The Coupled Motion of Containers and their Sloshing Liquid Loads

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April 16, 2005

Dissertation submitted in partial fulfillment of the requirements for an MMath degree at School of Mathematics, University of East Anglia, Norwich.

Abstract

A mathematical approach is undertaken to determine the possible behaviour of fluid in containers of fairly simple geometry. The containers are initially assumed to be fixed, with the only motion under investigation being the oscillation of the fluid in the containers concerned. After this work has been carried out a simple linearised theory is used to obtain the coupled motion of the fluid and container, the container assumed to be on a smooth surface with no further constraints. Finally, in addition to the containers being allowed to move on a smooth surface, similar methods are used to approximate the same motion with the added constraint of a spring-like apparatus providing a restoring force to the system. All the motions described above are assumed time periodic and the valid frequencies of oscillation are determined in each case.

Contents

1 Introduction

From common experience, the motion of fluids is known to affect most aspects of life to some degree. For example, the motion of coffee in a cup sat on the table of a moving train, given the right impetus, can turn from a refreshing beverage to a minor annoyance. On a larger scale, the movement of waves due to the tides, storms and tsunamis have a more dire consequence if not properly addressed, but the possibility of the power of the tides being harnessed to generate electricity is of interest. The above examples are a few of many situations which display the importance of fluid dynamics as a whole.

More relevant to this project is the motion of liquids and the containers that they occupy considered as a coupled system where the liquid has a free surface within the container, if the container is closed. Examples of this sort of behaviour are the motion of liquid being transported, either terrestrially over water, in the air or through space. These coupled motions can significantly effect the motion of the liquid carrier, which in turn may detrimentally affect the motion of the carrier. A relatively small scale example of this is the action of fuel in the fuel tank of a motorcycle when using a roundabout, where a motion is set up in the liquid fuel by the varying inclination of the fuel tank to the vertical. Another such example, but on a larger scale, is that of an tanker carrying large quantities of liquid on a rough sea due to a storm. Both of these scenarios show that the motion of the fuel-container systems can become problematic if not properly managed and that the changes in the fluid behaviour due to the motion of the carrier can induce a motion of the carrier, which in turn may give rise to another motion on the fluid and so on.

There has been a lot of research into the motion of fluids which occupy fixed containers. Examples include Paterson [1] who studies the motion of fluids in fixed rectangular and upright cylindrical containers, where potential theory is used to find frequencies for which periodic motions may occur. Lamb [2] also includes many examples of oscillatory motions of liquids in fixed containers, including containers in the shape of triangular prisms and quotes results for containers of non constant cross section. The subject of fluid oscillations in fixed containers is also given attention by Evans and Linton [3] who use potential theory and complex variable theory to determine the time periodic motions which may occur in a horizontal cylinder with semicircular cross section and the hemisphere. Moiseev and Petrov [4] also extend this list to include the time periodic oscillations in shapes as diverse as coaxial cylinders and the torus, utilising methods including potential theory and the Ritz method. In his solo work, Moiseev [5] has also studied the oscillations of liquids in elastic beams under torsion which have hollow regions to hold fluid (the cavities being first totally filled and then partially filled with fluid, allowing a free surface in the latter case) and the same problem with the beam also undergoing deflection. He also studies the properties, such as stability and ability to use the Ritz Method, of the equations used in general. Finally, the paper by Davis and Weidman [6] gives a thorough investigation into the values for the natural sloshing frequencies of fixed fluid filled containers whose walls have differing inclinations to the horizontal. The fluid has a free surface and the wavelengths in the fluid are considered small compared with the horizontal measurement of the resting free surface. The containers considered by them include those with a gap at the bottom, as well as those with closed continuous beds.

The case of forced oscillations of a fluid in a container has also been the subject of research. The work by Chester [7] analyses theoretically the behaviour of liquids in containers which are forced to oscillate in a given horizontal direction with a prescribed frequency; the results obtained then being verified by physical observations. Miles [8] gives an account of vertical oscillations of cylinders with prescribed periodic displacement. In his work, he also considers the effects of capillary motion for sufficiently narrow cylindrical containers. His paper also discusses the properties of cross waves (waves which may be generated by a vibrating vertical plate partially immersed in a fluid filled basin) generated in a rectangular wavetank, given that the wavemaker undergoes a regular periodic motion, considering ultimately the effect of capillarity.

The work undertaken in this dissertation involves a more subtle generalisation of the above problems. It is based on the paper by Cooker [9] and is an extension of the work therein. The aim of this project is to give a theoretical treatment of the coupled oscillation of a hollow container and its inviscid liquid load for several container geometries and under the restriction of different forces. Potential theory is used to obtain results which culminate in the determination of the possible frequencies of oscillation. Section 2 of this paper states the problem more accurately in general terms and lays the foundation for the work that follows.

The main body of the project picks up straight from the end of the theoretical work in the paper by Cooker [9], where a container of hyperbolic vertical section is treated. The periodic translational motion for containers of suitable hyperbolic shape are covered in Section 3, first for the case of the fixed tank then for the unrestrained tank and finally for the tank restrained by springs.

Section 4 takes its lead from the work in Section 3 and describes the motion, in the same scenarios for the hyperbolic containers, for a container that is a hollow triangular prism. The cross section of this tank is a right-angled triangle with sides of equal inclination to the horizontal. The interrelation between the problems of Sections 3 and 4 is then discussed.

The penultimate section, Section 5, extends the two-dimensional problems of Sections 3 and 4 and introduces sloshing in a rigid rotating tank under the restriction of a restoring torque. The tank is assumed to have uniform mass distribution and to be freely pivoted at its midpoint. It is seen that the frequency of oscillation satisfies the same sort of conditions as the frequency for the translating prism in Section 4

Section 6 gives a summary of the results obtained and also gives some open questions which are related to the work in this dissertation.

2 The General Problem

Suppose there is an immovable container of arbitrary shape resting on a smooth horizontal surface which contains fluid, at rest, with horizontal free surface. Further, suppose that the fluid is disturbed and that it settles into a time

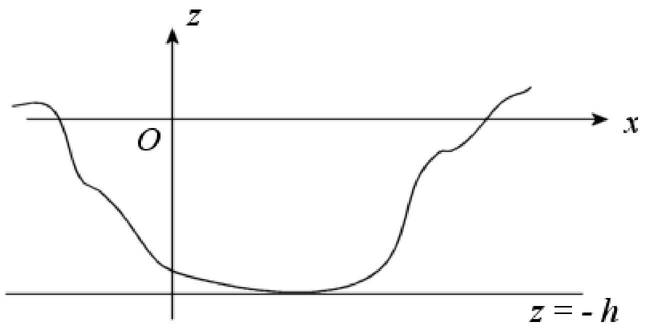

Fig 1: A general tank

periodic motion. In this general situation there are a few conditions which must be assumed and others which must be satisfied to get a solution to the fluid motion. Firstly, a set of Cartesian axes labelled x, y and z are set in three dimensional space so that z increases vertically upwards with $z = 0$ as the level of the undisturbed free surface of the fluid and $z = -h$ the surface on which the tank can slide smoothly. The x and y axes may be assigned arbitrarily (whilst remaining perpendicular to the z axis and to each other) unless the container has some form of symmetry; if the tank has a uniform cross section, the x and z axes are set parallel to the common cross section through the tank with x increasing to the right. This ensures that the flow can be treated as two-dimensional, with each section given by $y =$ constant having the same flow pattern. Also, if this arrangement shows some form of mirror symmetry of the container in the xz plane then the line of symmetry, assuming it is vertical, is set so as to coincide with the z axis. This is summarised visually in Fig. 1 above, where the tank has walls parallel to the x axis.

Now that the spatial concerns have been addressed, the fluid flow may be analysed in an unambiguous fashion. The only fluid motion which will be considered hereafter is a steady, time periodic motion so the time $t = 0$ is chosen at a convenient instant after the fluid has settled into such a motion. The motion of the fluid for $t \geq 0$ is assumed to be irrotational and incompressible, so the existence of a velocity potential $\phi(x, y, z, t)$ is guaranteed where the velocity field for the flow, denoted here by $\mathbf{u}$, satisfies $\mathbf{u} = \nabla\phi$. The incompressibility of the fluid also gives rise to the condition that $\nabla^2\phi = 0$ in the region of flow $\forall t \geq 0$. The usual linearised boundary conditions for flows with a free surface with amplitude small compared to the fluid depth ($\phi_{tt} + g\phi_z = 0$ on $z = 0$) and for flows in a container ($\mathbf{n} \cdot \nabla\phi = 0$ where $\mathbf{n}$ is an outward pointing normal to a solid wall bounding the region of flow) are also to be satisfied.

Suppose now that the tank is allowed to move, restricted by external forces or otherwise. The axes (as set out in the case of the fixed tank) remain in the same place in space, i.e. they are in an inertial frame of reference. The motion is firstly assumed to be horizontal and parallel to the x axis. It is also assumed that the periodic fluid motion induces a regular periodic motion to occur for the tank. The (vertical) amplitude of the fluid motion is assumed small compared with h. The approximation on the tank motion which shall be utilised from this point on is that the motion is periodic in time with the amplitude of the motion of the tank small compared to the tank's width in the x direction. It may seem that the motion of the tank and the motion of the fluid are being treated as separate, but they will be coupled by the following conditions on ϕ. If the motion of the tank is such that the tank's displacement is given by $X(t) = X_0 \sin(\omega t)$ with constant amplitude X_0 and frequency of oscillation ω and $X(0) = 0$ then ϕ must satisfy $\nabla^2\phi = 0$ with $\phi_{tt} + g\phi_z = 0$ on $z = 0$ as before. The main difference is the insistence that the normal component of the fluid velocity matches the normal component of the velocity of the tank wall, which gives the required coupled motion of the system. In these cases, there will be an equation of the form $F = M(\mathrm{d}^2X/\mathrm{d}t^2)$, derived from Newton's second law of motion where F is the net force of the fluid and any restoring springs resolved to the right and M is the dry mass of the tank. This will generally give a condition on ω^2 and give a closed system of equations for the unknown variables X_0 and ω^2 and the other known physical quantities. However, this relation may be highly complicated and may not yield solutions in closed form (the interested reader may check this is the case, following the general methods utilised later, for a moving rectangular tank which is assumed to have walls at $x = 0$, $x = a$, $y = \pm b/2$ and $z = -h$ for constants a, b and h all greater than zero. The difficulty with this problem is the satisfaction of the conditions at the vertical tank walls).

