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Laplace Adomian Decomposition and Modify Laplace Adomian Decomposition Methods for Solving Linear Volterra Integro-Fractional Differential Equations with Constant Multi-Time Retarded Delay

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Abstract

In this work, we present Laplace transform with series Adomian decomposition and modify Adomian decomposition methods for the first time to solve linear Volterra integro-differential equations of the fractional order in Caputo sense with constant multi-time Retarded delay. This method is primarily based on the elegant mixture of Laplace transform method, series expansion method and Adomian polynomial with modifications. The proposed technique will transform the multi-term delay integro-fractional differential equations into some iterative algebraic equations, and it is capable of reducing computational analytical works where the kernel of difference and simple degenerate types. Analytical examples are presented to illustrate the efficiency and accuracy of the proposed methods.

Keywords: Caputo fractional derivative, Delay differential equations, Integro-differential equation, Laplace transform, Adomian decomposition method and Modify Adomian decomposition method.

1. Introduction

The idea of this work is to solve linear Volterra Integro-Fractional Differential Equations (VIFDE's) in Caputo sense with constant multi-time Retarded Delay (RD) in the general form:

$$ {}_{a}^{C}D_{t}^{\alpha_n} u(t) + \sum_{i=1}^{n-1} P_i(t)\, {}_{a}^{C}D_{t}^{\alpha_{n-i}} u(t) + P_0(t) u(t-\tau) = f(t) + \lambda \sum_{j=1}^{m} \int_{0}^{t} \mathcal{K}_j(t,x)\, u(x-\tau_j)\, dx\,, \quad t \in [0,b] \qquad \dots(1) $$

For $\alpha_n > \alpha_{n-1} > \alpha_{n-2} > \alpha_{n-3} > \cdots > \alpha_1 > \alpha_0 = 0$, with initial conditions which are given: $u^{(k)}(0) = u_k$; $k = 0,1,2, \dots, \mu - 1$, $(\mu = \lceil \alpha_n \rceil)$ and $\mu - th$ historical continuity differentiable functions $u(t) = \varphi(t)$ for $t \in [\bar{a}, 0]$, where $\bar{a} = -\max\{\tau, \tau_j : j = 1,2, \dots, m\}$. Connected, where $u(t)$ is the solution of equation (1) which is the unknown function and $\mathcal{K}_j \in C(S \times \mathbb{R}, \mathbb{R})$, $S = \{(t, x) : 0 \le x \le t \le b\}$ for all $j = 1, 2, \cdots, m$ and given $f, P_i \in C([0, b], \mathbb{R})$, for all $i = 0,1, \dots, n-1$ where $u(t) \in \mathbb{R}$, ${}^{C}D_t^{\alpha_i} u(t)$ is the α_i-fractional Caputo-derivative order of u on $[0, b]$ and all $\alpha_i, \in \mathbb{R}^+$ for $(i \neq 0)$, $n_{\alpha_i - 1} < \alpha_i \le n_{\alpha_i}$, $n_{\alpha_i} = \lceil \alpha_i \rceil$, for all $i = 1,2, \dots, n$. Furthermore, the quantities $\tau_j \in \mathbb{R}^+$ for all $j = 1,2, \dots, m$ are called the time-lags (delay).

Such equation LVIFDE-RD's it is relatively a new subject in mathematics so there are only few of techniques for solving it and the exact analytic solution has not, thus approximation technique must be used for treating it.

The author Adomian [1], introduced The Adomian decomposition technique (ADM) which possess great potential in solving different kinds of the linear/nonlinear functional equation. This method assumes that the unknown function $u(t)$ can be expressed with the aid of a sum of a limitless number of components $u_i(t)$ described by the decomposition series. Each term of the series is obtained from a polynomial generated by a power series expansion of an analytical function. Adomian and Rach [2] additionally Wazwaz [3] have investigated the noise terms phenomena of the self-canceling where the sum of all factors are vanishing in the limit. In [4], the noise terms are described as the same terms with an opposite sign that appear within the elements say $u_0(t)$ and $u_1(t)$ which that exists only in particular types of non-homogenous equations. Further, it used to be formally justified that if terms in $u_0(t)$ are vanishing by terms in $u_1(t)$, even though. $u_1(t)$ includes further terms, then the closing non-canceled terms in $u_0(t)$ may additionally represent the exact solution of the problem.

The main objective of this work is to use the combined Laplace transform-Adomian decomposition method with noise term phenomenon in solving the higher fractional order of linear VIDE's with constant multi-time Retarded delay problem where the kernel of difference and simple degenerate types.

This paper is prepared as follows: Section 2 presents the definition and some important property; section 3 solve linear Volterra integro-differential equation of fractional order with constant multi-time Retarded delay using Laplace-Adomian decomposition method; our results illustrated throughout examples in section 4. Finally, section 5 includes a discussion for this method.

2. Basic definitions and some property

In this section, some preliminaries and notations related to fractional calculus and Laplace operation are given. For more details, see [5,6,7,8,9]:

Definition 2.1: A real valued function u defined on $[a, b]$ be in the space $C_\delta[a, b]$, δ-any real number, if there exists a real number $\ell > \delta$, such that $u(t) = (t - a)^\ell u_c(t)$, where $u_c \in C[a, b]$, and it is said to be in the space $C_\delta^n[a, b]$ if and only if $u^{(n)} \in C_\delta[a, b], n$-positive integer number with zero.

Definition 2.2: Let $u \in C_\delta[a,b], \delta \geq -1$ with any positive arbitrary real number α. Then the Riemann-Liouville (R-L) fractional integral operator ${}_aJ_t^\alpha$ of order α of a function u, is defined as:

$$ {}_aJ_t^\alpha u(t) = \begin{cases} \int_a^t \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} u(\xi)d\xi, & \alpha > 0 \\ u(t) \quad \text{whenever} & \alpha = 0 \end{cases} $$

Definition 2.3: Let $\alpha \geq 0$, and $m = \lceil \alpha \rceil$. the Riemann-Liouville fractional derivative operator ${}_a^RD_t^\alpha$, of order α and $u \in C_{-1}^m[a,b]$ and defined as:

$$ {}_a^RD_t^\alpha u(t) = \begin{cases} D_t^m[{}_aJ_t^{m-\alpha} u(t)], & \alpha > 0 \\ u(t) \quad \text{whenever} \quad \alpha = 0 \\ u^{(m)}(t), \text{If } \alpha = m(\in \mathbb{N}) \text{ and } u \in C^m[a,b] \end{cases} $$

Definition 2.4: The Caputo fractional derivative operator ${}_a^CD_t^\alpha$ of order $\alpha \in \mathbb{R}^+$ of a function $u \in C_{-1}^m[a,b]$ and $m-1 < \alpha \leq m, m \in \mathbb{N}$ is defined as:

$$ {}_a^CD_t^\alpha u(t) = \begin{cases} {}_aJ_t^{m-\alpha}[D_t^m u(t)], & \alpha > 0 \\ u(t) \quad \text{whenever} \quad \alpha = 0 \\ u^{(m)}(t), \text{If } \alpha = m(\in \mathbb{N}) \text{ and } u \in C^m[a,b] \end{cases} $$

Hence, we have the following properties:

- For $\alpha \geq 0$ and $\beta > 0$, then ${}_aJ_t^\alpha (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1}$.
- For all $\alpha \geq 0, \beta \geq 0$ and $u(t) \in C_\delta[a,b], \delta \geq -1$, then:
$$ {}_aJ_t^\alpha\, {}_aJ_t^\beta u(t) = {}_aJ_t^\beta\, {}_aJ_t^\alpha u(t) = {}_aJ_t^{\alpha+\beta} u(t) $$
- ${}_a^RD_t^\alpha \mathcal{A} = \mathcal{A}\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}$ and ${}_a^CD_t^\alpha \mathcal{A} = 0$; $\mathcal{A}$ is any constant; ($\alpha \geq 0, \alpha \notin \mathbb{N}$)
- ${}_a^RD_t^\alpha u(t) = D_t^m\, {}_aJ_t^{m-\alpha} u(t) \neq {}_aJ_t^{m-\alpha} D_t^m u(t) = {}_a^CD_t^\alpha u(t)$; $m = \lceil \alpha \rceil$.
- Assume that $u \in C_{-1}^m[a,b]$; $\alpha \geq 0, \alpha \notin \mathbb{N}$ and $m = \lceil \alpha \rceil$ then ${}_a^CD_t^\alpha u(t)$ is continuous on $[a,b]$, and $[{}_a^CD_t^\alpha u(t)]_{t=a} = 0$.
- Let $\alpha \geq 0$, $m = \lceil \alpha \rceil$ and $u \in C^m[a,b]$, then, the relation between the Caputo derivative and Riemann-Liouville (R-L) integral are formed:
$$ {}_a^CD_t^\alpha[{}_aJ_t^\alpha u(t)] = u(t) \;;\quad a \leq t \leq b \;;\quad {}_aJ_t^\alpha\,[{}_a^CD_t^\alpha u(t)] = u(t) - \sum_{k=0}^{m-1} \frac{u^{(k)}(a)}{k!}(t-a)^k $$
- ${}_a^CD_t^\alpha u(t) = {}_a^RD_t^\alpha\big[u(t) - T_{m-1}[u;a]\big]$, $(m-1 < \alpha \leq m)$ and $T_{m-1}[u;a]$ denotes the Taylor polynomial of degree $m-1$ for the function u, centered at a.
- Let $\alpha \geq 0$; $m = \lceil \alpha \rceil$ and for $u(t) = (t-a)^\beta$ for some $\beta \geq 0$. Then:
$$ {}_a^CD_t^\alpha u(t) = \begin{cases} 0 & if\ \beta \in \{0,1,2,\cdots,m-1\} \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(t-a)^{\beta-\alpha} & if\ \beta \in \mathbb{N}\ and\ \beta \geq m \\ & or\ \beta \notin \mathbb{N}\ and\ \beta > m-1 \end{cases} $$

Form the thought of the fractional derivative: The Caputo's definition is a modification of the Riemann-Liouville (R-L) definition and has the benefit of dealing properly with the initial value problem so we undertake Caputo’s definition in this papers.

Definition (2.5): [10, 11] The Laplace transforms of a function $u(t)$ of real variable $t \in \mathbb{R}^+$, denoted by $U(s)$, is defined by the equation

$$U(s) = \mathcal{L}\{u(t); s\} = \int_0^{\infty} e^{-st} u(t)\ dt \qquad \dots (2)$$

and its inverse is given for $t \in \mathbb{R}^+$ by the formula, symbolically written as: $\mathcal{L}^{-1}\{U(s); t\} = u(t)$.

In references [9,10,12,13,14] and [6,8] respectively can be founding the proves of all lemmas (1-4) and lemma (5-i, 5-ii), about the Laplace transform with several properties such that important for our work.

Lemma (1): The Laplace transform is related to the transform of the $n-th$ derivative of a function, where $U(s)$ is a Laplace of $u(t)$:

$$\mathcal{L}\left\{\frac{d^n u(t)}{dt^n}\right\} = s^{\,n} U(s) - \sum_{k=0}^{n-1} s^{n-k-1}\ u^{(k)}(0)$$
$$= s^{\,n} U(s) - \sum_{k=0}^{n-1} s^{n}\ u^{(n-k-1)}(0) \qquad \dots (3)$$

Lemma (2): The Laplace transform of the convolution of two functions is the product of their Laplace transforms. Thus $U(s)$ and $V(s)$ are the Laplace transforms of $u(t)$ and $v(t)$ respectively, then:

$$\mathcal{L}\{(u * v)(t)\} = \mathcal{L}\left\{\int_0^t u(t-x) v(x) dx\right\} = U(s)V(s) \qquad \dots (4)$$

especially:

$$\mathcal{L}\left\{\int_0^t u(x) dx\right\} = \frac{1}{s}\ U(s) \qquad \dots (5)$$

Lemma (3): If $U(s)$ is the Laplace of $u(t)$ and t^n is a power function of order $n \in \mathbb{Z}^+$, then:

$$\mathcal{L}\{t^n u(t)\} = (-1)^n\ \frac{d^n}{ds^n}\ \mathcal{L}\{u(t)\} = (-1)^n\ \frac{d^n}{ds^n}\ U(s) \qquad \dots (6)$$

Lemma (4): let $U(s)$ be the Laplace of $u(t)$ then:

$$\left.\begin{aligned} \mathcal{L}\left\{\int_0^t t\, u(x) dx\right\} &= -\frac{d}{ds}\left(\frac{1}{s}\ U(s)\right) \\ \mathcal{L}\left\{\int_0^t x\, u(x) dx\right\} &= -\frac{1}{s}\frac{d}{ds}\ U(s) \end{aligned}\right\} \qquad \dots (7)$$

Lemma (5):

(i) The Laplace transform of the R-L Fractional integral for order $\alpha \in \mathbb{R}^+$, $J_t^{\alpha}\, u(t) = {}_0J_t^{\alpha}\, u(t)$, using the convolution property, gives:

$$\mathcal{L}\{J_t^{\alpha}\, u(t)\} = \mathcal{L}\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)} * u(t)\right\} = \mathcal{L}\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\} \mathcal{L}\{u(t)\} = s^{-\alpha}\, U(s) \qquad \dots (8)$$

(ii) The Laplace transform of Caputo Fractional of order α $(m-1 < \alpha \le m)$ and $m = \lceil \alpha \rceil$, ${}_0^cD_t^{\alpha}\, u(t) = {}^cD_t^{\alpha}\, u(t)$, can be obtained as follows:

$$\mathcal{L}\{\,{}^{c}D_t^{\alpha}\,u(t)\} = \mathcal{L}\{J_t^{m-\alpha} D_t^{m} u(t)\} = s^{-(m-\alpha)}\,\mathcal{L}\{u^{(m)}(t)\}$$
$$= s^{-(m-\alpha)}\left[s^{m}\,U(s) - \sum_{k=0}^{m-1} s^{m-k-1}\,u^{(k)}(0)\right]$$
$$= s^{\alpha}\,U(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1}\,u^{(k)}(0) \qquad \dots (9)$$

Laplace transform of a constant delay function is explained in the following important new-lemma:

Lemma (6): Let $u(t)$ be a continuous differentiable function on a closed bounded interval $[0,b]$, $b \in \mathbb{R}^{+}$ and let τ be a constant delay such that:

$$u(t) = \varphi(t), \qquad for\ \ -\tau \le t < 0 \qquad \dots (10)$$

Then the Laplace transform of a $\tau -$ delay function is given by:

$$\mathcal{L}\{u(t-\tau)\} = e^{-s\tau}\,[U(s) + Q(s,\tau)] \qquad \dots (11)$$

where

$$Q(s,\tau) = \int_{-\tau}^{0} e^{-sx}\varphi(t)dt$$

and

$$\mathcal{L}\{u(t)\} = U(s).$$

If the historical function $\varphi(t)$ is defined by power function $t^{n}, (n \in \mathbb{Z}^{+})$ we have:

$$\mathcal{L}\{u(t-\tau)\} = e^{-s\tau}\,U(s) + \sum_{p=0}^{n}(-1)^{n-p}\,p!\,\binom{n}{p}\frac{\tau^{n-p}}{s^{p+1}} - \frac{n!}{s^{n+1}}\,e^{-s\tau} \qquad \dots (12)$$

