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# Saturated simplicial complexes

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**Abstract**

Among shellable complexes a certain class has maximal modular homology, and these are the so-called *saturated* complexes. We extend the notion of saturation to arbitrary pure complexes and give a survey of their properties. It is shown that saturated complexes can be characterized via the $p$-rank of incidence matrices and via the structure of links. We show that rank-selected subcomplexes of saturated complexes are also saturated, and that order complexes of geometric lattices are saturated.


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## 1. Introduction

Let $\Delta$ be a simplicial complex on the vertex set $\Omega$. The standard simplicial homology theory is concerned with the $\boldsymbol{Z}$-module $\boldsymbol{Z}\Delta$ with basis $\Delta$ and the boundary map

$$\tau \mapsto \sigma_1 - \sigma_2 + \sigma_3 - \cdots \pm \sigma_k,$$

which assigns to the face $\tau$ the alternating sum of the co-dimension 1 faces of $\tau$. This defines a homological sequence over $\boldsymbol{Z}$ and hence over any domain with identity.

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In [18] we started to investigate the same module with respect to a different homomorphism. This is the *inclusion map* $\partial : \mathbf{Z}\Delta \to \mathbf{Z}\Delta$ given by

$$\partial : \quad \tau \mapsto \sigma_1 + \sigma_2 + \sigma_3 + \cdots + \sigma_k .$$

Then $\partial^2 \neq 0$ unless $\Delta$ is trivial. However, when coefficients are taken modulo an integer $p$ then a simple calculation shows that in fact $\partial^p = 0$. One may attempt therefore to build a generalized *modular homology theory* of simplicial complexes, in particular when $p$ is a prime. This kind of homology appears to be mentioned first in Mayer [15] in 1942; further historical remarks and references can be found in [1,18]. Among the more recent papers note also Tikaradze [28] and Berger et al. [2].

The goal of this paper is to investigate complexes which have nice properties in modular homology. That such complexes exist is not obvious: modular homology is not homotopy invariant nor is it a topological invariant. Even among shellable complexes there are examples of complexes with the same $h$-vector but with different modular homology. The behaviour of the modular homology of non-shellable complexes is even more erratic, see Section 3.3 later.

The key to understanding the topological properties of complexes with good modular homology is the study of the links in the complex. More precisely, the crucial property is for links to admit cycles which are null both for standard and modular homology. For one-dimensional complexes (graphs) these are *even cycles* . In arbitrary dimension complexes with this property include Coxeter complexes or more generally, two-colourable triangulations of spheres, which could be considered as ‘generalized even cycles’.

This observation leads us to conjecture that complexes with enough ‘generalized even cycles’ will have properties in modular homology that are not dissimilar to standard homology. In particular, any complex in which links admit Coxeter-type reflection groups as automorphisms would be a candidate for this class of complexes. In this regard we mention also Borovik’s recent survey of Coxeter matroids [9]. Coxeter complexes also illustrate the fact that a topological space, here the sphere, can have a rich structure in modular homology depending on such triangulations.

Another approach to complexes with nice modular homological properties is completely algebraic: It has been shown in [18] that the modular homology of every shellable complex can be embedded into a well-understood module constructed purely from the shelling of the complex. It follows in particular that the modular Betti numbers for an arbitrary shellable complex are bounded by functions of its $h$-vector only. More generally, a complex whose modular Betti numbers attain this bound is called *saturated*, and such complexes are the principal subject of this article.

It is interesting to look again at the situation in dimension one: a connected graph is saturated if and only if is bipartite, that is, all its cycles are of even length. This is not accidental: we shall show that the topological and algebraic approach both lead to the same class of shellable complexes. Our main results are Theorems 4.4 and 6.3 which characterize saturated complexes via the structure of links and via the $p$-rank of incidence matrices, respectively.

As we show, the modular homology of *saturated* complexes behaves in some respects quite similarly to standard simplicial homology over fields. In fact, there is a surprising analogy between the modular homology of saturated complexes and the standard simplicial homology of Cohen–Macaulay complexes. For example,

- The standard homology of a Cohen–Macaulay complex over a field is completely determined by the last component of the $h$-vector of the complex. Similarly, the modular homology of a saturated complex is completely determined by its $h$-vector; however now *all* components of the $h$-vector are significant.
- The order complex of a geometric lattice is both Cohen–Macaulay and saturated.
- It is well-known that the type-selected subcomplex of a balanced Cohen–Macaulay complex again is Cohen–Macaulay. We will show that the same holds for saturated complexes.
- The Steinberg module appears among the top modular homologies of a saturated complex, just as for standard homology, where the Steinberg module is the unique top homology of a Cohen–Macaulay complex.

For modular homology these properties are proved in Theorems 5.1 and 6.12. The Steinberg modules will be treated in a forthcoming paper. Thus, the loss of homotopy invariance in modular homology is not too unsatisfactory if we are looking at saturated complexes. At the same time, the fundamental advantage of modular homology over standard simplicial homology is that the *inclusion map commutes with the action of all automorphisms*. In particular, all modular homology modules are modules for the full automorphism group of the complex. This is far from true for standard homology. Applications of such group actions can be found in [20,21].

In Section 2, we collect the prerequisites from previous papers as far as they are needed here. In Section 3, we extend the definition of saturation from shellable complexes to pure complexes in general. This section also contains many important examples. In Section 4, we prove one of the main results, the Null-Link Theorem which gives a topological characterization of saturation. In Section 5, it is shown that geometric lattices are saturated for all primes, and Section 6 gives additional applications, including the fact that the rank selection of a saturated complex remains saturated.

## 2. Prerequisites

In this section, we shall introduce the main notation for this paper. It follows closely our papers [18,19] and it may be useful to consult these papers for further details. However, we hope that the notes in the following section will render this paper reasonably self-contained.

### 2.1. Simplicial complexes and modules

Let $\Omega$ be a finite set and let $\Delta \subseteq 2^{\Omega}$ be a simplicial complex on the vertex set $\Omega$. Thus whenever $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. As we consider no other complexes often the word 'simplicial' is omitted. The elements of $\Delta$ are called *simplices* or *faces* and the maximal faces are the *facets* of $\Delta$. If $\Delta = 2^{\Omega}$ then $\Delta$ is the *simplex* on $\Omega$. If $0 \leqslant k$ we let $\Delta_k := \{\sigma \in \Delta : |\sigma| = k\}$. The *dimension* of $\sigma \in \Delta$ is $\dim \sigma := |\sigma| - 1$ and the *dimension* of $\Delta$ is the maximum of $\dim \sigma$ for $\sigma \in \Delta$. The complex is called *pure* of dimension $k$ if all facets of $\Delta$ have dimension $k$. If $\sigma \in \Delta$ then the *star* is the complex $\mathrm{star}_\Delta(\sigma)$ whose facets are those facets of $\Delta$ which contain $\sigma$, and the *link* is the subcomplex $\mathrm{link}_\Delta(\sigma) \subset \mathrm{star}_\Delta(\sigma)$ of all faces which have empty intersection with $\sigma$.

Now let $F$ be a field and let $0 \leqslant k$ be an integer. Then we denote by $M_k^\Delta$ the $F$-vector space with basis $\Delta_k$. We put $M^\Delta := \oplus_{0 \leqslant k}\ M_k^\Delta$ so that

$$M^\Delta = \{ \sum_{\sigma \in \Delta} f_\sigma \sigma \ : \ f_\sigma \in F \}$$

and in particular $\Delta \subseteq M^\Delta$ by identifying $\sigma$ with $1\sigma$. Clearly, if $\Delta' \subseteq \Delta$ is another complex then $M^{\Delta'} \subseteq M^\Delta$ and we set $M^* := M^{2^\Omega}$, the module attached to the complete simplex on $\Omega$. If $|\Omega| = m$ we may also write $M^m := M^*$. Now consider the linear map $\partial : M^* \to M^*$ defined on a basis of $M^*$ by

$$\partial : \quad \Omega \supseteq \sigma \mapsto \sum \tau,$$

where the summation runs over all $\tau \subset \sigma$ with $\dim\ \tau = \dim\ \sigma - 1$. This map is called the *inclusion map*. Note that it restricts to a map $\partial : M^\Delta \to M^\Delta$ precisely as $\Delta$ is a complex. Thus attached to $\Delta$ there is the sequence

$$\mathcal{M}^\Delta : \quad 0 \xleftarrow{\partial} M_0^\Delta \xleftarrow{\partial} M_1^\Delta \xleftarrow{\partial} M_2^\Delta \ \cdots \ \xleftarrow{\partial} M_{k-1}^\Delta \xleftarrow{\partial} M_k^\Delta \xleftarrow{\partial} \ \cdots \ \xleftarrow{\partial} 0$$

and such sequences are the subject of this paper.

The *support* of the element $f = \sum_{\sigma \subseteq \Omega} f_\sigma \sigma \in M^*$ is the set

$$\mathrm{supp}\ f = \bigcup \{\sigma \ : \ f_\sigma \neq 0\}$$

and its *weight* is the number

$$\mathrm{wt}(f) = |\{\sigma \ : \ f_\sigma \neq 0\}|.$$

We say that two elements $f,\ g \in M^*$ are *disjoint* if $\mathrm{supp}\ f \cap \mathrm{supp}\ g = \emptyset$. If $g = \sum_{\tau \subseteq \Omega} g_\tau \tau$ then we put

$$f \cup g := \sum_{\sigma, \tau \subseteq \Omega} f_\sigma g_\tau (\sigma \cup \tau).$$

This union product turns $(M^*,\ \cup)$ into an associative algebra with $\emptyset$ as identity. The fundamental relationship between the union product and the inclusion map is the *product rule*

$$\partial(f \cup g) = f \cup \partial(g) + \partial(f) \cup g,$$

which holds whenever $f$ and $g$ are disjoint. It is therefore natural to write $f' = \partial(f)$ and $f^{(k)} = (f^{(k-1)})'$.

For every $\sigma \subseteq \Omega$ there is a *decomposition* relative to $\sigma$: Let $f \in M^*$ and let $\sigma$ be any subset of $\Omega$. Then we may decompose $f$ with respect to $\sigma$ uniquely as

$$f = \sum_{\tau \subseteq \sigma} f^\tau \cup \tau$$

in such a way that for all $\tau \subseteq \sigma$ we have that $f^{\tau} \in M^*$ is disjoint from $\tau$. If $\Delta$ is a complex on $\Omega$, if $f \in M^{\Delta}$ and if $\sigma \subseteq \Omega$ is arbitrary one may note more precisely that each $f^{\tau}$ belongs to $M^{\Lambda} \subseteq M^{\Delta}$ where $\Lambda = \mathrm{link}_{\Delta}(\tau)$. In a fashion, $f^{\tau}$ can be regarded as the *linearized link* of $\tau$ with respect to *f*. Applying the product rule it is a simple matter to verify the following fact which will be needed later:

**Lemma 2.1.** *Let $i > 0$ be an integer, let $f \in M^*$ and $\sigma \subseteq \Omega$. Then*

$$f^{(i)} = \sum_{\tau \subseteq \sigma} \left( \sum_{k=0}^{i} k! \binom{i}{k} \sum_{\tau \subseteq \rho \subseteq \sigma : |\rho \setminus \tau| = k} (f^{\rho})^{(i-k)} \right) \cup \tau.$$

**Proof.** Establish this when *f* is a set in $\Omega$. The result then follows by linearity. $\square$

## 2.2. Shellability

For shellability one is interested in the inductive process of gluing an $\Sigma^n$-simplex onto a given pure $(n-1)$-dimensional complex $\Gamma$. If this is done in such a way that the intersection $\Gamma \cap \Sigma^n$ is a pure $(n-2)$-dimensional complex of *k* facets then the resulting complex $\Gamma \overset{k}{\cup} \Sigma^n$ is said to be a *k-gluing* of $\Sigma^n$ onto $\Gamma$. Thus a pure $(n-1)$-dimensional complex $\Delta$ is *shellable* if its facets can be arranged as a *shelling sequence*

$$\sigma_1,\ \sigma_2,\ \sigma_3, \ldots,\ \sigma_{i-1},\ \sigma_i,\ \ldots$$

in such a way that for all $i = 1, \ldots$ the composition

$$(2^{\sigma_1} \cup 2^{\sigma_2} \cup \cdots \cup 2^{\sigma_{i-1}}) \overset{k_i}{\cup} 2^{\sigma_i}$$

is a $k_i$-gluing for some $k_i$. In this case we set

$$h_j := |\{ i \ : \ k_i = j \}|$$

for $j = 1, \ldots, n$ and call $(h_0,\ h_1,\ \ldots,\ h_n)$ the *h-vector* of $\Delta$. If we put $f_j := |\Delta_j|$ then $(f_0,\ f_1,\ \ldots,\ f_n)$ correspondingly is the *f-vector* of $\Delta$. As is well-known, see [4,10], these two important quantities are related by

$$f_i = \sum_{j=0}^{n} \binom{n-j}{i-j} h_j. \tag{2.1}$$

By inverting this relation we have equivalently

$$h_j = \sum_{i=0}^{n} (-1)^{j+i} \binom{n-i}{j-i} f_i. \tag{2.2}$$

*2.3. The modular homology of complexes*

We recall the main definitions from [18,19] on modular homology. Assume that $\Delta \subseteq 2^\Omega$ is a complex of dimension $n-1$. Denote by $\Delta_k$ the set of faces $\sigma \in \Delta$ with $|\sigma| = k$. Further, let $F$ be a field of characteristic $p > 0$. Then it is a simple matter to verify that $\partial^p$ is the zero map on $M^\Delta$. For any $j$ and $0 < i < p$ consider the sequence

$$\cdots \xleftarrow{\partial^*} M^\Delta_{j-p} \xleftarrow{\partial^*} M^\Delta_{j-i} \xleftarrow{\partial^*} M^\Delta_{j} \xleftarrow{\partial^*} M^\Delta_{j+p-i} \xleftarrow{\partial^*} M^\Delta_{j+p} \xleftarrow{\partial^*} \cdots$$

in which $\partial^*$ is the appropriate power of $\partial$. It is convenient to regard this as an infinite sequence by setting $M^\Delta_\ell = 0$ for $\ell < 0$ and $n < \ell$. This sequence is determined by any arrow $M^\Delta_l \leftarrow M^\Delta_r$ in it and so is denoted by $\mathcal{M}^\Delta_{(l,r)}$. (Of course, $(l, r)$ stands for left-right.) The unique arrow $M^\Delta_a \leftarrow M^\Delta_b$ for which we have $0 \leqslant a + b < p$ is the *initial arrow* and $M^\Delta_b$ is the 0-*position* of $\mathcal{M}^\Delta_{(l,r)}$. (For instance, if $p = 5$ then $\mathcal{M}^\Delta_{(-2,2)} = \mathcal{M}^\Delta_{(2,3)} = \mathcal{M}^\Delta_{(3,7)}$ has initial arrow $M^\Delta_{-2} \leftarrow M^\Delta_2$.) The position of any other module in $\mathcal{M}^\Delta_{(l,r)}$ is counted from this 0-position and $(a, b)$ is the *type* of $\mathcal{M}^\Delta_{(l,r)}$. Further, the *weight* of $\mathcal{M}^\Delta_{(l,r)}$ is the integer $w = w_{(l,r)}$ with $0 < w \leqslant p$ with $w \equiv l + r - n \pmod p$.

