μ.*

Proof. Let $\{N_\xi\colon \xi < \kappa\} \subseteq \mathcal{N}$ be a (κ, λ)-Rothberger family. We have $\mu(T) > 0$, so for every $\xi < \kappa$ there is a compact set F_ξ such that $F_\xi \subseteq T \setminus N_\xi$ and $\mu(F_\xi) > 0$. It is clear that no point of T belongs to λ many among the sets F_ξ. □

Part (1) of the following result is due to Cichoń [7].

Corollary 3.4. *Suppose that $\aleph_0 < \mathrm{cf}(\kappa)$ and $\kappa \geqslant \lambda$.*

(1) *For any θ, $\mathrm{pc}_\theta(\kappa, \lambda)$ holds if and only if there is no (κ, λ)-Rothberger family for the ideal $\mathcal{N}_\theta$ of the null subsets of 2^θ.*
(2) $\mathrm{pc}(\kappa, \lambda)$ *if and only if there is no (κ, λ)-Rothberger family for $\mathcal{N}_\kappa$.*
(3) *There is θ such that there is a (κ, λ)-Rothberger family for $\mathcal{N}_\theta$ iff there is in fact a (κ, λ)-Rothberger family for $\mathcal{N}_\kappa$.*

Proof. (1) follows from Theorem 3.2 and Lemma 3.3; (2) is a consequence of (1) and Theorem 2.8. (3) is a consequence of (1) and (2). □

Corollary 3.5. *Let μ be a totally finite Radon measure on a space T, and let $\mathcal{N}$ be the ideal of μ-null sets*:

(1) *If $\kappa = \mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N})$ then κ is not a calibre of μ.*
(2) *If $\kappa = \mathrm{non}(\mathcal{N}) = |T|$ then κ is not a calibre of μ.*
(3) *If κ is regular, μ is homogeneous and $\kappa > \mathrm{non}(\mathcal{N})$ then κ is a calibre of μ.*

Proof. If either $\kappa = \mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N})$ or $\kappa = \mathrm{non}(\mathcal{N}) = |T|$ then we can write T as an increasing union of κ many null sets. This gives a (κ, κ)-Rothberger family for $\mathcal{N}$ so κ is not a calibre of μ by Lemma 3.3.

We can argue for (3) as follows. First note that the assumptions imply that κ is uncountable. If κ is not a calibre of μ then (it is not a precalibre of the measure algebra of μ by Lemma 2.5 and) by Lemma 3.2(2) there is a set $A \in \Sigma$ of positive measure which

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is an increasing union of κ many null sets $\{N_\xi\colon \xi < \kappa\}$. Since μ is homogeneous we can assume that in fact $A = T$ (indeed, the measure μ restricted to A has the same non, see Section 1.3).

Take a set $Z \subseteq T$ which is not null and $|Z| = \mathrm{non}(\mathcal{N})$. Since $\mathrm{non}(\mathcal{N}) < \kappa$ there must be $\xi < \kappa$ such that $Z \subseteq N_\xi$, which is impossible. $\square$

Recall that for any uncountable κ we have $\mathrm{add}(\mathcal{N}_\kappa) = \aleph_1$, see, e.g., Theorem 2.1. in [29]. Therefore part (1) of Corollary 3.5 is interesting mostly when $\kappa = \aleph_1$.

Corollary 3.6.

(1) *If κ is regular and* $\mathrm{non}(\mathcal{N}_\kappa) < \kappa$ *then κ is a precalibre of all measure algebras.*
(2) *$\aleph_1$ is a precalibre of all measure algebras if and only if* $\mathrm{cov}(\mathcal{N}_{\aleph_1}) > \aleph_1$.

Proof. (1) follows from Corollary 3.5 (3) and Theorem 2.8; (2) is a consequence of Corollary 3.5(1), Theorem 2.8 and Theorem 3.2 combined with the homogeneity of $\mu_{\aleph_1}$. $\square$

In connection with the above considerations we mention the following result due to D.H. Fremlin.

Theorem 3.7. *If $\kappa <$* $\mathrm{cov}(\mathcal{N}_\kappa)$ *then κ is a measure precalibre of all measure algebras.*

Note that for κ of uncountable cofinality the result follows directly from Theorem 3.2. The case $\mathrm{cf}(\kappa) = \aleph_0$ requires an additional nontrivial argument, see 524M of [24] for details. Combining (the easier part of) Theorem 3.7 with Corollary 3.6 and Corollary 3.5 we can obtain the following:

Corollary 3.8. *If κ is regular and* $\mathrm{non}(\mathcal{N}_\kappa) < \mathrm{cov}(\mathcal{N}_\kappa)$ *then*

(a) *κ is a precalibre of all measure algebras*; *and*
(b) *every regular λ is a calibre of μ_κ.*

The next result (with two different proofs) can be found in Argyros and Tsarpalias [2, Theorem 4.1] (see also [9, Theorem 6.18], and Shelah [40, Theorem 1.3]). It is a generalisation of the fact that under CH the cardinal $\aleph_1$ is not a precalibre of measure algebras.

Theorem 3.9. *If κ is a strong limit cardinal of countable cofinality and $\kappa^+ = 2^\kappa$ then κ^+ is not a calibre of μ_κ.*

Proof. The point is that under such assumptions $\mathrm{non}(\mathcal{N}_\kappa) = 2^\kappa$ see [16, 6.17e] and the argument for 6.18d. Hence we can apply Corollary 3.5(2). $\square$

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We can now discuss what the possibilities for $\mathrm{pc}(\kappa,\lambda)$ are when $\lambda \leqslant \kappa \leqslant \mathfrak{c}$. The following theorem is due to Cichoń and Pawlikowski and was proved as a claim within the proof of Theorem 3.1 of [8].

Theorem 3.10. *Suppose that* **V** *is any universe of set theory, and c is a Cohen real over* **V**. *Then in* $\mathbf{V}[c]$ *there is a* $(\mathfrak{c},\aleph_1)$*-Rothberger family for the ideal* $\mathcal{N}_{\aleph_0}$ *(and hence, by Lemma* 3.3 $(\mathfrak{c},\aleph_1)$ *is not a calibre of the Lebesgue measure).*

The following corollary will be useful in Section 5.

Corollary 3.11. *It is consistent that* $\mathfrak{c} > \aleph_1$ *and* $(\mathfrak{c},\aleph_1)$ *is not a calibre of the Lebesgue measure.*

Proof. Start with **V** which fails CH and add a Cohen real over **V**. Hence $\mathbf{V}[G]$ will fail CH and satisfy $\neg\,\mathrm{pc}_{\aleph_0}(\mathfrak{c},\aleph_1)$, by Theorem 3.10. □

Theorem 3.10 suggests a consideration of the situation when a Cohen subset is added to a regular cardinal $\lambda > \aleph_0$. Must $\neg\,\mathrm{pc}_\lambda(2^\lambda,\lambda^+)$ hold in the extension? The proof in [8] uses the Borel structure of 2^ω, but there are alternative proofs for which it is not immediate if one needs to be at ω. However, it turns out that $\mathfrak{c}^+$ is always a precalibre of measure algebras (see Section 4), hence if we add a Cohen subset to $\aleph_1$ over a model of GCH we shall not obtain a $(2^{\aleph_1},\aleph_2)$-Rothberger family of $\mathcal{N}_{\aleph_1}$ in the extension and we shall even have $\mathrm{pc}_{\aleph_1}(2^{\aleph_1},{\aleph_1}^+)$.