Besides translation, the other main type of motion that a tank can undertake is rotational motion about some axis or axes of rotation. Considering the motion of a general tank about a vertical axis due to a net torque imposed upon the fluid/container system, it is seen that the problem must be considered as three-dimensional. The general tank of Fig. 1 is assumed to have a series of strings under torsion, or some similar device, which supports the tank so that the lowest points of the tank (as in the fixed case) is the lowest point of the tank throughout the motion of the tank-fluid system. The torsion in the strings is assumed to give rise to a torque which acts in opposition to the torque imposed on the tank by the fluid motion inside the tank, with the net torque allowing regular periodic motion of the tank with small amplitude. In this case, an equation of motion for the system will be of the form $I\ddot{\theta} = T_R$ where I is the moment of inertia of the dry tank, θ is the angular displacement of an end wall of the tank from its equilibrium position and T_R is the net torque on the system. This equation is just an equivalent form of Newton's Second Law, but for rotational motion. As in the translational case, the equation of motion couples the motions of the tank and the fluid to give the overall motion of the system.

3 The Family of Hyperbolic Containers

The present aim is to construct a fluid flow, which is physically acceptable, given only a particular velocity potential. This velocity potential will be seen to give rise to a two dimensional flow which occurs in a container whose cross section in a plane of constant y is a hyperbola. The work in this section is important as it forms the basis for the work in the following sections.

3.1 Preliminary work for fixed tank

Suppose, for constants β, α and a, with $a > 0$, a velocity potential

$$\phi = \left(\beta x \left(z + \frac{a}{2}\right) + \alpha\right) \cos(\omega t)$$

is given, where it is assumed that $\cos(\omega t)$ gives an appropriate time periodic factor for the flow in a fixed container. This choice of time dependence will be seen to be relevant, when the tank is allowed to move.

The stream function, denoted by ψ, which is associated with this ϕ is found via the relations $\phi_x = \psi_z$, $\phi_z = -\psi_x$ where the subscripts denote partial differentiation. The relations on ϕ, ψ may be recast to give

$$\psi = \int \phi_x \, \mathrm{d}z \qquad \psi = -\int \phi_z \, \mathrm{d}x$$

It may be verified that the first integral gives $\psi = \left(\beta[z + (a/2)]^2 \cos(\omega t)\right)/2 + f(x,t)$ and the second gives $\psi = -(\beta x^2 \cos(\omega t))/2 + g(z,t)$ where f, g are arbitrary functions of their respective variables. Combining these gives

$$\psi = \frac{\beta([z + (a/2)]^2 - x^2) \cos(\omega t)}{2} + G(t)$$

where $G(t)$ is an arbitrary function of time. As G is an arbitrary function, any constants may be incorporated into it, so the stream lines of the flow (and the possible positions for the containers) may be found by letting $\psi = 0$ without loss of generality. Further, if $G(t) = [\beta D \cos(\omega t)]/2$ for some constant D, then it is seen that the equation for the stream function reduces to

$$z = f(x) = -\frac{a}{2} + \sqrt{x^2 - c^2 + \frac{a^2}{4}} \tag{1}$$

where D has been chosen to be equal to $c^2 - a^2/4$ with parameter c. Equation (1) is seen to give a family of hyperbolic containers if $|c| \leq a/2$. In the remainder of this section, c is assumed to satisfy $0 \leq c \leq a/2$ with no loss in generality. In equation (1), the term $+\sqrt{\dots}$ is used instead of $\pm\sqrt{\dots}$ to give an equation valid for the upper branch of the pair of hyperbolae under consideration, which asymptote to lines parallel to $z = x - a/2$ and $z = -x - a/2$. Thus (1) determines the boundary of a flow which corresponds to a confined region of fluid.

Now that the basic shapes of the containers for the fluid flow have been determined, it is helpful to select the free surface of the undisturbed fluid in the container to coincide with the plane $z = 0$. The amplitude of the oscillatory motion is assumed small, so that the fluid level is never markedly different from its undisturbed value. Thus an approximation to the width of the container at the free surface, valid for all time, may be derived from (1) by letting z equal zero. This gives $x^2 - c^2 = 0$ as the equation for x at the free surface of the fluid throughout its motion. From this it is clear that at the free surface, $x \in [-c, c]$. [N.B. It is important to remember that a relatively simple linear theory is being employed here, so the only predictions which can be made are those for which x is not greater in absolute value than its value at $z = 0$].

We now determine the maximum depth of the hyperbolic container in terms of the c parameter discussed above. Letting $x = 0$ in (1) gives directly

$$h = \frac{a}{2} - \sqrt{\frac{a^2}{4} - c^2} \tag{2}$$

from which it is seen that the quantity a is given by

$$a = \frac{h^2 + c^2}{h} \tag{3}$$

Here $z = -h$ is the equation of the line in $x\,z$ space which is level with the maximum depth of the container. Thus the general hyperbolic container has the form shown in Figure 2 below.

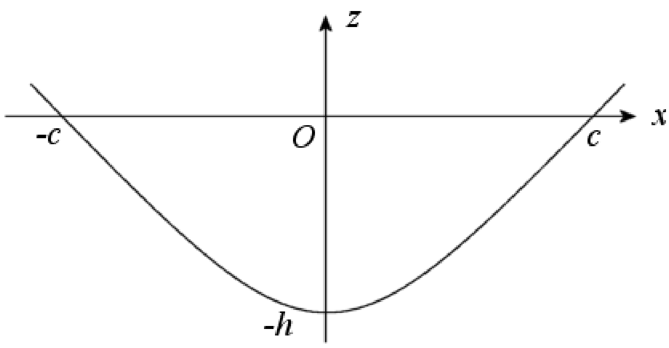

Fig 2: Hyperbolic tank

Bernoulli's equation applies throughout the domain of this flow for all times $t \geq 0$ due to the property of irrotationality. Thus for constant density ρ the pressure $p = p(x, z, t)$ is given by

$$\frac{p}{\rho} + \phi_t + \frac{1}{2} (\nabla\phi)^2 + gz = 0 \tag{4}$$

where any arbitrary functions of t have been absorbed into the ϕ_t term and g is the acceleration due to gravity. To keep in line with the linear theory which is being adopted, it is postulated that the $(\nabla\phi)^2$ in (4) is negligible compared with ϕ_t or gz. Also, as the amplitude of the fluid motion is supposed small compared to the maximum depth of the container, on the free surface $z = \zeta(x, t)$, equation (4) gives

$$\left.\frac{\partial\phi}{\partial t}\right|_{z=\zeta} + g\zeta = 0. \tag{5}$$

The above has used the assumption that the pressure at the free surface is constant, so it may be included in the ϕ_t term without loss of generality. Expanding ϕ_t about $z = 0$ as a Taylor series expansion shows that $\phi_t(x, \zeta, t) \approx \phi_t(x, 0, t)$. Thus an approximation for the free surface behaviour is obtained via the equation

$$\zeta = -\frac{1}{g}\frac{\partial\phi}{\partial t}\bigg|_{z=0} \tag{6}$$

which, in this case and under the assumption that the fluid has undisturbed level at $z = 0$, gives the free surface to be

$$\zeta = \frac{\beta a\omega}{2g}x\cos(\omega t). \tag{7}$$

whence it is seen that $\alpha = 0$. Also the velocity potential is given by

$$\phi = \beta x\left(z + \frac{a}{2}\right)\cos(\omega t) \tag{8}$$

Hence the free surface in the oscillatory motion of the fluid is a straight line in xz space which oscillates about the point $(0, 0)$.

The condition to be satisfied at the free surface is a combination of the kinematic condition and Bernoulli's equation. It is given by

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0 \text{ on } z = 0 \tag{9}$$

which is an approximation used in this linear theory for the conditions on the free surface $z = \zeta(x, t)$. A simple substitution of the given ϕ into (9) shows that ω satisfies

$$\omega^2 = \frac{2gh}{c^2 + h^2} \tag{10}$$

Equation (10) shows that for $c \ll h$ the frequency of oscillation of the system is given by $\omega \approx \sqrt{(}2g/h)$ and for $c \gg h$ the frequency of oscillation is $\omega \approx \sqrt{2g/c}$. Thus the frequency of oscillation for a narrow, deep tank (i.e. where $c \ll h$) is small compared to the frequency for a tank where $c \approx h$ and the same is true for a wide, shallow tank (where $c \gg h$).