Since $\partial^p = 0$ we have $(\partial^*)^2 = 0$ and so $\mathcal{M}^\Delta_{(l,r)}$ is a homological sequence. The homology at $M^\Delta_{j-i} \leftarrow M^\Delta_j \leftarrow M^\Delta_{j+p-i}$ is denoted by

$$H^\Delta_{j,i} := (\operatorname{Ker} \partial^i \cap M^\Delta_j)/\partial^{p-i}(M^\Delta_{j+p-i})$$

and $\beta^\Delta_{j,i} := \dim H^\Delta_{j,i}$ is the corresponding Betti number. The rank of $\partial^{r-l} : M^\Delta_l \leftarrow M^\Delta_r$ is denoted by $\mathrm{rk}^\Delta_p(l, r)$. The Euler characteristic $\chi^\Delta_{(l,r)}$ of $\mathcal{M}^\Delta_{(l,r)}$ is

$$\pm\,\chi^\Delta_{(l,r)} := \cdots + f_{j-p} - f_{j-i} + f_j - f_{j+p-i} + f_{j+p} \;\ldots,$$

where the parity can be calibrated on the zero position of $\mathcal{M}^\Delta_{(l,r)}$.

If $\mathcal{M}^\Delta_{(l,r)}$ has at most one non-vanishing homology then it is said to be *almost exact* and the only non-trivial homology then is denoted by $H^\Delta_{(l,r)}$. If $\mathcal{M}^\Delta_{(l,r)}$ is almost exact for every choice of $l$ and $r$ then $\mathcal{M}^\Delta$ is *almost p-exact*. In general, when referring to a particular sequence $\mathcal{M}^\Delta_{(l,r)}$, the homology at position $t$ is denoted by $H^\Delta_t$ and $\beta^\Delta_t := \dim H^\Delta_t$ is the Betti number of $\mathcal{M}^\Delta_{(l,r)}$ at position $t$. The switching between parameter pairs such as $(j, i)$ and position is a useful notational tool which we will employ wherever this can be done without creating a misunderstanding.

Let $\Delta$ be any complex of dimension $n-1$ and suppose that $\mathcal{M}^\Delta_{(l,r)}$ has type $(a, b)$. Then we put

$$d^n_{(l,r)} := \begin{cases} \left\lfloor \frac{n-a-b}{p} \right\rfloor & \text{if } n-a-b \not\equiv 0 \pmod p, \\ \infty & \text{if } n-a-b \equiv 0 \pmod p. \end{cases}$$

### 2.4. The modular homology of the simplex

At the beginning of the discussion of simplicial modular homology stands the homology of the simplex. The simplex on $n$ vertices is denoted by $\Sigma^n$ and the sequences associated to it are denoted by

$$\mathcal{M}^n := \mathcal{M}^{\Sigma^n} \quad\text{and}\quad \mathcal{M}^n_{(l,r)} := \mathcal{M}^{\Sigma^n}_{(l,r)}.$$

The main results on the structure of the $p$-modular homology of $\Sigma^n$ and on its Betti numbers can be found in [1,14,16]. Fundamental is a simple branching rule:

**Theorem 2.2.** *Let $F$ have characteristic $p \geqslant 2$, let $0 < i < p$ and let $k \leqslant n$ be arbitrary. Then $H^n_{k,i} \cong H^{n-1}_{k,i+1} \oplus H^{n-1}_{k-1,i-1}$ is an isomorphism of* $\mathrm{Sym}_{n-1}$*-modules.*

From this theorem the following facts, while not immediately obvious, can nevertheless all be derived without much effort:

**Theorem 2.3.** *Let $p$ be a prime and let $0 < i < p$. Then for each $k \leqslant n$ the following holds*:

(i) *Middle Term Condition*: *$H^n_{k,i} = 0$ unless $n - p < 2k - i < n$. In particular,*
(ii) *Almost Exactness*: *The sequence $\mathcal{M}^n$ is almost $p$-exact.*
(iii) *Irreducibility*: *$H^n_{k,i}$ is an irreducible* $\mathrm{Sym}_n$*-module iff $2k - i = n - 1$.*
(iv) *Brauer Character*: *On each $H^n_{k,i}$ the Brauer character is given by $\chi(g, H^n_{k,i}) = \pm\sum_{t=-\infty}^{+\infty} \mathrm{fix}_{k-pt}(g) - \mathrm{fix}_{k-i-pt}(g)$ where $\mathrm{fix}_k(g)$ denotes the number of $k$-sets fixed by $g \in \mathrm{Sym}_n$. (The sign is determined by $\chi(1, H^n_{k,i}) \geqslant 0$.)*
(v) *Betti Number*: *For all $l, r$ with $0 < r - l < p$ the Betti numbers of $\mathcal{M}^n_{(l,r)}$ are zero in all positions $\neq d^n_{(l,r)}$ while $\beta^n_{(l,r)} = |\sum_{t=-\infty}^{+\infty} \binom{n}{l-pt} - \binom{n}{r-pt}|$ in position $d^n_{(l,r)}$.*

The expression in the last part of the theorem are ordinary binomial coefficients and so the sum in particular is finite. As we have seen, for each $l, r$ with $0 < r-l < p$ the sequence $\mathcal{M}^n_{(l,r)}$ is almost exact and its only non-trivial homology is the

$$\textit{Fibonacci module} \quad \cdots = H^n_{(r-p,l)} = H^n_{(l,r)} = H^n_{(r,l+p)} = \cdots$$

associated to this $(l, r)$-sequence. This terminology follows Ryba's paper [24] which contains a different and very explicit construction of these modules for the particular case $p = 5$. The dimension of the Fibonacci module is the Betti number

$$\beta^n_{(l,r)} = \dim H^n_{(l,r)}.$$

A fundamental property of Fibonacchi modules has been stated in [24,1]: If $H^n_{j,i}$ is non-zero then it is generated by classes of the shape $[c_{j,i-1}]$ where $c_{j,i-1}$ has the form $c_{j,i-1} = v \cup s$, with $v = (\alpha_1 - \beta_1) \cup \cdots \cup (\alpha_{j-i+1} - \beta_{j-i+1})$, where $s$ is an $(i-1)$-face $s = \{\gamma_1, \ldots, \gamma_{i-1}\}$ and where $\alpha_i$, $\beta_i$ and $\gamma_i$ are pairwise distinct vertices. Note that the terms in $v$ represent the alternately signed faces of an $(j - i)$-dimensional octahedron. Therefore the terms of

$c_{j,i}$ correspond to an $(i-1)$-fold cone over this octahedron. By a convenient abuse of terminology we shall therefore call $c_{j,i-1}$ a *generalized octahedron*, or just an *octahedron*. It is easy to check that $\partial^i(c_{j,i-1}) = 0$ over every field and so octahedra can be considered as analogues of ‘constants’ for $\partial$. We shall return to this in Section 4.4.

The Fibonacci modules are of key importance for the entire theory of modular simplicial homology. Evaluating the above formula for the Betti numbers the following emerges. If $p = 2$ then $\beta^n_{(l,r)} = 0$, as expected, for this is the standard homology of the simplex taken mod 2; if $p = 3$ then $\beta^n_{(l,r)} = 0$ or 1. For $p = 5$ all Betti numbers are ordinary Fibonacci numbers. For $p > 5$ the expression for $\beta^n_{(l,r)}$ amounts to linear recurrence relations of degree $(p-1)/2 > 2$.

Further examples of branching rules will appear later. Such rules apply for instance to the suspension and the cone over a complex. Theorem 6.6 later in fact generalizes Theorem 2.2.

## 3. The definition of saturation

Let $\Delta$ be a pure complex over a finite vertex set $\Omega$ and let $p \geqslant 2$ be a prime. In [19] we have introduced the notion of saturation for a shellable complex in relation to the prime *p*. There we already showed that saturation for shellable complexes has several equivalent definitions. The purpose of this section is to expand further on the combinatorial and algebraic significance of saturation. We shall then extend the definition of saturation to arbitrary complexes.

### 3.1. The embedding property for shellable complexes

Let $\Delta$ be pure and shellable of dimension $n-1$ with *h*-vector $h(\Delta) = (h_0, h_1, \ldots, h_n)$. Fix two integers $(l, r)$ with $0 < r - l < p$ and as in Section 2 we consider the sequence

$$\mathcal{M}^{\Delta}_{(l,r)}: \cdots \xleftarrow{\partial^*} M^{\Delta}_{j-p} \xleftarrow{\partial^*} M^{\Delta}_{j-i} \xleftarrow{\partial^*} M^{\Delta}_{j} \xleftarrow{\partial^*} M^{\Delta}_{j+p-i} \xleftarrow{\partial^*} M^{\Delta}_{j+p} \xleftarrow{\partial^*} \cdots .$$

As before we let $w_{(l,r)}$ be the weight of $\mathcal{M}^{\Delta}_{(l,r)}$ and we let $m_{(l,r)} := \min\{d^n_{(l,r)},\ d^{n+1}_{(l,r)}\}$ be the *middle* of the sequence. One of the main results in [18] was the following *embedding property:*

**Theorem 3.1** (*Embedding Theorem*). *The homology of $\mathcal{M}^{\Delta}_{(l,r)}$ is zero in the initial positions, that is*

$$H^{\Delta}_t = 0 \quad for\ all\ t < m := m_{(l,r)}, \tag{3.1}$$

*while in all other positions there is a canonical embedding*

$$H^{\Delta}_{m+s} \hookrightarrow \bigoplus_{j=w+(s-1)p+1}^{w+sp} h_j H^{n-j}_{(l-j,r-j)} \quad for\ w := w_{(l,r)}\ and\ all\ s \geqslant 0. \tag{3.2}$$

Here of course $hH$ stands for the direct sum $H \oplus H \oplus \cdots \oplus H$ of $h$ summands. From this theorem the saturation of shellable complexes is defined as follows:

**Definition 3.2.** The shellable complex $\Delta$ is $(l,r)$-*saturated* relative to $p$ if (3.2) is an isomorphism for all $s \geqslant 0$. Further, $\Delta$ is *saturated* if $\Delta$ is $(l,r)$-saturated for all $(l,r)$.

We mention that there are shellable complexes which are $(l,r)$-saturated for certain parameters $(l,r)$ but not for others, and that there are complexes which are saturated for $p=2$ but for no other primes. Examples of such situations shall appear later.

### 3.2. Formal Betti numbers and saturation in general

The module on the right-hand side of (3.2) depends on the $h$-vector of $\Delta$ only. If $\Delta$ is a not necessarily shellable complex we may still define its $h$-vector as a function of its $f$-vector by the use of the relation (2.2) in Section 2.3. Hence the module on the right-hand side of (3.2) above can be defined for an arbitrary pure complex, at least if all the $h_j$ are non-negative. This motivates the next definition:

**Definition 3.3** (*Formal Betti numbers*). Let $\Delta$ be an arbitrary pure complex of dimension $n-1$ with $h$-vector $h(\Delta)=(h_0,\ h_1,\ \ldots,\ h_n)$. For given $(l,r)$ we set $m:=m_{(l,r)}$ and $w:=w_{(l,r)}$. Now let the *formal Betti numbers* of $\mathcal{M}^{\Delta}_{(l,r)}$ be given as

$$\beta_t^{h(\Delta)} := 0 \quad \text{for all } t<m$$

and

$$\beta_{m+s}^{h(\Delta)} := \sum_{j=w+(s-1)p+1}^{w+sp} h_j\,\beta^{n-j}_{(l-j,r-j)} \quad \text{for all } s \geqslant 0,$$

where $\beta^{n-j}_{(l-j,r-j)}$ is the appropriate Betti number of the $(n-j-1)$-simplex given in Theorem 2.3.

