# On Orbits of the Ring $Z_n^m$ under Action of the Group $SL(m, Z_n)$

P. Novotný, J. Hrivnák

*We consider the action of the finite matrix group $SL(m,Z_n)$ on the ring $Z_n^m$. We determine orbits of this action for n arbitrary natural number. It is a generalization of the task which was studied by A. A. Kirillov for m = 2 and n prime number.*

*Keywords: ring, finite group.*

## 1 Introduction

The important role of symmetries in classical and quantum physics is well known. We focus on so called discrete quantum physics; this means that the corresponding Hilbert space is finite dimensional [1, 2]. Well known are also $2\times2$ Pauli matrices. Besides spanning real Lie algebra su(2), they form a fine grading of sl(2, C). The fine gradings of a given Lie algebra are preferred bases which yield quantum observables with additive quantum numbers.

The generalized $n\times n$ Pauli matrices were described in [3]. For $n = 3$ these $3\times3$ Pauli matrices form one of four non-equivalent gradings of sl(3,C). Other fine gradings are Cartan decomposition and the grading which corresponds to Gell-Mann matrices [4, 5]. The symmetries of the fine grading of sl($n$, C) associated with these generalized Pauli matrices were studied only recently in [6]. This work pointed out the importance of the finite group $SL(2, Z_n)$ as the group of symmetry of the Pauli gradings. The additive quantum numbers, mentioned above, form in this case the finite associative additive ring $Z_n\times Z_n$. The action of $SL(2, Z_n)$ on $Z_n\times Z_n$ then represents the symmetry transformations of Pauli gradings of sl($n$, C). The orbits of this action form such points in $Z_n\times Z_n$ which can be reached by symmetries.

For the purpose of so called graded contractions [7], it became convenient to study the action of $SL(2, Z_n)$ on various types of Cartesian products of $Z_n$ [8]. Note that the orbits of $SL(2, Z_p)$ on $Z_p^2$, where $p$ is a prime number were, considered in [9] §16.3. The purpose of this paper is to generalize this result to orbits of $SL(m,Z_n)$ on $Z_n^m$ where $m$, $n$ are arbitrary natural numbers.

## 2 Action of the group $SL(m, Z_n)$

Throughout the paper we shall use the following notation: N:={1, 2, 3, …} denotes the set of all natural numbers and P:={2, 3, 5, …} denotes the set of all prime numbers. Let $n$ be a natural number, then the set $\{0, 1, \dots, n-1\}$ forms, together with operations $+_{\mathrm{mod}\, n}$, $\times_{\mathrm{mod}\, n}$, an associative commutative ring with unity. We will denote this ring, as usual, by $Z_n$. It is well known that for $n$ prime the ring $Z_n$ is a field.

Let us consider $m$, $n$ to be arbitrary natural numbers. We denote by

$$Z_n^m = \underbrace{Z_n \times Z_n \times \ldots \times Z_n}_{m}$$

the Cartesian product of $m$ rings $Z_n$. It is clear that $Z_n^m$ with operations $+_{\mathrm{mod}\, n}$, $\times_{\mathrm{mod}\, n}$ defined elementwise is an associative commutative ring with unity again. It contains divisors of zero and we call its elements **row vectors** or **points**. Furthermore we call the zero element (0, …, 0) **zero vector** and denote it simply by 0.

We denote by $Z_n^{m,m}$ the set of all $m\times m$ matrices with elements in the ring $Z_n$. For $k \in N$ and $A \in Z_n^{m,m}$ we will denote by $(A)_{\mathrm{mod}\, k}$ a matrix which arose from matrix A after application of operation modulo $k$ on its elements.

In the following we shall frequently use a product on the set $Z_n^{m,m}$ defined as matrix multiplication together with operation modulo $n$, i.e.

$$A, B \in Z_n^{m,m} \to (AB)_{\mathrm{mod}\, n}. \qquad (2.1)$$

This product is, due to the associativity of matrix multiplication, associative again and the set $Z_n^{m,m}$ equipped with this product forms a semigroup. If we take matrices $A, B \in Z_n^{m,m}$, such that $\det(A) = \det(B) = 1 \pmod n$, then $\det((AB)_{\mathrm{mod}\, n}) = 1 \pmod n$ holds. It follows that the subset of $Z_n^{m,m}$ formed by all matrices with the determinant equal to unity modulo $n$ is a semigroup.

