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Multiple Equilibria in Tullock Contests

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Abstract
We find the sufficient conditions for the existence of multiple equilibria in Tullock-type contests and show that asymmetric equilibria may arise even under symmetric prize and cost structures. We also identify contests in the literature where multiple equilibria exist under reasonably weak conditions.

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rent-seeking, contest, asymmetric equilibrium, multiple equilibria

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1. Introduction

Contests are a type of games in which players expend costly efforts (resources) in order to win prize(s). The effort expenditures by players determine their respective probabilities of winning a prize. The function that maps efforts into probabilities of winning is called a contest success function (CSF). One of the most frequently used CSFs in the contest literature is a lottery CSF of Tullock (1980); in which the probability of winning equals the ratio of a player's effort to the sum of all players' efforts.[1]

In this paper we consider a Tullock-type contest in which players' outcome-contingent payoffs are linear functions of prizes, own effort, and the effort of the rival. Under this structure we find the sufficient conditions for the existence of multiple equilibria in this contest. We show that asymmetric equilibria may arise even under symmetric prize and cost structures. We also identify several contests in which multiple equilibria may arise under very general conditions.

The existing literature documents that asymmetry in prize valuation (Nti, 1999), cost structure (Paul and Wilhite, 1990), and effectiveness in influencing the CSF (Gradstein, 1995) can result in asymmetric equilibrium. In this paper, however, we show that even under symmetric set up one may obtain asymmetric equilibria in Tullock-type contests.

Szidarovszky and Okuguchi (1997) prove the existence and uniqueness of the symmetric equilibrium for a simple Tullock contest. Cornes and Hartley (2005) extend the analysis and argue that multiple equilibria may exist in contests with increasing returns CSFs. Yamazaki (2008) reaffirms this result for contests in which players are asymmetric in terms of value, effectiveness and budget constraints. In this paper we show that the uniqueness of equilibrium

[1] Tullock's lottery CSF is widely employed because a number of studies have provided axiomatic justification for it (Skaperdas 1996; Clark and Riis 1998). Also, Baye and Hoppe (2003) identified conditions under which a variety of rent-seeking contests, innovation tournaments, and patent-race games are equivalent to the Tullock contest.

crucially depends on the specification of the cost and spillover parameters in the payoff function. Under very general restrictions, even under a standard lottery CSF multiple equilibria may exist in symmetric Tullock contests.

The finding that multiple equilibria may arise in simple Tullock-type contests is important for a number of reasons. First, in multi-stage or repeated games the existence of multiple non-payoff equivalent equilibria means that one can condition equilibrium selection in the subgame based on past behavior. This allows for a wide range of payoffs to be supported as subgame perfect equilibria. Second, in the presence of multiple equilibria, comparative statics have to be conditioned on a particular equilibrium since different equilibria may lead to different comparative statics results. Finally, the existence of multiple equilibria is important for designing both static and dynamic contests. A contest designer needs to account for the full profile of equilibria and corresponding comparative statics in order to achieve a given objective.

2. Contest Model and Equilibria

We consider a Tullock-type contest involving two risk-neutral players and two prizes. The players, denoted by i and j, value the winning prize as $W > 0$ and the losing prize as $L \in \mathbb{R}$, with $W > L$. Players simultaneously expend efforts $x_i \geq 0$ and $x_j \geq 0$. The probability that player i is the winner is decided by a lottery CSF:

$$p_i(x_i, x_j) = \begin{cases} x_i/(x_i + x_j) & \text{if } x_i + x_j \neq 0 \\ 1/2 & \text{if } x_i = x_j = 0 \end{cases} \qquad (1)$$

The outcome contingent payoff for player i is a linear function of prizes, own effort, and the effort of the rival:

$$\pi_i(x_i, x_j) = \begin{cases} W + \alpha_1 x_i + \beta_1 x_j & \text{with probability } \quad p_i(x_i, x_j) \\ L + \alpha_2 x_i + \beta_2 x_j & \text{with probability } \ 1 - p_i(x_i, x_j) \end{cases} \qquad (2)$$

where α_1, α_2 are cost and β_1, β_2 are spillover parameters with restrictions $\alpha_1 < 0$ and $\alpha_2 \leq 0$.

Define the contest described by (1) and (2) as $\Gamma(i, j, \Omega)$, where $\Omega = \{W, L, \alpha_1, \alpha_2, \beta_1, \beta_2\}$ is a set of parameters. Under complete information the expected payoff for player i is:

$$E\left(\pi_i(x_i, x_j)\right) = \frac{x_i}{x_i + x_j}\left(W + \alpha_1 x_i + \beta_1 x_j\right) + \frac{x_j}{x_i + x_j}\left(L + \alpha_2 x_i + \beta_2 x_j\right) \tag{3}$$

where $(x_i, x_j) \neq (0,0)$. For $x_i = x_j = 0$, the expected payoff is $E\left(\pi_i(x_i, x_j)\right) = (W + L)/2$.