Proof:

By taking Laplace transform of τ-delay function $u(t-\tau)$, as in definition (1), and applying the change of variable by $t-\tau = x$ we obtain:

$$\mathcal{L}\{u(t-\tau)\} = \int_0^{\infty} e^{-st}u(t-\tau)dt = e^{-s\tau}\int_{-\tau}^{\infty} e^{-sx}u(x)dx$$
$$= e^{-s\tau}\left[\int_{-\tau}^{0} + \int_0^{\infty}\right] e^{-sx}\,u(x)dx \qquad \dots (13)$$

Use by part integral method for solving first integral in (13) after instead $u(x)$ by historical function (H.F.) $\varphi(x)$, which is defined $x^{n}, n \in \mathbb{Z}^{+}$,we get:

$$Q(s,\tau) = \int_{-\tau}^{0} e^{-sx}\varphi(x)dx = \int_{-\tau}^{0} e^{-sx}\,x^{n}\,dx$$
$$= e^{s\tau}\sum_{p=0}^{n}(-1)^{n-p}\,p!\,\binom{n}{p}\frac{\tau^{n-p}}{s^{p+1}} - \frac{n!}{s^{n+1}} \qquad \dots (14)$$

And the second integral part in (13) is the Laplace transform of $u(x)$, thus

$$\int_0^\infty e^{-sx}u(x)dx = U(s) \qquad \dots(15)$$

Substitution equations (14) and (15) into equation (13) we obtain:

$$\mathcal{L}\{u(t-\tau)\} = e^{-s\tau}\,U(s) + \sum_{p=0}^{n}(-1)^{n-p}\,p!\binom{n}{p}\frac{\tau^{n-p}}{s^{p+1}} - \frac{n!}{s^{n+1}}\,e^{-s\tau}$$

which completes the proof. Note that, in general, for historical function (H.F.) which is defined:

$$\varphi(x) = \sum_{r=1}^{R} a_r x^{n_r} \;\; ; \;\{R \in \mathbb{Z}^+ \;, n_r \in \mathbb{Z}^+ \;, a_r \in \mathbb{Z}^+\} \qquad \dots(16)$$

Then the formula (12) becomes:

$$\mathcal{L}\{u(t-\tau)\} = e^{-s\tau}\,U(s) + \sum_{r=1}^{R} a_r\left[\sum_{p=0}^{n_r}(-1)^{n_r-p}p!\binom{n_r}{p}\frac{\tau^{n_r-p}}{s^{p+1}} - \frac{n_r!}{s^{n_r+1}}e^{-s\tau}\right]$$

3. Analysis Technique of Method

In this section we try to find general solution form of linear VIFDE's with multi-time RD by applying the Laplace transform with aid of the Adomain decomposition techniques for two different types of kernel: difference and simple degenerate kernel.

3.1 Apply the LADM for Solving Linear VIFDE-RD's of Difference Kernel

Firstly, consider the VIFDE-RD's of difference kernel type $\mathcal{K}_j\,(t,x) = \mathcal{K}_j\,(t-x)$ for all $j = 1,2,\dots,m$. Moreover, take $P_i(t)$ as a power function, say $C_i t^{\ell_i}$, $C_i \in \mathbb{R}$ and ℓ_i be any nonnegative integer numbers for all i. Apply Laplace transform on both sides of equation (1):

$$\mathcal{L}\{{}_a^C D_t^{\alpha_n}u(t)\} + \sum_{i=1}^{n-1}\mathcal{L}\{P_i(t)\,{}_a^C D_t^{\alpha_{n-i}}u(t)\} + \mathcal{L}\{P_0(t)\,u(t-\tau)\} = \mathcal{L}\{f(t)\} + \sum_{j=1}^{m}\lambda\mathcal{L}\left\{\int_0^t \mathcal{K}_j\,(t-x)\;u(x-\tau_j)dx\right\} \qquad \dots(17)$$

First, using Caputo fractional differentiation property of Laplace transform (9) with initial conditions $u_k = u^{(k)}(0)$, where $m_{\alpha_n} - 1 < \alpha_n \le m_{\alpha_n}$, we obtain

$$\mathcal{L}\{{}^C D_t^{\alpha_n}u(t)\} = s^{\alpha_n}\,U(s) - \sum_{k=0}^{m_{\alpha_n}-1} s^{\alpha_n-k-1}\;u_k \qquad \dots(18)$$

Second, for all $i = 1,2,\dots.n-1$, using equation (6) and then applying equation (9), where ℓ_i is the order of $P_i(t)$ for each i, and $m_{\alpha_{n-i}} - 1 < \alpha_{n-i} \le m_{\alpha_{n-i}}$. We get:

$$\mathcal{L}\{P_i(t)\ {}^{C}D_t^{\alpha_{n-i}}u(t)\} = C_i(-1)^{\ell_i}\frac{d^{\ell_i}}{ds^{\ell_i}}\left(s^{\alpha_{n-i}}\,U(s)\right) - C_i\ (-1)^{\ell_i}\frac{d^{\ell_i}}{ds^{\ell_i}}\sum_{k=0}^{m_{\alpha_{n-i}}-1}\left[s^{\alpha_{n-i}-k-1}\ u_k\right] \quad \dots(19)$$

Third, using equation (6) and then applying the Lemma (6), using 11 and 12 respectively, we obtain:

$$\mathcal{L}\{P_0(t)u(t-\tau)\} = C_0(-1)^{\ell_0}\frac{d^{\ell_0}}{ds^{\ell_0}}\left[e^{-s\tau}\big(U(s)+Q(s,\tau)\big)\right] \quad \dots(20,A)$$

As a special case, where ℓ_0 is the order of $P_0(t)$ and q is the order of historical polynomial function, $\varphi(t)$ is $t^q, q \in \mathbb{Z}^+$,we have

$$\mathcal{L}\{P_0(t)\ u(t-\tau)\} = C_0(-1)^{\ell_0}\frac{d^{\ell_0}}{ds^{\ell_0}}\left[e^{-s\tau}U(s)\right] + C_0(-1)^{\ell_0}\frac{d^{\ell_0}}{ds^{\ell_0}}\left[\sum_{p=0}^{q}(-1)^{q-p}\ p!\binom{q}{p}\frac{\tau^{q-p}}{s^{p+1}}\right] - C_0(-1)^{\ell_0}\frac{d^{\ell_0}}{ds^{\ell_0}}\left[\frac{q!}{s^{q+1}}e^{-s\tau}\right] \quad \dots(20,B)$$

Fourth, using the definition of Laplace transformation, we get:

$$\mathcal{L}\{f(t)\} = F(s) \quad \dots(21)$$

At last, for all $j = 1,2, \dots m$ we apply equation (4) with Lemma (6), 11 and 12 respectively to obtain:

$$\mathcal{L}\left\{\int_0^t \mathcal{K}_j\,(t-x)\ \ u\big(x-\tau_j\big)dx\right\} = \mathcal{K}_j(s)\ e^{-s\tau_j}\left[\Big(U(s)+Q\big(s,\tau_j\big)\Big)\right] \quad \dots(22,A)$$