To finalise this section let us consider the possibilities when $\mathfrak{c} = \aleph_2$. Employing the fact that $\mathrm{non}(\mathcal{N}_{\aleph_0}) = \mathrm{non}(\mathcal{N}_{\aleph_1}) = \mathrm{non}(\mathcal{N}_{\aleph_2})$ (see Remark 1.10(3)), Corollary 3.5(2) and Corollary 3.6 we can draw the following conclusions. They show that all combinations between $\mathrm{pc}(\aleph_1,\aleph_1)$ and $\mathrm{pc}(\aleph_2,\aleph_2)$ follow from various assumptions about cov and non. See Table 1.

The assumptions of the second line of the table hold in the iterated Sacks model, see, e.g.,[4]. In Chapter 7.3.B [4] presents a forcing with perfect trees whose countable support iteration of length ω_2 over a model of GCH gives a model of the third line of the table. Adding $\aleph_2$ random reals to a model of GCH gives a model satisfying the assumptions of the last line of the table (see the remark after Theorem 5.5 of [29]). However we do not know of a model in which the assumptions of the first line hold. This also leaves open the problem of the "mixed types", see Problem 7.4.

Table 1

Assumptions	$\mathrm{pc}(\aleph_1,\aleph_1)$	$\mathrm{pc}(\aleph_2,\aleph_2)$
$\mathrm{cov}(\mathcal{N}_{\aleph_1}) = \aleph_2$ and $\mathrm{non}(\mathcal{N}_{\aleph_0}) = \aleph_2$	yes	no
$\mathrm{cov}(\mathcal{N}_{\aleph_0}) = \aleph_1$ and $\mathrm{non}(\mathcal{N}_{\aleph_0}) = \aleph_1$	no	yes
$\mathrm{cov}(\mathcal{N}_{\aleph_0}) = \aleph_1$ and $\mathrm{non}(\mathcal{N}_{\aleph_0}) = \aleph_2$	no	no
$\mathrm{cov}(\mathcal{N}_{\aleph_1}) = \aleph_2$ and $\mathrm{non}(\mathcal{N}_{\aleph_0}) = \aleph_1$	yes	yes

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4. When $\kappa > \mathfrak{c}$

There are many cardinals above the continuum that are precalibres of every measure algebra. For instance, $\mathfrak{c}^+$ is such a cardinal and in fact it is a calibre of all ccc compact spaces (the latter statement follows using Remark 2.9). This a particular case of a result due to Argyros and Tsarpalias [2], Theorem 2.5 see also [9], Theorem 6.21). We formulate their theorem in the (less general) measure-theoretic terms.

Theorem 4.1. *Suppose κ is a cardinal such that both κ and $\mathrm{cf}(\kappa)$ are $\aleph_1$-inaccessible. Then κ is a precalibre of measure algebras.*

The proof we give of Theorem 4.1 is simpler than that of the original. First, we prove it for κ regular, using a well-known method. Then, taking advantage of the regularity of $\mathrm{cf}(\kappa)$, Theorem 4.1 follows from the more general Theorem 4.3 below.

Lemma 4.2. *If κ is a regular $\aleph_1$-inaccessible cardinal then κ is a precalibre of measure algebras.*

Proof. The proof uses Theorem 2.8 and Lemma 2.5. We consider positive measure subsets F_ξ of 2^κ ($\xi < \kappa$), so we can assume that every F_ξ is a closed set depending only on the coordinates in a countable set $J_\xi \subseteq \kappa$. Having a family $\{J_\xi\colon \xi < \kappa\}$ of countable sets and using the assumption on the $\aleph_1$-inaccessibility of κ, we can apply Theorem 1.12 to get a Δ-system of size κ contained in $\{J_\xi\colon \xi < \kappa\}$. Let us then assume that $X \subseteq \kappa$ is a set of size κ such that $J_\xi \cap J_\eta = J$ for some fixed set J whenever $\xi \neq \eta \in X$. Since J is countable there are only $\leqslant \mathfrak{c}$ many closed subsets of 2^J, so, using the fact that $\mathrm{cf}(\kappa) > \mathfrak{c}$, we can find a closed set $H \subseteq 2^J$ and a set $Y \subseteq X$ still of size κ such that $\pi_J[F_\xi] = H$ for every $\xi \in Y$.

It follows that $\bigcap_{\xi\in Y} F_\xi \neq \emptyset$. Indeed, to find an element in this intersection, take any $s \in H$ and choose $t_\xi \in F_\xi$ with $\pi_J(t_\xi) = s$. Define $t \in 2^\kappa$ so that it is s on J and t_ξ on $J_\xi \setminus J$, which is possible since the sets $J_\xi \setminus J$ for $\xi \in Y$ are pairwise disjoint. Then $t \in F_\xi$ for every $\xi \in Y$. □

As an example of the use of Lemma 4.2, combining it with the fourth line of the table at the end of Section 3, we obtain that if $\mathfrak{c} = \aleph_2$, $\mathrm{cov}(\mathcal{N}_{\aleph_1}) = \aleph_2$, $\mathrm{non}(\mathcal{N}_{\aleph_0}) = \aleph_1$ and $2^{\aleph_n} = \aleph_{n+1}$ for every $n \geqslant 2$ then $\mathrm{pc}(\aleph_n, \aleph_n)$ for every $n < \omega$.

The following Theorem 4.3 has been independently proved by Fremlin [24], see 524K, and it is likely to be known otherwise as well.

Theorem 4.3. *Suppose that κ is an $\aleph_1$-inaccessible cardinal and $\mathrm{cf}(\kappa)$ is a precalibre of measure algebras. Then so is κ.*

The converse of Theorem 4.3 is easily seen to be true even without the assumption of $\aleph_1$-inaccessibility of κ, see Observation 4.5.

Our proof of the next theorem, with minimal changes, gives another proof of Theorem 4.3. We state Theorem 4.4 in terms of measure precalibres in order to give an explicit partial answer to a question of Haydon (Problem 7.3).

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Theorem 4.4. *Suppose that κ is an $\aleph_1$-inaccessible cardinal of countable cofinality. Then κ is a measure precalibre.*

Proof. We shall apply Theorem 1.13, starting by an application of Lemma 1.14. Let κ be given as in the assumptions of the theorem and let $\langle\kappa_n\colon n<\omega\rangle$ be as provided by Lemma 1.14. Suppose that $\varepsilon>0$ and we are given a family $\{B_\alpha\colon \alpha<\kappa\}$ of subsets of 2^κ each of which has measure $>\varepsilon$. Without loss of generality each B_α is a closed set determined by a countable set of coordinates J_α.