We shall now give a combinatorial interpretation of these formal Betti numbers. The Euler characteristic of

$$\mathcal{M}^{\Delta}_{(l,r)}:\quad \cdots \xleftarrow{\partial^*} M^{\Delta}_{j-p} \xleftarrow{\partial^*} M^{\Delta}_{j-i} \xleftarrow{\partial^*} M^{\Delta}_{j} \xleftarrow{\partial^*} M^{\Delta}_{j+p-i} \xleftarrow{\partial^*} M^{\Delta}_{j+p} \xleftarrow{\partial^*} \cdots$$

satisfies

$$\begin{aligned} \pm\,\chi^{\Delta}_{(l,r)} &= \cdots + f_{j-p} - f_{j-i} + f_j - f_{j+p-i} + \cdots \\ &= \cdots + \beta^{\Delta}_{j-p} - \beta^{\Delta}_{j-i} + \beta^{\Delta}_{j} - \beta^{\Delta}_{j+p-i} + \cdots \end{aligned} \tag{3.3}$$

by the Euler–Poincaré equation. Using (2.1) we may now formally substitute the $f_j$ by the components of the $h$-vector and collect terms in ascending order of index. Using the expressions for Betti numbers of simplices from Theorem 2.3 and taking $h_i=0$ if $i<0$ or $i>n$, it turns out that the Euler characteristic takes the shape

$$\pm\chi^{\Delta}_{(l,r)} = \cdots + \beta^{h(\Delta)}_{j-p} - \beta^{h(\Delta)}_{j-i} + \beta^{h(\Delta)}_{j} - \beta^{h(\Delta)}_{j+p-i} + \cdots \tag{3.4}$$

(A rigorous proof of this claim involves tedious and trivial calculations which we shall avoid. An example is given in [19, p. 384].)

Comparing (3.3) and (3.4) one may make the naïve conjecture that each *actual* Betti number $\beta_j^{\Delta}$ in (3.4) is equal to the corresponding *formal* Betti number $\beta_j^{h(\Delta)}$ in (3.3). This is evidently not true in general. (The boundary of a simplex gives the simplest example of the situation when formal and actual Betti numbers differ.) However, the next result follows immediately from the Embedding Theorem:

**Theorem 3.4.** *Let $\Delta$ be a shellable complex. Then the following are equivalent*:
(i) *$\Delta$ is $(l, r)$-saturated, and*
(ii) *the corresponding actual and formal Betti numbers in $\mathcal{M}_{(l,r)}^{\Delta}$ coincide.*

Recall that (2.2) defines the *h*-vector formally for an arbitrary complex. This leads us to the definition of saturation for a general pure simplicial complex:

**Definition 3.5** (*Saturated complexes*). Let $\Delta$ be an arbitrary pure complex and let $F$ be a field of characteristic $p > 0$. Then $\Delta$ is $(l, r)$-*saturated* (*in characteristic* $p$) if and only if all actual and formal Betti numbers in $\mathcal{M}_{(l,r)}^{\Delta}$ coincide. Further, $\Delta$ is *saturated* (*in characteristic* $p$) if $\Delta$ is $(l, r)$-*saturated* for all $(l, r)$.

It follows from Theorem 2.3 that the simplex is saturated for all primes, and this fact is the basis of induction for all the complexes we examine.

*Comments*: (1) In this definition we allow $F$ to have characteristic 2, when modular homology coincides with standard homology. According to the definition here $\Delta$ is saturated in characteristic 2 if and only if its homology is concentrated in the top dimension, that is $\beta_n^{\Delta} = h_n$, $\beta_i^{\Delta} = 0$ for $i < n$. In particular, complexes which are Cohen–Macaulay over **GF**(2) are saturated in characteristic 2.

(2) There are non-shellable complexes which are saturated: These include the order complex of the posets on pages 599 and 600 of the survey [7]. The first is a triangulation of the real projective plane with $f = (1, 13, 36, 24)$ and $h = (1, 10, 13, 0)$. It is Cohen–Macaulay over all fields of characteristic $\neq 2$ but it is not Cohen–Macaulay over $\boldsymbol{Z}$ or **GF**(2). It is also saturated for $p > 2$ and not saturated in characteristic 2.

The second is the triangulation of the *dunce hat* with $f = (1, 17, 52, 36)$ and $h = (1, 14, 21, 0)$. It is acyclic and Cohen–Macaulay over $\boldsymbol{Z}$ but not (lexicographically) shellable. It is also saturated in characteristic 3.

Other examples include the well-known non-shellable triangulations of 3-balls such as the *knotted hole ball* described by Furch in 1924 and the *2-roomed house* constructed by Bing in 1964. Both are saturated in characteristic 3 and Cohen–Macaulay with $f = (1, 380, 1929, 2722, 1172)$ and $f = (1, 480, 2511, 3586, 1554)$ respectively. For this see [12,13,30]. See the next section for more examples and remarks.

### 3.3. Remarks and examples

**Remark 1.** Modular homology, just as standard homology, depends on the topology of the complex, not only on the *h*-vector. For example, the natural triangulations of the

2-dimensional torus $T$, see Munkres [22, p. 17–18], and of the Klein bottle $K$ share the same $f$-vector $f(T) = f(K) = (1, 9, 27, 18)$, and so have the same $h$-vector $h(T) = h(K) = (1, 6, 12, -1)$. Nevertheless, the 3-modular homologies of $T$ and $K$ are different: $T$ is $(1, 3)$-saturated but $K$ is not $(1, 3)$-saturated. Similar examples are known [18, p. 362] even for shellable complexes.

**Remark 2.** The behaviour of the modular homology of non-shellable complexes seems to be extremely erratic. In particular, the *embedding property* may not hold, and homologies can be non-trivial even in positions $t < m$ to the left of the middle. An example for this is the 7-dimensional analogue $\Delta$ of the Möbius band given by

$$\begin{aligned}\Delta = \{&\{1, 2, 3, 4, 5, 6, 7, 8\}, \{2, 3, 4, 5, 6, 7, 8, 9\}, \{3, 4, 5, 6, 7, 8, 9, 10\}, \{4, 5, 6, 7, 8, 9, 10, 11\}, \{5, 6, 7, 8, 9, 10, 11, 12\},\\ &\{6, 7, 8, 9, 10, 11, 12, 13\}, \{7, 8, 9, 10, 11, 12, 13, 14\}, \{8, 9, 10, 11, 12, 13, 14, 15\}, \{9, 10, 11, 12, 13, 14, 15, 16\},\\ &\{10, 11, 12, 13, 14, 15, 16, 17\}, \{11, 12, 13, 14, 15, 16, 17, 18\}, \{12, 13, 14, 15, 16, 17, 18, 19\}, \{13, 14, 15, 16, 17, 18, 19, 20\},\\ &\{14, 15, 16, 17, 18, 19, 20, 21\}, \{15, 16, 17, 18, 19, 20, 21, 22\}, \{16, 17, 18, 19, 20, 21, 22, 23\}, \{17, 18, 19, 20, 21, 22, 23, 24\},\\ &\{7, 18, 19, 20, 21, 22, 23, 24\}, \{6, 7, 19, 20, 21, 22, 23, 24\}, \{5, 6, 7, 20, 21, 22, 23, 24\}, \{4, 5, 6, 7, 21, 22, 23, 24\},\\ &\{3, 4, 5, 6, 7, 22, 23, 24\}, \{2, 3, 4, 5, 6, 7, 23, 24\}, \{1, 2, 3, 4, 5, 6, 7, 24\}\ \}\end{aligned}$$

with $f(\Delta) = (1, 24, 168, 504, 840, 840, 504, 168, 24)$. For $p = 3$ the sequence $\mathcal{M}^{\Delta}_{(1,2)}$ is exact while $\mathcal{M}^{\Delta}_{(1,3)}$ and $\mathcal{M}^{\Delta}_{(2,3)}$ have non-zero Betti numbers $\beta_{3,2} = 1$, $\beta_{4,1} = 24$ and $\beta_{3,1} = 1$, $\beta_{5,2} = 24$, respectively. The middle position of the sequence $\mathcal{M}^{\Delta}_{(1,3)}$ is $m_{(1,3)} = 2$. However, $\beta_{3,2} = 1$ occurs in position 1.

**Example 3.** It is instructive to work out the formal Betti numbers in terms of the $h$-vector for some low-dimensional complexes. Here we do this for a complex of dimension 7. So let $\Delta$ have $h$-vector $(h_0, h_1, \ldots, h_8)$. In the following table the formal Betti numbers for $p = 3$ are given (We suppress superscripts and write $\beta_{i,j}$ instead of $\beta^{h(\Delta)}_{i,j}$.)

| (l,r) | w | | | | |
|---|---|---|---|---|---|
| (1,2) | 1 | $\beta_{4,2} = h_0$; | $\beta_{5,1} = h_2 + h_3$; | $\beta_{7,2} = h_5 + h_6$; | $\beta_{8,1} = h_8$ |
| (1,3) | 2 | $\beta_{4,1} = h_0 + h_1$; | $\beta_{6,2} = h_3 + h_4$; | $\beta_{7,1} = h_6 + h_7$ | |
| (2,3) | 3 | $\beta_{5,2} = h_1 + h_2$; | $\beta_{6,1} = h_4 + h_5$; | $\beta_{8,2} = h_7 + h_8$ | |

Similarly, for $p = 5$ the formal Betti numbers are the following:

| $(l, r)$ | $w$ | | | |
|---|---|---|---|---|
| (1,2) | 5 | $\beta_{6,4} = 8h_1 + 8h_2 + 5h_3 + 2h_4$; | | $\beta_{7,1} = h_6 + h_7$ |
| (1,3) | 1 | $\beta_{3,2} = 21h_0$; | $\beta_{6,3} = 8h_2 + 8h_3 + 5h_4 + 2h_5$; | $\beta_{8,2} = h_7 + h_8$ |
| (1,4) | 2 | $\beta_{4,3} = 34h_0 + 13h_1$; | | $\beta_{6,2} = 5h_3 + 5h_4 + 3h_5 + h_6$ |
| (1,5) | 3 | $\beta_{5,4} = 21h_0 + 13h_1 + 5h_2$; | | $\beta_{6,1} = 2h_4 + 2h_5 + h_6$ |
| (2,3) | 2 | $\beta_{3,1} = 21h_0 + 8h_1$; | $\beta_{7,4} = 3h_3 + 3h_4 + 2h_5 + h_6$; | $\beta_{8,1} = h_8$ |
| (2,4) | 3 | $\beta_{4,2} = 34h_0 + 21h_1 + 8h_2$; | | $\beta_{7,3} = 3h_4 + 3h_5 + 2h_6 + h_7$ |
| (2,5) | 4 | $\beta_{5,3} = 21h_0 + 21h_1 + 13h_2 + 5h_3$; | | $\beta_{7,2} = 2h_5 + 2h_6 + h_7$ |
| (3,4) | 4 | $\beta_{4,1} = 13h_0 + 13h_1 + 8h_2 + 3h_3$; | | $\beta_{8,4} = h_5 + h_6 + h_7 + h_8$ |
| (3,5) | 5 | $\beta_{5,2} = 13h_1 + 13h_2 + 8h_3 + 3h_4$; | | $\beta_{8,3} = h_6 + h_7 + h_8$ |
| (4,5) | 1 | $\beta_{4,4} = 13h_0$; | | $\beta_{5,1} = 5h_2 + 5h_3 + 3h_4 + h_5$ |

Several observations can be made. For instance, the coefficients of the $h_i$ always belong to the set of values taken by the dimension of the Fibonacci module in the corresponding characteristic. Thus they belong to $\{0,\ 1\}$ for characteristic $p = 3$ and to $\{0,\ 1, 2,\ 3,\ 5,\ 8,\ \dots\}$ for characteristic $p = 5$. Similarly, the 'highest' Betti number $\beta_{8,1}$ is the same in both characteristics, with obvious patterns for $\beta_{8,2},\dots, \beta_{8,p-1}$. To some of these observations we shall return later.

**Example 4.** Let $\Gamma$ be a graph with $n$ vertices and $m$ edges. As a one-dimensional complex $\Gamma$ is shellable if and only if it is connected, in which case $m \geqslant n - 1$. It is easy to note that $\Gamma$ is saturated in characteristic $p$ if and only if its incidence matrix has $p$-rank $n - 1$. It is also easy to check then that even cycles are saturated for every $p$ while odd cycles are not saturated for $p > 2$. In fact, we shall see later that a graph is saturated for $p > 2$ if and only if it is bipartite.

**Example 5.** Finite Coxeter complexes and spherical buildings are saturated for every prime $p$, see [19]. It is easy to check that the sporadic $C_3$-geometry for $A_7$ with $f = (1, 57, 315, 315)$ and $h = (1, 54, 204, 56)$, constructed by Neumaier (see [23, p. 50]), is saturated in characteristic 3.

## 4. The topological condition for saturation

In this section, we shall be interested in the geometric and topological aspects of saturation. In particular,

*if $\Gamma$ is a saturated $(n - 1)$-dimensional complex and if $\Delta := \Gamma \overset{k}{\cup} \Sigma^n$ is a k-gluing, under what conditions on the gluing is it true that also $\Delta$ is saturated?*

A comprehensive answer to this question would in particular classify all shellable saturated complexes. For $p = 2$ the modular homology coincides with standard simplicial homology. Thus, the homology of $\Delta$ and $\Gamma$ is the same for $k < n$ and for $k = n$ all but the top homology is the same, with the top homology increased by 1. However, for $p > 2$ the situation is rather more complicated. First, we shall pose the problem above in a slightly more general form, when $\Gamma$ itself may not be saturated.