As a special case, where the historical function $\varphi(t)$ is $t^q, q \in \mathbb{Z}^+$,we have:

$$\mathcal{L}\left\{\int_0^t \mathcal{K}_j\,(t-x)\ \ u\big(x-\tau_j\big)dx\right\} = \mathcal{K}_j(s)\left[e^{-s\tau_j}U(s) + \sum_{p=0}^{q}(-1)^{q-p}p!\binom{q}{p}\frac{\tau_j^{\,q-p}}{s^{p+1}} - \frac{q!}{s^{q+1}}e^{-s\tau_j}\right] \quad \dots(22,B)$$

Finally, substitution the equations $(18,19,20{:}\,A, 21,22{:}\,A)$ into the equation (17) and after some simple manipulations, to get the following formula:

$$s^{\alpha_n}\,U(s) = F^*(s) + \lambda\sum_{j=1}^{m}\mathcal{K}_j(s)\ e^{-s\tau_j}\ \ U(s) + W\big(s,U(s)\big) \quad \dots(23)$$

Where

$$W\big(s,U(s)\big) = -\left[\sum_{i=1}^{n-1} C_i(-1)^{\ell_i}\frac{d^{\ell_i}}{ds^{\ell_i}}[s^{\alpha_{n-i}}U(s)] + C_0(-1)^{\ell_0}\frac{d^{\ell_0}}{ds^{\ell_0}}[e^{-s\tau}U(s)]\right] \quad \dots (24)$$

and

$$F^*(s) = F(s) + \lambda\sum_{j=1}^{m}\mathcal{K}_j(s)\, e^{-s\tau_j}\, Q\big(s,\tau_j\big) + \sum_{k=0}^{m_{\alpha_n}-1} s^{\alpha_n-k-1}\ u_k + \sum_{i=1}^{n-1} C_i(-1)^{\ell_i}\frac{d^{\ell_i}}{ds^{\ell_i}}\left[\sum_{k=0}^{m_{\alpha_{n-i}}-1} s^{\alpha_{n-i}-k-1}\ u_k\right] - C_0(-1)^{\ell_0}\frac{d^{\ell_0}}{ds^{\ell_0}}[e^{-s\tau}Q(s,\tau)] \quad \dots (25)$$

If historical function is power function $t^q, q \in \mathbb{Z}^+$, putting the equations $(18,19,20\colon B, 21,22\colon B)$ into equation (17), to obtain the following equations (26) instead of (25):

$$F^*(s) = F(s) + \lambda\sum_{j=1}^{m}\mathcal{K}_j(s)\left[\sum_{p=0}^{q}(-1)^{q-p}\, p!\binom{q}{p}\frac{{\tau_j}^{q-p}}{s^{p+1}} - \frac{q!}{s^{q+1}}e^{-s\tau_j}\right] + \sum_{k=0}^{m_{\alpha_n}-1} s^{\alpha_n-k-1}\ u_k + \sum_{i=1}^{n-1} C_i(-1)^{\ell_i}\frac{d^{\ell_i}}{ds^{\ell_i}}\left[\sum_{k=0}^{m_{\alpha_{n-i}}-1} s^{\alpha_{n-i}-k-1}\ u_k\right] - C_0(-1)^{\ell_0}\frac{d^{\ell_0}}{ds^{\ell_0}}\left[\sum_{p=0}^{q}(-1)^{q-p}\, p!\binom{q}{p}\frac{\tau^{q-p}}{s^{p+1}} - \frac{q!}{s^{q+1}}e^{-s\tau}\right] \quad \dots (26)$$

According to the decomposition method which consists of decomposing the unknown function $u(t)$ into a sum of components defined by the decomposition series

$$u(t) = u_0(t) + u_1(t) + \cdots + u_r(t) + \cdots = \sum_{r=0}^{\infty} u_r(t) \quad \dots (27)$$

Taking Laplace transform to each components in equation (27) and letting $U_r(s) = \mathcal{L}\{u_r(t)\}$, $\forall\ r = 0,1,\dots$. Thus

$$U(s) = U_0(s) + U_1(s) + \cdots + U_r(s) + \cdots = \sum_{r=0}^{\infty} U_r(s) \qquad \ldots (28)$$

After substituting equation (28) with (23) and (24), it leads to the following recursive relation:

$$\left.\begin{aligned} U_0(s) &= \frac{1}{\psi(s)} F^*(s) \\ U_{k+1}(s) &= \frac{1}{\psi(s)} \left[W_k\big(s, U(s)\big) + \lambda \sum_{j=1}^{m} \mathcal{K}_j\,(s)\, e^{-s\tau_j}\, U_k(s) \right]_{for\ k\geq 0} \end{aligned}\right\} \qquad \ldots (29)$$

$$W_k\big(s, U(s)\big) = -\left[\sum_{i=1}^{n-1} C_i (-1)^{\ell_i} \frac{d^{\ell_i}}{ds^{\ell_i}} [s^{\alpha_{n-i}} U_k(s)] + C_0 (-1)^{\ell_0} \frac{d^{\ell_0}}{ds^{\ell_0}} [e^{-s\tau} U_k(s)] \right] \qquad \ldots (30)$$

If the historical function is any continuous differentiable function $\varphi(t)$,thus $F^*(s)$ is defined in equation (25) and if historical function is power function $t^q, q \in \mathbb{Z}^+$ then $F^*(s)$ is take the formula (26), with $\psi(s) = s^{\alpha_n}$ and applying the inverse Laplace transform to equation (29) and putting in the equation (27) gives $u(t)$ the solution of linear VIFDE-RD's of difference kernel.

3.2 Apply the LADM for Solving Linear VIFDE's of Simple Degenerate Kernel:

Laplace-Adomian decomposition technique can also be used to solve the VIFDE's with constant multi-time Retarded delays which the kernels are simple degenerate type, formed as: $\mathcal{K}_j(t, x) = c_j t^{k_j^1} + d_j x^{k_j^2}$ for all $k_j^1, k_j^2 \in \mathbb{Z}^+$ and c_j , $d_j \in \mathbb{R}$ for all $j = 1,2, \ldots, m$. Apply Laplace transform on both sides of equation (1):

$$\mathcal{L}\{{}_a^C D_t^{\alpha_n} u(t)\} + \sum_{i=1}^{n-1} \mathcal{L}\{P_i(t) {}_a^C D_t^{\alpha_{n-i}} u(t)\} + \mathcal{L}\{P_0(t)\, u(t-\tau)\}$$
$$= \mathcal{L}\{f(t)\} + \sum_{j=1}^{m} \lambda \mathcal{L}\left\{ \int_0^t \left[c_j t^{k_j^1} + d_j x^{k_j^2}\right]\ u(x-\tau_j) dx \right\} \qquad \ldots (31)$$