By Theorem 1.13 there are sequences $\langle I_n\colon n<\omega\rangle$ and $\langle R_n\colon n<\omega\rangle$ such that

(i) $I_n\in[\kappa]^{\kappa_n}$ and the sets in $\langle I_n\colon n<\omega\rangle$ are pairwise disjoint,
(ii) if $\alpha\neq\beta\in I_n$ then $J_\alpha\cap J_\beta=R_n$ (hence each R_n is countable) and
(iii) if $n<m$ and $\alpha\in I_n,\beta\in I_m$ then $J_\alpha\cap J_\beta\subseteq R_m$.

For $n<\omega$ let $\pi_n\colon 2^\kappa\to 2^{R_n}$ be the natural projection. Fix for a moment $n<\omega$ and for $\alpha\in I_n$ let $F^n_\alpha=\pi_n[B_\alpha]$. Hence each F^n_α is a closed subset of 2^{R_n}. There are at most $\mathfrak{c}$ closed subsets of 2^{R_n}, as R_n is countable. Since $\mathfrak{c}=2^{\aleph_0}<\kappa_n=\mathrm{cf}(\kappa_n)$ by the choice of κ_n, and this holds for any n, we may in addition assume that

(iv) for each $n<\omega$ the set F^n_α $(\alpha\in I_n)$ is a fixed closed set F_n in 2^{R_n}.

As $\mu_\kappa(B_\alpha)>\varepsilon$ for every α we have in particular that $\mu_\kappa(\pi_n^{-1}[F_n])>\varepsilon$ for every $n<\omega$. Since $\aleph_0$ is a measure precalibre we may without loss of generality assume that

(v) the family $\{\pi_n^{-1}[F_n]\colon n<\omega\}$ is centred.

Let us again fix $n<\omega$ and consider any $m>n$. For any $j\in R_m\setminus R_n$ we have that (by (ii))

$$\bigl|\{\alpha\in I_n\colon j\in J_\alpha\}\bigr|\leqslant 1.$$

By throwing away from each I_n those α for which there is $m>n$ such that for some $j\in R_m\setminus R_n$ we have $j\in J_\alpha$ (so countably many such α) we may further assume

(vi) if $n<m$ and $\alpha\in I_n$ then $J_\alpha\cap R_m\subseteq R_n$.

We claim that (the many times trimmed by now) family $\{B_\alpha\colon \alpha\in\bigcup_{n<\omega}I_n\}$ is centred, which suffices to prove the theorem.

By (v) we may choose and fix $y\in\bigcap_{n<\omega}\pi_n^{-1}[F_n]$. We now try to define $x\in 2^\kappa$ so that $x\in B_\alpha$ for every $\alpha\in\bigcup_{n<\omega}I_n$. We put $x(\xi)=y(\xi)$ whenever $\xi\in\bigcup_{n<\omega}R_n$. Consider now $n<\omega$ and $\alpha\in I_n$. By our choice of y

$$\pi_n(y)\in F_n=\pi_n[B_\alpha],$$

so we can find $x_\alpha\in B_\alpha$ such that $\pi_n(y)=\pi_n(x_\alpha)$. Our intention is to let

$$x(\xi)=x_\alpha(\xi)\quad\text{for every }\xi\in J_\alpha \tag{$*$}$$

and to have $x(\xi)=0$ for all other ξ.

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If such an element x really exists then $x \in B_\alpha$ for every $\alpha \in \bigcup_{n<\omega} I_n$ (by $(*)$, as every B_α is determined by the coordinates in J_α) and the proof is complete. So we check the consistency of the above definition of x.

If $\xi \notin \bigcup_{n<\omega} R_n$ then by (iii) $\xi \in J_\alpha$ for at most one α and $x(\xi)$ is well defined. Consider now $\xi \in \bigcup_{n<\omega} R_n$ and let m be the first $m < \omega$ for which $\xi \in R_m$.

Suppose there is $n < m$ and $\alpha \in I_n$ such that $\xi \in J_\alpha$. Then by (vi), $\xi \in J_\alpha \cap R_m \subseteq R_n$, a contradiction. If there is $n > m$ and $\alpha \in I_n$ such that $\xi \in J_\alpha$ then by (vi) $\xi \in R_n$ so $x(\xi) = y(\xi) = x_\alpha(\xi)$.

In conclusion, $x(\xi)$ is well defined for every ξ. □

The reader has probably noticed that by starting with a family of sets of positive measure and replacing the fact that $\aleph_0$ is a measure precalibre by the assumption that $\mathrm{cf}(\kappa)$ is a precalibre of measures, the above proof gives the proof of Theorem 4.3. As a final note about singular cardinals we give the following simple observation.

Observation 4.5. Suppose that κ is a precalibre of measure algebras (measure precalibre). Then so is $\mathrm{cf}(\kappa)$.

Proof. The proof in both instances is along the same lines, so we concentrate on precalibres of measure algebras. Suppose for contradiction that the claim is not true and that κ demonstrates this. Clearly κ is singular, let $\theta = \mathrm{cf}(\kappa) < \kappa$ and let $\langle \kappa_\alpha \colon \alpha < \theta \rangle$ be an increasing sequence of regular cardinals converging to κ, with $\kappa_0 > \theta$. Let $\{F_\alpha \colon \alpha < \theta\}$ exemplify that θ is not a precalibre of measure algebras, so without loss of generality each F_α is a subset of 2^θ of positive measure and $\bigcap_{\alpha<\theta} F_\alpha = \emptyset$. We now form a family of κ many subsets of 2^κ by taking for each α κ_α many copies of the inverse projection of F_α in 2^κ. This family contradicts the assumption that κ is a precalibre of measure algebras. □

A small twist on the above proof gives a family of κ *distinct* sets that show that κ is not a precalibre of measure algebras, in case one wishes to insist in having distinct sets in the definition of precalibre. The distinction between these notions seems to be blurred in the literature and we have not found another instance but the above where the difference could matter.

Under suitably simple assumptions on the cardinal arithmetic (GCH) the results presented so far enable us to completely classify which cardinals are precalibres of measure algebras.

Corollary 4.6. *Under* GCH *exactly one of the following holds for any uncountable cardinal* κ:

(1) $\kappa = \tau^+$ *for some* τ *and then* $\mathrm{pc}(\kappa,\kappa) \iff \mathrm{cf}(\tau) > \aleph_0$; *or*
(2) κ *is a limit cardinal and* $\mathrm{cf}(\kappa) = \aleph_0$, *in which case* $\mathrm{mpc}(\kappa,\kappa)$ *and* $\neg\,\mathrm{pc}(\kappa,\kappa)$; *or*
(3) κ *is weakly inaccessible, in which case* $\mathrm{pc}(\kappa,\kappa)$; *or*
(4) κ *is a singular limit cardinal with* $\theta = \mathrm{cf}(\kappa) > \aleph_0$ *and then* $\mathrm{pc}(\theta,\theta) \iff \mathrm{pc}(\kappa,\kappa)$.

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Proof. (1) If $\mathrm{cf}(\tau) = \aleph_0$ then $\neg\,\mathrm{pc}(\kappa, \kappa)$ by Theorem 3.9. If $\mathrm{cf}(\tau) > \aleph_0$ then from GCH implies that κ is $\aleph_1$-inaccessible and we have $\mathrm{pc}(\kappa, \kappa)$ by Lemma 4.2.

(2) follows from Theorem 4.4 since under GCH every limit cardinal is $\aleph_1$-inaccessible.