By taking first order condition in (3), player i's best response function (BRF) is

$$x_i^{BRF} = -x_j + \sqrt{\frac{\{(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2)\}x_j^2 - \{W - L\}x_j}{\alpha_1}} \tag{4}$$

if $x_j \leq (W - L)/(-\alpha_2 - \beta_1 + \beta_2)$, and otherwise, $x_i^{BRF} = 0$. And the corresponding unique symmetric equilibrium is:[2]

$$x_i^* = x_j^* = x = \frac{(W - L)}{-(3\alpha_1 + \alpha_2) - (\beta_1 - \beta_2)} \tag{5}$$

The slope of the BRF is derived as:

$$\frac{\partial x_i^{BRF}}{\partial x_j} = -1 + \frac{2\{(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2)\}x_j - \{W - L\}}{2\sqrt{\alpha_1\left[\{(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2)\}x_j^2 - \{W - L\}x_j\right]}} \tag{6}$$

It is clear that the slope, as well as, the curvature of the BRF is different for different values of the cost and spillover parameters. The BRF is a parabola, and if the curvatures of the two BRFs are large enough, then the two parabolas may intersect in multiple points, generating multiple equilibria. Therefore, in addition to the symmetric equilibrium (5), the contest $\Gamma(i, j, \Omega)$ can generate two asymmetric equilibria (see Figures 1 and 2). The additional restriction $(5\alpha_1 - \alpha_2) - (\beta_1 - \beta_2) > 0$ guarantees a large enough curvature of the BRF to generate asymmetric equilibria $\{x_i^* = \overline{x}; x_j^* = \underline{x}\}$ and $\{x_i^* = \underline{x}; x_j^* = \overline{x}\}$, where

[2] This particular equilibrium is derived in Chowdhury and Sheremeta (2010), who show that the needed restrictions for this equilibrium are: $\alpha_1 < 0$, $\alpha_2 \leq 0$, $\beta_2 - \alpha_1 \geq 0$, and $-(3\alpha_1 + \alpha_2) - (\beta_1 - \beta_2) > 0$. They also show that when the BRF is positive, then the first order condition is necessary and sufficient for equilibrium.

$$\overline{x} = \frac{1}{2}\frac{(W-L)}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)}\left[1+\sqrt{\frac{(5\alpha_1-\alpha_2)-(\beta_1-\beta_2)}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)}}\right] \tag{7}$$

$$\underline{x} = \frac{1}{2}\frac{(W-L)}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)}\left[1-\sqrt{\frac{(5\alpha_1-\alpha_2)-(\beta_1-\beta_2)}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)}}\right] \tag{8}$$

By imposing further incentive compatibility restriction $\sqrt{\frac{(5\alpha_1-\alpha_2)-(\beta_1-\beta_2)}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)}} \geq \frac{(4\alpha_1-\alpha_2)-(\beta_1-2\beta_2)}{(2\alpha_1-\alpha_2)-\beta_1}$ we ensure that the players are willing to expend equilibria efforts, i.e. $E\big(\pi_i(x_i^* = \overline{x}, x_j^* = \underline{x})\big) \geq L$ and $E\big(\pi_j(x_i^* = \overline{x}, x_j^* = \underline{x})\big) \geq L$. These results are summarized in the following Proposition.[3]

Proposition: In contest $\Gamma(i,j,\Omega)$, if $-(3\alpha_1+\alpha_2)-(\beta_1-\beta_2) > 0$ and $\beta_2-\alpha_1 \geq 0$ then there exists a symmetric equilibrium defined by (5). Furthermore, if $(5\alpha_1-\alpha_2)-(\beta_1-\beta_2) > 0$ and $\sqrt{\frac{(5\alpha_1-\alpha_2)-(\beta_1-\beta_2)}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)}} \geq \frac{(4\alpha_1-\alpha_2)-(\beta_1-2\beta_2)}{(2\alpha_1-\alpha_2)-\beta_1}$ then in addition there exist two asymmetric equilibria defined by (7) and (8).