**Definition 2.1**: For $m, n \in N$, $n \geq 2$ we define

$$SL(m, Z_n) := \{A \in Z_n^{m,m} \,|\, \det A = 1 \pmod n\}.$$

Now we show that $SL(m,Z_n)$ with operation (2.1) forms a group. Because $SL(m, Z_n)$ is a semigroup, it is sufficient to show that there exists a unit element and a right inverse element. Unit matrix is clearly the unit element. In order to find a right inverse element consider the following equation

$$AA^{\mathrm{adj}} = \det(A)I. \qquad (2.2)$$

The symbol $A^{\mathrm{adj}}$ denotes the adjoint matrix defined by $(A^{\mathrm{adj}})_{i,j} := (-1)^{i+j} \det A(j,i)$, where $A(j,i)$ is the matrix obtained from matrix A by omitting the $j$-th row and the $i$-th column. The equation (2.2) holds for an arbitrary matrix, hence it holds for matrices from $SL(m, Z_n)$, and evidently holds after application of operation modulo $n$ on both sides. Consequently, for $A \in SL(m, Z_n)$, we have

$AA^{\mathrm{adj}} = I \pmod n$, i.e. $(AA^{\mathrm{adj}})_{\mathrm{mod}\, n} = I$.

Therefore $A^{\mathrm{adj}}$ is the right inverse element corresponding to matrix A, and consequently $SL(m, Z_n)$ is a group.

 

The group $SL(m, Z_n)$ is finite and its order was computed by You Hong and Gao You in [10] (see also [11], p. 86). If $n \in N, n \geq 2$ is written in the form $n = \prod_{i=1}^{r} p_i^{k_i}$, where $p_i$ are distinct primes, then according to [10], the order of $SL(m, Z_n)$ is

$$\left|SL(m, Z_n)\right| = n^{m^2-1} \prod_{i=1}^{r} \prod_{j=2}^{m} \left(1 - \frac{1}{p_i^j}\right). \qquad (2.3)$$

Let $G$ be a group and $X \neq 0$ a set. Recall that a mapping $\psi: G \times X \to X$ is called a **right action** of the group $G$ on the set X if the following conditions hold for all elements $x \in X$:

1. $\psi(gh, x) = \psi(g, \psi(h, x))$ for all $h, g \in G$.
2. $\psi(e, x) = x$, where $e$ is the unit element of $G$.

Let $\psi$ be an action of a group $G$ on a set X. A subset of $G$, $\{g \in G | \psi(g, a) = a\}$ is called a **stability subgroup** of the element $a \in X$. A subset of X, $\{b \in X | \exists\, g \in G, b = \psi(g, a)\}$ is called an **orbit** of the element $a \in X$ with respect to the action $\psi$ of group $G$.

Let us note that if $\psi$ is an action of a group $G$ on a set X then relation $\sim$ defined by formula

$$a, b \in X, \quad a \sim b \Leftrightarrow \exists\, g \in G, \psi(g, a) = b \qquad (2.4)$$

is an equivalence on the set X and the corresponding equivalence classes are orbits.

**Definition 2.2**: For $m, n \in N$, $n \geq 2$ we define a right action $\psi$ of the group $SL(m, Z_n)$ on the set $Z_n^m$ as right multiplication of the row vector $a \in Z_n^m$ by the matrix $A \in SL(m, Z_n)$ modulo $n$:

$$\psi(A, a) := (aA)_{\bmod n}.$$

Henceforth we will omit the symbol mod $n$ and write this action simply as $a$A.