It is not obvious which terms in (4) may be assumed negligible on the grounds of relative magnitudes. The expression given for the velocity potential ϕ and the derived expression for the free surface in (7) for $|x| \leq a/2$ give the following

$$\phi_t = -\frac{\omega\beta a}{2} x \sin(\omega t) \implies |\phi_t| \leq \frac{\omega\beta a^2}{4} \tag{11}$$

$$g\zeta = \frac{\omega\beta a}{2} x \sin(\omega t) \implies |g\zeta| \leq \frac{\omega\beta a^2}{4} \tag{12}$$

$$\frac{(\nabla\phi)^2}{2} = \frac{\beta^2}{2}\left(x^2 + \frac{a^2}{4}\right)\cos^2(\omega t) \implies |(\nabla\phi)^2/2| \leq \frac{\beta^2 a^2}{4}. \tag{13}$$

where $x < c^2 < a^2/4$ so that x assumes a value in the region of flow. Now (7) with the assumption that the amplitude of the free surface is small compared to $a/2$ shows that $|\mathrm{d}\zeta/\mathrm{d}x| \ll 1$, i.e. $[\omega\beta a/(2g)] \ll 1$. Also $|(\nabla\phi)^2/2| \ll |\phi_t|$ iff $\beta \ll \omega$ from (11) and (13). But this assumption gives $|\mathrm{d}\zeta/\mathrm{d}x| \ll a\omega^2/(2g)$. But it has been seen that $\omega = \sqrt{2g/a}$, so the assumption on $(\nabla\phi)^2$ is valid. This means that it is acceptable to omit the non-linear term $(\nabla\phi)^2$ and (4) gives

$$p = p(x, z, t) = -\rho\phi_t - \rho g z$$

as the relevant equation for the pressure distribution in the region of flow.

Turning to the calculation of the force due to the fluid motion, it is apparent that performing a z integral of the pressure given by (4) would not be an ideal approach. Instead, an x integral of the pressure is used as follows. The wall of the container is known to be given by $z = f(x)$ from (1) and the horizontal component of the liquid force on the boundary of the container, denoted hereafter by F_l, is then given by

$$F_l = b\int_{-c}^{c} p^* \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x$$

where the pressure on the bed $p^* = p(x, f(x), t)$ is found from (4) and b is the width of the tank in the y direction. This definite integral for the force evaluates to

$$F_l = \frac{2\beta\rho b c^3}{3}\sqrt{\frac{2g}{a}}\sin\left(t\sqrt{\frac{2g}{a}}\right). \tag{14}$$

As a summary for this subsection, the important functions ϕ, ζ and F_l corresponding to the velocity potential, free surface position and liquid force for this problem have been found to be

$$\phi = \beta x \left(z + \frac{a}{2}\right) \cos\left(t\sqrt{\frac{2g}{a}}\right) \tag{15}$$

$$\zeta = \beta x \sqrt{\frac{a}{2g}} \sin\left(t\sqrt{\frac{2g}{a}}\right) \tag{16}$$

$$F_l = \frac{2\beta\rho b c^3}{3} \sqrt{\frac{2g}{a}} \sin\left(t\sqrt{\frac{2g}{a}}\right) \tag{17}$$

where $\sqrt{a}\beta(\sqrt{2g})^{-1} \ll 1$ to keep in line with the assumption of small amplitude fluid motion.

3.2 Unrestrained moving tank

Since the foundations of the fixed case are in place, the discussion now moves to the description of the hyperbolic container which is free to move parallel to the x axis without the restriction of a spring-like apparatus which provides a restoring force. Assume as set out in the general case of Section 2, that the displacement of the tank, measured in a frame of reference outside the moving tank/fluid system is given by $X(t) = X_0 \sin(\omega t)$. The normal component of fluid velocity on the wall of the tank is given by $\mathbf{n} \cdot \nabla\phi$. Using the general container geometry given in Section 3.1, the problem to solve now is essentially the same as in Section 3.1 but with the condition that the normal component of fluid velocity on the tank wall must match the normal component of the tank wall velocity. Thus if $\mathbf{n}$ is a normal vector to the container, the condition

$$\mathbf{n} \cdot \nabla\phi = (\mathbf{i} \cdot \mathbf{n}) \frac{\mathrm{d}X}{\mathrm{d}t}$$

must hold on the container wall. We introduce the velocity potential

$$\phi = \left\{X_0 \omega x + \beta x \left(z + \frac{a}{2}\right)\right\} \cos(\omega t) \tag{18}$$

which is seen to satisfy $\nabla^2\phi = 0$ in the region of flow and to give the correct horizontal velocity to the tank. The additional term $X_0\omega\cos(\omega t)$ is to accommodate the motion of the tank whose displacement is $X_0 \sin(\omega t)$. The time $t = 0$

here corresponds to a time where the fluid has regular periodic motion and the tank is passing through its equilibrium position, heading instantaneously to the right. Thus at $t = 0$, the spatial axes for the problem are such that the tank and liquid occupy the region $-c \le x \le c$, $-b/2 \le y \le b/2$, $f(x) \le z \le 0$ where $z = f(x)$ is the position of the tank bed. Neglecting the motion of the container walls, the boundary conditions are applicable on $z = f(x)$ $\forall t \ge 0$ (this is the same $f(x)$ as in Section 3.1).

The only boundary condition left to satisfy is $\phi_{tt} + g\phi_z = 0$ on $z = 0$. This relationship yields

$$\beta = \frac{2X_0\omega^3}{2g - a\omega^2} \tag{19}$$

which gives an explicit expression for β, which remains arbitrary because X_0 is arbitrary. Equation (19) shows that resonance occurs if a forcing frequency given by $\omega^2 = 2g/a$ is imposed upon the system. The latter condition is expected on physical grounds; the forcing frequency equals the natural frequency of the system, and hence resonance occurs, if $\omega^2 = 2g/a$.

The free surface of the fluid at general time $t \ge 0$ is found from (7) and is, for the moving tank

$$\zeta = \frac{X_0\omega^2}{g}\left\{1 + \frac{a\omega^2}{2g - a\omega^2}\right\} x\sin(\omega t) \tag{20}$$

and is seen to oscillate about the y axis for all times $t \ge 0$.

For the equation of motion, if F_l denotes the force on the container due to the movement of the fluid and M is the dry mass of the container, a direct application of Newton's Second Law of Motion gives

$$F_l = -M\omega^2 X_0 \sin(\omega t)$$

in which the only undetermined quantity is F_l. Using the fact that, from (4), $p = p(x, z, t) = -\rho\phi_t - \rho g z$ is the approximation which is consistent with the horizontal free surface boundary condition in (9). As in the fixed case, $F_l = b\int_{-c}^{c} p^* f'(x)\mathrm{d}x$ where $p^* = p(x, f(x), t)$, it may be verified that the equation which ω satisfies is

$$\omega^2 \left(Ma + ma - \frac{4b\rho c^3}{3} \right) - 2g(M+m) = 0$$

and this has solution

$$\Omega^2 = \frac{3(1+\mu)}{3(1+\mu) - C} \tag{21}$$

where $\Omega^2 = a\omega^2/2g$, $\mu := m/M$ is the (dimensionless) ratio of the mass of fluid to the mass of the container and $C := 4b\rho c^3/Ma$ is a dimensionless quantity. The relation for ω^2 just derived means that for each choice of a, b, c and ρ there is only one admissible frequency (a negative frequency is the same as its positive counterpart for all intents and purposes) and that this frequency is greater than the natural frequency of the system. The coupled oscillations of the tank and fluid must be in antiphase to sustain this particular motion. As $M \to \infty$ for fixed m, $\mu \to 0$ and $C \to 0$, so equation (21) gives $\Omega^2 \to 1$. This reult is the same as the fixed case in Section 3.1. So this mode of oscillation is sustained with a frequency which is close to the frequency for the fixed container, if the container is sufficiently heavy. The summary for this subsection recaps the important results discovered for clarity, where it is supposed that $B := C/(1+\mu)$ is another dimensionless quantity:

$$\phi = X_0 \sqrt{\frac{2g}{a(1-B)}} \left\{ 1 - \frac{2}{aB} \left(z + \frac{a}{2} \right) \right\} x \cos \left(t \sqrt{\frac{2g}{a(1-B)}} \right) \tag{22}$$

$$\zeta = \frac{2X_0}{a(1-B)} \left\{ 1 - \frac{1}{B} \right\} x \sin \left(t \sqrt{\frac{2g}{a(1-B)}} \right) \tag{23}$$

$$F_l = -\frac{2MX_0}{a(1-B)} \sin \left(t \sqrt{\frac{2g}{a(1-B)}} \right) \tag{24}$$

and the scenario described above is shown in Figure 3, where the double headed arrow denotes the instantaneous velocity of the system when the free surface is in the position shown by the dashed line.

In Figure 3 the tank is free to slide due to the forces of the oscillating fluid in the container. As the fluid moves to the left, the tank is forced to the right,

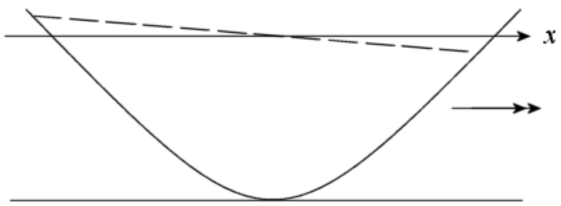

Fig 3: The unrestrained hyperbolic tank

until the fluid is at the point of maximum runup. At the time corresponding to maximum runup on the left, the tank is instantaneously at rest. The fluid then returns from this extreme to its equilibrium level. At this point, the tank is moving with its maximum speed to the left. The fluid continues its motion smoothly until the time of maximum runup on the right hand side of the container, at which time the container is at rest. The fluid then returns smoothly to its equilibrium level, at which time the container is moving at its greatest speed to the right. This is followed by the smooth transition to the situation where maximum fluid runup occurs on the left. The coupled system undergoes such cycles for all times $t \geq 0$.