### *4.1. Gluing sequences*

Let now $p$ be a prime $>2$. We assume that $\Gamma$ is an arbitrary pure complex of dimension $n - 1$ and we suppose that $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ is a $k$-gluing for some $k \leqslant n$. As in Section 3 we fix two integers $(l, r)$ with $0 < r - l < p$. To compare the homologies of $\mathcal{M}^{\Gamma}_{(l,r)}$ and $\mathcal{M}^{\Delta}_{(l,r)}$ we set $d := d^{n}_{(l,r)}$ and $u := d^{n+k}_{(l,r)}$. In Theorems 4.1 and 4.2 of [18] we have shown the following.

**Theorem 4.1.** *Good cases*: *If $u = d,\ \infty$ or if $H^{\Gamma}_{u-1} = 0$ then $\mathcal{M}^{\Gamma}_{(l,r)}$ and $\mathcal{M}^{\Delta}_{(l,r)}$ have the same homology in all positions except possibly in position u in which case*

$$H^{\Delta}_{u} \simeq H^{\Gamma}_{u} \oplus H^{n-k}_{(l-k,r-k)}.$$

*Bad cases*: *If $d \neq u < \infty$ and $H^{\Gamma}_{u-1} \neq 0$ then $\mathcal{M}^{\Gamma}_{(l,r)}$ and $\mathcal{M}^{\Delta}_{(l,r)}$ have the same homology in all positions except possibly in positions u and $u-1$ in which case the homologies are related via the* 5*-term exact gluing sequence*

$$\mathcal{GS}: 0 \longleftarrow H^{\Delta}_{u-1} \longleftarrow H^{\Gamma}_{u-1} \oplus H^{n}_{u-1} \longleftarrow H^{n}_{u-1} \oplus H^{n-k}_{(l-k,\,r-k)} \xleftarrow{\bar\theta} H^{\Delta}_{u} \longleftarrow H^{\Gamma}_{u} \longleftarrow 0.$$

The map $\bar\theta$ here is crucial. In [19] we showed that in either case $\bar\theta$ has the property $\bar\theta(H^{\Delta}_{u}) \subseteq H^{n-k}_{(l-k,r-k)}$. From exactness at $H^{\Delta}_{u}$ we obtain the embedding

$$H^{\Delta}_{u} \hookrightarrow H^{\Gamma}_{u} \oplus H^{n-k}_{(l-k,r-k)}$$

which occurred in Theorem 3.1.

**Definition 4.2.** The gluing $\Delta := \Gamma \overset{k}{\cup} \Sigma^n$ is $(l,r)$-*saturated over* $\Gamma$ if the conclusion of the first part of Theorem 4.1 holds, that is, if

$$H^{\Delta}_{t} \simeq H^{\Gamma}_{t} \text{ for } t \neq u \quad \text{and} \quad H^{\Delta}_{u} \simeq H^{\Gamma}_{u} \oplus H^{n-k}_{(l-k,r-k)}.$$

Further, $\Delta$ is *saturated over* $\Gamma$ if $\Delta := \Gamma \overset{k}{\cup} \Sigma^n$ is $(l,r)$-saturated for all $(l,r)$.

We have therefore a first answer to the question at the beginning of this section:

**Proposition 4.3.** (i) *The gluing $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ is $(l,r)$-saturated over $\Gamma$ unless we are in one of the bad cases of Theorem* 4.1. *If the latter happens then the gluing is $(l,r)$-saturated if and only if $\bar\theta(H^{\Delta}_{u}) = H^{n-k}_{(l-k,r-k)}$.*

(ii) *Assume that $\Gamma$ is shellable and that $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ is $(l,r)$-saturated over $\Gamma$. Then $\Delta$ is $(l,r)$-saturated iff $\Gamma$ is $(l,r)$-saturated.*

### *4.2. The topological characterization*

To formulate one of the main results of this paper we need additional topological background material, see Björner [3,4] or Stanley [27].

Let $\Gamma$ be a pure complex of dimension $n-1$, let $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ be a $k$-gluing and let $\sigma$ denote the vertex set of $\Sigma^n$. Then the *restriction* $\Re$ of $\sigma$ is the set of all vertices $\beta \in \sigma$ such that $\sigma \setminus \{\beta\}$ is contained in $\Gamma$. So $\Re$ is a $(k-1)$-dimensional face of $\Sigma^n$ and one should regard it as the *outer face* in the gluing. Its complement $\tau := \sigma \setminus \Re$ then is the *inner face* in the gluing. The subcomplexes $\mathrm{star}_{\Delta}(\tau)$ and $\mathrm{link}_{\Delta}(\tau)$ are as defined earlier. In particular, the dimension of $\mathrm{link}_{\Delta}(\tau)$ is $n-|\tau|-1$. The definitions are illustrated in Fig. 1.

In Fig. 1 the restriction of the gluing is $\Re = \{\beta_1,\ \beta_2\}$ and the inner face is $\tau = \{\beta_3, \beta_4\}$. It is useful to regard $2^{\sigma}$ and $\Gamma$ as subcomplexes of $\Delta$. So we could also say that $\Re = \{\delta_1,\ \delta_2\}$ and $\tau = \{\delta_3, \delta_4\}$. Also, $\mathrm{link}_{\Delta}(\tau)$ is the cyclic graph on the vertices $(\delta_1,\ \delta_2,\ \delta_5,\ \delta_6)$.

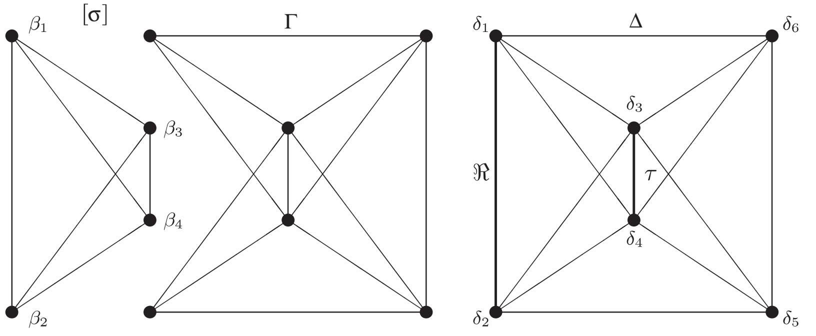


Fig. 1. Gluing $\Sigma^4$ onto $\Gamma$.

The main theorem now follows. When saying that $\Re$ is a *1-cycle of* $\Delta$ *relative to link*$_\Gamma(\tau)$ we mean that there is some $f \in M^\Lambda \subseteq M^\Delta$ with $\partial(\Re + f) = 0$ where $\Lambda := \mathrm{link}_\Gamma(\tau)$.

**Theorem 4.4** (*Null-Link Theorem*). *Let* $\Gamma$ *be a complex and let* $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ *be a gluing with restriction* $\Re$ *and inner face* $\tau$. *Suppose that* $p > 2$. *Then* $\Delta$ *is saturated over* $\Gamma$ *if and only if* $\Re$ *is a* 1-*cycle of* $\Delta$ *relative to* $\mathrm{link}_\Gamma(\tau)$.

In one direction this result is Theorem 4.1 in [19]. The converse is rather involved and will be proved in the next section. First we shall concentrate on the topological significance of the theorem and its combinatorial interpretations. We begin with a further definition which explains the name of the theorem:

**Definition 4.5.** Let $\Lambda$ be a pure complex with facets $\tau_1, \ldots, \tau_m$. Then $\Lambda$ is *null with respect to* $\partial$ *over F*, or just *null* , if there are non-zero $c_1, \ldots, c_m \in F$ such that $\partial(c_1\tau_1 + \cdots + c_m\tau_m) = 0$.

Let $\Lambda$ be a pure complex and let $\Lambda^*$ be a subcomplex of $\Lambda$. Then $\Lambda^*$ is a *part* of $\Lambda$ if all facets of $\Lambda^*$ are also facets of $\Lambda$. We can now give an equivalent formulation of Theorem 4.4:

**Corollary 4.6.** *Let* $\Gamma$ *be a complex and let* $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ *be a gluing with restriction* $\Re$ *and inner face* $\tau$. *Suppose that* $p > 2$. *Then* $\Delta$ *is saturated over* $\Gamma$ *if and only if* $\Re$ *belongs to a part* $\Lambda^*$ *of* $\mathrm{link}_\Delta(\tau)$ *such that* $\Lambda^*$ *is null.*

**Proof.** Let *f* be as in Theorem 4.4 and let $\Re + f = \Re + c_1\lambda_1 + \cdots + c_m\lambda_m$. Now let $\Lambda^*$ be the complex with facets $\Re, \lambda_1, \ldots, \lambda_m$. The rest is evident. □

We will call a null-part of $\Lambda = \mathrm{link}_{\underline{\Delta}}(\tau)$ coming through $\Re$ a *null-continuation* (or *null-extension*) of $\Re$ in $\Lambda$ and denote it by $\overline{\Re}$. Thus, $\Delta$ is saturated over $\Gamma$ if and only if $\Re$ has a null-extension.

Nullness is a most fundamental concept for modular homology. It forms the link between the algebraic and the topological theory in modular simplicial geometry. Note first that nullness with respect to the inclusion map is in no obvious relationship to nullness with respect to the boundary map. However, whenever we deal with *complexes whose links are null in both senses* then deep connections between algebraic and topological properties can be made. Let us say here that a complex is *uniform null* if it is null with regard to the inclusion map and the simplicial boundary map. Note that this property depends on the characteristic of $F$. For one-dimensional complexes for instance, an even cycle is uniform null over every field, while an odd cycle is null in either sense only over fields of characteristic 2.

An important class of uniform null complexes are $n$-dimensional octahedra, also known as cross polytopes or as duals of $n$-dimensional cubes, see Section 4.4 later. As we have noted in Section 2.4, these play a crucial role as generators of the homology of the simplex, see also Theorem 5.2 in [1].

A wider class of uniform null complexes, comprising octahedra and Coxeter complexes, are *bi-colourable pseudo-manifolds without boundary*. These are pure complexes in which each facet can be given one of two colours such that every co-dimension 1 face is contained in exactly two facets and where these facets have different colours. Here it is clear that the coefficients in Definition 4.5 can be taken as $c_i = \pm 1$ according to the bi-colouring. It may be interesting to note that any union of at most $(p-1)/2$ such bi-colourable pseudomanifolds remains uniform null, by adjusting the coefficients $c_i$ in the obvious way.

In [19] we have shown that Coxeter complexes and buildings are saturated; this depended crucially on the nullness of links. The same property will appear later on in this paper in applications of the Null-Link Theorem. To mention are also the Coxeter matroids considered in Borovik's article [9]. Also there we expect to reveal the same deeper relationship between modular and standard homology. We summarize the comments from above in the following corollary, already noted in [18]:

**Corollary 4.7.** *Let* $\Delta = \Gamma \overset{k}{\cup} [\sigma]$ *with inner face* $\tau$ *and suppose that* $p > 2$. *Suppose that* $\mathrm{link}_\Delta(\tau)$ *is a* 2*-colourable pseudomanifold without boundary*. (*In particular*, *suppose that* $\mathrm{link}_\Delta(\tau)$ *is a* 2*-colourable triangulation of a sphere*.) *Then* $\Delta$ *is saturated relative to* $\Gamma$.

### *4.2.1. Some examples: graphs and 1-shellable complexes*

Zero-dimensional complexes are collections of some isolated $v$ vertices, with $h$-vector $(v,\ 0)$. These are always saturated. Pure 1-dimensional complexes are graphs without isolated vertices. As a complex, a graph is shellable if and only if it is connected. We illustrate the Null-Link Theorem by the following simple examples.

**Example 1.** Let $\Gamma$ be the pentagon, i.e. the graph of five vertices and five edges, and let $\sigma$ be one of its diagonals. Then for the gluing $\Delta = \Gamma \cup [\sigma]$ we have $k = 2$, $\tau = \emptyset$ and $\mathrm{link}_\Gamma(\tau) = \Gamma$. There are a three-cycle and a four-cycle through $\sigma$ in $\Gamma \cup [\sigma]$. Choose $\Lambda$ to be the four-cycle. Then $\Lambda$ is null and so $\Delta = \Gamma \cup [\sigma]$ is saturated over $\Gamma$ (but not saturated, as $\Gamma$ is not saturated). In general, the following result may be checked, see also Proposition 6.5 later:

**Corollary 4.8.** *A connected graph* (*as a one-dimensional complex*) *is saturated in characteristic* $p > 2$ *if and only if it is bipartite*.

**Example 2.** The case of $k$-gluing with $k = 1$ is also easy. Indeed, here $|\Re| = k = 1$ and so $\Lambda = \mathrm{link}_\Delta(\tau)$ is just a collection of vertices. Take $\Lambda^*$ to be a pair of vertices. Evidently $\Lambda^*$ is null, and so $\Delta = \Gamma \overset{1}{\cup} [\sigma]$ *is always saturated over* $\Gamma$.

**Example 3.** We call a shellable complex $\Delta$ with $m$ facets 1-*shellable* if $h(\Delta) = (1,\ m-1,\ 0, \ldots, 0)$. Here $k = 1$ for every gluing and so one may consider such complexes as *generalized trees*. The next result follows from the previous example and from the definition of saturated complexes:

**Corollary 4.9.** *Every* 1-*shellable complex* $\Delta$ *is saturated for every* $p > 2$. *Moreover*, *every sequence* $\mathcal{M}^{\Delta}_{(l,r)}$ *is almost p-exact with homology*

$$H^{\Delta}_{(l,r)} \simeq H^{n}_{(l,r)} \ \oplus \ (m-1)H^{n-1}_{(l-1,r-1)}$$

*in the middle*.