# 3 Orbits for $n = p$ prime number

The purpose of this section is to describe orbits of the ring $Z_p^m$ under the action of the group $SL(m, Z_p)$, where $p$ is prime. Trivially, for $m = 1$ is $SL(1, Z_p) = \{(1)\}$ and any orbit has the form $\{a\}$ for $a \in Z_p$. Consequently we will further consider $m \geq 2$. It is clear that the zero element can be transformed by the action of $SL(m, Z_p)$ to itself only, thus it forms a one-point orbit and its stability subgroup is the whole $SL(m, Z_p)$. Let us take a nonzero element, for instance $(0, \ldots, 0, 1) \in Z_p^m$, and find its orbit. An arbitrary matrix A from $SL(m, Z_p)$ acts on this element as follows

$$(0, \ldots, 0, 1) \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m-1,1} & A_{m-1,2} & \cdots & A_{m-1,m} \\ A_{m,1} & A_{m,2} & \cdots & A_{m,m} \end{pmatrix} =$$

$$= (A_{m,1}, A_{m,2}, \ldots A_{m,m}) \pmod p.$$

Thus the orbit of element $(0, \ldots, 0, 1)$ contains the last row of any matrix from $SL(m, Z_p)$. It follows from $\det(A) = 1$ that these rows cannot be zero and we show that they can be equal to an arbitrary nonzero element from $Z_p^m$. Let $(A_{m,1}, A_{m,2}, \ldots A_{m,m}) \in Z_p^m$ be a nonzero element, which means $\exists\, j \in \{1, 2, \ldots, m\}$ such that $A_{mj} \neq 0$, then matrix A can be chosen with the determinant equal to 1. Without loss of generality consider $j = 1$:

$$A = \begin{pmatrix} 0 & & & \\ \vdots & & B & \\ 0 & & & \\ A_{m,1} & A_{m,2} & \ldots & A_{m,m} \end{pmatrix},$$

where $B = \mathrm{diag}\left(1, \ldots, 1, (-1)^{1+m}(A_{m,1})^{-1}\right)$.

Here $(A_{m,1})^{-1}$ denotes the inverse element to $A_{m,1}$ in the field $Z_p$.

We conclude that in the case of $n = p$ prime there are only two orbits:

1. one-point orbit represented by the zero element $(0, \ldots, 0, 0)$
2. $(p^m - 1)$-point orbit $Z_p^m \setminus \{0\}$ represented by the element $(0, \ldots, 0, 1)$

# 4 Orbits for $n$ natural number

We consider an arbitrary natural number $n$ of the form

$$n = \prod_{i=1}^{r} p_i^{k_i},$$

where $p_i$ are distinct primes and $k_i$ are natural numbers.

The action of the group $SL(m, Z_n)$ on the ring $Z_n^m$ was established in definition 2.2 as a right multiplication of a row vector from $Z_n^m$ by a matrix from $SL(m, Z_n)$ modulo $n$. We define an equivalence induced by this action on the ring $Z_n^m$ according to (2.4). Elements $a = (a_1, a_2, \ldots, a_m)$, $b = (b_1, b_2, \ldots, b_m) \in Z_n^m$ are equivalent $a \sim b$ if and only if there exists $A \in SL(m, Z_n)$ such that $aA = b$ i.e.

$$\sum_{j=1}^{m} a_j A_{i,j} = b_i (\bmod n), \quad \forall i \in \{1, 2, \ldots, m\}. \qquad (4.1)$$

**Definition 4.1**: Let $\sim$ be the equivalence on $Z_n^m$ defined by (4.1). For any divisor $d$ of $n$, we will denote by $\mathrm{Or}_{m,n}(d)$ the class of equivalence (orbit) containing the point $(0, \ldots, 0, (d)_{\bmod n})$, i.e.

$$\mathrm{Or}_{m,n}(d) = \{a \in Z_n^m | a \sim (0, \ldots, (d)_{\bmod n})\}. \qquad (4.2)$$

Note that the orbit $\mathrm{Or}_{m,n}(n)$ contains only the zero vector, because the zero vector can be transformed by the action of $SL(m, Z_n)$ only to itself. We shall see later that any orbit in $Z_n^m$ has the form (4.2).