3.3 Moving tank with springs

Now that the basic work for a fixed hyperbolic container has been covered, in a parallel fashion to the preliminary work, it is possible to compare the results to be obtained for a moving container. Assuming that there is a smooth horizontal plane at $z = -h$ and that there is a spring-like apparatus which obeys Hooke's Law throughout the tank's motion. For a further simplification of analysis, it is assumed that the only tank motions under consideration are in the x direction, the direction of x increasing being taken as the positive direction for displacement.

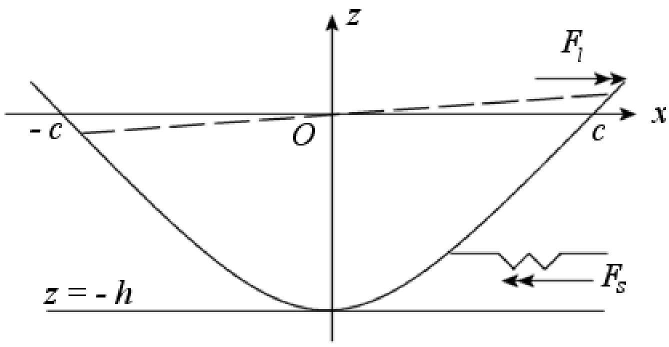

Fig 4: Sketch of moving tank

It is assumed as in the other cases that $t = 0$ corresponds to a state of the system where the motion has become periodic and where the tank is heading to the right. It is postulated that the displacement of the tank from its equilibrium position is given by $X(t) = X_0 \sin(\omega t)$ where it is assumed that $|X_0| \ll c$, to keep in line with the linear theory. A reasonable consequence of this is that the conditions on the moving boundary can be replaced by conditions applied on the boundary as if the tank were fixed at its equilibrium position. The only alteration is the introduction of a restoring force due to a Hookean spring, or similar apparatus. The equation for the velocity potential ϕ is the same as in equation (18) and the corresponding expressions for the parameter β and free surface ζ are the same as equations (19) and (20) respectively.

Now it is an appropriate time to derive the equation of motion for the system as a whole. From the definition sketch of the moving tank on Page 18 it is clear that Newton's Second Law gives, on resolving to the right

$$F_l - F_s = M \frac{\mathrm{d}^2 X}{\mathrm{d}t^2}$$

where F_s is the restoring force due to the spring-like apparatus. Thus assuming that the spring has spring constant K (measured in Nm^{-1}, for instance) then Newton's Second Law and Hooke's Law give:

$$F_l - KX_0 \sin(\omega t) = -M\omega^2 X_0 \sin(\omega t) \tag{25}$$

The force F_l must now be calculated for a tank free to move. Using the x integral approach as on Page 13 it is seen that an expression for F_l is given by

$$F_l = b\rho\omega \int_{-c}^{c} \left\{ \frac{X_0 \omega x^2}{\sqrt{x^2 - c^2 + (a^2/4)}} + \beta x^2 \right\} \mathrm{d}x \sin(\omega t)$$

where it may be checked that (e.g. see M. R. Murray "Mathematical Handbook of Formulas and Tables") on omitting the arbitrary constant of integration and assuming $\sigma^2 \geq 0$.

$$\int \frac{x^2}{\sqrt{x^2 + \sigma^2}} \mathrm{d}x = \frac{x\sqrt{x^2 + \sigma^2}}{2} - \frac{\sigma^2}{2} \ln\left(x + \sqrt{x^2 + \sigma^2}\right)$$

Thus on using this result with $\sigma^2 := -c^2 + (a^2/4) \geq 0$ gives the liquid force to be

$$F_l = X_0 \omega^2 b\rho \left\{ \left[\frac{ac}{2} + \frac{1}{2}\left(\frac{a^2}{4} - c^2\right) \ln\left(\frac{a - 2c}{a + 2c}\right) \right] + \frac{4\omega^2 c^3}{3(2g - a\omega^2)} \right\} \sin(\omega t)$$

For the sake of simplicity, allowing the constant term in square brackets to be denoted by A, (25) gives

$$\omega^2 b\rho A + \frac{4b\rho\omega^4 c^3}{3(2g - a\omega^2)} - K = -M\omega^2 \tag{26}$$

Thus rearranging (26) gives a quartic in ω, or equally (and more usefully) a quadratic in ω^2. Introducing the dimensionless quantities

$$\Omega^2 = \frac{a\omega^2}{2g}; \quad D = \frac{mgh}{K(h^2 + c^2)}; \quad E = \frac{Mgh}{K(h^2 + c^2)}; \quad F = \frac{mgh^2 c^3}{KA(h^2 + c^2)^2}$$

the quadratic in ω^2 reduces to the dimensionless form

$$\Omega^4 - \alpha\Omega^2 + \gamma = 0 \tag{27}$$

where

$$\alpha = \frac{6D + 6E + 3}{6E - 8F + D}; \qquad \gamma = \frac{3}{6E - 8F + D}$$

The discriminant for (27) is given by

$$\Delta = \alpha^2 - 4\gamma$$

which rearranges to

$$\Delta = 4D^2 + 8DE + \frac{8D}{3} + \frac{32F}{3} + (2E-1)^2$$

which is positive as all of the dimensionless variables D, E and F are. Thus two real values for Ω^2 are guaranteed to exist on solving (27) for Ω^2, assuming of course that the leading coefficient in (27) is non zero. It is easily verified that $(2D + 2E + 1) \neq \pm\sqrt{\Delta}$, so the values of Ω^2 which are solutions of (27) are distinct. A graph showing the behaviour of Ω for various values of h is given in Figure 5 below. The assumption is that $g = 10$, $M = 10$, $K = 20$ and $c = 10$. The graph shows that generally Ω assumes two distinct values for each value of h, which is predicted in the work above. The graph also shows a minimum value for the parameter Ω with respect to h occurs for $h \approx 6$. This stationary value for Ω is in stark contrast to the work of previous sections, where Ω is shown graphically to be a strictly increasing function of the variable on the horizontal axis.

3.4 Summary

The conclusion for this subsection is that unlike the previous work undertaken there are two distinct frequencies for which a physically acceptable motion of the system can occur. Also it is a good time to note that as $K \to 0$, equation (26) can be rearranged to give equation (21), so the motion in subsection 3.2 is a special case of this example. Further the work of subsection 3.1 for the fixed tank can be obtained by letting $K \to 0$ and then $M \to \infty$ with m fixed. The two behaviours of the system are shown in Figures 6 and 7, where the free surface is a dashed line and the velocity of the tank is a double-headed arrow.

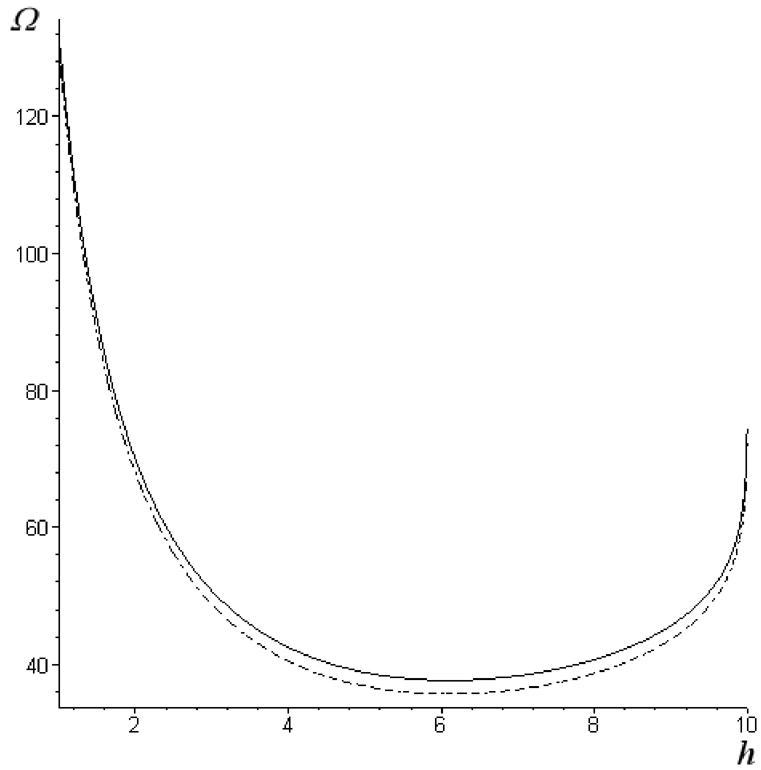

Fig 5: Graph of Ω against h

4 The Tank with "Vee-Shaped" Cross Section

The work in this section is a special case of the work in the preceding section of this dissertation. In effect, the results derived in this section can easily be obtained from the corresponding work in Section 3 on letting $2c = a$ and $2h = a$. In this section, the motion of a container whose cross section in the xz plane is a right angled triangle. The walls of this container are described by the equations $z = -(a/2) \pm x$ in xz space and the tank is assumed to have triangular end walls at $y = -b/2$ and $y = b/2$, so giving a closed tank. Making the assumption that the mean free surface is horizontal at $z = 0$ then at the mean free surface

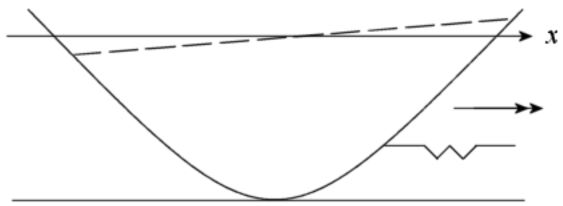

Fig 6: Low frequency behaviour

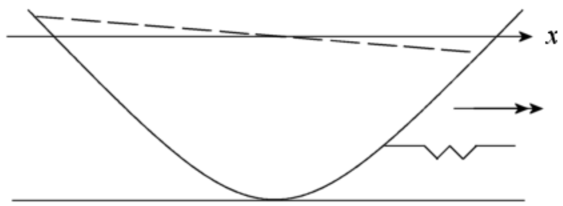

Fig 7: High frequency behaviour

the width of the still water surface in the tank is a. The work in this section will stem from the use of the same velocity potential introduced in Section 3. As in that Section, the fluid motion is assumed to have no variation in y and the flow is assumed to be irrotational and incompressible. The amplitude of the free surface is presumed to be negligible in magnitude compared with $a/2$.