### *4.3. The Proof of the Null-Link Theorem*

In this section we shall prove the Null-Link Theorem. In the first part we give a condensed version of the proof for sufficiency as presented in [19]. This lets us introduce all the techniques needed to complete the proof of necessity in the second part.

#### *4.3.1. Sufficiency of nullness*

As we noticed previously, the property of $\Delta = \Gamma \overset{k}{\cup} [\sigma]$ being saturated is completely determined by the map $\bar{\theta}$ in the exact gluing sequence $\mathcal{GS}$. We shall recall briefly the definition of this map.

1. *The definition of the map* $\bar{\theta}$*:* For $\Delta = \Gamma \overset{k}{\cup} [\sigma]$ let $\mathrm{A} := \Gamma \cap [\sigma]$ be the part of the boundary of $[\sigma]$ generated of the $k$ faces of dimension $(n-2)$. Associated to $\Gamma \overset{k}{\cup} [\sigma]$ is the Mayer–Vietoris sequence

$$0 \longleftarrow \mathcal{D} \overset{\psi}{\longleftarrow} \mathcal{C} \oplus \mathcal{B} \overset{\phi}{\longleftarrow} \mathcal{A} \longleftarrow 0$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ denote the modular homological sequences $\mathcal{M}^{\mathrm{A}}_{(l,r)}$, $\mathcal{M}^{n}_{(l,r)}$, $\mathcal{M}^{\Gamma}_{(l,r)}$ and $\mathcal{M}^{\Delta}_{(l,r)}$, respectively.

To define the maps $\phi$ and $\psi$ note that there are natural embeddings $\mathcal{B} \leftarrow \mathcal{A} \rightarrow \mathcal{C}$ and $\mathcal{B} \rightarrow \mathcal{D} \leftarrow \mathcal{C}$ and for $a \in \mathcal{A}$ we indicate its images in $\mathcal{B}$ and $\mathcal{C}$ by $a_B$ and $a_C$ respectively. The same convention applies to $b \in \mathcal{B}$ and $c \in \mathcal{C}$. The homomorphisms $\phi$ and $\psi$ are now given by $\phi(a) := (-a_C,\ a_B)$ and $\psi(c, b) := c_D + b_D$, see also [22, p. 143].

Now the gluing sequence $\mathcal{GS}$ is just an interval of the long homological sequence

$$\cdots \longleftarrow H^{\Delta}_{u-1} \longleftarrow H^{\Gamma}_{u-1} \oplus H^{n}_{u-1} \longleftarrow H^{\mathrm{A}}_{u-1} \overset{\bar{\theta}}{\longleftarrow} H^{\Delta}_{u} \longleftarrow H^{\Gamma}_{u} \oplus H^{n}_{u} \longleftarrow \cdots$$

between $H^A_{u-2} = 0$ and $H^A_u = 0$, see Theorem 3.1 of [18]. Here $\bar{\theta}$ is the usual connecting map. Its definition is standard and may be found in any textbook of homological algebra or algebraic topology, see for example [22] or [19, p. 389]. For short one can say that $\bar{\theta}$ is induced by $\theta = \phi^{-1}\partial^*\psi^{-1}$ where $\psi^{-1}(d)$ is *any* pre-image of $d$ and where $\partial^*$, as before, stands for whatever power of $\partial$ is needed in the context.

Thus $\bar{\theta}$ maps $H^{\Delta}_u$ into $H^{\mathrm{A}}_{u-1}$. To understand this map better we need to explain the structure of the module $H^{\mathrm{A}}_{u-1}$.

2. *The structure of $H^{\mathrm{A}}_{u-1}$:* For this we need some notation. As above, let $\Re$ be the restriction of the gluing $\Delta = \Gamma \overset{k}{\cup} [\sigma]$ and let $\tau = \sigma \setminus \Re$ be the inner face. Let $\mathrm{T} = [\tau] \simeq \Sigma^{n-k}$. Suppose that $H^{\Delta}_u$ corresponds to $H^{\Delta}_{j,i}$ when switching from positions to the two-parameter notation of modular homology, see Section 2.3.

As was mentioned in Section 2.4, Fibonacci modules are generated by octahedra. This means that any element in $H^T_{j,i} \neq 0$ arises as a linear combination of classes $[e]$ of the following shape: The element $e \in M^T_j$ is of the form $e = v \cup s$ where $v = (\alpha_1 - \beta_1) \cup \cdots \cup (\alpha_{j-i+1} - \beta_{j-i+1})$, where $s$ is an $(i-1)$-face of the form $s = \{\gamma_1, \ldots, \gamma_{i-1}\}$ and where the $\alpha_i$, $\beta_i$ and $\gamma_i$ are pairwise distinct. Note that $\partial^i(v \cup s) = 0 = \partial^i(s)$ and $\partial(v) = 0$. The next result has been proved in [18,19]:

**Lemma 4.10.** *The module $H^{\mathrm{A}}_{u-1}$ is isomorphic to $H^n_{u-1} \oplus H^{\mathrm{T}}_{(l-k,r-k)}$. Moreover, the isomorphism $H^{\mathrm{A}}_{u-1}/H^n_{u-1} \simeq H^{\mathrm{T}}_{(l-k,r-k)}$ is of the form $[\partial^i(\Re \cup e)] \leftrightarrow [e] \in H^{\mathrm{T}}_{j-k,i}$ where $e \in M^{\mathrm{T}}_{j-k}$ is an octahedron with $\partial^i(e) = 0$.*

3. *Completing the proof of sufficiency:* As we noticed before in Proposition 4.3, $\Delta$ is saturated over $\Gamma$ if and only if $\bar{\theta}$ has the maximal possible image $\bar{\theta}(H^{\Delta}_u) \simeq H^{\mathrm{T}}_{(l-k,r-k)} \simeq H^{n-k}_{(l-k,r-k)}$. Thus, to show that $\Delta$ is saturated over $\Gamma$ it is enough to prove that for every octahedron $e = s \cup v \in M^{\mathrm{T}}_{j-k}$ with $\partial^i(e) = 0$ one can find some $[h] \in H^{\Delta}_u$ such that $\theta(h) = \partial^i(\Re \cup e)$. Suppose that there is $f \in M^{\mathrm{link}_\Gamma \tau}_k \subset M^{\Delta}$ such that $\partial(\Re + f) = 0$. Now take $h := (\Re + f)_D \cup e_D \in M^{\Delta}$. Since $\partial^i(e) = 0$, and $e_D$ and $(\Re + f)_D$ have non-intersecting supports, also $\partial^i(h) = 0$ and so the corresponding class $[h]$ is in $H^{\Delta}_u$. Then

$$\begin{aligned}
\theta(h) &= \phi^{-1}\partial^i\psi^{-1}[(\Re + f) \cup e]_D \\
&= \phi^{-1}\partial^i([\Re \cup s \cup v]_B,\ [f \cup s \cup v]_C) \\
&= \phi^{-1}(\partial^i(\Re \cup s) \cup v_B,\ \partial^i(f \cup s) \cup v_C) \\
&= \phi^{-1}(\partial^i(\Re \cup s) \cup v_B,\ -\partial^i(\Re \cup s) \cup v_C) \\
&= \partial^i(\Re \cup s) \cup v_A \\
&= \partial^i(\Re \cup s \cup v) \\
&= \partial^i(\Re \cup e).
\end{aligned}$$

The equality $\partial^i(f \cup s) = -\partial^i(\Re \cup s)$ follows from the fact that $\partial^i s = 0$ and so $\partial^i((\Re + f) \cup s) = (\Re + f) \cup \partial^i s = 0$. This completes one direction of the proof.

*4.3.2. Necessity of nullness*

We keep the notation of the previous section. Suppose that $\Delta$ is saturated over $\Gamma$ and let $\Lambda = \mathrm{link}_\Gamma\tau$ where $\tau$ is the inner face of the gluing. To establish necessity we need to show that *if* $\bar\theta(H_u^\Delta) \simeq H^{\mathrm{T}}_{(l-k,r-k)} \neq 0$ *then there exists some* $f \in M_k^\Lambda$ *such that* $\partial(\Re + f) = 0$. As we have noticed above, when $\Delta$ is saturated, $\bar\theta(H_u^\Delta)$ is spanned by elements of the form $[\partial^i(\Re \cup e)]$ where $e \in M^{\mathrm{T}}_{j-k}$ is an octahedron with $\partial^i(e) = 0$. That is, for every such $e$ there exists an element $[h] \in H_u^\Delta$ such that $\bar\theta[h] = [\partial^i(\Re \cup e)]$. We shall look first at the more general situation when only *some* octahedra are images under $\bar\theta$:

**Lemma 4.11.** *Let* $\Delta = \Gamma \overset{k}{\cup} [\sigma]$ *be an arbitrary k-gluing. Let* $e \in M^{\mathrm{T}}$ *be an octahedron with support* $\eta = \mathrm{supp}\,(e) \in \mathrm{T} \subseteq \Delta$. *Suppose that* $[\partial^i(\Re \cup e)] \in \bar\theta(H_u^\Delta)$. *Then there exists an element* $f \in M_k^{\mathrm{link}_\Gamma(\eta)}$ *such that* $\partial(\Re + f) = 0$.

In other words, *if an octahedron e is in the image of the gluing map* $\bar\theta$, *then there is a part of* $\mathit{link}_\Delta(\mathit{supp}\,(e))$ *that is null and comes through the restriction* $\Re$. To proof the lemma we need a new tool. This is the idea of ‘division by octahedra’ developed in a section below which will be entirely independent of this material.

**Proof.** Let $[h] \in H_u^\Delta$ be such that $\bar\theta[h] = [\partial^i(\Re \cup e)]$. As $\theta = \phi^{-1}\partial^i\psi^{-1}$, it is easy to check that *there is* $g \in \mathcal{M}_j^\Gamma$ *such that* $\partial^i g = \partial^i(e \cup \Re)$. Note that this statement is non-trivial: while $\partial^i(\Re \cup e) \in \mathcal{M}^\Gamma$ we have $e \cup \Re \notin \mathcal{M}^\Gamma$. Now set $f := -g/e$ as will be explained in the next section. It follows then from Theorem 4.15 that $f \in M^{\mathrm{link}_\Gamma(\eta)}$ and $\Re' = -f'$. The result follows. □

For the remainder of the necessity proof we shall need octahedra $e \in M^{\mathrm{T}}$ with the maximal support $\mathit{supp}(e) = \mathrm{T}$. The existence of such octahedra follows from the next fact which is a simple exercise in the ‘middle-term condition’ of Theorem 2.3:

**Lemma 4.12.** *For every simplex* $\Sigma^n$ *and every* $p > 2$, *the Fibonacci module* $H^n_{(l,r)}$ *is* $\neq 0$ *if* $\mathcal{M}^n_{(l,r)}$ *has weight* $p-1$. *In this case* $H^n_{(l,r)}$ *is generated by octahedra e of maximal support, that is* $\mathrm{supp}\,(e)$ *is equal to the vertex set of* $\Sigma^n$. *In particular,* $H^n_{(l,r)} = H^n_{m,1}$ for $n = 2m$ *and* $H^n_{(l,r)} = H^n_{m,2}$ *for* $n = 2m - 1$.

To finish the proof, note that the saturation of $\Delta$ over $\Gamma$ means that $\Delta$ is $(l, r)$-saturated over $\Gamma$ for every $(l, r)$. Using Lemma 4.12 choose $(l, r)$ such that the homology $H^{\mathrm{T}}_{(l-k,r-k)} \simeq \bar\theta(H_u^\Delta)$ should be generated by octahedra $e$ of the maximal support $\tau$. Now choose any such octahedron $e$ and use Lemma 4.11 above. This completes the proof of the Null-Link Theorem. □

*4.3.3. Some corollaries*

We continue to keep the notation of the previous section. The proof of the Null-Link Theorem implies two important corollaries. The first states that if a gluing is saturated in some special sequence then it is saturated globally:

**Corollary 4.13.** *Let* $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ *be a k-gluing. Let* $\mathcal{M}^{\Delta}_{(l,r)}$ *be any sequence of weight* $w \equiv k-1 \pmod{p}$. *Then* $\Delta$ *is saturated over* $\Gamma$ *if and only if it is* $(l,r)$*-saturated over* $\Gamma$.

**Proof.** The sequence $\mathcal{M}^{\mathrm{T}}_{(l-k,r-k)}$ has weight $p-1$ and so is generated by octahedra of maximal support. Hence $\Re$ can be null-extended and so, by the Null-Link Theorem, $\Delta$ is saturated over $\Gamma$. □

Now let $\Delta$ be saturated over $\Gamma$. We shall describe the 'new' elements that arise in the homology of $\Delta$ under the gluing. Recall that a null-part of $\Lambda = \mathrm{link}_{\Delta}(\tau)$ that comes through the restriction $\Re$ was denoted by $\overline{\Re}$ and called a *null-continuation* of $\Re$ in $\Lambda$.

**Theorem 4.14** (*Structure of generators*). *Let* $\Delta$ *be saturated over* $\Gamma$ *so that in particular*

$$H^{\Delta}_u \simeq H^{\Gamma}_u \oplus H^{n-k}_{(l-k,r-k)}, \quad \textit{where } u := d^{n+k}_{(l,r)},$$

*for any* $(l,r)$. *Then the vector space* $H^{\Delta}_u / H^{\Gamma}_u$ *is spanned by elements of the form* $[\overline{\Re} \cup e]$, *where* $[e]$ *runs over all generators of* $H^{\mathrm{T}}_{(l-k,r-k)}$. (*In particular, e could be taken as an octahedron*).