**Definition 4.2**: A **greatest common divisor** of the element $a = (a_1, a_2, \ldots, a_m) \in Z_n^m$ and the number $n \in N$ is the greatest common divisor of all components of the element $a$ and the number $n$ in the ring of integers Z. We denote it by

$$\gcd(a, n) := \gcd(a_1, a_2, \ldots, a_m, n). \qquad (4.3)$$

**Lemma 4.3**: The action of the group $SL(m, Z_n)$ on the ring $Z_n^m$ preserves the greatest common divisor of an arbitrary element $a \in Z_n^m$ and the number $n$, i.e.

$$\gcd(a\mathrm{A}, n) = \gcd(a, n) \quad \forall a \in \mathrm{Z}_n^m, \ \forall \mathrm{A} \in SL(m, \mathrm{Z}_n).$$

**Proof**: It follows from
$a\mathrm{A} = \left( \sum_{i=1}^m a_i A_{i,1}, \ldots, \sum_{i=1}^m a_i A_{i,m} \right)$ and
$\gcd(a,n) \mid \sum_{i=1}^m a_i A_{i,j}$, $\forall j \in \{1, 2, \ldots, m\}$ that
$\gcd(a, n) \mid \gcd(a\mathrm{A}, n)$, i.e. the greatest common divisor cannot decrease during this action. If we take an element $a\mathrm{A}$ and a matrix $\mathrm{A}^{-1}$ we obtain
$\gcd(a\mathrm{A}, n) \mid \gcd(a\mathrm{A}\mathrm{A}^{-1}, n) = \gcd(a, n)$ and together with the first condition we have $\gcd(a\mathrm{A}, n) = \gcd(a, n)$. QED

**Corollary 4.4**: For any divisor $d$ of $n$ the orbit $\mathrm{Or}_{m,n}(d)$ is a subset of $\{a \in \mathrm{Z}_n^m \mid \gcd(a, n) = d\}$.

We will show that the orbit $\mathrm{Or}_{m,n}(1)$ is equal to the set $\{a \in \mathrm{Z}_n^m \mid \gcd(a, n) = 1\}$. From corollary 4.4 we know that $\mathrm{Or}_{m,n}(1)$ is the subset of $\{a \in \mathrm{Z}_n^m \mid \gcd(a, n) = 1\}$ and we prove that they have the same number of elements. At first we determine the number of points in $\mathrm{Or}_{m,n}(1)$. For this purpose we determine the stability subgroup of the element (0, …, 0, 1). It is obviously formed by matrices of the form

$$\mathrm{A} = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m-1,1} & A_{m-1,2} & \cdots & A_{m-1,m} \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \det(\mathrm{A}) = 1 \,(\mathrm{mod}\, n).$$

Expansion of this determinant gives
$1 = \det(\mathrm{A}) = (-1)^{m+m} \det \mathrm{A}(m, m) = \det \mathrm{A}(m, m) \ (\mathrm{mod}\, n)$.

Therefore the stability subgroup of the point (0, …, 0, 1) is:

$$S := \left\{ \mathrm{A} = \begin{pmatrix} & & & A_{1,m} \\ & \mathrm{B} & & A_{2,m} \\ & & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in SL(m, \mathrm{Z}_n) \mid \mathrm{B} \in SL(m-1, \mathrm{Z}_n) \right\},$$

and its order is

$$|S| = n^{m^2 - m - 1} \prod_{i=1}^{r} \prod_{j=2}^{m-1} (1 - p_i^{-j}). \tag{4.4}$$

According to the Lagrange theorem, the product of the order and the index of an arbitrary subgroup of a given finite group is equal to the order of this group. If we define on the group $SL(m, \mathrm{Z}_n)$ a left equivalence induced by the stability subgroup $S$ by formula

$$\mathrm{A}, \mathrm{B} \in SL(m, \mathrm{Z}_n) \quad \mathrm{A} \approx_S \mathrm{B} \Leftrightarrow \mathrm{A}\mathrm{B}^{-1} \in S,$$

then we obtain equivalence classes of the form $S\mathrm{B} = \{\mathrm{AB} \mid \mathrm{A} \in S\}$, $\mathrm{B} \in SL(m, \mathrm{Z}_n)$, i.e. right cosets from $SL(m, \mathrm{Z}_n)/S$. The number of these cosets is, by definition, the index of subgroup $S$. These cosets correspond one-to-one with the points of the orbit which includes the point (0, …, 0, 1). Therefore the index of the stability subgroup $S$ is equal to the number of points in this orbit. A similar calculation can be done for an arbitrary point in an arbitrary orbit. Thus we have the following proposition.