Thus, as a same step in section 3.1 we get same equations (17, 18, 19, 20 (A, B) and 21) with replacing last equation (22 (A, B)) to the following formulas. Now we apply equation (7) with Lemma (5), (equations 11 and 12) respectively, and using Leibniz's formula for higher derivative of multiplication functions [15], then after some manipulating we obtain for all $j = 1,2, \ldots, m$:

$$\mathcal{L}\left\{\int_0^t \left[c_j t^{k_j^1} + d_j x^{k_j^2}\right] u(x-\tau_j)dx\right\}$$

$$= \frac{e^{-s\tau_j}}{s}\left\{\left[c_j\left(\sum_{r=0}^{k_j^1} r!\binom{k_j^1}{r}\frac{1}{s^r}{\tau_j}^{k_j^1-r}\right) + d_j{\tau_j}^{k_j^2}\right] + \left[d_j\sum_{r=0}^{k_j^2-1}(-1)^{r+k_j^2}{\tau_j}^r\binom{k_j^2}{r}\frac{d^{k_j^2-r}}{ds^{k_j^2-r}}\right]\right.$$

$$\left. + \left[c_j\sum_{r=0}^{k_j^1-1}(-1)^{r+k_j^1}r!\binom{k_j^1}{r}\frac{1}{s^r}\left(\sum_{p=0}^{k_j^1-r-1}(-1)^p{\tau_j}^p\binom{k_j^1-r}{p}\frac{d^{k_j^1-r-p}}{ds^{k_j^1-r-p}}\right)\right]\right\}U(s)$$

$$+\frac{1}{s}\left\{c_j\left[\sum_{r=0}^{k_j^1}(-1)^{r+k_j^1}r!\binom{k_j^1}{r}\frac{1}{s^r}\frac{d^{k_j^1-r}}{ds^{k_j^1-r}}\right]\right.$$

$$\left.+d_j\left[(-1)^{k_j^2}\frac{d^{k_j^2}}{ds^{k_j^2}}\right]\right\}H_j^q(s) \qquad \dots(32)$$

Where

$$H_j^q(s) = \begin{cases} e^{-s\tau_j}Q(s,\tau_j)\;; \quad if \text{ the (HF) any continuos differentiable function} \\ \displaystyle\sum_{p=0}^{q}(-1)^{q-p}\,p!\binom{q}{p}\frac{{\tau_j}^{q-p}}{s^{p+1}} - \frac{q!}{s^{q+1}}e^{-s\tau_j} \quad ; \qquad if \quad \varphi(t) = t^q \end{cases}$$

and

$$Q(s,\tau_j) = \int_{-\tau_j}^{0} e^{-sx}\varphi(x)dx$$

Finally, substitution the equations $(18,19,20{:}\,A,21,32)$ into the equation (31) and after some simple manipulations, we get the following formula:

$$s^{\alpha_n}U(s) = F^*(s) + W(s,U(s))$$

$$+\lambda\sum_{j=1}^{m}\frac{e^{-s\tau_j}}{s}\left[c_j\left(\sum_{r=0}^{k_j^1}r!\binom{k_j^1}{r}\frac{{\tau_j}^{k_j^1-r}}{s^r}\right)\right.$$

$$\left.+d_j{\tau_j}^{k_j^2}\right]U(s) \qquad \dots(33)$$

Where

$$W\big(s,U(s)\big) = \lambda\sum_{j=1}^{m}\frac{e^{-s\tau_j}}{s}\left\{\left[c_j\sum_{r=0}^{k_j^1-1}(-1)^{r+k_j^1}r!\binom{k_j^1}{r}\frac{1}{s^r}\left(\sum_{p=0}^{k_j^1-r-1}(-1)^p{\tau_j}^p\binom{k_j^1-r}{p}\frac{d^{k_j^1-r-p}}{ds^{k_j^1-r-p}}\right)\right] + \left[d_j\sum_{r=0}^{k_j^2-1}(-1)^{r+k_j^2}{\tau_j}^r\binom{k_j^2}{r}\frac{d^{k_j^2-r}}{ds^{k_j^2-r}}\right]\right\}U(s) - \sum_{i=1}^{n-1}C_i(-1)^{\ell_i}\frac{d^{\ell_i}}{ds^{\ell_i}}[s^{\alpha_{n-i}}U(s)] - C_0(-1)^{\ell_0}\frac{d^{\ell_0}}{ds^{\ell_0}}[e^{-s\tau}U(s)] \quad \dots(34)$$

If the historical function is any continuous differentiable function $\varphi(t)$,thus:

$$F^*(s) = F(s) + \lambda\sum_{j=1}^{m}\frac{1}{s}\left\{c_j\left[\sum_{r=0}^{k_j^1}(-1)^{r+k_j^1}r!\binom{k_j^1}{r}\frac{1}{s^r}\frac{d^{k_j^1-r}}{ds^{k_j^1-r}}\right] + d_j\left[(-1)^{k_j^2}\frac{d^{k_j^2}}{ds^{k_j^2}}\right]\right\}e^{-s\tau_j}Q(s,\tau_j) + \sum_{k=0}^{m_{\alpha_n}-1}s^{\alpha_n-k-1}\,u_k + \sum_{i=1}^{n-1}C_i(-1)^{\ell_i}\frac{d^{\ell_i}}{ds^{\ell_i}}\left[\sum_{k=0}^{m_{\alpha_{n-i}}-1}s^{\alpha_{n-i}-k-1}\,u_k\right] - C_0(-1)^{\ell_0}\frac{d^{\ell_0}}{ds^{\ell_0}}[e^{-s\tau}Q(s,\tau)] \quad \dots(35)$$

If the historical function is $\varphi(t) = t^q, q \in \mathbb{Z}^+$ thus:

$$F^*(s) = F(s) + \lambda\sum_{j=1}^{m}\frac{1}{s}\left\{c_j\left[\sum_{r=0}^{k_j^1}(-1)^{r+k_j^1}r!\binom{k_j^1}{r}\frac{1}{s^r}\frac{d^{k_j^1-r}}{ds^{k_j^1-r}}\right] + d_j\left[(-1)^{k_j^2}\frac{d^{k_j^2}}{ds^{k_j^2}}\right]\right\}\left[\sum_{p=0}^{q}\binom{q}{p}(-1)^{q-p}\,p!\,\frac{{\tau_j}^{q-p}}{s^{p+1}} - \frac{q!}{s^{q+1}}e^{-s\tau_j}\right] + \sum_{k=0}^{m_{\alpha_n}-1}s^{\alpha_n-k-1}\,u_k + \sum_{i=1}^{n-1}C_i(-1)^{\ell_i}\frac{d^{\ell_i}}{ds^{\ell_i}}\left[\sum_{k=0}^{m_{\alpha_{n-i}}-1}s^{\alpha_{n-i}-k-1}\,u_k\right] - C_0(-1)^{\ell_0}\frac{d^{\ell_0}}{ds^{\ell_0}}\left[\sum_{p=0}^{q}\binom{q}{p}(-1)^{q-p}\,p!\,\frac{\tau^{q-p}}{s^{p+1}} - \frac{q!}{s^{q+1}}e^{-s\tau}\right] \quad \dots(36)$$

After substitution equation where (33) and (34) into (28), then the following recursive relation is:

$$\left.\begin{aligned} U_0(s) &= \frac{1}{\psi(s)}F^*(s) \\ U_{k+1}(s) &= \frac{1}{\psi(s)}\left\{W_k\big(s,U(s)\big) + \lambda\sum_{j=1}^{m}\frac{e^{-s\tau_j}}{s}\left[c_j\left(\sum_{r=0}^{k_j^1} r!\binom{k_j^1}{r}\frac{{\tau_j}^{k_j^1-r}}{s^r}\right) + d_j{\tau_j}^{k_j^2}\right]U_k(s)\right\} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad for\ \ k \geq 0 \end{aligned}\right\} \quad \ldots(37)$$

where

$$\begin{aligned} & W_k\big(s,U(s)\big) \\ &= \lambda\sum_{j=1}^{m}\frac{e^{-s\tau_j}}{s}\left\{\left[c_j\sum_{r=0}^{k_j^1-1}(-1)^{r+k_j^1} r!\binom{k_j^1}{r}\frac{1}{s^r}\left(\sum_{p=0}^{k_j^1-r-1}(-1)^p{\tau_j}^p\binom{k_j^1-r}{p}\frac{d^{k_j^1-r-p}}{ds^{k_j^1-r-p}}\right)\right]\right. \\ &+ \left.\left[d_j\sum_{r=0}^{k_j^2-1}(-1)^{r+k_j^2}{\tau_j}^r\binom{k_j^2}{r}\frac{d^{k_j^2-r}}{ds^{k_j^2-r}}\right]\right\}U_k(s) - \sum_{i=1}^{n-1}C_i(-1)^{\ell_i}\frac{d^{\ell_i}}{ds^{\ell_i}}\left[s^{\alpha_{n-i}}U_k(s)\right] \\ &- C_0(-1)^{\ell_0}\frac{d^{\ell_0}}{ds^{\ell_0}}\left[e^{-s\tau}U_k(s)\right] \end{aligned} \quad \ldots(38)$$