4.1 The Fixed Tank

This first subsection of Section 4 gives the basic results for the case of a fixed tank whose shape is described in the above introduction. The position of the tank and orientation of the axes is shown in Figure 8 below.

The velocity potential for this problem is given by (8) by assumption and

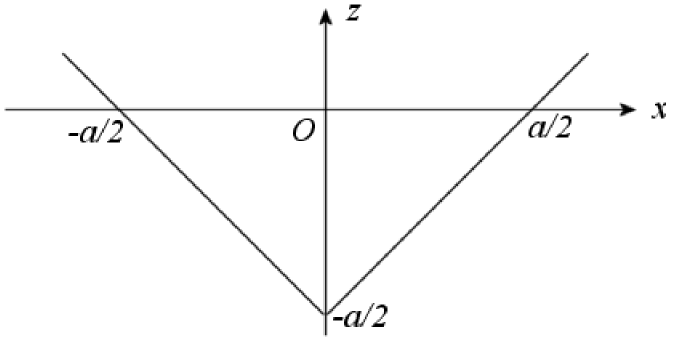

Fig 8: The hollow prism-shaped tank

so, in parallel to the discussion for the hyperbolic containers in Section 3, the free surface is found, using the relation (6), to be

$$\zeta = \frac{\beta a\omega}{2g} x \cos(\omega t)$$

which is the same as (7). Thus the free surface is a line in xz space or a plane in 3 dimensional space which oscillates in time and always passes through the origin of the axes as they have been set out in space. As the velocity potential being used is (8), $\nabla^2\phi = 0$ is clearly satisfied and so it remains to satisfy $\phi_{tt} + g\phi_z = 0$ on $z = 0$. It is thus found that

$$\omega^2 = \frac{2g}{a}$$

Compared to the working in Section 3 the force due to the moving liquid and acting to the right (the direction of x increasing), denoted by F_l, is easier to calculate. Unlike in the previous case, the fluid force is calculated from the z integral of $p = p(x, z, t)$ where the pressure p is found from the linearised version of Bernoulli's equation, i.e.

$$p = -\rho\phi_t - \rho g z$$

(the non-linear term $(\nabla\phi)^2$ is omitted for the same reason as in the general hyperbolic case). Thus if P_R denotes the pressure exerted on the right-hand wall of the container (described by $z = -(a/2) + x$ for general time t in xz space) and P_L is the corresponding pressure exerted on the left-hand wall (described by $z = -(a/2) - x$) then F_l is found from

$$F_l = b\left\{\int_0^{a/2} P_R \frac{\mathrm{d}f(x)}{\mathrm{d}x}\,\mathrm{d}x + \int_{-a/2}^{0} P_L \frac{\mathrm{d}f(x)}{\mathrm{d}x}\,\mathrm{d}x\right\}.$$

The function $f(x)$ is such that $z = f(x)$ is the position of the tank bed. This gives, after some elementary manipulations

$$F_l = \frac{\beta m a \omega}{3}\sin(\omega t) \tag{28}$$

4.2 Moving Container with no External Constraints

The work in this subsection generalises the work of Section 4.2 thus allowing the case of a moving container with the same shape as in Section 4.1 to be considered. It can be seen that the work of Section 4.1 is a special case of this motion on allowing X_0, the amplitude of the tank motion, to tend to zero.

The same conditions on the flow, of irrotationality and incompressibility, as in the previous subsection are assumed. Thus the only change to the problem which will arise at this stage is due to the fact that the no normal flow condition in the x direction must be changed. To accommodate the motion of the trough gives rise to the same situation as if there was an outflow of liquid from one side of the tank and an influx of fluid from the other. These changes must occur with the same velocity as that with which the tank moves. Thus the no normal flow condition is replaced by the stipulation that $\mathbf{n}\cdot\nabla\phi = \dot{X}(\mathbf{n}\cdot\mathbf{i})$, where $\mathbf{n}$ is a unit outward pointing normal on the tank wall at a general position and $\dot{X}$ is the time derivative of $X(t)$. Thus a choice of velocity potential

$$\phi = \left\{X_0\omega x + \beta x\left(z + \frac{a}{2}\right)\right\}\cos(\omega t)$$

seems valid. The condition $\phi_{tt} + g\phi_z = 0$ at $z = 0$ gives

$$\beta = \frac{2X_0\omega^3}{2g - a\omega^2}$$

whence it is seen that resonance occurs if the system is forced to oscillate at its natural frequency, which is expected on physical grounds. The free surface is given by

$$\zeta = \left[\frac{X_0\omega^2}{g}x + \frac{2aX_0\omega^4}{g(2g-a\omega^2)}x\right]\sin(\omega t)$$

on using equation (7)

As X_0 is arbitrary in the sense that $0 \leq |X_0| \ll a/2$, the only remaining unknown variable for this system is ω. This is found from the overall equation of motion of the system, which is easily derived from Newton's Second Law of Motion. If F_l is the force due to the moving liquid in the trough and M is the dry mass of the trough then Newton's Second Law states that $F_l = M\ddot{X}$. The liquid force is again calculated by performing an x integral of $p = p(x, z, t)$. This simple calculation ultimately yields the relation

$$\omega^2(3Ma + 2ma) - (6Mg + 6mg) = 0$$

where m is the mass of fluid in the trough. On letting $\mu = m/M$ and be a dimensionless quantity, ω is given by

$$\Omega^2 = \left(1 + \frac{\mu}{3+2\mu}\right) \tag{29}$$

The relationship between ω and μ is shown graphically in Figure 9 where $\Omega^2 = a\omega^2(2g)^{-1}$ is a relevant dimensionless quantity. The dashed line has equation $\Omega = \sqrt{3/2}$, which the curve $\Omega = \Omega(\mu)$ asymptotes to as $\mu \to \infty$.

Thus several things now become apparent. Firstly, on physical grounds, the tank motion and fluid motion must be in antiphase for this motion to occur. Also, letting $M \to \infty$ means $\mu \to 0$ and so $\omega^2 \to 2g/a$. This shows that if the tank is made heavier, the motion occurring is closer to the motion for the fixed tank in Section 4.1. Also, as $m \to \infty$, $\omega^2 \to 3g/a$ and so the frequency for this type of behaviour must lie between the extremal values of $\omega = \sqrt{2g/a}$ and $\omega = \sqrt{3g/a}$. This is reinforced by the graph of Ω against μ, where a rapid increase in the values assumed by Ω is seen to occur for $\mu < 10$ followed by a

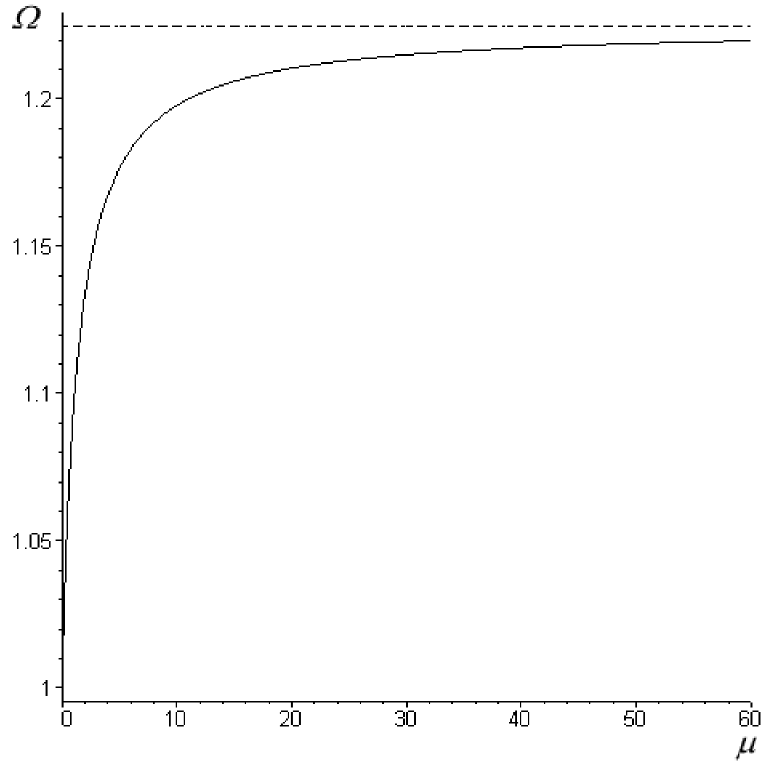

Fig 9: A plot of Ω against μ

smaller rate of increase of Ω with μ for $\mu \geq 10$. The situation is shown visually in Figure 10 below where the double headed arrow shows the velocity of the tank and the dashed line shows the position of the free surface. Note that if the tank is travelling to the left, the diagram corresponding to Figure 10 would simply be the mirror image (in a vertical mirror through the apex) of Figure 10.