**Proposition 4.5**: The number of elements in an orbit is equal to the order of the group $SL(m, \mathrm{Z}_n)$ divided by the order of the stability subgroup of an arbitrary element in this orbit.

Using (2.3) and (4.4) we obtain that the number of points in the orbit $\mathrm{Or}_{m,n}(1)$ is equal to

$$\left|\mathrm{Or}_{m,n}(1)\right| = n^m \prod_{i=1}^{r} (1 - p_i^{-m}). \tag{4.5}$$

Now we will determine the number of all elements in $\mathrm{Z}_n^m$ that have the greatest common divisor with the number $n$ equal to unity. This number is equal to the Jordan function.

**Definition 4.6**: For $m \in \mathrm{N}$ a mapping $\varphi_m : \mathrm{N} \to \mathrm{N}$ defined by

$$\varphi_m(n) = \left| \{a \in \mathrm{Z}_n^m \mid \gcd(a, n) = 1\} \right| \tag{4.6}$$

is called the **Jordan function** of the order $m$.

We present, without proof, some basic properties of the Jordan function which can be found in [12].

**Proposition 4.7**: For the Jordan function $\varphi_m$ of the order $m \in \mathrm{N}$ and for any $n \in \mathrm{N}$ holds:

1. $$\varphi_m(n) = n^m \prod_{p \mid n,\, p \in \mathrm{P}} (1 - p^{-m}) \tag{4.7}$$

2. $$\sum_{d \mid n,\, d \in \mathrm{N}} \varphi_m(d) = n^m \tag{4.8}$$

3. $$\varphi_m\!\left(\frac{n}{d}\right) = \left| \{a \in \mathrm{Z}^m_{\frac{n}{d}} \mid \gcd(a, \tfrac{n}{d}) = 1\} \right| = \left| \{a \in \mathrm{Z}_n^m \mid \gcd(a, n) = d\} \right|. \tag{4.9}$$

The number of all elements in $\mathrm{Z}_n^m$, which are co-prime with $n$, given by the first property of the Jordan function $\varphi_m(n)$ (4.7), is equal to the number of points in the orbit $\mathrm{Or}_{m,n}(1)$. Therefore the orbit $\mathrm{Or}_{m,n}(1)$ is formed by all elements in $\mathrm{Z}_n^m$ which are co-prime with $n$.

**Proposition 4.8**: For $m, n \in \mathrm{N}$, $m \geq 2$ holds

$$\mathrm{Or}_{m,n}(1) = \{a \in \mathrm{Z}_n^m \mid \gcd(a, n) = 1\}.$$

## *4.1 Orbits for $n = p^k$ power of a prime*

Let us now consider $n$ of the form $n = p^k$, where $p$ is a prime number and $k \in \mathrm{N}$, and determine orbits in this case.

**Definition 4.1.1**: For $j \in \mathrm{N}$, $j \leq k$, we define a mapping $\mathrm{F}^j : \mathrm{Z}^m_{p^k} \to \mathrm{Z}^m_{p^k}$ by the formula

$$\mathrm{F}^j(a) = (p^j \cdot a)_{\mathrm{mod}\, p^k} \text{ for any } a \in \mathrm{Z}^m_{p^k}.$$

**Lemma 4.1.2**: Let $a$ and $b$ be two equivalent elements from $\mathrm{Z}^m_{p^k}$ and $j \leq k$. Then the elements $\mathrm{F}^j(a)$ and $\mathrm{F}^j(b)$ are equivalent as well.

**Proof**: Let $a, b \in \mathrm{Z}^m_{p^k}$, $a \sim b$. It follows from the definition of equivalence $\sim$ that there exists a matrix $\mathrm{A} \in SL(m, \mathrm{Z}_{p^k})$ such that $a\mathrm{A} = b$. Consequently $\mathrm{F}^j(a\mathrm{A}) = \mathrm{F}^j(b)$, where

$$\mathrm{F}^j(a\mathrm{A}) = (p^j a\mathrm{A})_{\mathrm{mod}\, p^k} = (p^j a)_{\mathrm{mod}\, p^k} (\mathrm{A})_{\mathrm{mod}\, p^k} = \mathrm{F}^j(a)\mathrm{A}.$$

Since we have $F^j(a)A = F^j(b)$ and therefore $F^j(a) \sim F^j(b)$.