If the historical function is any continuous differentiable function $\varphi(t)$,then $F^*(s)$ is defined in equation (35) and if historical function is power function $t^q, q \in \mathbb{Z}^+$ then $F^*(s)$ is take the formula (36), with $\psi(s) = s^{\alpha_n}$ with applying the inverse Laplace transform to equation (37) and substitution in the equation (27) gives $u(t)$ the solution of linear VIFDE-RD's of simple degenerate kernel.

3.3 Apply the Modify LADM for Solving Linear VIFDE-RD's:

Wazwaz [3] has been developed the Adomian decomposition method and this change minimizes the step size of calculation with effectiveness if comparing together. To apply this modification, assume that the function $f(t)$ can be divided into the sum of two parts, namely $f_1(t)$ and $f_2(t)$, therefore, we set

$$f(t) = f_1(t) + f_2(t) \quad \ldots(39)$$

In view of (39), we introduce a qualitative change in the formation of the recurrence relations: (29) of difference kernel and (37) of simple degenerate kernel. To minimize the step-size of calculations, we identify the zeros component $U_0(s)$ by one part of $F(s)$, namely, $F_1(s)$ or $F_2(s)$ which is the Laplace transform of $f_1(t)$ or $f_2(t)$, respectively. The other part $F(s)$ can be added to the component $U_1(s)$ among the other terms.

For Difference kernel the equations (29 and 30) with equations (25 and 26) the modified recurrence relations are:

$$\left.\begin{aligned} U_0(s) &= \frac{1}{\psi(s)} F_1^*(s) \\ U_1(s) &= \frac{F_2^*(s)}{\psi(s)} + \frac{1}{\psi(s)} \left\{ W_0\big(s, U(s)\big) + \lambda \sum_{j=1}^{m} \mathcal{K}_j\,(s)\, e^{-s\tau_j}\, U_0(s) \right\} \\ U_{k+1}(s) &= \frac{1}{\psi(s)} \left\{ W_k\big(s, U(s)\big) + \lambda \sum_{j=1}^{m} \mathcal{K}_j\,(s)\, e^{-s\tau_j}\, U_k(s) \right\} for\ k \geq 1 \end{aligned}\right\} \ldots \quad (40)$$

Where $\psi(s) = s^{\alpha_n}$, $W_k\big(s, U(s)\big)$ is defined in equation (30) and $F_1^*(s)$ is formed as in equation (25) or (26) except at the first part $F(s)$, changing it to $F_1^*(s) = \mathcal{L}\{f_1(t)\}$ while $F_2^*(s) = \mathcal{L}\{f_2(t)\}$.

For Simple Degenerate kernel the equations (37 and 38) with equations (35 and 36) the modified recurrence relations are:

$$\left.\begin{aligned} U_0(s) &= \frac{1}{\psi(s)} F_1^*(s) \\ U_1(s) &= \frac{F_2^*(s)}{\psi(s)} + \frac{1}{\psi(s)} \left\{ W_0\big(s, U(s)\big) + \lambda \sum_{j=1}^{m} \frac{e^{-s\tau_j}}{s} \left[c_j \left(\sum_{r=0}^{k_j^1} r! \binom{k_j^1}{r} \frac{{\tau_j}^{k_j^1 - r}}{s^r} \right) + d_j {\tau_j}^{k_j^2} \right] U_0(s) \right\} \\ U_{k+1}(s) &= \frac{1}{\psi(s)} \left\{ W_k\big(s, U(s)\big) + \lambda \sum_{j=1}^{m} \frac{e^{-s\tau_j}}{s} \left[c_j \left(\sum_{r=0}^{k_j^1} r! \binom{k_j^1}{r} \frac{{\tau_j}^{k_j^1 - r}}{s^r} \right) + d_j {\tau_j}^{k_j^2} \right] U_k(s) \right\} k \geq 1 \\ &\ldots (41) \end{aligned}\right\}$$

Where $\psi(s) = s^{\alpha_n}$, $W_k\big(s, U(s)\big)$ is defined in equation (38) and $F_1^*(s)$ is formed as in equation (35 or 36) except at the first part $F(s)$, changing it to $F_1^*(s) = \mathcal{L}\{f_1(t)\}$,while $F_2^*(s) = \mathcal{L}\{f_2(t)\}$.

4. Analytic Examples:

Here, some example of Linear VIFDE's with constant multi-time Retarded delay which solved by Laplace-Adomian decomposition and Modify Laplace-Adomian decomposition methods.

Example (1): Consider the Linear VIFDE's with variable coefficients of constant multi-time R-D with difference kernel type:

$${}_0^C D_t^{0.3} u(t) - t u(t - 0.3) = f(t) + \int_0^t [(t - x)^2 u(x - 0.5) + 2(t - x) u(x - 0.3)] dx$$

where

$$f(t)=\frac{6}{\Gamma(3.7)}t^{2.7}-\frac{1}{60}t^{6}-\frac{1}{20}t^{5}-\frac{73}{80}t^{4}+\frac{511}{600}t^{3}-\frac{243}{1000}t^{2}+\frac{27}{1000}t$$

with initial condition $u(0)=0$ and historical function $\varphi(t)=t^3$.

Since here we have: $\mathcal{K}_1(t,x)=(t-x)^2$; $\mathcal{K}_2(t,x)=2(t-x)$ with constant time-delays $\tau=\tau_0=0.3$, $\tau_1=0.5$, $\tau_2=0.3$ and $q=3$. Also we can putting $\alpha_1=0.3$; $m_{\alpha_1}=1$; $P_0(t)=-t$; $\ell_0=1$; $C_0=-1$. So the Laplace transforms of kernels are given

$$\mathcal{L}\{\mathcal{K}_1(t)\}=\frac{2}{s^3} \quad ; \quad \mathcal{L}\{\mathcal{K}_2(t)\}=\frac{2}{s^2}$$