(3) follows similarly from Lemma 4.2 and (4) from Theorem 4.3 and Observation 4.5. □

We now move away from GCH and present a measure-theoretic version of a theorem due to Shelah [40]. Shelah's original assumptions were

$$\beth_2 \leqslant \theta = \theta^{\aleph_0} < \mathrm{cf}(\kappa) \leqslant \kappa \leqslant 2^{\theta}$$

and conclusion that for every family of κ positive measure sets in 2^{κ}, there is an independent subfamily of size κ. Consequently $\mathrm{pc}(\kappa, \kappa)$. It turns out that the conclusion about the precalibres can be obtained under weaker assumptions, as we do in Theorem 4.7 below. It is in fact also possible to slightly weaken the assumptions of the original theorem, and in fact one can view Shelah's proof (or our rendition of it) as consisting of two parts: one in which one uses a part of the assumptions to get the conclusions about the precalibres, and the other where the rest of the assumptions are used to get the full independence. It seems also that the original proof is somewhat harder to read than what we make of it here, so we decided to present it as well, in Section 6. It will build on the proof we give below.

Theorem 4.7. *Suppose that θ and κ are cardinal numbers such that*

$$\theta = \theta^{\aleph_0} < \mathrm{cf}(\kappa) \leqslant \kappa \leqslant 2^{\theta}.$$

Then κ is a precalibre of measure algebras.

Note 4.8. Clearly, the assumptions of Theorem 4.7 imply that $\mathfrak{c} = 2^{\aleph_0} < \mathrm{cf}(\kappa)$.

It might also be worthwhile to compare the assumptions of this theorem with those of Lemma 4.2. If $\theta = \theta^{\aleph_0}$ then for any $n < \omega$ we have $(\theta^{+n})^{\aleph_0} = \theta^{+n}$, so if θ and κ of Theorem 4.7 are close to each other in the sense that $\kappa = \theta^{+n}$ for some n, then the assumption $\theta = \theta^{\aleph_0}$ implies that $\mathrm{cf}(\kappa) = \kappa$ is $\aleph_1$-inaccessible, hence the conclusion already follows by Lemma 4.2. However, moving κ away from θ it is perfectly possible that for some $\lambda \in (\theta, \kappa)$ we have, for example, that $\lambda^{\aleph_0} \geqslant \kappa$. By König's lemma this will happen any time that κ is the successor of a singular cardinal of countable cofinality. As an example, we could have

$$\theta = 2^{\aleph_0} = \aleph_1, \qquad 2^{\aleph_1} = \aleph_{\omega}^{++}, \qquad \kappa = \aleph_{\omega}^{+},$$

which is the situation obtained when $\aleph_{\omega}^{++}$ Cohen subsets are added to $\aleph_1$ over a model of GCH. In this situation Lemma 4.2 and Theorem 4.3 do not apply but Theorem 4.7 does.

We also observe that many, even regular, cardinals might not satisfy either the assumptions of Lemma 4.2 or the assumptions of Theorem 4.7. For instance, successors of singulars of countable cofinality in a model of GCH will fail both sets of assumptions, as is to be expected from Theorem 3.9. The assumptions of Theorem 3.9 may also fail. Magidor [32] proved starting from the existence of an infinite sequence of supercompact cardinals that for every $0 < n < \omega$ there is a model of ZFC in which $\aleph_{\omega}$ is a strong limit cardinal but $2^{\aleph_{\omega}} = \aleph_{\omega+n}$, hence for $n \geqslant 3$, $\kappa = \aleph_{\omega}^{+}$ in such a model does not satisfy the

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assumptions of any of Lemma 4.2, Theorem 4.7 or Theorem 3.9. We do not know if κ is a precalibre of measure algebras in such a model.

We note that there are many later and more refined consistency results about the failure of the singular cardinal hypothesis, of which Magidor's theorem is the first instance. One may consult the introduction to Shelah's book [39] for a survey. See also the comments about Problem 7.1 below.

Proof. We consider a family $\{B_\alpha\colon \alpha<\kappa\}$ of subsets of $\{0,1\}^\kappa$ with positive measure. We can assume that every B_α is a closed set determined by the coordinates in a countable set $J_\alpha\subseteq\kappa$. Further assume that every J_α is infinite and has a 1–1 enumeration $J_\alpha=\{i(\alpha,n)\colon n<\omega\}$, as the situation of κ many among the J_αs being finite can be handled in a much easier manner using a Δ-system argument.

Since $\theta=\theta^{\aleph_0}$ we can apply the Engelking–Karłowicz lemma to find a family $\{f_\gamma\colon \gamma<2^\theta\}$ of functions from θ into θ, with the property that for every sequence $\langle\gamma_n\colon n<\omega\rangle\subseteq 2^\theta$ and $\langle\zeta_n\colon n<\omega\rangle\subseteq\theta$ there is $\zeta<\theta$ such that $f_{\gamma_n}(\zeta)=\zeta_n$ for every n.

Using the above functions we define for $\zeta<\theta$

$$X_\zeta=\bigl\{\alpha<\kappa\colon f_{i(\alpha,n)}(\zeta)=n \text{ for all } n\bigr\}.$$

We have $\bigcup_{\zeta<\theta}X_\zeta=\kappa$ by the choice of f_γs. Since $\theta<\mathrm{cf}(\kappa)$ there is $\zeta<\kappa$ such that $|X_\zeta|=\kappa$.

For every $\alpha<\kappa$ we define a mapping π_α, where

$$\pi_\alpha:\{0,1\}^\kappa\to\{0,1\}^\omega,\qquad \pi_\alpha(x)(n)=x\bigl(i(\alpha,n)\bigr)\quad\text{for every } n.$$

Then $F_\alpha=\pi_\alpha[B_\alpha]$ is a closed subset of $\{0,1\}^\omega$. Using $\mathfrak{c}<\mathrm{cf}(\kappa)$ we can as well assume that $F_\alpha=F$ for every $\alpha<\kappa$. Thus we have $\pi_\alpha[B_\alpha]=F$; note also that $\pi_\alpha^{-1}[F]=B_\alpha$ for every $\alpha<\kappa$. Namely if $\pi_\alpha(x)\in F$ then $\pi_\alpha(x)=\pi_\alpha(y)$ for some $y\in B_\alpha$; as B_α is determined by the coordinates in J_α this implies that $x\in B_\alpha$.

We claim that $\bigcap_{\alpha\in X_\zeta}B_\alpha\neq\emptyset$. Indeed, take any $t\in F$ and attempt to define $x\in\{0,1\}^\kappa$ so that $x(i(\alpha,n))=t(n)$ for every n and every $\alpha\in X_\zeta$ (and $x(\xi)=0$ for other ξ). Note that if $\alpha,\beta\in X_\zeta$ and $i(\alpha,n)=i(\beta,k)$ then $n=k$, so the definition is consistent and hence we can fix such an x. For every $\alpha\in X_\zeta$ we have $\pi_\alpha(x)=t\in F$ so $x\in\pi_\alpha^{-1}[F]=B_\alpha$ and we are done. □

5. Some applications

We now mention some applications of precalibres. Although the applications are mostly in topological measure theory, we start by a purely combinatorial notion isolated by Fremlin.