QED

**Proposition 4.1.3**: Any orbit in the ring $Z^m_{p^k}$ has the form

$$\mathrm{Or}_{m,p^k}(p^j) = \{a \in Z^m_{p^k} \mid \gcd(a, p^k) = p^j\},\ 0 \le j \le k,$$

and consists of $\left|\mathrm{Or}_{m,p^k}(p^j)\right| = \varphi_m(p^{k-j})$ points.

**Proof**: From Lemma 4.1.2 it is clear that $F^j$ maps the orbit $\mathrm{Or}_{m,p^k}(1)$ into the orbit $\mathrm{Or}_{m,p^k}(p^j)$ and from Corollary 4.4 we have

$$F^j(\mathrm{Or}_{m,p^k}(p^j)) \subset \mathrm{Or}_{m,p^k}(p^j) \subset \{a \in Z^m_{p^k} \mid \gcd(a, p^k) = p^j\}.$$

Conversely,

$$\{a \in Z^m_{p^k} \mid \gcd(a, p^k) = p^j\} = \{p^j a \mid a \in Z^m_{p^{k-j}}, \gcd(a, p^{k-j}) = 1\}$$

$$\subset \{(p^j a)_{\mathrm{mod}\, p^k} \mid a \in Z^m_{p^k}, \gcd(a, p^k) = 1\} = F^j(\mathrm{Or}_{m,p^k}(1)).$$

Thus we have

$$F^j(\mathrm{Or}_{m,p^k}(1)) = \mathrm{Or}_{m,p^k}(p^j) = \{a \in Z^m_{p^k} \mid \gcd(a, p^k) = p^j\}.$$

QED

## 4.2 Orbits for $n = pq$, $\gcd(p,q) = 1$

Let us now consider $n$ of the form $n = pq$, where $pq \in N$ are co-prime numbers. In this case it will be very useful to apply the Chinese remainder theorem [13].

**Theorem 4.2.1**: (Chinese remainder theorem)

Let $a_1, a_2 \in Z$. Let $p_1, p_2 \in N$ be co-prime numbers. Then there exists $x \in Z$, such that

$$x = a_i (\mathrm{mod}\ p_i), \quad \forall i = 1, 2.$$

If $x$ is a solution, then $y$ is a solution if and only if

$$x = y (\mathrm{mod}\ p_1 p_2).$$

**Definition 4.2.2**: For $p, q \in N$, $\gcd(p, q) = 1$ we define a mapping $G: Z^m_{pq} \to Z^m_p \times Z^m_q$ by the formula

$$G(a) := \left((a)_{\mathrm{mod}\, p}, (a)_{\mathrm{mod}\, q}\right) \text{ for any } a \in Z^m_{pq},$$

and a mapping $g: SL(m, Z_{pq}) \to SL(m, Z_p) \times SL(m, Z_q)$ by the formula

$$g(A) := \left((A)_{\mathrm{mod}\, p}, (A)_{\mathrm{mod}\, q}\right) \text{ for any } A \in SL(m, Z_{pq}).$$

It is clear from definition that G, g are homomorphisms and the Chinese remainder theorem implies that G, g are one-to-one correspondences. Thus we have the following proposition.

**Proposition 4.2.3**: The mapping G is an isomorphism of rings and the mapping g is an isomorphism of groups.

Further we determine orbits on the Cartesian product of rings $Z^m_p \times Z^m_q$. For this purpose we define the action of the Cartesian product of groups $SL(m, Z_p) \times SL(m, Z_q)$ on ring $Z^m_p \times Z^m_q$ by the formula

$$aA = (a_1, a_2)(A_1, A_2) = \left((a_1A_1)_{\mathrm{mod}\, p}, (a_2A_2)_{\mathrm{mod}\, q}\right)$$

for any $a = (a_1, a_2) \in Z^m_p \times Z^m_q$ and any $A = (A_1, A_2) \in SL(m, Z_p) \times SL(m, Z_q)$.