Equation (26) yields:

$$F^*(s)=\frac{6}{s^{3.7}}-\frac{12}{s^7}e^{-0.5s}-\frac{12}{s^6}e^{-0.3s}-\frac{24}{s^5}e^{-0.3s}-\frac{1.8}{s^4}e^{-0.3s}$$

with $\psi(s)=s^{0.3}$, using the first part recursive relation (29) we obtain:

$$U_0(s)=\frac{6}{s^4}-\frac{12}{s^{7.3}}e^{-0.5s}-\frac{12}{s^{6.3}}e^{-0.3s}-\frac{24}{s^{5.3}}e^{-0.3s}-\frac{1.8}{s^{4.3}}e^{-0.3s}$$

for $k=0$ and using equation (30), we have:

$$W_0\big(s,U(s)\big)=0.3e^{-0.3s}U_0(s)-e^{-0.3s}\frac{d}{ds}U_0(s)$$

Applying the second part recursive relation (29) with $k=0$, we get:

$$\begin{aligned}U_1(s)&=\frac{1}{\psi(s)}\left[W_0\big(s,U(s)\big)+\sum_{j=1}^{2}\mathcal{K}_j(s)\,e^{-s\tau_j}\,U_0(s)\right]\\&=\frac{12}{s^{7.3}}e^{-0.5s}+\frac{12}{s^{6.3}}e^{-0.3s}+\frac{24}{s^{5.3}}e^{-0.3s}+\frac{1.8}{s^{4.3}}e^{-0.3s}-\frac{138}{s^{6.6}}e^{-0.6s}\\&-\frac{22.14}{s^{5.6}}e^{-0.6s}-\frac{13.2}{s^{7.3}}e^{-0.8s}-\frac{1.08}{s^{4.6}}e^{-0.6s}-\frac{123.6}{s^{7.6}}e^{-0.6s}-\frac{48}{s^{9.6}}e^{-0.8s}\\&-\frac{135.6}{s^{8.6}}e^{-0.8s}-\frac{24}{s^{10.6}}e^{-s}-\frac{24}{s^{8.6}}e^{-0.6s}\end{aligned}$$

We see that the phenomena of the self-canceling "noise" term, [3,8], $\pm\frac{12}{s^{7.3}}e^{-0.5s}$; $\pm\frac{12}{s^{6.3}}e^{-0.3s}$; $\pm\frac{24}{s^{5.3}}e^{-0.3s}$ and $\pm\frac{1.8}{s^{4.3}}e^{-0.3s}$ appear in $U_0(s)$ and $U_1(s)$, cancelling this terms from the zeroth Laplace component $U_0(t)$, thus $U_0(s)=\frac{6}{s^4}$, and taking the inverse of Laplace transform of it gives the exact solution $u(t)=\mathcal{L}^{-1}\{U(s)\}=t^3$ that satisfies with the equation.

Example (2): Consider the Linear VIFDE's with variable coefficients of constant multi-time Retarded delays with simple degenerate kernel, $\mathcal{K}(t,x)=(2t+x)$ and constant coefficients

$${}_0^C D_t^{0.8}u(t)+2u(t-1)=f(t)+\int_0^t(2t+x)u(x-0.5)\,dx$$

where

$$f(t)=\frac{-6}{\Gamma(2.2)}t^{1.2}+\frac{11}{4}t^{4}-4t^{3}-\frac{33}{8}t^{2}+12\,t-6$$

with initial condition and historical function: $u(0)=0$ and $\varphi(t)=-3t^{2}$, respectably.

Since here we have only one kernel: $\mathcal{K}_1(t,x)=2t+x$ that is simple degenerate type $m=1$ and $k_1^1=k_1^2=1$; $c_1=2, d_1=1$, with constant time-delays $\tau=\tau_0=1$, $\tau_1=0.5$ and $P_0(t)=2$ so $\ell_0=0$; $C_0=2$. Also, since $\alpha_1=0.8$; $m_{\alpha_1}=1$. From equation (36) we yield:

$$F^{*}(s)=\frac{-6}{s^{2.2}}+\frac{66}{s^{5}}e^{-0.5s}+\frac{9}{s^{4}}e^{-0.5s}-\frac{12}{s^{3}}e^{-s}$$

with $\psi(s)=s^{0.3}$, using the first part recursive relation (37) we obtain:

$$U_0(s)=\frac{-6}{s^{3}}+\frac{66}{s^{5.8}}e^{-0.5s}+\frac{9}{s^{4.8}}e^{-0.5s}-\frac{12}{s^{3.8}}e^{-s}$$

for $k=0$ and using equation (38), we have:

$$W_0\big(s,U(s)\big)=-\frac{3}{s}e^{-0.5s}\frac{d^{2}}{ds^{2}}U_0(s)-2e^{-s}U_0(s)$$

By applying the second part recursive relation (29) with $k=0$, we get:

$$U_1(s)=\frac{1}{\psi(s)}\left\{W_0\big(s,U(s)\big)+\frac{e^{-s\tau_1}}{s}\left[c_1\left(\sum_{r=0}^{k_1^1}r!\binom{k_1^1}{r}\frac{{\tau_1}^{k_1^1-r}}{s^{r}}\right)+d_1{\tau_1}^{k_1^2}\right]U_0(s)\right\}$$
$$=\frac{-66}{s^{5.8}}e^{-0.5s}-\frac{9}{s^{4.8}}e^{-0.5s}+\frac{12}{s^{3.8}}e^{-s}-\cdots$$

We see that the phenomena of the self-canceling “noise” term, [5,7], $\pm\frac{66}{s^{5.8}}e^{-0.5s}$; $\pm\frac{9}{s^{4.8}}e^{-0.5s}$ and $\pm\frac{12}{s^{3.8}}e^{-s}$ appear in $U_0(s)$ and $U_1(s)$, Cancelling this terms from the zeroth Laplace component $U_0(t)$, and taking the inverse of Laplace transform of it, gives the exact solution $u(t)=\mathcal{L}^{-1}\{U(s)\}=-3t^{2}$ that satisfies the equation.

Example (3): Consider the Linear VIFDE’s with variable coefficients of constant multi-time Retarded delays with difference kernel type:

$${}_0^{C}D_t^{1.2}u(t)-3t\,{}_0^{C}D_t^{0.4}u(t)+\frac{1}{2}u(t-0.6)$$
$$=f(t)+\int_0^t\left[e^{t-x}u(x-0.1)-(t-x)^{2}u(x-0.7)\right]dx$$

where:

$$f(t) = \frac{2}{3\Gamma(1.8)} t^{0.8} - \frac{2}{\Gamma(2.6)} t^{2.6} + \frac{119}{300} e^{t} + \frac{1}{90} t^{5} - \frac{7}{180} t^{4} - \frac{251}{900} t^{3} + \frac{1}{2} t^{2} + \frac{2}{5} t - \frac{251}{300}$$

With initial condition: $u(0) = -1$; $u'(0) = 0$ and initial function: $\varphi(t) = \frac{1}{3} t^2 - 1$.

Now, from the above problem we have: $\mathcal{K}_1(t,x) = e^{t-x}$; $\mathcal{K}_2(t,x) = -(t-x)^2$

with constant time-lags $\tau = \tau_0 = 0.6$, $\tau_1 = 0.1$, $\tau_2 = 0.7$ and $P_1(t) = -3t$; and $P_0(t) = 1/2$ that is $\ell_1 = 1$, $C_1 = -3$; $\ell_0 = 0$, $C_0 = 1/2$. Also, since $\alpha_2 = 1.2$, $m_{\alpha_2} = 2$; $\alpha_1 = 0.4$, $m_{\alpha_1} = 1$. So the Laplace transform of kernels are formed

$$\mathcal{L}\{\mathcal{K}_1(t)\} = \frac{1}{s-1} \quad ; \quad \mathcal{L}\{\mathcal{K}_2(t)\} = -\frac{2}{s^3}$$

Applying equation (33) to obtain $F^*(s)$ and after some simple manipulations we can form:

$$F^*(s) = \frac{2}{3s^{1.8}} - s^{0.2} + \frac{1.8}{s^{1.6}} - \frac{5.2}{s^{3.6}} - \frac{2}{3s^3(s-1)} e^{-0.1s} + \frac{1}{s(s-1)} e^{-0.1s} + \frac{4}{3s^6} e^{-0.7s} - \frac{2}{s^4} e^{-0.7s} + \frac{1}{3s^3} e^{-0.6s} - \frac{1}{2s} e^{-0.6s}$$