3. Examples of Multiple Equilibria

Next we consider several contests in which multiple equilibria may exist. In the 'lazy winner' contest of Chowdhury and Sheremeta (2010) the winner faces lower marginal cost than the loser, i.e. $\Gamma(i,j,\{W,0,\alpha_1,\alpha_2,0,0\})$ with $|\alpha_1| < |\alpha_2|$. The payoff function for player i is:

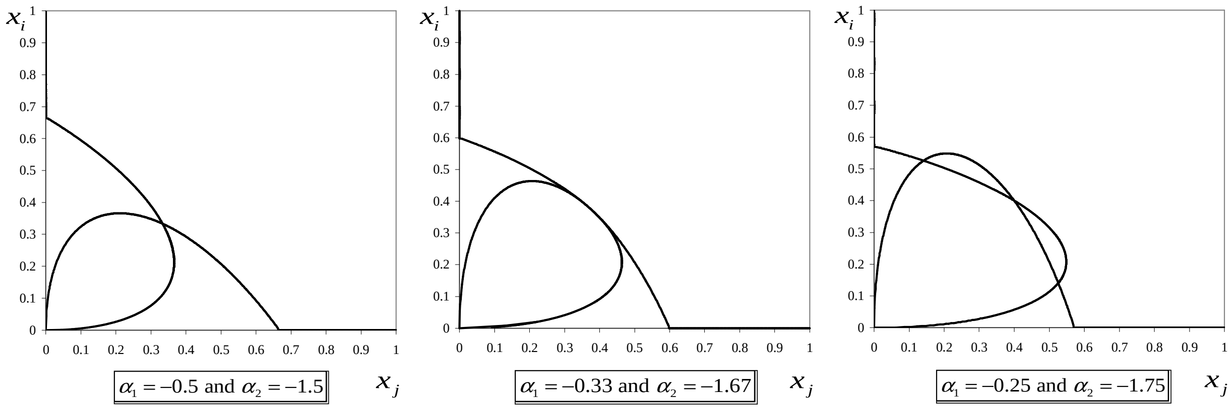
$$\pi_i(x_i,x_j) = \begin{cases} W+\alpha_1 x_i & \text{with probability} \quad p_i(x_i,x_j) \\ \alpha_2 x_i & \text{with probability} \quad 1-p_i(x_i,x_j) \end{cases} \tag{9}$$

Under symmetric equilibrium, according to the proposition, both players expend equal efforts $x_i^* = x_j^* = W/(-3\alpha_1-\alpha_2)$. However, this contest can also generate multiple equilibria if the

[3] This type of equilibria is informally described by Schelling (1971) in the context of racial segregation. The proposition matches in flavour with Schelling's conjecture on multiple equilibria (see Figure 19). Schelling shows that the symmetric equilibrium in his setting is a stable equilibrium, but the two asymmetric equilibria are unstable.

difference between the cost parameters is sufficiently high, i.e. $5\alpha_1 > \alpha_2$.[4] In Figure 4.1 we plot the BRFs for different values of marginal costs. When $\alpha_1 = -0.25$ and $\alpha_2 = -1.75$, the BRFs intersect three times, indicating one symmetric and two asymmetric equilibria. This result comes from the perceptive behavior of the players. One player gives more weight to the fact that the loser has a higher marginal cost and thus expends a low effort in equilibrium. On the other hand, the other player envisions a lower marginal cost of winning and expends a higher effort.

Figure 1: BRFs and Equilibria in 'Lazy Winner' Contest (W = 1)



Multiple equilibria can also arise in contests with spillovers (Chung, 1996; Chowdhury and Sheremeta, 2010). Consider, for example, a general 'input spillover' contest, where the effort expended by player j partially benefits player i and vice versa. Such a contest can be written as $\Gamma(i, j, \{W, 0, -1, -1, \beta_1, \beta_2\})$, where $\beta_1 \geq 0$, $\beta_2 \geq 0$, and $\beta_1 - \beta_2 < 4$. The payoff function of 'input spillover' contest takes the form:

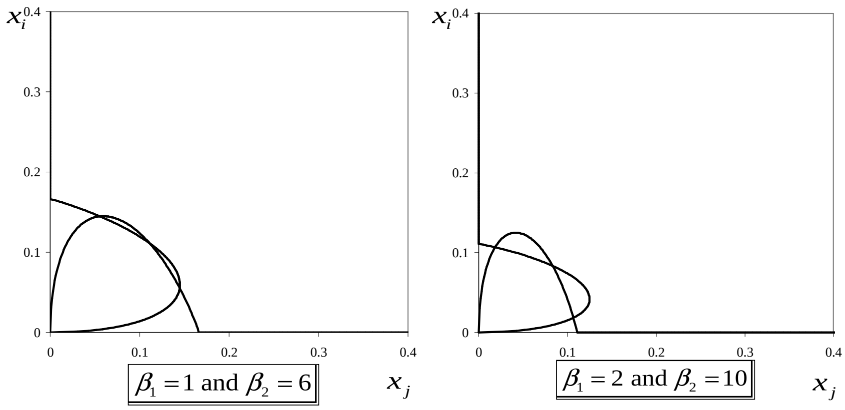
$$\pi_i(x_i, x_j) = \begin{cases} W - x_i + \beta_1 x_j & \text{with probability } p_i(x_i, x_j) \\ -x_i + \beta_2 x_j & \text{with probability } 1 - p_i(x_i, x_j) \end{cases} \tag{10}$$