**Proof.** Note that $\overline{\Re}$ and $e$ are disjoint and that $[\overline{\Re} \cup e] \in H^{\Delta}_u$. Then compare dimensions. □

*Note*: This corollary gives us the precise structure of the generators of the homology of buildings and geometric lattices among others. This is an important part of the theory of modular homology, in particular when group actions on complexes are considered. This will be discussed in a forthcoming paper [21].

### 4.4. Division by octahedra

In this section, we shall take the analogy between the inclusion map and differentiation a little further. In one respect this material is technical and is needed only to complete the proof in the previous section. On the other hand, the division considered here has some interesting algebraic properties which may make it worth investigation in its own right.

When differentiating functions, $f' = 0$ means that $f = c$ is constant. If $c' = 0$ with $c \neq 0$, and if $f, h$ satisfy $f' = (ch)'$ then $h' = (f/c)'$. Are there similar relations for the inclusion map? To answer this question we have to consider the structure of the kernel of $\partial$. This depends on the characteristic of the field $F$. For instance, if $\mathrm{char}\, F = 0$ then $\mathrm{Ker}\, \partial^m \cap \mathcal{M}^*_k$ is spanned by elements of the form

$$c_{k,m} = (\alpha_1 - \beta_1) \cup (\alpha_2 - \beta_2) \cup \cdots \cup (\alpha_t - \beta_t) \cup \{\gamma_1, \gamma_2, \ldots, \gamma_m\},$$

where $k = t + m$, see [16,1,25]. (Note, we suppress set brackets, and write $\alpha_i$, $\beta_i$ instead of $\{\alpha_i\}$, $\{\beta_i\}$ more properly.) As we pointed out in Section 2.4, for $t = k$ the terms in $c_{k,0}$ represent the alternately signed faces of a $k$-dimensional octahedron and for $t < k$ they represent a cone over such an octahedron. For this reason we called $c_{k,m}$ a generalized octahedron, or just an octahedron. It is easy to check that $\partial^{m+1}(c_{k,m}) = 0$ over every field and so octahedra could be considered as analogues of 'constants' for $\partial$. However, for this to work we need to define 'division by constants'.

This can be done under the following restricted circumstances. Let $\Delta$ be an arbitrary complex with vertex set $\subseteq \Omega$. Suppose that $F$ has characteristic $p > 2$ and let

$$c = \sum_{\tau \subseteq \Omega} c_\tau \, \tau \in M^\Delta, \quad c_\tau \in F,$$

be such that $c^{(p-1)} = 0$. Set $\sigma := \operatorname{supp} c$ and let $f$ be an arbitrary element in $M^\Delta$. Decompose $f$ with respect to $\sigma$ as in Section 2.1,

$$f = \sum_{\tau \subseteq \sigma} f^\tau \cup \tau$$

and set

$$[f, c] := \sum_{\tau \subseteq \sigma} c_\tau f^\tau.$$

Note that $[f, c] \in M^\Delta$ as each $f^\tau$ belongs to $M^\Delta$. If $\operatorname{wt}(c) \neq 0 \in F$ we can define, writing again $c'$ for $\partial(c)$ and $c^{(j)}$ for $\partial^j(c)$,

$$f/c := \frac{1}{\operatorname{wt}(c)} \left( [f, c] + \frac{1}{1!2!}[f, c']' + \frac{1}{2!3!}[f, c'']'' + \cdots + \frac{1}{(p-2)!(p-1)!}[f, c^{(p-2)}]^{(p-2)} \right).$$

In particular, if $c$ is an octahedron then its weight is a power of 2 and so $f/c$ exists by the assumptions on the characteristic of $F$. The quotient has interesting properties such as $(c \cup f)/c = f$ and $(f_1 + f_2)/c = f_1/c + f_2/c$. Furthermore,

**Theorem 4.15.** *Let $F$ be a field of characteristic $p > 2$ and let $0 < i < p$ be an integer. Let $\Delta$ be a complex with vertex set $\subseteq \Omega$ and suppose that $c \in M^\Delta$ is an octahedron with $c^{(i)} = 0$. Let $f, h \in M^\Delta$ be such that $h$ is disjoint from $c$ and $f^{(i)} = (c \cup h)^{(i)}$. Then $f/c$ belongs to $M^\Delta$ and $h' = (f/c)'$. Moreover, if $\sigma := \operatorname{supp} c$ is a face of $\Delta$ then $f/c \in M^{\operatorname{link}_\Delta \sigma}$.*

**Proof.** The proof is straightforward. Any octahedron $c \in M^\Delta$ such that $c^{(i)} = 0$ has the form

$$c = (\alpha_1 - \beta_1) \cup (\alpha_2 - \beta_2) \cup \cdots \cup (\alpha_t - \beta_t) \cup \gamma,$$

where $\gamma = \emptyset$ if $i = 1$ and $\gamma := \{\gamma_1, \gamma_2, \ldots, \gamma_{i-1}\}$ for $i \geqslant 2$. Note that $\gamma^{(i-1)} = (i-1)!$ and $\gamma^{(i)} = 0$. Then $(c \cup h)^{(i)} = (\alpha_1 - \beta_1) \cup \ldots \cup (\alpha_t - \beta_t) \cup (\gamma \cup h)^{(i)}$ where

$$(\gamma \cup h)^{(i)} = \sum_{k=0}^{i} \binom{i}{k} \gamma^{(i-k)} \cup h^{(k)} = i!h' + g_1 h'' + g_2 h''' + \cdots, \quad (g_k \in M^\Delta).$$

On the other hand, according to Lemma 2.1,

$$(c \cup h)^{(i)} = f^{(i)} = \sum_{\tau \subseteq \sigma} \left( \sum_{k=0}^{i} k! \binom{i}{k} \sum_{\tau \subseteq \rho \subseteq \sigma \,:\, |\rho \setminus \tau| = k} (f^{\rho})^{(i-k)} \right) \cup \tau.$$

Let $\tau$ be any of the $\mathrm{wt}(c) = 2^t$ many faces which appear in the decomposition of $(\alpha_1 - \beta_1) \cup \cdots \cup (\alpha_t - \beta_t)$. Let $\mathrm{sign}(\tau) = \pm 1$ be the coefficient of $\tau$. Comparing the previous relations, we have

$$\begin{aligned} \mathrm{sign}(\tau) h' &= \sum_{k=0}^{i} \frac{1}{(i-k)!} \left( \sum_{\tau \subseteq \rho \subseteq \sigma \,:\, |\rho \setminus \tau| = k} f^{\rho} \right)^{(i-k)} \\ &= \sum_{k=0}^{i} \frac{1}{k!} \left( \sum_{\tau \subseteq \rho \subseteq \sigma \,:\, |\rho| = t+i-k} f^{\rho} \right)^{(k)}. \end{aligned}$$

Now take the alternating sum of these relations:

$$2^t h' = \sum_{\tau} \mathrm{sign}(\tau) \sum_{k=0}^{i} \frac{1}{k!} \left( \sum_{\tau \subseteq \rho \subseteq \sigma \,:\, |\rho| = t+i-k} f^{\rho} \right)^{(k)}.$$

The result follows after collecting terms. Also the last assertion is easily verified. □

**Example 1.** Let $p > 2$ and $c = (\{\alpha\} - \{\beta\}) \cup \{\gamma\}$. Then $\sigma = \{\alpha, \beta, \gamma\}$. Since $c' = \{\alpha\} - \{\beta\}$ and $c'' = 0$ we have

$$f/c = \frac{f^{\{\alpha\gamma\}} - f^{\{\beta\gamma\}}}{2} + \frac{(f^{\{\alpha\}} - f^{\{\beta\}})'}{4}.$$

**Example 2.** Let $p > 3$ and $c = (\{\alpha\} - \{\beta\}) \cup \{\gamma\varepsilon\}$. Then $\sigma = \{\alpha, \beta, \gamma, \varepsilon\}$ and $c' = (\{\alpha\} - \{\beta\}) \cup (\{\gamma\} + \{\varepsilon\}) = \{\alpha, \gamma\} - \{\beta, \gamma\} + \{\alpha, \varepsilon\} - \{\beta, \varepsilon\}$, $c'' = 2(\{\alpha\} - \{\beta\})$, $c''' = 0$. Thus we have

$$f/c = \frac{f^{\{\alpha\gamma\varepsilon\}} - f^{\{\beta\gamma\varepsilon\}}}{2} + \frac{(f^{\{\alpha\gamma\}} - f^{\{\beta\gamma\}} + f^{\{\alpha\varepsilon\}} - f^{\{\beta\varepsilon\}})'}{4} + \frac{(f^{\{\alpha\}} - f^{\{\beta\}})''}{12}.$$

## 5. Geometric lattices are saturated

It is well known that the order complex of the proper part of a geometric lattice is Cohen–Macaulay. Here we shall use the Null-Link Theorem to prove a somehow similar result about saturation. If $(L, <)$ is a partially ordered set then the faces of the *order complex* $\Delta(L)$ are the linearly ordered subsets $\alpha_1 < \alpha_2 < \cdots < \alpha_t$, with $\alpha_i \in L$.

**Theorem 5.1.** *Let $L = \hat{L} \cup \hat{0} \cup \hat{1}$ be a finite geometric lattice with proper part $\hat{L}$. Then the order complex $\Delta(\hat{L})$ of $\hat{L}$ is saturated for every $p > 2$.*

Evidently, $\Delta(L)$ is the bi-cone over $\Delta(\hat{L})$ and hence $\Delta(L)$ is also saturated, as will be shown in Theorem 6.6. Note the contrast to the case of standard simplicial homology where $\Delta(L)$ is always acyclic in view of its contractibility. The following proof is based on ideas from [5].

**Proof.** Let $\alpha_1, \ldots, \alpha_r$ be the atoms of $L$ and for $\gamma, \delta \in L$ let $\gamma \prec \delta$ denote that $\gamma$ is covered by $\delta$. To each such pair we associate the label

$$\lambda(\gamma \prec \delta) := \min\{i \;:\; \alpha_i \;\vee\; \gamma = \delta\}.$$

To every unrefinable chain $\sigma \;:\; \gamma_0 \prec \gamma_1 \prec \cdots \prec \gamma_k$ now associate the sequence

$$\lambda(\sigma) = \Big(\lambda(\gamma_0 \prec \gamma_1),\; \lambda(\gamma_1 \prec \gamma_2),\; \ldots,\; \lambda(\gamma_{n-1} \prec \gamma_k)\Big) \in \boldsymbol{Z}^k.$$

There are no repetitions in $\lambda(\sigma)$ and the maximal chains $\gamma_0 \prec \gamma_1 \prec \cdots \prec \gamma_n$ are the facets of $\Delta(L)$. If we arrange these in lexicographical order $\sigma_1, \sigma_2, \ldots, \sigma_m, \ldots$ then this is a shelling of $\Delta(L)$ since $L$ is a geometric lattice, see [5].

Thus let $\Gamma_m := \sigma_1 \cup \cdots \cup \sigma_m$ and $\Delta := \Gamma \overset{k}{\cup} \sigma_{m+1}$. We need to show that $\Delta$ is saturated over $\Gamma$. Just as in Theorem 4.4 let $\Re$ be the restriction of the gluing $\Gamma \overset{k}{\cup} \sigma_{m+1}$ and let $\tau = \sigma_{m+1} \setminus \Re$ be the inner face of the gluing. It follows from [5] that $\Re$ is completely determined by the *descent set* of $\lambda := \lambda(\sigma_{m+1})$. To be more precise, let us say that $\lambda = (\lambda_1, \ldots, \lambda_n)$ has a *descent* at position $i$, where $0 < i < n$, if $\lambda_i > \lambda_{i+1}$. The sequence $\lambda$ then is said to have *descent set* $D(\lambda) = \{i \;:\; \lambda_i > \lambda_{i+1}, 0 < i < n\}$. Let $k = |D(\lambda)|$. The restriction $\Re = \Re(\sigma_{m+1}) \in \Delta(L)$ is the $k$-chain of elements

$$\gamma_i := \bigvee_{j=1}^{i} \alpha_{\lambda_j} \in L \quad \text{for} \quad i \in D(\lambda)$$

or, in other words, the face $\Re = \{\gamma_i \;:\; i \in D(\lambda)\}$.

Let us decompose $\lambda = (\lambda_1, \ldots, \lambda_n)$ into parts according to the rule:

- every group of neighbouring descents elements, together with the immediately subsequent element, form a part, and
- any element not in some part of the previous type forms a one-element part.

We shall denote this partition by $\Pi(\lambda) = \{\pi_i \;:\; \bigcup \pi_i = \lambda\}$. For example, the sequence $1\bar{4}\bar{3}25$ produces the partition $\{\{1\}, \{2, 3, 4\}, \{5\}\}$, while $\bar{2}13\bar{5}4$ gives $\{\{1, 2\}, \{3\}, \{4, 5\}\}$. (Here bars mark descents.)