A family $\mathcal{D}$ of finite subsets of κ is said to be ε-*dense open* for $\varepsilon\in(0,1)$ if $\mathcal{D}$ is closed under subsets and for any finite $F\subseteq\kappa$ there is $F'\subseteq F$ with $F'\in\mathcal{D}$ and $|F'|\geqslant\varepsilon|F|$.

We say that κ is a λ-*Fremlin cardinal* iff whenever $\mathcal{D}$ is a 1/2-dense open family of finite subsets of κ, there is $A\in[\kappa]^\lambda$ such that all finite subsets of A are in $\mathcal{D}$. By a result of Fremlin [20], the definition of κ being a λ-Fremlin cardinal does not change if 1/2 in the above is replaced by any $\varepsilon\in(0,1)$.

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There is a connection with precalibres which can be explained as follows, see [20]; other properties of Fremlin cardinals are discussed in Apter and Džamonja [1]; see also Džamonja and Plebanek [14].

Theorem 5.1. *If* $\mathrm{cf}(\kappa) > \aleph_0$ *and* κ *is* λ*-Fremlin then* $\mathrm{pc}(\kappa, \lambda)$.

Proof. Suppose that the family $\{F_\alpha\colon \alpha < \kappa\}$ witnesses that $\mathrm{pc}(\kappa, \lambda)$ fails, where $\mathrm{cf}(\kappa) > \aleph_0$. We can assume that there is $\varepsilon > 0$ such that all F_α have measure at least ε. Let $\mathcal{D}$ be given by

$$\mathcal{D} \stackrel{\mathrm{def}}{=} \left\{ d \text{ finite} \subseteq \kappa\colon \bigcap_{\alpha \in d} F_\alpha \neq \emptyset \right\}.$$

Then $\mathcal{D}$ is ε-dense open. Indeed, for any finite $a \subseteq \kappa$ we have

$$\left\| \sum_{\xi \in a} \chi_{F_\xi} \right\| \geqslant \int \sum_{\xi \in a} \chi_{F_\xi} \, \mathrm{d}\mu_\kappa \geqslant \varepsilon |a|,$$

which implies that there is $d \subseteq a$, $|d| \geqslant \varepsilon |a|$ such that $\bigcap_{\xi \in d} F_\xi \neq \emptyset$ (here $\|\cdot\|$ denotes the supremum norm). By the choice of $\mathcal{D}$ it follows that κ is not λ-Fremlin. $\square$

For any set Γ, the Corson space $\Sigma(\mathbb{R}^\Gamma)$ is defined as the set

$$\Sigma\left(\mathbb{R}^\Gamma\right) = \left\{ x \in \mathbb{R}^\Gamma\colon \left| \left\{ \gamma \in \Gamma\colon x(\gamma) \neq 0 \right\} \right| \leqslant \aleph_0 \right\}$$

endowed with the subspace topology. A topological space K is called a *Corson compactum* if K is homeomorphic to a compact subset of $\Sigma(\mathbb{R}^\kappa)$ for some κ. The following Theorem 5.2 was proved by Kunen and van Mill [31] in the special case $\kappa = \aleph_1$; the result shows that precalibres of measure algebras are closely related to the question of what the Maharam types of measures defined on Corson compacta are. The proof of (i) $\Rightarrow$ (ii) is standard and well known; the argument for the reverse implication is taken from [36].

Theorem 5.2. *The following are equivalent for any cardinal* κ:

(i) *there is a Corson compact space* K *carrying a Radon measure of Maharam type* κ;
(ii) $\mathrm{pc}(\kappa, \aleph_1)$ *does not hold.*

Proof. (i) $\Rightarrow$ (ii). Let μ be a probability Radon measure of type κ on a Corson compact space K. We can assume that K is a subset of $\Sigma(\mathbb{R}^\Gamma)$ for some Γ. For $\gamma \in \Gamma$ let $C_\gamma = \{x \in K\colon x(\gamma) \neq 0\}$.

Claim 5.3. *Letting* $G = \{\gamma \in \Gamma\colon \mu(C_\gamma) > 0\}$ *we have* $|G| \geqslant \kappa$.

Proof. Let

$$K_G = \left\{ x \in K\colon x(\gamma) = 0 \text{ for all } \gamma \in \Gamma \setminus G \right\}.$$

Note that $\gamma \in \Gamma \setminus G$ means that $\mu(\{x \in K\colon x(\gamma) = 0\}) = 1$ and so K_G is an intersection of a family of closed sets of full measure and therefore $\mu(K_G) = 1$, since the measure is

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Radon. Since μ is of type κ, the topological weight of K_G is at least κ, so $|G| \geqslant \kappa$. Here we use the following simple observation: If K is a topological space of weight κ then every Radon measure on K has Maharam type at most κ. $\square$

Take any $\gamma \in G$. We have $\mu(C_\gamma) > 0$ and

$$C_\gamma = \bigcup_{n \geqslant 1} \bigl\{x \in K\colon \bigl|x(\gamma)\bigr| \geqslant 1/n\bigr\},$$

so there is $n_\gamma \geqslant 1$ such that letting $D_\gamma = \{x \in K\colon |x(\gamma)| \geqslant 1/n_\gamma\}$ we have $\mu(D_\gamma) > 0$.

Now the family $\{D_\gamma\colon \gamma \in G\}$ witnesses that μ does not have calibre $(\kappa, \aleph_1)$, which suffices by Lemma 2.5.

(ii) $\Rightarrow$ (i). We shall again use Lemma 2.5, as well as Theorem 2.8. Hence our assumptions allow us to choose a family $\{C_\xi\colon \xi < \kappa\}$ of compact positive measure subsets of $\{0,1\}^\kappa$ witnessing that $(\kappa, \aleph_1)$ is not a calibre of the product measure $\mu = \mu_\kappa$.

Using the fact that μ has Maharam type κ on every set of positive measure, we may use induction on $\xi < \kappa$ to find compact sets F_ξ such that for every ξ we have $F_\xi \subseteq C_\xi$ and

$$\inf\bigl\{\mu(A \mathbin{\triangle} F_\xi)\colon A \in \mathcal{A}_\xi\bigr\} > 0, \tag{$*$}$$

where $\mathcal{A}_\xi$ is the Boolean algebra of sets generated by the family $\{F_\alpha\colon \alpha < \xi\}$.