4.3 Trough Restrained with Springs

This final subsection of Section 4 deals with the case of a container (whose shape is as discussed earlier) which is free to move in the x direction under the restraining forces of a spring-like system. It may be verified that the work of

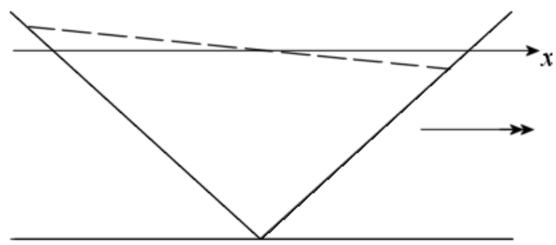

Fig 10: The unrestrained trough

Section 4.2 is a special case of the following work on allowing the spring constant K to tend to zero. This in turn implies that the fixed tank of Section 4.1 is also a special following work. The situation under consideration is shown in Figure 11 below.

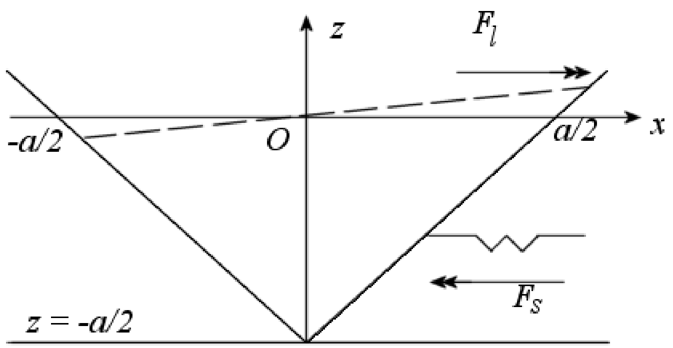

Fig 11: The restrained trough

The velocity potential is the same as in the previous subsection, but for completeness, it is included here. The velocity potential for this problem is

$$\phi = X_0 \omega x \cos(\omega t) + \beta \left(z + \frac{a}{2}\right) x \cos(\omega t)$$

which satisfies the conditions of no normal flow vertically on the base of the tank, $\nabla^2\phi = 0$ in the region of flow and $\mathbf{n} \cdot \nabla\phi = (\mathbf{n} \cdot \mathbf{i})\dot{X}$, where $\mathbf{n}$ is a unit normal to the tank at the tank walls and $\dot{X}$ is the first time derivative of X. The last condition ensures the matching of fluid and tank velocities at the tank wall, which is also required in Section 4.2. The remaining condition, which combines Bernoulli's equation and the kinematic condition at the free surface, is $\phi_{tt} + g\phi_z = 0$ at $z = 0$. This gives an explicit expression for β, which is found to be

$$\beta = \frac{2X_0\omega^3}{2g - a\omega^2}$$

where it again seen that forced oscillation at the frequency ω satisfying $a\omega^2 = 2g$ leads to resonance.

The free surface, as may have been expected from the sketch for the system now under consideration, is planar in three dimensional space. It is given by (7), and thus explicitly

$$\zeta = \frac{X_0\omega^2}{g}\left\{1 + \frac{2a\omega^2}{2g - a\omega^2}\right\} x\sin(\omega t) \tag{30}$$

which is seen to be a straight line which oscillates in time about the point $(0, 0)$ in xz space.

The unknown frequency ω is yet to be determined and this is found from the overall equation of motion for the system. This equation will effectively close the system, leaving X_0 as the only possible variable. If the dry mass of the trough is denoted by M, the enclosed mass of fluid is denoted by m, F_l is the force exerted by the fluid on the walls of the trough and F_s is the restoring force from the spring-like apparatus, Newton's Second Law gives:

$$F_l - F_s = M\frac{\mathrm{d}X(t)}{\mathrm{d}t}$$

on resolving forces in the direction of x increasing (c.f. equation(25)). Since the spring-like apparatus is supposed to obey Hooke's Law for all times $t \geq 0$, it is known that $F_s = KX(t)$. Also, F_l assumes the same form as in Section 4.2, i.e.

$$F_l = mX_0\omega^2 \left(\frac{6g + a\omega^2}{3(2g - a\omega^2)} \right) \sin(\omega t)$$

Thus from Newton's Second Law, after some manipulation

$$\omega^4(3Ma + 2ma) - 3\omega^2(2Mg + 2mg + Ka) + 6Kg = 0$$

The introduction of the dimensionless quantities

$$\Omega := \frac{a\omega^2}{2g}; \qquad E := \frac{Mg}{Ka}; \qquad F := \frac{mg}{2Ka}$$

reduces the quadratic in Ω^2 to give the simpler form

$$2\Omega^4(3E - 4F) - 3\Omega^2(2E + 4F + 1) + 3 = 0$$

The above equation has discriminant Δ, given by

$$\Delta = 9(2E + 4F + 1)^2 - 24(3E - 4F)$$

which rearranges to give

$$\Delta = 9\left(2E + 4F - 1\right)^2 + 8F$$

and as all of the dimensionless quantities are strictly positive, it is seen that $\Delta > 0$ for any choice of $E > 0$ and $F > 0$. This is the relation acquired in the case of the general hyperbolic tank moving under the restriction of a spring when $2c = a$. Thus for any choice of tank mass, tank dimensions and fluid mass, two distinct real values for Ω^2 are guaranteed to exist as $\pm\sqrt{\Delta} \neq 6(2E+2F+1)$. These are found from the usual formula for the roots of quadratic equations and are

$$\Omega^2 = \frac{3(2E + 4F + 1)}{4(3E - 2F)} \pm \frac{\sqrt{9(2E + 4F - 1)^2 + 8E)}}{4(3E - 2F)}$$

from which it is clear that the corresponding values for ω^2 can be determined.

The expressions for the dimensionless variables E and F show that $F = \mu E$ where μ is as in Section 3.2. This means that $E \propto F$ and so the lines of constant

Ω are straight lines in EF space. The quadratic in Ω^2 can be rearranged to give

$$F = \frac{3(\Omega^2 - 1)}{2(2\Omega^2 - 3)}E - \frac{3(\Omega^2 - 1)}{4\Omega^2(2\Omega^2 - 3)} \tag{31}$$

from which it is seen that $\Omega \neq 1$ (or else $F \equiv 0$), so $\omega \neq 2g/a$. Also, resonance may occur if $\Omega^2 = 3/2$ which means that the system would undergo resonance if $\omega^2 = 3g/a$, which is the upper bound for the frequency in Section 4.2. A graph of the lines in EF space for various fixed values of Ω is included in Figure 12. The line with negative gradient corresponds to $\Omega^2 = 1.25$, so it

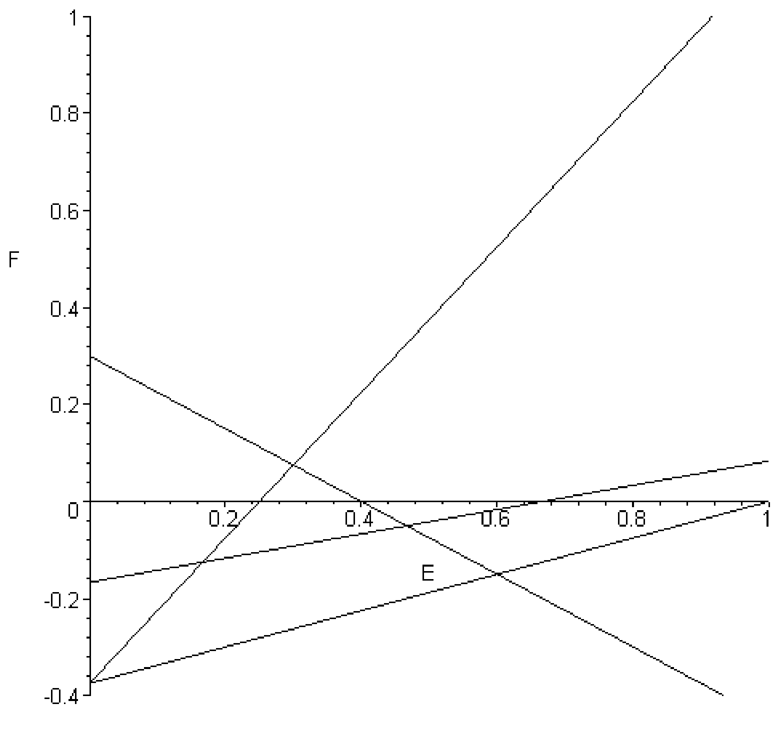

Fig 12: Lines of constant Ω

represents $1 < \Omega^2 < 1.5$. The line with the steepest positive gradient describes the behaviour of F with E for $\Omega^2 = 2$, so is representative of the behaviour for $\Omega^2 > 1.5$. The line with the next steepest positive slope is found from (31) by letting $\Omega^2 = 0.75$, so representing the case where $\Omega^2 < 1$.Finally, the remaining line is found from (31) by taking $\Omega^2 = 0.5$, which is a representative of the set of Ω^2 values for which $0 \leq \Omega^2 < 1$. Thus each of these lines gives the behaviour of Ω^2 for either $0 < \Omega^2 < 1$, $1 < \Omega^2 < 1.5$ or $1.5 < \Omega^2$. Note that from (4.3) if $\Omega^2 \rightarrow$ then $F \rightarrow 3E/2$ and if $\Omega^2 \rightarrow \infty$, $F \rightarrow \infty$. This shows that values of $\Omega^2 \ll 1$ are obtained if $F/E \approx 3/2$ (i.e. $\mu \approx 3/2$) and that values of $\Omega^2 \gg 1$ are obtained if $F \gg E$ (i.e. $\mu \gg 1$). Further, from equation (30), it may be seen that, when $2x = a$

$$\zeta = \frac{X_0 \Omega^2}{2} \left\{ 1 + \frac{\Omega^2}{2(1 - \Omega^2)} \right\} \sin(\omega t)$$

so that the approximation $2|\zeta| \ll a$ is satisfied iff

$$\frac{2a}{X_0} \geq \Omega^2 + \frac{\Omega^4}{1 - \Omega^2}$$

Where two distinct values for ω exist, comparison of these values shows that one of them has a lower absolute value than the other. The former is termed here the "low frequency" and the latter the "high frequency" (N.B. This classification is regardless of signs, only the relative magnitudes of the frequencies). The behaviours at these frequencies are shown in the following diagrams, Figures 13 and 14, where the double-headed arrow indicates the tanks motion and the dashed line indicates the position of the free surface. Once again, if the tank is heading to the left, the behaviours are the mirror images of Figures 13 and 14 in a vertical mirror.