The Young subgroup of $S_n$ associated with $\Pi(\lambda)$ will be denoted by $G(\lambda) = S(\pi_1) \times S(\pi_2) \times \cdots \times S(\pi_t)$. For $i \in D(\lambda)$ and $g \in G(\lambda)$ let now $\rho_g$ be the simplex

$$\rho_g = \left\{ \bigvee_{j=1}^{i} \alpha_{g(\lambda_j)} \;:\; i \in D(\lambda) \right\}$$

and set

$$h := \sum_{g \in G(\lambda)} \operatorname{sign}(g)\, \rho_g,$$

$$f := \sum_{1 \neq g \in G(\lambda)} \operatorname{sign}(g)\, \rho_g = h - \Re.$$

Let $\Lambda \subseteq \Delta(L)$ be the $(k-1)$-dimensional complex generated by the faces which appear in $h$, $\Lambda := \langle \rho_g : g \in G(\lambda) \rangle$ and let $\Lambda^* := \langle \rho_g : 1 \neq g \in G(\lambda) \rangle$ be the subcomplex corresponding to $f$. Note the following:

- In view of the lexicographical ordering of chains we have $\Lambda \subseteq \Delta$ and $\Lambda^* \subseteq \Gamma$. Moreover, it is easy to note that actually $\Lambda \subseteq \operatorname{link}_\Delta(\tau)$ and $\Lambda^* \subseteq \operatorname{link}_\Gamma(\tau)$.
- As $G(\lambda) = S(\pi_1) \times S(\pi_2) \times \cdots \times S(\pi_t)$, the complex $\Lambda$ is the direct product of Coxeter complexes of non-trivial symmetric groups $S(\pi_i)$. Therefore $\Lambda$ is 2-colourable and so $\partial h = 0$ for any field of characteristic $p > 0$, see [19, p. 391].

Thus $\Re$ is a 1-cycle of $\Delta$ relative to $\operatorname{link}_\Gamma(\tau)$ and by Theorem 4.4 we know that $\Delta$ is saturated over $\Gamma$. The result follows by induction. □

**Example.** In Fig. 2 the initial part of the shelling of the 2-dimensional Coxeter complex $A_3 = \Delta(\widehat{\mathcal{B}}_4)$ is presented. The sequences $\lambda$ are precisely the permutations of the symmetric group $S_4$:

$$1234 < 12\bar{4}3 < 1\bar{3}24 < 13\bar{4}2 < 1\bar{4}23 < 1\bar{4}\bar{3}2 < \bar{2}134 < \bar{2}1\bar{4}3 < \cdots .$$

The first non-trivial case $k = 2$ occurs for $\lambda = 1\bar{4}\bar{3}2$ when $\Pi(\lambda) = \{\{1\}, \{2, 3, 4\}\}$ and $G(\lambda) = S(1) \times S(\{2, 3, 4\}) \simeq S_3$. Here $\Lambda$ is the Coxeter complex of $S_3$, i.e. just a hexagon. We may see from figure that in our case $\tau = \{1\}$ and $\Lambda = \operatorname{link}_\Delta(\tau)$. Note that this always holds when $L$ is a Boolean algebra but that this may fail in general.

The second case of $k = 2$ occurs for $\lambda = \bar{2}1\bar{4}3$ when $\Pi(\lambda) = \{\{1, 2\}, \{3, 4\}\}$ and $G(\lambda) = S(\{1, 2\}) \times S(\{3, 4\}) \simeq S_2 \times S_2$. Here $\tau = \{1, 2\}$ and $\Lambda = \operatorname{link}_\Delta(\tau)$ is the direct square of the Coxeter complex of $S_2$, i.e. just a square.

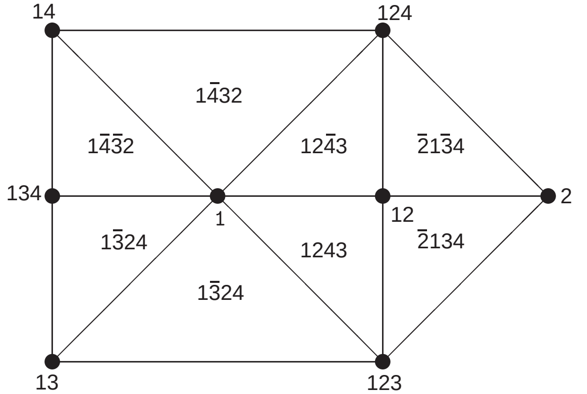


Fig. 2. An initial part of the lexicographic shelling of $A_3$.

## 6. Other applications

In this section, we shall discuss some application of the results in Section 4. Obviously, if the modular Betti numbers are known then we can determine the rank of the inclusion maps $\partial : M_k^\Delta \to M_{k-1}^\Delta$ and this is the first area of applications. Secondly we shall look at standard constructions such as forming cones and suspensions where it is possible to compute the modular homology directly. In the last section we deal with rank selected order complexes.

### *6.1. On the p-rank of incidence matrices*

Let again $\Delta$ be a complex of dimension $n-1$ and let $s \leqslant t \leqslant n$ be integers. Then we may define a $\{0,\ 1\}$-incidence matrix $I = I^\Delta(s,t)$ of size $f_s \times f_t$ which records the containment relation between the elements of $\Delta_s$ and $\Delta_t$. Thus

$$I_{\sigma\tau} = \begin{cases} 1 & \text{if } \sigma \subseteq \tau, \\ 0 & \text{if } \sigma \nsubseteq \tau. \end{cases}$$

When $\Delta = \Sigma^n$ is the simplex of dimension $n-1$ we denote the corresponding matrix by $I^n(s,t)$. As the $I^\Delta(s,t)$ are representations of the complex their algebraic properties are of importance. In Wilson [29] for instance, the invariant factors (or Smith form) of $I^n(s,t)$ has been determined. To obtain results of this kind for other important complexes is of very considerable interest.

A partial answer in this direction are formulae for the rank of $I^\Delta(s,t)$ when considered as a matrix over $GF(p)$. This quantity we shall denote by $\mathrm{rk}_p^\Delta(s,t)$. (Of course, if $t-s<p$ then $\mathrm{rk}_p^\Delta(s,t)$ is the rank of the map $\partial^{t-s} : M_t^\Delta \to M_s^\Delta$.) For the simplex $\Sigma^n$ there is the well-established result

$$\mathrm{rk}_p^n(s,t) = \left| \sum_{k=0}^{\infty} \binom{n}{s-pk} - \binom{n}{t-p-pk} \right|$$

for all $s,\ t$ with $s+t<n$, see [11,16,29]. As has been noticed in [8, p. 152] “it is an interesting problem whether the general form of Wilson’s results has any extension...”. Here we shall find a (partial) solution to this problem.

An expression similar to the above can be formed for arbitrary shellable complexes and these can in fact be used as further algebraic characterizations of saturation, as we shall see now.

In the spirit of Section 3.2 we establish a formal expression for the $p$-rank of complexes associated to a given $h$-vector:

**Definition 6.1** (*Formal p-Rank*). Let $h(\Delta) = (h_0,\ h_1,\ \ldots,\ h_n)$ be the $h$-vector of a complex of dimension $n-1$. Then

$$\mathrm{rk}_p^{h(\Delta)}(s,t) := \sum_{i=0}^{n} h_i \cdot \mathrm{rk}_p^{n-i}(s-i,t-i) \tag{6.1}$$

is the *formal p-rank* associated to $h(\Delta)$ .

Now the following two closely connected observations can be made:

**Theorem 6.2.** *Let $\Delta$ be a shellable $(n-1)$-dimensional complex with h-vector $h(\Delta)$. Let $p > 2$ be a prime and suppose that $s < t \leqslant n$ are non-negative integers with $t - s < p$. Then*

$$\mathrm{rk}_p^{\Delta}(s,t) = \mathrm{rk}_p^{h(\Delta)}(s,t) \quad \mathit{if}\ s + t < n. \tag{6.2}$$

**Theorem 6.3.** *Let $\Delta$ be a $(n-1)$-dimensional complex (possibly non-shellable), and let $p > 2$ be a prime. Then $\Delta$ is saturated in characteristic p if and only if*

$$\mathrm{rk}_p^{\Delta}(s,t) = \mathrm{rk}_p^{h(\Delta)}(s,t) \quad \mathit{for\ all}\ s < t \leqslant n\ \mathit{such\ that}\ t - s < p. \tag{6.3}$$

We shall prove both theorems simultaneously:

**Proof.** First, let $\Delta$ be an arbitrary shellable complex with *f*-vector $(f_0, f_1, \ldots, f_n)$. In view of the condition $0 < t - s < p$ we may look at $\mathrm{rk}_p^{\Delta}(s,t)$ as the *p*-rank of the map $\partial^{t-s}\colon M_t^{\Delta} \to M_s^{\Delta}$. According to Theorem 3.1, in the sequence $\mathcal{M}_{(s,t)}^{\Delta}$ all homologies to the left from the middle are trivial. Equivalently, see [18, Corollary 5.6], for $s + t < n$, we have

$$\mathrm{rk}_p^{\Delta}(s,t) = f_s - f_{t-p} + f_{s-p} - f_{t-2p} + f_{s-2p} - f_{t-3p} + \cdots . \tag{6.4}$$

The result follows now from the formula

$$h_k = \sum_{i=0}^{n} (-1)^{i+k} f_i \binom{n-i}{k-i} \tag{6.5}$$

after substituting it into (6.1) above. This proves Theorem 6.2.

Now let $\Delta$ be saturated. Hence its Betti numbers are

$$\beta_{m+s}^{\Delta} := \sum_{j=w+(s-1)p+1}^{w+sp} h_j \beta_{(l-j,r-j)}^{n-j} \qquad \text{for all } s \geqslant 0.$$

For $s + t \geqslant n$ we need to take these into account when evaluating the rank:

$$\mathrm{rk}_p^{\Delta}(s,t) = \sum_{k=0} (f_{s-kp} - f_{t-p-kp}) - (\beta_{s-kp,p-t+s}^{\Delta} - \beta_{t-p-kp,t-s}^{\Delta}). \tag{6.6}$$

Also

$$\mathrm{rk}_p^{n}(s,t) = \sum_{k=0} \binom{n}{s-pk} - \binom{n}{t-p-pk} \pm \beta_{(s,t)}^{n}, \tag{6.7}$$

where the sign of the Betti number is determined by its position in the sequence $\mathcal{M}_{(s,t)}^{n}$. Now put (6.5) and (6.7) into the right-hand side of (6.3). After transforming dimensions into positions we obtain (6.6). Thus, for saturated $\Delta$ the relation (6.3) holds also for $s + t \geqslant n$. Finally, since Betti numbers are completely determined by ranks, (6.3) implies saturation of $\Delta$. $\square$

If $p \geqslant n$ then Theorem 2.3(v) implies that $\mathrm{rk}_p^n(s,t) = \min\{\binom{n}{s}, \binom{n}{t}\}$. Therefore if $p, q > n$ are primes then $\mathrm{rk}_p^n(s,t) = \mathrm{rk}_q^n(s,t)$ and we may derive the following:

**Corollary 6.4.** *Let $\Delta$ be an $(n-1)$-dimensional complex. If $\Delta$ is saturated for some $p > n$ then $\Delta$ is saturated for every $q > n$.*

An interesting special case illustrating this corollary arises for $n = 2$ when the complex is a graph. As we have seen previously,

- a connected graph $\Gamma$ is saturated in characteristic $p$ if and only if its incidence matrix has $p$-rank one less than the number of vertices of $\Gamma$;
- a connected graph $\Gamma$ is saturated for all $p > 2$ if and only if it is bipartite.

From these we derive immediately the well-known fact which follows also from results of [8].

**Proposition 6.5.** *Let $\Delta$ be a connected graph on $v$ vertices with vertex-edge incidence matrix $I^\Delta(1,2)$. Then $\Delta$ is bipartite if and only if for any $0 \leqslant p \neq 2$ we have* $\mathrm{rank}_p\, I^\Delta(1,2) = v-1$.

In fact, the Theorem 6.3 above could be considered as a multi-dimensional generalization of this fact!

### 6.2. Cones and suspensions

For some classes of simplicial complexes the modular homology can be computed by general constructions. These include cones and suspensions. Let $\Delta$ be an $(n-1)$-dimensional simplicial complex on the vertex set $\Omega$ and let $\alpha, \beta \notin \Omega$ be new vertices. Then the *cone* over $\Delta$ is the $n$-dimensional complex

$$C\Delta = \Delta \cup \{\alpha \cup \sigma \,:\, \sigma \in \Delta\}$$

and the *suspension* over $\Delta$ is the $n$-dimensional complex

$$S\Delta = \Delta \cup \{\alpha \cup \sigma \,:\, \sigma \in \Delta\} \cup \{\beta \cup \sigma \,:\, \sigma \in \Delta\}.$$

It is well-known that cones are acyclic in standard homology, see [22]. The modular homology of cones is more complicated:

**Theorem 6.6.** *Let $\Delta$ be a pure complex. Then for every $0 \leqslant k \leqslant n$ and $0 < i < p$ we have*

$$H_{k,i}^{C\Delta} \simeq H_{k,i+1}^{\Delta} \oplus H_{k-1,i-1}^{\Delta}.$$

(*We put* $H_{k,0}^{\Delta} = H_{k,p}^{\Delta} = 0$.)

If $p = 2$ then we have $H_{k,1}^{C\Delta} = 0$ for all $k$. Also the case 3 is special. Here we have $H_{k,1}^{C\Delta} = H_{k,2}^{\Delta}$ and $H_{k-1,2}^{C\Delta} = H_{k-2,1}^{\Delta}$. Thus, in a sense, the 3-modular homology is preserved but shifted:

**Corollary 6.7.** *Let $p = 3$. Then the modular homology in the sequence $\mathcal{M}^{C\Delta}_{(l,r)}$ coincides with the homology in $\mathcal{M}^{\Delta}_{(l+1,r+1)}$.*

Clearly, Theorem 6.6 generalizes Theorem 2.3. As we have seen already, the 3-modular Betti numbers of $\Sigma^n$ are either 0 or 1. Similarly, all eight non-zero 5-modular Betti numbers of $\Sigma^n$ are among $\varphi(n-1)$, $\varphi(n)$ or $\varphi(n+1)$, where $\varphi(n)$ are elements of the sequence 1, 1, 2, 3, 5, 8, 13, 21, ... of Fibonacci numbers (so that, for example, $\varphi(8) = 21$). Now we look at suspensions. Here the $h$-vector of $S\Delta$ satisfies $h^{S\Delta}_k = h^{\Delta}_{k-1} + h^{\Delta}_k$ for all $k \leqslant n$. It is well-known that $H^{S\Delta}_k = H^{\Delta}_{k-1}$ in standard homology. Again, the modular case is more complicated.