Under symmetric equilibrium, both players expend equal efforts $x_i^* = x_j^* = W/(4 - \beta_1 + \beta_2)$. Figure 2 displays the BRFs and the resulting equilibria for different values of β_1 and β_2. When

[4] The incentive compatibility restriction also holds. The two asymmetric equilibria, defined by the proposition, are given by $\{x_i^* = \overline{x}; x_j^* = \underline{x}\}$ and $\{x_i^* = \underline{x}; x_j^* = \overline{x}\}$, where $\overline{x} = \frac{1}{2}\frac{\sqrt{\alpha_1-\alpha_2}+\sqrt{5\alpha_1-\alpha_2}}{\sqrt{(\alpha_1-\alpha_2)^3}}W$ and $\underline{x} = \frac{1}{2}\frac{\sqrt{\alpha_1-\alpha_2}-\sqrt{5\alpha_1-\alpha_2}}{\sqrt{(\alpha_1-\alpha_2)^3}}W$.

spillover gain of the loser is sufficiently higher than the spillover gain of the winner, we arrive at the case of multiple equilibria. In particular, any combination of β_1 and β_2, such that $\beta_1 - \beta_2 < -4$, will generate one symmetric and two asymmetric equilibria. In any asymmetric equilibrium, one player expends very high effort, increasing the chance of winning, while the other player expends very low effort, ensuring a significant spillover benefit from losing. This scenario resembles R&D contests in countries where property rights are not protected by the government and the spillover in case of losing is very high. Therefore, there is a strong incentive to free ride on the effort of the others.

Figure 2: BRFs and Equilibria in 'Input Spillover' Contest (*W* = 1)



One can apply our analysis to show that multiple equilibria can also arise in contests of Amegashie (1999), Glazer and Konrad (1999), and Matros and Armanios (2009). For example, Glazer and Konrad (1999) study a contest $\Gamma(i, j, \{(1-t)w, 0, -(1-t), -1,0,0\})$ in which the non-negative profit of a rent-seeker is taxed by a tax rate $t \in (0,1)$. It is easy to show that when the tax rate is excessively high (i.e. more than 80%) then, besides the symmetric equilibrium, multiple equilibria exist. In the endogenous prize value contest by Amegashie (1999), the winner's prize value is a linear function of own effort expended, i.e. $\Gamma(i, j, \{W, 0, -(1-m), -1,0,0\})$ where $m \in (0,1)$ shows the impact of own effort on prize value. If this impact is high enough, then following the aforementioned logic this contest induces multiple equilibria. Finally, Matros and Armanios (2009) examine a contest where either the winner or the loser or both can be reimbursed. A two-payer version of the contest can be written as $\Gamma(i, j, \{W, 0, (\alpha - 1), (\gamma - 1), 0, 0\})$ where $\alpha \in (0,1)$ and $\gamma \in (0,1)$ are the reimbursement parameters. Using our proposition, it is straightforward to show that when $5\alpha - 4 > \gamma$ then, in addition to the symmetric equilibrium, two asymmetric equilibria exist.[5]

There are other contest settings that can produce multiple equilibria. For example, Baye et al. (2005) use an all-pay auction to analyze several litigation systems in which the winner or the loser compensates a part of the rival's legal expenditure. By modeling such litigation contests as Tullock-type contests, one can show that certain legal systems, such as the 'Continental system of litigation,' can produce multiple equilibria.

4. Conclusion

In this paper, we construct a two-player Tullock contest under complete information and find the sufficient conditions for the existence of multiple equilibria in this setting. We show that asymmetric equilibria may arise even under symmetric prize and cost structures. We also identify several contests in which multiple equilibria may arise under very general conditions. The findings of this paper can be applied to areas of contest design, R&D spillovers, litigations and repeated games, where multiple equilibria may arise. One can also extend the analysis in the current study in terms of incomplete information, the number of players, risk aversion, and non-linear CSFs. We leave these questions for future research.

[5] Following the same procedure, one can derive multiple equilibria in Cohen and Sela (2005), where only the winner is reimbursed. This has been independently shown by Matros (2009).

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