4.4 Summary

The above working shows that for general situation in which a trough with the given triangular cross section undertakes a coupled motion with its fluid flow with a restoring force provided by springs, one of two distinct behaviours maybe observed. The work in Section 4.3 is related to the work in Section 4.2 by letting $K \rightarrow 0$ in the quadratic for ω^2. There is a similar relationship between the work

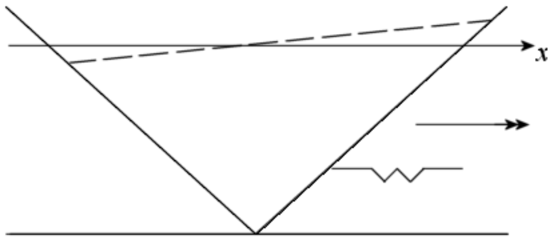

Fig 13: Low frequency behaviour

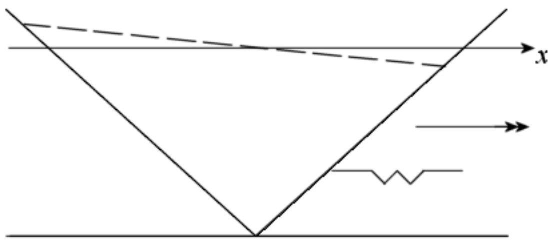

Fig 14: High frequency behaviour

of Sections 4.2 and 4.1 on letting $M \to \infty$, taking due care where necessary to avoid terms in equations becoming infinite. Overall, the work in this section is a special case of that in Section 3 on letting $2c \to a$, but one worthy of investigation. It is also worthwhile to note here that the treatment of this area by Lamb was for a fixed triangular tank in which the streamlines were shown to be hyperbolas which asymptote to lines parallel to the tank walls. Thus his treatment was in the reverse order to the one here, but this approach shows greater generality in the treatment of the trough with triangular cross section as a special case of the hyperbolic containers.

5 The Rotating Trough

This section is a simple generalisation of the work undertaken thus far. In all of the earlier sections, the only type of motion considered was a regular translational motion, assumed periodic in time, of a tank and its fluid contents. This section will set out a basic theory for a tank, much the same as described in Section 4, but the problem considered will be three dimensional in nature. The flow is still assumed to have the properties of irrotationality and the fluid is supposed to be incompressible and inviscid. The amplitude of the fluid oscillation is also assumed to be negligible compared with the vertical depth of the container.

Assume that a trough is given with the same shape as the trough in Section 4, but whose length in the y direction is much more than its width in the x direction. The fluid occupies the region $-a/2 \leq x \leq a/2$, $-d \leq y \leq d$, $-a/2 \pm x \leq z \leq 0$. The walls are positioned at $y = \pm d$ and $z = -a/2 \pm x$ and the length d used here is analogous to the length b of the tank in the y direction in the preceding sections. The cross section for constant value of y is the same as in Figure 8 in Section 4.1 and a plan view of the resting position of the trough under consideration here is given in Figure 15 below. In this diagram, the z axis is such that it points up out of the plane of the paper.

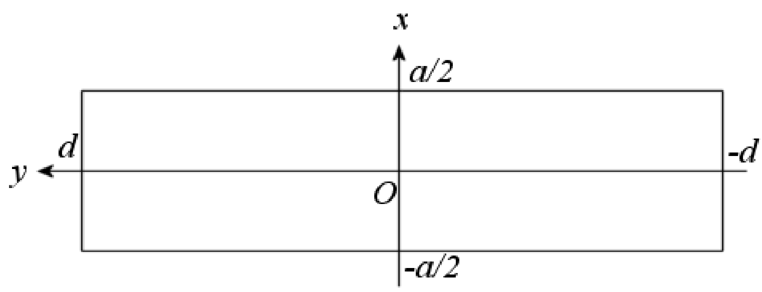

Fig 15: Plan view of trough

Further, the tank rests on a smooth pivot located at the point $(x, y, z) = (0, 0, -a/2)$ in three dimensional Cartesian coordinate space, or is subject to torsion due to a system of wires which suspend the trough and keep the bottom of the trough horizontal. With the axes remaining in the same position in space, i.e. not moving with the trough, the fluid in the trough is set into motion and allowed to settle to a regular periodic motion. The fluid motion induces the tank to move so that at time $t = 0$ the tank occupies its equilibrium position heading instantaneously clockwise as viewed from above. A restoring torque that is proportional to the angle through which the tank has travelled clockwise from its equilibrium position opposes the motion of the tank. If Γ is the restoring torque and θ is the angle through which the tank has moved clockwise from its resting position then

$$\Gamma = -K\theta$$

where $K > 0$ is the constant of proportionality. (As a side note, in SI units, K is measured in Nm rad^{-1}). The angle θ is so small that the approximation $\theta = \sin(\theta)$ holds well. The displacement of interest is the displacement of the end wall of the trough given by $y = d$ and this is derived now. If the wall described by $y = d$ moves with a displacement given by $X(t) = X_0 \sin(\omega t)$ in the x direction where $2|X_0| \ll a$ then a fair approximation for $X(t)$ is

$$X(t) = \frac{X_0}{d} \sin(\omega t)$$

where y is constant. For a general y value, where $X_0 = X_1 y/d$ where $2|X_1| \ll a$, this result generalises to

$$X(t) = X_1 \frac{y}{d} \sin(\omega t) \tag{32}$$

The only problem now is to determine a valid velocity potential. As $a \ll d$ due to the trough being much longer than it is wide, each section through the trough in a plane of constant y gives the two dimensional oscillatory motion discussed in Section 4.3. For each such section

$$\phi = \frac{2X_0\omega^3}{2g - a\omega^2}\left(z + \frac{a}{2}\right) x\cos(\omega t) \tag{33}$$

gives the valid velocity potential, where again ω is the frequency of oscillation. However, for the rotating trough, $X_0 \propto y$, thus the velocity potential in equation (33) is seen to be a solution of Laplace's equation in three spatial dimensions. Substituting the expression for X_0 into (33) leads to the required velocity potential

$$\phi = \frac{2X_1}{d(2g - a\omega^2)}\left(z + \frac{a}{2}\right) xy\cos(\omega t)$$

The free surface for this flow is found from (7) and is given by

$$\zeta = A_0 xy\sin(\omega t) \tag{34}$$

where A_0 is the constant amplitude of the fluid motion. The behaviour of the free surface at $t = \pi(2\omega)^{-1}$ is shown qualitatively in Figure 16. Note that for all times $t \geq 0$ the free surface has a saddle at the point $x = 0$, $y = 0$.

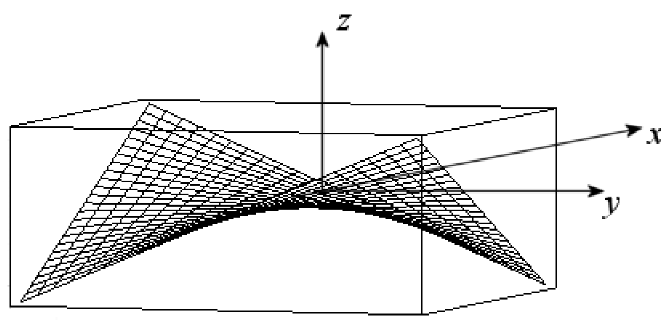

Fig 16: Free surface at extremal position

The one remaining thing to be accomplished is closure of the system of equations. This is accomplished by using the equation of motion for the coupled system. The equation of motion under consideration here will be the rotational counterpart of Newton's Second Law. If I is the moment of inertia of the dry tank and L is the total moment about the z axis due to the net liquid torque acting on the trough then the equation of motion is given by

$$I\frac{\mathrm{d}^2\vartheta}{\mathrm{d}t^2} = L + \Gamma$$

where $\vartheta = d\sin(\theta)\sin(\omega t)$ gives the displacement of the wall at $y = d$ with $\vartheta \approx X_1\sin(\omega t)$ as θ is assumed small. The quantity L is easily found from the expression for F_l in Section 4.3 using the equation

$$L = \int_{-d}^{d} yF_l(y)\mathrm{d}y$$

Here $X(t) = X_0\sin(\omega t)$ has been replaced with $X(t) = X_0(y)\sin(\omega t)$. Thus the equation of motion gives

$$-I\omega^2 X_1\sin(\omega t) = 2md^2X_1\omega^2\left(\frac{6g - a\omega^2}{9(2g - a\omega^2)}\right)\sin(\omega t) - KX_1d\sin(\omega t)$$

This leads to the relation

$$\omega^4(9Ia + 2md^2a) - 3\omega^2(6Ig + 4md^2g + 9Ka) + 18Kg = 0 \qquad (35)$$

where m is the mass of fluid in the container. Deploying the dimensionless variable $\Omega^2 = a\omega^2/(2g)$ equation (35) gives

$$2\Omega^4\left(\frac{9Ig + 2md^2g}{Ka}\right) - 3\Omega^2\left(\frac{6Ig + 4md^2g}{Ka} + 1\right) + 9 = 0$$

and introducing $\alpha = md^2g/(Ka)$ and $\gamma = Ig/(Ka)$ as dimensionless variables, this simplifies further to give the dimensionless equation

$$2\Omega^4 - 3\frac{6\alpha + 4\gamma + 3}{9\alpha + 2\gamma}\Omega^2 + 9 = 0$$

The discriminant for this dimensionless quadratic equation in Ω^2 is

$$\Delta = \frac{9}{4}\left(\frac{6\alpha + 12\gamma + 9}{9\alpha + 2\gamma}\right)^2 - \frac{18}{9\alpha + 2\gamma}$$

which rearranges to give

$$\Delta = \frac{81(\alpha + 1)^2 + (18\gamma + 25/2)^2 + 324\alpha\gamma - 55)}{(9\alpha + 2\gamma)^2}$$

Thus there is a possibility for two real values for Ω^2, but only those solutions to the quadratic in Ω^2 which are positive give the possible frequencies of oscillation for the system. The graph in Figure 17 shows the behaviour of Ω where $\gamma = 1$ and $m \propto d$ such that $\alpha = d^3$ for values of d in the interval $[0, 5]$. Here the dashed line gives the low frequency behaviour and the solid line gives the high frequency behaviour. Note that in this case the low frequency case is only observable if $d > 1$.