**Theorem 6.8.** *Let $\Delta$ be a pure complex. Then for every $0 \leqslant k \leqslant n$ and $0 < i < p$ we have*

$$H^{S\Delta}_{k,i} \simeq H^{C\Delta}_{k,i} \oplus H^{\Delta}_{k-1,i} \simeq H^{\Delta}_{k,i+1} \oplus H^{\Delta}_{k-1,i} \oplus H^{\Delta}_{k-1,i-1}.$$

Note that for $p = 2$ we have $H^{S\Delta}_{k,1} = H^{\Delta}_{k-1,1}$, just as for the standard homology. Also the case $p = 3$ is special. Here we have $H^{S\Delta}_{k,1} = H^{\Delta}_{k,2} \oplus H^{\Delta}_{k-1,1}$ and $H^{S\Delta}_{k,2} = H^{\Delta}_{k-1,2} \oplus H^{\Delta}_{k-1,1}$.

**Proof.** The proofs of the theorems are very similar, the first essentially being identical to the proof of Theorem 5.2 in [1]. So we will give the details only for the slightly more complicated case of suspension.

Let $f$ be any element of $M^{S\Delta}_k$ and let $\alpha \neq \beta$ be the new vertices of $S\Delta$. When applying the decompositions of Section 2.1 we suppress unnecessary brackets and write $f^{\alpha}$ instead of $f^{\{\alpha\}}$, etc. Now suppose that $\partial^i(f) = 0$. Then $f$ can be written uniquely as $f = \alpha \cup f^{\alpha} + \beta \cup f^{\beta} + g$ where $f^{\alpha}, f^{\beta} \in M^{\Delta}_{k-1}$ and $g \in M^{\Delta}_k$. In fact, since $\partial^i(f) = \alpha \cup \partial^i(f^{\alpha}) + \beta \cup \partial^i(f^{\beta}) + i\partial^{i-1}(f^{\alpha} + f^{\beta}) + \partial^i(g) = 0$, we have $\partial^i(f^{\alpha}) = \partial^i(f^{\beta}) = 0$, $\partial^{i-1}(if^{\alpha} + if^{\beta} + g') = 0$ and so $\partial^{i+1}(g) = 0$. Now define the map $\Phi : H^{S\Delta}_{k,i} \mapsto H^{\Delta}_{k,i+1} \oplus H^{\Delta}_{k-1,i} \oplus H^{\Delta}_{k-1,i-1}$ by putting

$$\Phi : [f] \mapsto ([g],\ [if^{\beta}],\ [if^{\alpha} + if^{\beta} + g']).$$

We need to show that this map is well-defined, and that it is surjective and injective. To show that it is well-defined suppose that $[f] = [h]$ with $h = \alpha \cup h^{\alpha} + \beta \cup h^{\beta} + m$. So there exists $F = \alpha \cup F^{\alpha} + \beta \cup F^{\beta} + G \in M^{S\Delta}_{k+p-i}$ such that

$$\begin{aligned} \partial^{p-i} F &= f - h \\ &= \alpha \cup (f^{\alpha} - h^{\alpha}) + \beta \cup (f^{\beta} - h^{\beta}) + (g - m) \\ &= \alpha \cup \partial^{p-i}(F^{\alpha}) + \beta \cup \partial^{p-i}(F^{\beta}) - i\partial^{p-i-1}(F^{\alpha} + F^{\beta}) + \partial^{p-i}(G), \end{aligned}$$

implying that $f^{\alpha} - h^{\alpha} = \partial^{p-i}(F^{\alpha})$, $f^{\beta} - h^{\beta} = \partial^{p-i}(F^{\beta})$ and so $[if^{\beta}] = [ih^{\beta}]$. Further, $g - m = \partial^{p-i-1}(G' - iF^{\alpha} - iF^{\beta})$, so that $[g] = [m]$. Finally, applying $\partial$ to the equation above gives

$$\begin{aligned}(f-h)' &= \alpha\cup(f^\alpha-h^\alpha)'+(f^\alpha-h^\alpha)+\beta\cup(f^\beta-h^\beta)'\\ &\quad +(f^\beta-h^\beta)+(g-m)'\\ &= \partial^{p-i+1}F\\ &= \alpha\cup\partial^{p-i+1}(F^\alpha)+\beta\cup\partial^{p-i+1}(F^\beta)\\ &\quad -(i-1)\partial^{p-i}(F^\alpha+F^\beta)+\partial^{p-i+1}(G)\end{aligned}$$

implying that

$$\begin{aligned}f^\alpha-h^\alpha+f^\beta-h^\beta+(g-m)' &= \partial^{p-i+1}(G)-(i-1)\partial^{p-i}(F^\alpha+F^\beta)\\ &= \partial^{p-i+1}(G)-(i-1)(f^\alpha-h^\alpha+f^\beta-h^\beta).\end{aligned}$$

Hence $if^\alpha+if^\beta+g'=\partial^{p-i+1}(G)+(ih^\alpha+ih^\beta+m')$ and so $[if^\alpha+if^\beta+g']=[ih^\alpha+ih^\beta+m']$. Therefore $\Phi$ is well-defined.

To show that $\Phi$ is injective, suppose that $\Phi[f]=([0],\ [0],\ [0])]$. Then there exists $G\in M^\Delta_{k+p-i-1}$ such that $\partial^{p-i-1}G=g$, there exists $F\in M^\Delta_{k+p-i-1}$ such that $\partial^{p-i}F=if^\beta$ and there exists $E\in M^\Delta_{k+p-i}$ such that $\partial^{p-i+1}E=if^\alpha+if^\beta+g'$. Now take $J=\alpha\cup(E'-G)+iE+(\beta-\alpha)\cup F\in M^{S\Delta}_{k+p-i}$ and check $\partial^{p-i}J=if$, so that $[f]=[0]$.

Finally, to show that $\Phi$ is surjective, suppose that $([g],\ [h],\ [e])\in H^\Delta_{k,i+1}\oplus H^\Delta_{k-1,i}\oplus H^\Delta_{k-1,i-1}$. Take $f=i^{-1}\{\alpha e+(\beta-\alpha)h+ig-\alpha g'\}\in M^{S\Delta}_k$ and check that $\partial^i f=0$ and $\Phi[f]=([g],\ [h],\ [e])$. This completes the proof in the case of suspension.

For the proof of Theorem 6.6 let $f$ be in $M^{C\Delta}_k$ with $\partial^i(f)=0$. If $\alpha$ is the new vertex we write $f=\alpha\cup f^\alpha+g$ so that $\partial^{i-1}(if^\alpha+g')=0$ and $\partial^{i+1}(g)=0$. Now define the map $\Phi: H^{C\Delta}_{k,i}\to H^\Delta_{k,i+1}\oplus H^\Delta_{k-1,i-1}$ by putting

$$\Phi:[f]\mapsto([g],\ [if^\alpha+g']).$$

Repeating the arguments above (or looking at [1]), show that this map is well-defined, surjective and injective. □

**Theorem 6.9.** *Let $\Delta$ be an arbitrary complex.*

(a) *$\Delta$ is shellable if and only if $C\Delta$ is shellable, and $C\Delta$ is shellable if and only if $S\Delta$ is shellable.*
(b) *If $\Delta$ is saturated then $C\Delta$ and $S\Delta$ are saturated.*
(c) *If $\Delta$ is shellable and if either $C\Delta$ or $S\Delta$ are saturated then $\Delta$ is saturated.*

**Proof.** The first part is well-known. Part (b) is simple: If $h(\Delta)=(h_0,h_1,\ldots,h_n)$ then $h(C\Delta)=(h_0,h_1,\ \ldots,h_n,0)$ and $h(S\Delta)=(h_0,h_0+h_1,h_1+h_2,\ldots,h_{n-1}+h_n,h_n)$. Now evaluate the formal Betti numbers for $\mathcal{C}\Delta$ or $S\Delta$ according to Definition 3.3 and compare these to the actual Betti numbers obtained from Theorems 6.6 and 6.8.

For part (c) note that if $\sigma_1,\ldots,\sigma_m$ is a shelling sequence for $\Delta$ then $\alpha\cup\sigma_1,\ldots,\alpha\cup\sigma_m$ is a shelling sequence for $C\Delta$ and $\alpha\cup\sigma_1,\beta\cup\sigma_1,\ldots,\alpha\cup\sigma_m,\beta\cup\sigma_m$ is a shelling sequence for $S\Delta$. For the cone note that the links of the inner faces of consecutive gluings are the same in the two shelling sequences. The result thus follows from Theorem 4.4. For the suspension $S\Delta$ take its shelling sequence $\alpha \cup \sigma_1, \beta \cup \sigma_1, \ldots, \alpha \cup \sigma_m, \beta \cup \sigma_m$ and note that at each odd gluing, say $2i - 1$, the link of the inner face is exactly the same as the link of the inner face in the $i$-th gluing for $\Delta$. $\square$

As an immediate corollary we have the result mentioned previously:

**Corollary 6.10.** *Cross-polytopes are saturated in every characteristic.*

### *6.3. Rank-selection*

Let $S = \{0, \ldots, n-1\}$ be an $n$-element set. Then a *balanced* complex on the vertex set $\Omega$ is a pure $(n-1)$-dimensional complex $\Delta$ with a partition $\Omega = \bigcup_{s \in S} \Omega_s$ such that $|\sigma \cap \Omega_s| = 1$ for every facet $\sigma$ of $\Delta$ and every $s \in S$. It is convenient to think of $S$ as a set of colours, the condition being that every facet has exactly one vertex of each colour. For instance, if $\mathcal{P}$ is a ranked partially ordered set then the order complex $\Delta(\mathcal{P})$ formed by all linearly ordered subsets of $\mathcal{P}$ is balanced. (Note, balanced complexes are called *numbered* by Bourbaki and *completely balanced* by Stanley.)

Let $\Delta$ be such a balanced complex. For a face $\tau$ of $\Delta$ let its *type* be $t(\tau) := \{s \in S : \tau \cap \Omega_s \neq \emptyset\} \subseteq S$. For every subset $J$ of $S$ we may define the complex $\Delta_J := \{\tau \in \Delta : t(\tau) \subseteq J\}$ and this is a pure $(|J| - 1)$-dimensional complex, called a *type-selected subcomplex* of $\Delta$.

For the order complex $\Delta(\mathcal{P})$ of a ranked poset $\mathcal{P}$ the $\Omega_s$ can be taken to be the elements of rank $s$ in $\mathcal{P}$. Here it is common to call $\Delta_J$ *rank-selected* and we shall use this term also for general balanced complexes. For instance, buildings with non-linear diagram are examples of balanced complexes that are not order complexes.

Let $\Omega_J := \bigcup_{s \in J} \Omega_s$ and for $\tau \in \Delta$ let $\tau_J := \tau \cap \Omega_J$. Note that the facets of $\Delta_J$ are of the form $\sigma_J$ where $\sigma$ is a facet of $\Delta$. (It is possible, of course, that different facets of $\Delta$ produce the same facet of $\Delta_J$.) The correspondence $\tau \to \tau_J$ can be extended naturally to a linear map $\phi_J : M^{\Delta} \to M^{\Delta_J}$ defined on the faces of $\Delta$ by setting

$$\phi_J(\tau) = \begin{cases} \tau_J & \text{if } \tau_J \neq \emptyset \text{ and} \\ 0 & \text{if } \tau_J = \emptyset. \end{cases}$$

**Theorem 6.11.** *Suppose that the $(n-1)$-dimensional balanced complex $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ has saturated homology relative to $\Gamma$. Let $J \subseteq S$ such that $\Delta_J \neq \Gamma_J$. Then $\Delta_J$ has saturated homology relative to $\Gamma_J$.*

**Proof.** It is sufficient to prove the theorem when $|J| = n - 1$. Let $x \in \Sigma^n$ be the vertex not in $\Sigma^n_J$, let $r := \mathrm{res}(\Sigma^n)$ be the restriction and let $t := \Sigma^n \setminus r$ be the complement of the gluing. Since $\Delta_J \neq \Gamma_J$ we have $x \in t$. Hence $r$ is also the restriction of the gluing $\Delta_J = \Gamma_J \cup \Sigma^{n-1}$ and its inner face is $t_J = t \setminus \{x\}$.

It follows from the Null-Link Theorem that there exists some $f \in M^{\Gamma}_k \subset M^{\Delta}$ such that $r + f \in M^{\mathrm{link}_{\Delta} t}$ and $\partial(r + f) = 0$. Since $r + f \in M^{\mathrm{link}_{\Delta_J} t_J}$ the result follows from Null-Link Theorem. $\square$

**Theorem 6.12.** *Every non-trivial rank-selected subcomplex of a shellable saturated complex is shellable and saturated.*

**Proof.** Indeed, if $\sigma_1, \ldots, \sigma_m$ is a shelling for $\Delta$ then the distinct elements in the sequence $\sigma_1 \cap J_\Delta, \ldots, \sigma_m \cap J_\Delta$ form a shelling for $\Delta_J$. Therefore the result follows from the previous theorem. $\square$

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