We take the algebra $\mathcal{A} = \bigcup_{\xi<\kappa} A_\xi$ and show that its Stone space K is the required space. The measure μ restricted to $\mathcal{A}$ uniquely defines a Radon measure ν on K which is generated by letting for every $A \in \mathcal{A}$, $\nu(\widehat{A}) = \mu(A)$, where $\widehat{A} \subseteq K$ is the clopen subset of K induced by A. Then it follows from $(*)$ above that ν is of type at least κ. The fact that K is Corson compact and ν has type at most (hence exactly) κ follows from the fact that the mapping

$$g : K \to \{0,1\}^\kappa, \quad g(p) = \bigl(\chi_{\widehat{F_\xi}}(p)\bigr)_{\xi<\kappa},$$

is 1–1, hence by its definition a homeomorphic embedding, while $g[K] \subseteq \Sigma(\mathbb{R}^\kappa)$ since there is no uncountable centred subfamily of $\{F_\xi\colon \xi < \kappa\}$. $\square$

Corollary 5.4. *It is consistent that $\mathfrak{c} > \aleph_1$ and there is a Corson compact space carrying a Radon measure of type $\mathfrak{c}$.*

Proof. Apply Corollary 3.11 and Theorem 5.2. $\square$

Note that by Theorem 5.2, since $\mathrm{pc}(\mathfrak{c}^+, \mathfrak{c}^+)$ by Lemma 4.2, every Radon measure on a Corson compactum is of type at most $\mathfrak{c}$. We might generalise Theorem 5.2 to the case of an arbitrary pair (κ, λ) (where $\lambda \leqslant \kappa$), replacing Corson compacta by λ-Corson compacta.

Let us also mention another interesting and simple construction of a Corson compact space resulting from a family that witnesses that $\mathrm{pc}(\kappa, \aleph_1)$ does not hold. Let $(\mathfrak{A}, \mu)$ be a measure algebra and suppose that $\{a_\xi\colon \xi < \kappa\} \subseteq \mathfrak{A}^+$ is a family without an uncountable centred subfamily. Then one obtains a Corson compact space by letting

$$K = \bigl\{x \in \{0,1\}^\kappa\colon \bigl\{a_\xi\colon x(\xi) = 1\bigr\} \text{ is centred}\bigr\}.$$

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See e.g., Plebanek [35] for some applications of this construction, where it is shown, for instance, that such a space K admits a strictly positive measure. Earlier such a construction was used by Marciszewski [33] to prove the following.

Theorem 5.5. *The following are equivalent.*

(i) *There is compact* $K \subseteq \Sigma(\mathbb{R}^{\omega_1})$ *such that* $\overline{\mathrm{conv}(K)} \nsubseteq \Sigma(\mathbb{R}^{\omega_1})$.
(ii) $\aleph_1$ *is not a precalibre of measure algebras.*

The following theorem summarizes some of the remarks above and results from Kunen and van Mill [31], Marciszewski [33] and Plebanek [35].

Theorem 5.6. $\mathrm{pc}(\aleph_1, \aleph_1)$ *is equivalent to each of the following*:

(i) *Every Radon measure on a Corson compact space has a separable support.*
(ii) *Every Radon measure on a Corson compactum is of countable type.*
(iii) $\overline{\mathrm{conv}(K)} \subseteq \Sigma(\mathbb{R}^{\kappa})$ *for every* κ *and every compact* $K \subseteq \Sigma(\mathbb{R}^{\kappa})$.
(iv) *Every Radon measure on a first countable space is of countable type.*

The class of compact spaces on which every measure is of countable type was also investigated by Džamonja and Kunen [12,13].

Calibres are also crucial in understanding the so-called Haydon problem about the equivalence between the existence of continuous surjections onto $[0,1]^{\kappa}$ and homogeneous measures of type κ. The question originated in R. Haydon's results on the isomorphism structure of Banach spaces, see [25,26]; cf. Fremlin [18,19] and Plebanek [36,38]. We recall here just one result along these lines, see [38] for details.

Theorem 5.7. *The following are equivalent for any* $\kappa \geqslant \aleph_2$:

(i) *there is a continuous surjection from* K *onto* $[0,1]^{\kappa}$ *iff* K *carries a homogeneous Radon measure of type* κ;
(ii) κ *is a measure precalibre.*

Finally, let us mention that calibre-like properties of measure algebras are even relevant to a question on Pettis integrability of Banach-valued functions with respect to Radon measures, see, e.g., Plebanek [37].

6. Shelah's theorem on independent families

A theorem we were inspired by when working on this paper is Shelah's theorem on independent sets in measure algebras from [40], as we explained in the introduction to Theorem 4.7. As we mentioned before, it also turned out that one can use the methods developed here to give a somewhat simpler proof and slightly weaken the assumptions of the original theorem of Shelah. The first part of the argument necessary to do this is almost

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the same as the one already familiar from the proof of Theorem 4.7. We decided to give also the rest of the argument for the sake of completeness, and this is what this section is devoted to.

Definition 6.1. A subfamily of a Boolean algebra $\mathfrak{A}$ is said to be independent iff every nontrivial finite Boolean combination of its elements is nonzero.

Recall that by the Balcar–Franék theorem [3] every complete Boolean algebra $\mathfrak{A}$ contains an independent family of size $|\mathfrak{A}|$. If $\mathfrak{A}$ is a measure algebra this fact follows easily from the Maharam theorem. The result discussed below says that for large κ, in fact every family of κ many distinct elements of some measure algebra contains an independent subfamily of full size. Note that every nonatomic measure algebra contains a linearly ordered subfamily of cardinality $\mathfrak{c}$ so it is not always possible to choose an independent subfamily among $\mathfrak{c}$ many elements of a measure algebra. Shelah's original assumptions for Fact 2.1 from [40] were

$$\beth_2 \leqslant \theta = \theta^{\aleph_0} < \mathrm{cf}(\kappa) \leqslant \kappa \leqslant 2^{\theta}$$

and conclusion that for every family of κ many distinct elements in $\mathfrak{A}_\kappa$, there is an independent family of size κ. We shall see that the assumptions may be somewhat relaxed.

Prior to the main theorem we enclose the following technical lemma from [40]. For every $Y \subseteq \kappa$ we write $\mathfrak{A}_\kappa[Y]$ for the family of all $B^{\cdot} \in \mathfrak{A}_\kappa$ for which $B \in \Sigma_\kappa$ is determined by the coordinates in Y.

Lemma 6.2. *Suppose that $\{a_\alpha\colon \alpha < \kappa\}$ is a family of distinct elements of $\mathfrak{A}_\kappa$, where $a_\alpha \in \mathfrak{A}[J_\alpha]$, with $J_\alpha \subseteq \kappa$ countable for every α. Then for every $Y \subseteq \lambda$, denoting*

$$\mathrm{ind}(Y) \stackrel{\mathrm{def}}{=} \left\{\alpha \in Y\colon \neg(\exists m < \omega)(\exists \beta_0, \ldots, \beta_{m-1} \in Y \cap \alpha) a_\alpha \in \mathfrak{A}_\kappa\left[\bigcup_{k<m} J_{\beta_k}\right]\right\},$$

we have $|\mathrm{ind}(Y)| + \mathfrak{c} \geqslant |Y|$.

Proof. The lemma follows easily from the fact that $|\mathfrak{A}_\kappa[J]| \leqslant \mathfrak{c}$ whenever $J \subseteq \kappa$ is countable. □

Theorem 6.3. *Suppose that θ and κ are cardinals satisfying*

(i) $\theta = \theta^{\aleph_0} < \mathrm{cf}(\kappa) \leqslant \kappa \leqslant 2^{\theta}$;
(ii) $\beth_2 < \mathrm{cf}(\kappa)$.

Then for every family of κ many distinct elements of some measure algebra there is an independent subfamily of size κ.