It follows from the definition of this action that orbits in $Z^m_p \times Z^m_q$ are Cartesian products of orbits in $Z^m_p$ and $Z^m_q$.

**Proposition 4.2.4**: Let $p, q \in N$ be co-prime numbers. Then the mapping G provides one-to-one correspondence between the orbits in $Z^m_{pq}$ and the Cartesian products of the orbits in $Z^m_p$ and $Z^m_q$. Moreover, if $p_1 | p$, $q_1 | q$ and the orbits $\mathrm{Or}_{m,p}(p_1)$, $\mathrm{Or}_{m,q}(q_1)$ are of the form

$$\mathrm{Or}_{m,p}(p_1) = \{a \in Z^m_p \mid \gcd(a, p) = p_1\},$$

$$\mathrm{Or}_{m,q}(q_1) = \{a \in Z^m_q \mid \gcd(a, q) = q_1\},$$

then

$$\mathrm{Or}_{m,pq}(p_1q_1) = G^{-1}\left(\mathrm{Or}_{m,p}(p_1) \times \mathrm{Or}_{m,q}(q_1)\right) = \{a \in Z^m_{pq} \mid \gcd(a, pq) = p_1q_1\}.$$

**Proof**: First, we prove that G and $G^{-1}$ preserve equivalence, i.e.

$$a \sim b \Leftrightarrow G(a) \sim G(b) \text{ for all } a, b \in Z^m_{pq}.$$

From the definition of equivalence we have

$$a \sim b \Leftrightarrow \exists A \in SL(m, Z_{pq}),\ aA = b \Leftrightarrow G(aA) = G(b),$$

where

$$G(aA) = \left((aA)_{\mathrm{mod}\, p}, (aA)_{\mathrm{mod}\, q}\right) = \left((a)_{\mathrm{mod}\, p}, (a)_{\mathrm{mod}\, q}\right)\left((A)_{\mathrm{mod}\, p}, (A)_{\mathrm{mod}\, q}\right) = G(a)g(A).$$

Because G and g are one-to-one correspondences we obtain

$$a \sim b \Leftrightarrow aA = b \Leftrightarrow G(a)g(A) = G(b) \Leftrightarrow G(a) \sim G(b).$$

Since the mapping G is an isomorphism and G, $G^{-1}$ preserve equivalence, the orbits in the ring $Z^m_{pq}$ correspond one-to-one with the orbits in the ring $Z^m_p \times Z^m_q$, and these are Cartesian products of orbits on $Z^m_p$ and $Z^m_q$.

Now remain to prove that the orbit $\mathrm{Or}_{m,pq}(p_1q_1)$ corresponds to the orbit $\mathrm{Or}_{m,p}(p_1) \times \mathrm{Or}_{m,q}(q_1)$. It follows from the Chinese remainder theorem that G maps the set $\{a \in Z^m_{pq} \mid \gcd(a, pq) = p_1q_1\}$ on the set

$$\{(a_1, a_2) \in Z^m_p \times Z^m_q \mid \gcd(a_1, p) = p_1, \gcd(a_2, q) = q_1\},$$

which is equal to the orbit $\mathrm{Or}_{m,p}(p_1) \times \mathrm{Or}_{m,q}(q_1)$. Therefore the set $\{a \in Z^m_{pq} \mid \gcd(a, pq) = p_1q_1\}$ forms an orbit and from Corollary 4.4 it follows that

$$\mathrm{Or}_{m,pq}(p_1q_1) = \{a \in Z^m_{pq} \mid \gcd(a, pq) = p_1q_1\}.$$

QED

As a corollary of Propositions 4.1.3 and 4.2.4 we obtain the following theorem.