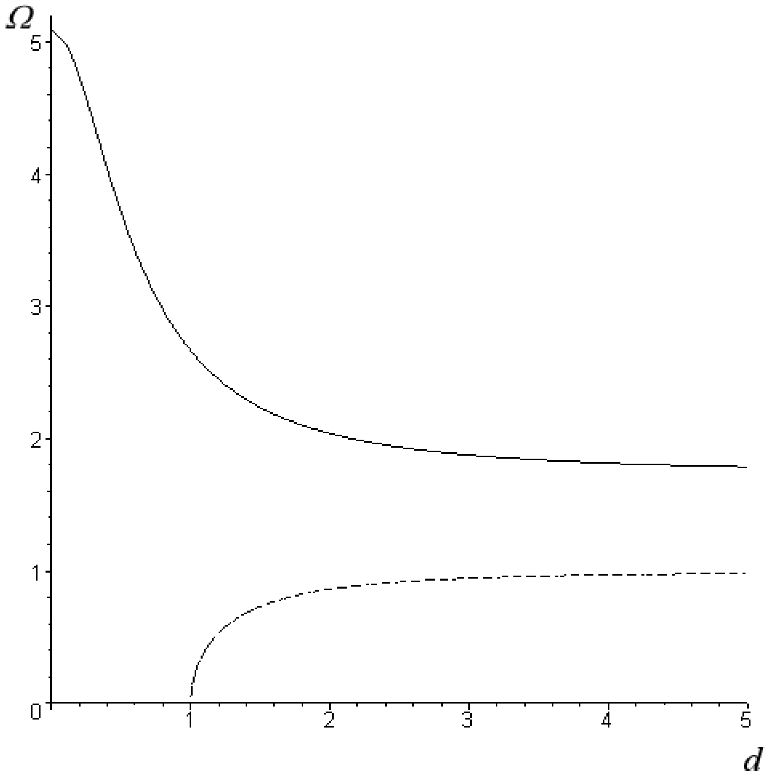

Fig 17: A graph of Ω against d.

6 Conclusion

In this project, various types of oscillatory motion of a fluid-container system have been investigated. Section 1 gives some references which were of use and which may benefit those interested in the background work for this project. Section 2 introduces the general hypothetical methods which are utilised throughout the project for a tank of arbitrary shape.

6.1 The general equation

It is important to note here that all of the results for Ω in sections 3 and 4 may be derived from the dimensionless quadratic

$$\Omega^4 - \eta\Omega^2 + \xi = 0 \tag{36}$$

in which

$$\eta = \frac{6(M+m)ga + 3Ka^2}{6(M+m)ga - 8\rho bc^3 g}; \qquad \xi = \frac{3Ka^2}{6(M+m)ga - 8\rho bc^3 g}$$

and the notation is as in section 3.3. Notice that the discriminant Δ of (36) is given by

$$\Delta = \frac{[6(1+\mu)ga - 3Ka^2M^{-1}]^2 + 72\rho a^2 bc^3 KgM^{-2}}{[6(1+\mu)ga - 8\rho bc^3 gM^{-1}]^2}$$

where $m = M\mu$. Thus $\Delta \geq 0$ for all $c \in [0, a/2]$, so from the general quadratic (36) there is always at least one real (positive) value for Ω^2, provided the denominator of Δ is nonzero. This fact is reinforced in the work of Sections 3 and 4.

Allowing K to tend to zero in equation (36) yields the results for Section 3.2 and also taking the limit as $M \to \infty$ gives the results found in Section 3.1. As has already been noted, the results in section 4 are analogous to the results in Section 3 on allowing c to tend to $a/2$, and this is also true in equation (36). Thus it is seen that (36) is the backbone of this dissertation, and that this underlying equation is both simple and elegant.

6.2 General recap of results

The main body of the theoretical work begins in section 3, where the general theoretical results for hyperbolic troughs undergoing translation motion in one direction are recorded.

Section 3.1 gives the framework for all of the following sections. In this section of the dissertation, the case of the fluid motion in a fixed hyperbolic container of uniform cross section is considered. It is shown that there is a unique frequency at which the fluid may oscillate whilst obeying the assumptions as set out in Section 2. The free surface is described in three dimensions as a plane which oscillates about the midpoint of the mean free surface for all time. An explicit expression for the liquid force acting on the container bed was also derived, using Bernoulli's equation.

Section 3.2 is concerned with the motion of a hyperbolic container which is free to move in one direction with no external constraints. The motion of the fluid gives rise to the motion of the container and the oscillations of the container are assumed small. Under the assumptions of Section 2 it was found that there is a unique frequency of oscillation at which this motion can be sustained and that the tank and fluid motions must be in antiphase.

The work of Section 3.3 is a generalisation of the work in the previous subsection and introduces the constraint on the system of a restoring force due to a Hookean spring. Under this constraint, there were found to be two frequencies of oscillation for the system. The lower frequency (in absolute value) corresponds to oscillations of the system in which the fluid and tank motions are in phase. The higher frequency value is the frequency of oscillation where the tank and fluid motions are in antiphase. The interesting discovery of minimum values for the frequencies of oscillation with respect to the tank depth h, for some intermediate value of h, was also shown graphically in this subsection. It isimportant to note that the work in Sections 3.1 and 3.2 are special cases of the work in Section 3.3 (on letting the spring constant tend to zero to recover Section 3.2. If in addition to this the mass of the tank is made to tend to infinity whilst the fluid mass remains constant, Section 3.1 is obtained). In the case of the restrained moving tank, it is found that there is an increase of frequency with depth, this increase being quite rapid initially, but decreases considerably for larger tank depths.

The work of Section 4 is a special case of the work in Section 3. The tank in this section has walls which coincide with the straight lines to which the hyperbolic containers of Section 3 asymptote. As might be expected, many of the results are similar to those of Section 3 (such as the expression for the free surface, which is unaltered), but here is one very interesting difference. In Section 4.3, where there is motion of the system subject to the constraints of springs obeying Hooke's law, there is a frequency at which resonance may occur. An explicit value was calculated for this resonant frequency and this coincides with the upper bound for the frequency in Section 4.2.

Section 5 gives a theoretical account of the motion on a liquid-container system under torsion. A velocity potential for this three dimensional problem was derived from the two dimensional velocity potential used in Sections 2 and 3. The free surface was found to be a hyperboloid of one sheet which oscillates in time and has a saddle a the middle of the mean free surface for all times considered. The lower frequency of oscillation for the system was found to be non complex only for large enough tank lengths. This lower frequency corresponds to a motion of tank and fluid which are in phase (the higher frequency, as before, gives a motion of the system in which the tank and fluid motions are out of phase).

6.3 Open questions for the interested

In this work, a simple linearised theory is used to give models for translational and rotational motions of fluid-container couples. The existence of a velocity potential is presumed, this is implied by the irrotationality of the flow. As the fluid is also assumed to be incompressible, the velocity potential thus satisfies Laplace's equation in the relevant number of spatial variables in any of the problems considered. Another consequence of irrotationality is that Bernoulli's equation for the pressure in the region of flow is applicable everywhere in the fluid. The last assumptions are that the free surface displacement is small compared to the tank depth and the amplitude of translational motion is small compared with the width of the tank. Thus some natural questions are:

- What happens if a viscous fluid is modelled?
- How does the behaviour of the systems alter if the fluid is compressible?
- What is the effect on the analysis of the systems if the free surface has an amplitude which is not negligible when compared with the tank depth?
- In the analysis, how large can X_0 (the amplitude of the tank oscillation) become whilst not affecting the accuracy of the theoretical results detrimentally?

Also, the general approach of moving from the fixed tank case to a moving case is to add $X_0 \omega x \sin(\omega t)$ to the velocity potential of the fixed case. The interested reader may check that for the most general case of motion for a fixed tank with walls at $x = 0, a$ and $z = -h$, which has velocity potential

$$\phi(x, z, t) = \sum_{n=1}^{\infty} A_{2n-1} \cos\left(\frac{(2n-1)\pi}{a} x\right) \cosh\left(\frac{(2n-1)\pi}{a}(z+h)\right) \cos(\omega t)$$

has a problem with the general approach pointed out in the above. This velocity potential is used to model the motion of a liquid in a fixed rectangular container which is independent of y for a suitable choice of starting time $t = 0$. So two more questions are:

- Is it possible to model the situation of a moving rectangular tank via the velocity potential for the fixed tank?
- How are the velocity potentials altered for a tank of arbitrary geometry on moving from the case of the fixed tank to that of an oscillating fluid-tank system?

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