Example 6.4. An example of a situation covered by Theorem 6.3 but not the original Shelah's theorem is when $2^{\aleph_0} = \aleph_2$, $\theta = 2^{\aleph_1} = \aleph_{57}$, $\beth_2 = 2^{\aleph_2} = \aleph_{99}$, while $2^{\theta} = \aleph_{\omega_1+1}$. Then any $\kappa \leqslant \aleph_{\omega_1+1}$ with cofinality $\aleph_{58}$ will satisfy the assumptions of Theorem 6.3 but not of the original theorem.

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Proof. (1) By Maharam's theorem we can suppose that we are given a family $\{a_\alpha\colon \alpha<\kappa\}$ of distinct elements of $\mathfrak{A}_\kappa$ and we need to find an independent subfamily of size κ. We shall work in the space 2^κ rather than in the algebra $\mathfrak{A}_\kappa$ itself. Let us fix a lifting $\varphi:\mathfrak{A}_\kappa\to\Sigma_\kappa$ and put $B_\alpha=\varphi(a_\alpha)$ for every $\alpha<\kappa$. Next we choose measurable sets B^0_α and B^1_α so that

$$B^0_\alpha\subseteq B_\alpha,\qquad B^1_\alpha\subseteq 2^\kappa\setminus B_\alpha,$$
$$B^{0\bullet}_\alpha=a_\alpha,\qquad B^{1\bullet}_\alpha=1-a_\alpha,$$
$$B^0_\alpha \text{ and } B^1_\alpha \text{ depend only on the coordinates in a countable set } J_\alpha\subseteq\kappa.$$

To choose B^0_α we apply Fact 1.5(5) to B_α and we similarly choose B^1_α.

(2) For the rest of the proof we consider disjoint pairs (B^0_α,B^1_α). We shall prove that there is $X\in[\kappa]^\kappa$ such that the pairs (B^0_α,B^1_α) for $\alpha\in X$ are independent, i.e.,

$$\bigcap_{\alpha\in I}B^{\varepsilon(\alpha)}_\alpha\neq\emptyset\quad\text{for every finite } I\subseteq X \text{ and every}\quad \varepsilon:I\to\{0,1\}.$$

This will prove the theorem since $\bigcap_{\alpha\in I}B^{\varepsilon(\alpha)}_\alpha\neq\emptyset$ implies that

$$\varphi\Bigl(\bigwedge_{\alpha\in I}a^{\varepsilon(\alpha)}_\alpha\Bigr)=\bigwedge_{\alpha\in I}\varphi\bigl(a^{\varepsilon(\alpha)}_\alpha\bigr)\neq\emptyset,$$

hence $\bigwedge_{\alpha\in I}a^{\varepsilon(\alpha)}_\alpha\neq 0$, and therefore the family $\{a_\alpha\colon\alpha\in X\}\subseteq\mathfrak{A}_\kappa$ is independent.

(3) Using Lemma 6.2 we can assume that for every $\alpha<\kappa$ and $\beta_0,\dots,\beta_{k-1}<\alpha$ we have $a_\alpha\neq B^\bullet$ whenever B depends on the coordinates in $\bigcup_{k<m}J_{\beta_k}$.

(4) Now we use the same argument as in the proof of Theorem 4.7, using the assumption that $\theta=\theta^{\aleph_0}$ to obtain X_ζ as there. Hence thanks to the assumption $\theta<\mathrm{cf}(\kappa)$ we can now pass to a subfamily of the original family if necessary and assume that $X_\zeta=\kappa$. This implies the following:

$$\text{if } i(\alpha,n)=i(\beta,k)\quad\text{then } n=k. \tag{$*$}$$

(5) Again, for every $\alpha<\kappa$ we define a mapping π_α, where

$$\pi_\alpha:\{0,1\}^\kappa\to\{0,1\}^\omega,\qquad \pi_\alpha(x)(n)=x\bigl(i(\alpha,n)\bigr)\quad\text{for every } n.$$

Then $F^0_\alpha=\pi_\alpha[B^0_\alpha]$ and $F^1_\alpha=\pi_\alpha[B^1_\alpha]$ are Borel subsets of $\{0,1\}^\omega$. Using $\mathfrak{c}<\mathrm{cf}(\kappa)$ we can as well assume that $F^0_\alpha=F^0$ and $F^1_\alpha=F^1$ for fixed F^0,F^1 and every $\alpha<\kappa$.

(6) We now come to the point of the argument where we shall need to use the assumption $\beth_2<\mathrm{cf}(\kappa)$. For each $\alpha<\kappa$ we define an ideal $\mathcal{N}_\alpha$ on ω. It is the ideal generated by the sets

$$Z^\alpha_\beta\stackrel{\text{def}}{=}\bigl\{n<\omega\colon i(\beta,n)=i(\alpha,n)\bigr\}\quad\text{for }\beta<\alpha.$$

By (3) the ideal $\mathcal{N}_\alpha$ is proper. Namely suppose that for some $\beta_0,\dots,\beta_{m-1}<\alpha$ we have $\bigcup_{l<m}Z^\alpha_{\beta_l}=\omega$. Then a_α belongs to $\mathfrak{B}[\{i(\beta_l,n)\colon l<m,\ n<\omega\}]$, contradicting (3).

As the number of possible ideals on ω is at most $\beth_2$, by our assumption $\mathrm{cf}(\kappa)>\beth_2$ for the rest of the proof we can fix a set $X\subseteq\kappa$ of size κ, such that for every $\alpha\in X$ $\mathcal{N}_\alpha=\mathcal{N}$, where $\mathcal{N}$ is a fixed proper ideal on ω.

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(7) We can at last prove that our family of pairs (B^0_α, B^1_α), $\alpha \in X$, is independent as defined in (2) above. So let us fix a finite set $I \subseteq X$ and a function $\varepsilon : I \to \{0,1\}$ and try to define $x \in 2^\kappa$ such that $x \in \bigcap_{\alpha\in I} B_\alpha^{\varepsilon(\alpha)}$. Let

$$N = \{n < \omega\colon\ i(\alpha,n) = i(\beta,n) \text{ for some } \alpha,\beta \in I, \alpha \neq \beta\},$$

$$R_\alpha = \{i(\alpha,n)\colon\ n \in N\}, \qquad R = \bigcup_{\alpha\in s} R_\alpha.$$

Let us denote by $\pi_N : 2^\omega \to 2^N$ the usual projection. For the sets $F^0, F^1 \subseteq 2^\omega$ defined in (5) we put

$$F^0_+ = \pi_N^{-1}\pi_N[F^0], \qquad F^1_+ = \pi_N^{-1}\pi_N[F^1].$$

Claim 6.5. $F^0_+ \cap F^1_+ \neq \emptyset$.