**Theorem 4.9**: Consider the decomposition of the ring $Z^m_n$, $m \ge 2$ into orbits with respect to the action of the group $SL(m, Z_n)$. Then

i) any orbit is equal to the orbit $\mathrm{Or}_{m,n}(d)$ for some divisor $d$ of $n$, i.e.

$$Z^m_n = \bigcup_{d|n} \mathrm{Or}_{m,n}(d);$$

ii) $\mathrm{Or}_{m,n}(d) = \{a \in \mathrm{Z}_n^m \mid \gcd(a,n) = d\}$;

iii) the number of points $\left|\mathrm{Or}_{m,n}(d)\right|$ in $d$-orbit is given by the Jordan function

$$\left|\mathrm{Or}_{m,n}(d)\right| = \varphi_m\left(\frac{n}{d}\right) = \left(\frac{n}{d}\right)^m \prod_{pd|n,\ p\in\mathrm{P}} (1 - p^{-m}).$$

# 5 Conclusion

We have stepwise determined the orbits on the ring $\mathrm{Z}_n^m$ with respect to the action of the group $SL(m, \mathrm{Z}_n)$. First, we proceeded in the same way as Kirillov in [9] and we obtained the orbits in the case of $n$ prime number. In this case there are only two orbits, the first is one-point orbit formed by the zero element and the second is formed by all nonzero elements. The next step was the case of $n = p^k$ power of prime. There we found $k+1$ orbits characterized by the greatest common divisor of their elements and number $n$. Finally the orbits for an arbitrary natural number $n$ were found. Our results are summarized in Theorem 4.9.

# 6 Acknowledgments

We would like to thank Prof. Jiří Tolar, Prof. Miloslav Havlíček and Doc. Edita Pelantová for numerous stimulating and inquisitive discussions.

# References

[1] Šťovíček, P., Tolar, J.: "Quantum Mechanics in a Discrete Space-Time." *Rep. Math. Phys*. Vol. **20** (1984), p. 157–170.

[2] Vourdas, A.: "Quantum Systems with Finite Hilbert Space." *Rep. Progr. Phys*. Vol. **67** (2004), p. 267–320.

[3] Patera, J., Zassenhaus, H.: The Pauli Matrices in $n$ Dimensions and Finest Gradings of Simple Lie Algebras of Type $\mathrm{A}_{n-1}$." *J. Math. Phys*. Vol. **29** (1988), p. 665–673.

[4] Gell-Mann, M.: "Symmetries of Baryons and Mesons." *Phys. Rev*. Vol. **125** (1962), p. 1067.

[5] Havlíček, M., Patera, J., Pelantová, E., Tolar, J.: "The Fine Gradings of sl(3, C) and Their Symmetries," in Proc. of XXIII. International Colloquium on Group Theoretical Methods in Physics, eds. A. N. Sissakian, G. S. Pogosyan and L. G. Mardoyan, JINR, Dubna, Vol. **1** (2002), p. 57–61.

[6] Havlíček, M., Patera, J., Pelantová, E., Tolar, J.: "Automorphisms of the Fine Grading of sl($n$, C) Associated with the Generalized Pauli Matrices." *J. Math. Phys*. Vol. **43**, 2002, p. 1083–1094.

[7] de Montigny, M., Patera, J.: "Discrete and Continuous Graded Contractions of Lie Algebras and Superalgebras." *J. Phys. A: Math. Gen.*, Vol. **24** (1991), p. 525–547 .

[8] Hrivnák, J.: "Solution of Contraction Equations for the Pauli Grading of sl(3, C)." Diploma Thesis, Czech Technical University, Prague 2003.

[9] Kirillov, A. A.: *Elements of the Theory of Representations*, Springer, New York 1976.

[10] You Hong, Gao You: "Computation of Orders of Classical Groups over Finite Commutative Rings." *Chinese Science Bulletin*, Vol. **39** (1994), No. 14, p. 1150–1154.

[11] Drápal, A.: *Group Theory – Fundamental Aspects* (in Czech), Karolinum, Praha 2000.

[12] Schulte, J.: "Über die Jordanische Verallgemeinerung der Eulerschen Funktion." *Resultate der Mathematik*, Vol. **36** (1999), p. 354–364.

[13] Graham, R. L., Kmoth, D. E., Patashnik, O.: *Concrete Mathematics*, Addison-Wesley, Reading, MA, 1994.

Ing. Petr Novotný
phone: +420 222 311 333
fujtajflik@seznam.cz

Ing. Jiří Hrivnák
phone: +420 222 311 333
hrivnak@post.cz

Department of Physics

Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering
Břehová 7
115 19 Prague 1, Czech Republic