To apply modify Laplace-Adomian decomposition method for difference kernel type we first split $F^*(s)$ into two parts, namely:

$$F_1^*(s) = \frac{2}{3s^{1.8}} - s^{0.2}$$

$$F_2^*(s) = \frac{1.8}{s^{1.6}} - \frac{5.2}{s^{3.6}} - \frac{2\ e^{-0.1s}}{3s^3(s-1)} + \frac{e^{-0.1s}}{s(s-1)} + \frac{4\ e^{-0.7s}}{3s^6} - \frac{2e^{-0.7s}}{s^4} + \frac{e^{-0.6s}}{3s^3} - \frac{e^{-0.6s}}{2s}$$

with $\psi(s) = s^{1.2}$ using the first part recursive relation (40) we obtain:

$$U_0(s) = \frac{1}{\psi(s)} F_1^*(s) = \frac{2}{3s^3} - \frac{1}{s}$$

and from equation (30) yields:

$$W_0\big(s, U(s)\big) = -3e^{-0.4s} \frac{d}{ds} U_0(s) - 1.2\ s^{-0.6} U_0(s) - \frac{1}{2}\ e^{-0.6s} U_0(s)$$

Applying the second part recursive relation (40), we get: $U_1(s) = 0$. Using the third part recursive relation (40), we obtain:

$$U_{k+1}(s) = 0, \qquad \text{for all } k \geq 1$$

It is obvious that each component of $u_r, r \geq 1$ is zero. The solution is:

$$u(t) = \mathcal{L}^{-1}\{U(s)\} = \frac{1}{3} t^2 - 1$$

Example (4): Consider the Linear VIFDE's with variable coefficients of constant multi-time Retarded delays with degenerate kernel, $\mathcal{K}(t,x) = -5t^2 + \frac{1}{5}x$, and variable coefficients on [0, 1]:

$$ {}_0^C D_t^{0.6} u(t) - \frac{t}{2} {}_0^C D_t^{0.3} u(t) + u(t-0.7) = f(t) + \int_0^t \left(-5t^2 + \frac{1}{5}x\right) u(x-3) dx $$

where

$$ f(t) = \frac{2}{\Gamma(2.4)} t^{1.4} - \frac{1}{\Gamma(2.7)} t^{2.7} + \frac{5}{3} t^5 - \frac{301}{20} t^4 + \frac{252}{5} t^3 - \frac{7}{5}\, t + \frac{149}{100} $$

With initial condition and initial function: $u(0) = 1$ and $\varphi(t) = 1 + t^2$, respectively. Since with constant time-lags $\tau = \tau_0 = 0.7\,, \tau_1 = 3$ and $P_1(t) = -\frac{1}{2}\, t$; and $P_0(t) = 1$ that is $\ell_1 = 1,\ C_1 = -\frac{1}{2}$; $\ell_0 = 0\,,\ C_0 = 1$. Also, since $\alpha_2 = 0.6,\ m_{\alpha_2} = 1;\ \alpha_1 = 0.3,$ $m_{\alpha_1} = 1$. Furthermore, $\mathcal{K}(t,x) = -5t^2 + \frac{1}{5}x$ that is simple degenerate type $m = 1$ and $k_1^1 = 2,\ k_1^2 = 1;\ c_1 = -5\,, d_1 = \frac{1}{5}$. From equation (36) we yield:

$$ F^*(s) = \frac{2}{s^{2.4}} - \frac{2.7}{s^{3.7}} + s^{-0.4} - 0.35 s^{0.3} + \frac{44.4}{s^2} e^{-3s} + \frac{59.8}{s^3} e^{-3s} + \frac{118.8}{s^4} e^{-3s} + \frac{238.8}{s^5} e^{-3s} + \frac{200}{s^6} e^{-3s} + \frac{2}{s^3} e^{-0.7s} + \frac{e^{-0.7s}}{s} $$

To apply MLAD method for simple degenerate kernel type we first split $F^*(s)$ into two parts, namely:

$$ F_1^*(s) = \frac{2}{s^{2.4}} + s^{-0.4} $$

$$ F_2^*(s) = \frac{-2.7}{s^{3.7}} - 0.35 s^{0.3} + \frac{44.4 e^{-3s}}{s^2} + \frac{59.8 e^{-3s}}{s^3} + \frac{118.8 e^{-3s}}{s^4} + \frac{238.8\, e^{-3s}}{s^5} + \frac{200\, e^{-3s}}{s^6} + \frac{2 e^{-0.7s}}{s^3} + \frac{e^{-0.7s}}{s} $$

with $\psi(s) = s^{0.6}$, using the first part recursive relation (41) we obtain:

$$ U_0(s) = \frac{1}{\psi(s)} F_1^*(s) = \frac{1}{s} + \frac{2}{s^3} $$

And equation (38) yields:

$$ W_0\big(s, U(s)\big) = \left[\frac{-0.15}{s^{0.7}} - e^{-0.7s}\right] U_0(s) + \left[\frac{10 e^{-3s}}{s^2} - \frac{s^{0.3}}{2} + \frac{29.8 e^{-3s}}{s}\right] U_0'(s) - \left[\frac{5 e^{-3s}}{s}\right] U_0''(s) $$

Applying the second part recursive relation (40), we get: $U_1(s) = 0$. Using the third part recursive relation (41), we obtain:

$$ U_{k+1}(s) = 0, \qquad \text{for all } k \geq 1 $$

It is obvious that each component of $u_r, r \geq 1$ is zero. The solution is:

$$u(t) = \mathcal{L}^{-1}\{U(s)\} = 1 + t^2$$

4. Discussion

In this paper, through the Laplace-Adomian decomposition (LAD) and modify Laplace-Adomian decomposition (MLAD) methods for solving linear Volterra integro-fractional differential equations of constant multi-time Retarded-delay type with variable coefficients has been successfully applied to finding the approximate solution. The results pointed the following:

1. In general, this technique in finding analytical solutions for this wide classes of linear VIFDE's-RD which was improved provides good results and effectiveness.
2. The Laplace-Adomian decomposition and modify Laplace-Adomian decomposition methods were applied for difference kernel and simple degenerate kernel in general cases and MLAD method provides more realistic series solutions that converge very rapidly than LAD method.
3. Sometimes the process of finding Laplace-Adomian decomposition method is not easy, so we use the Modifications.

Conflict of Interests.

There are non-conflicts of interest .

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الخلاصة

في هذا العمل نقدم تحويلات لابلاس مع طريقة أدوميان التحليلية المتسلسلة و كما اننا نعدل طريقة أدوميان التحليلية للمرة الاولى لحل معادلات فولتيرا التفاضلية–التكاملية الخطيه للرتب الكسرية كما في مفهوم كابوتو مع التأخير الحدي المتضاعف الثابت. هذه الطريقة تعتمد على مزيج ممتاز من طريقة تحويلات لابلاس، طريقة تحديد المتسلسلات، طريقة متعددات الحدود لادوميان مع التعديلات. أن التقنية المستخدمة تحول التأخير الحدي للمعادلات التفاضلية ذات التكاملات الكسرية الى معادلات جبرية متكررة عندما تكون نواة الفروق من نوع المنحل البسيط. و أخيراً أعطيت أمثلة لتوضيح فعالية و ديقة الطرق المقترحة.

الكلمات الدالة: مفهوم كابوتو ذات الرتب الكسرية، التأخير الحدي للمعادلات التفاضلية، تحويلات لابلاس، معادلات فولتيرا التفاضلية–التكاملية الخطيه ، طريقة متعددات الحدود لادوميان مع التعديلات