Proof. Indeed, otherwise taking $\alpha = \max(I)$ and $C = \pi_\alpha^{-1}[F^0_+]$ we would have $C^\bullet = a_\alpha$. Hence C is determined by the coordinates in R_α. But N is in the ideal $\mathcal{N}$ fixed in (6) and we have $\mathcal{N} = \mathcal{N}_\alpha$, so there are $\beta_0, \ldots, \beta_{k-1} < \alpha$ such that $N \subseteq \bigcup_{i<k-1} Z^\alpha_{\beta_i}$. Then $R_\alpha \subseteq \bigcup_{i<k-1} J_{\beta_i}$, and we get a contradiction with (3). □

Fix an element $t \in F^0_+ \cap F^1_+$; we define a desired element $x : \kappa \to \{0,1\}$ as follows:

- on R we let $x(i(\alpha,n)) = t(n)$ whenever $\alpha \in I$ and $i(\alpha,n) \in R$. Note that by $(*)$ of (4), this definition is consistent.
- Take any $\alpha \in I$ with $\varepsilon(\alpha) = 0$ (so that we want x in B^0_α). Since $t \in F^0_+$, there is $s \in F^0$ such that $s_{|N} = t_{|N}$. We can put $x(i(\alpha,n)) = s(n)$ for $n \notin N$. Then $x(i(\alpha,n)) = s(n)$ for every $n < \omega$, so $x \in \pi_\alpha^{-1}[F^0] = B^0_\alpha$, as required.
- For $\alpha \in I$ with $\varepsilon(\alpha) = 1$ we proceed analogously.

Thus x is defined so that $x \in \bigcap_{\alpha\in I} B_\alpha^{\varepsilon(\alpha)}$, and this finishes the proof. □

Analysing the argument above we can see that the requirement (ii) of Theorem 6.3 was applied only once, in (6) to make Claim 6.5 work. This enables us to derive the following conclusion (which is, in a sense, motivated by Claim 2.4(2) of [40]). Say that a family $\{a_\alpha\colon \alpha < \kappa\}$ in a measure algebra $(\mathfrak{A}, \mu)$ is *separated* if there is a constant $\delta > 0$ such that $\mu(a_\alpha \triangle a_\beta) \geqslant \delta$ whenever $\alpha \neq \beta$.

Corollary 6.6. *Suppose that θ and κ are cardinals satisfying $\theta = \theta^{\aleph_0} < \mathrm{cf}(\kappa) \leqslant \kappa \leqslant 2^\theta$ and let $\mathfrak{F}$ be a family of κ many distinct elements of some measure algebra. If either*

(i) *κ is $\aleph_1$-inaccessible; or*
(ii) *$\mathfrak{F}$ is separated;*

then $\mathfrak{F}$ contains an independent subfamily of size κ.

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Proof. We again deal with measurable sets in $\{0,1\}^\kappa$. Recall first that for a measurable set $B \subseteq \{0,1\}^\kappa$ there may be no minimal set $J \subseteq \kappa$ of indices with the property that B depends only on the coordinates in J. However, there is a (countable) set J^* such that whenever $C^\bullet = B^\bullet$ and C depends only on the coordinates in I then $J^* \subseteq I$, see Fremlin [21].

Now we proceed as in the proof of Theorem 6.3 with the following changes. First we shall note that if either (i) or (ii) hold then we can replace (3) of the proof of Theorem 6.3 by the requirement

(3)′ $a_\alpha \neq B^\bullet$ whenever B depends on the coordinates in $\bigcup_{\beta<\alpha} J_\beta$.

Indeed, for the set $Y = \bigcup_{\beta<\alpha} J_\beta$ we have $|Y| < \kappa$, so if $\aleph_1 \ll \kappa$ then $\mathfrak{A}_\kappa[Y]$ has only $|Y|^{\aleph_0} < \kappa$ elements. Similarly, if (ii) holds then $\mathfrak{A}_\kappa[Y]$ contains at most $|Y|$ elements a_ξ.

Next we replace (6) from the proof of Theorem 6.3 by the following. For every $\alpha < \kappa$ let J^*_α be the minimal set of coordinates for a_α, in the sense explained above. By (3)′ we have for every $\alpha < \kappa$

$$J^*_\alpha \nsubseteq \bigcup_{\beta<\alpha} J_\beta.$$

Now passing to a suitable subfamily we can assume that there is a natural number n^* such that for every $\alpha < \kappa$ we have

$$i(\alpha, n^*) \in J^*_\alpha \setminus \bigcup_{\beta<\alpha} J_\beta.$$

Having this property we can verify Claim 6.5 in the same way. □

7. Open problems

We list some open problems and partial solutions.

Problem 7.1 (*Fremlin*). Is it consistent that every regular κ is a precalibre of measure algebras?

Theorem 3.9 shows that if this is consistent then GCH fails at every strong limit of cofinality $\aleph_0$. (Recall that $\beth_\omega$ is such a strong limit). A positive answer to Problem 7.1 also implies the existence of $0^\sharp$. Jensen showed (see [11]) that if $0^\sharp$ does not exist then the singular cardinal hypothesis (SCH) is true, that is, for any singular cardinal κ the value of 2^κ is the least cardinal $\lambda \geqslant 2^{<\kappa}$ with $\mathrm{cf}(\lambda) > \kappa$. In particular, $2^\kappa = \kappa^+$ for every singular strong limit cardinal and so we obtain

Remark 7.2. If $0^\sharp$ does not exist then there is a regular cardinal which fails to be a precalibre of measure algebra.

Assuming various large cardinal hypotheses, many models make SCH false. One that seems particularly relevant given Theorem 4.1 and Theorem 3.9 was constructed by Cummings in [10], where (assuming the existence of a $\mathcal{P}_3\kappa$-hypermeasurable cardinal) a model is constructed in which $2^\kappa = \kappa^+$ if κ is a successor and $2^\kappa = \kappa^{++}$ if κ is a limit

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cardinal. One may consult [10] for further references. Calling Cummings's model V we may perform in V a forcing to collapse $\aleph_1$ followed by a forcing to add $\aleph_2$ random reals to obtain $V[G]$ in which $\mathfrak{c} = \aleph_2 = \mathrm{cov}(\mathcal{N}_{\aleph_1})$ while $\mathrm{non}(\mathcal{N}_{\aleph_0}) = \aleph_1$. Then by the table at the end of Section 3 the only regular cardinals that may fail to be precalibres of measure algebras in $V[G]$ are successors of singulars of countable cofinality, and Theorem 3.9 does not rule out that these cardinals are precalibres as well.

Theorem 4.4 gives a partial solution of the following

Problem 7.3 (*Haydon*). Let $\kappa = \sup_{n<\omega} \kappa_n$, where every κ_n is a measure precalibre of measure algebras. Does κ have the same property?

The table at the end of Section 3 suggests the following problem:

Problem 7.4. Is it consistent that $\mathrm{pc}(\aleph_2, \aleph_1)$ but $\neg\,\mathrm{pc}(\aleph_2, \aleph_2)$ and $\neg\,\mathrm{pc}(\aleph_1, \aleph_1)$?

Proofs of Lemma 4.2 and Theorem 4.7 show that there is a combinatorial property that suffices for a cardinal κ to be a precalibre of measure algebras, namely that for every family $\{I_\xi\colon\ \xi < \kappa\}$ of countably infinite subsets of κ there is $X \in [\kappa]^\kappa$ and enumerations $I_\xi = \{i(\xi, n)\colon\ n < \omega\}$ for $\xi \in X$ with the property that $i(\xi, n) = i(\eta, k)$ implies $n = k$. It might be interesting to see if this combinatorial property isolates a useful class of cardinals, and understanding how to force this property might be useful for Problem 7.1.

Uncited references

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