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Analysis of a Mathematical Model in a Food Web System Containing Scavenger Species

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Abstract

In this paper, the dynamics of scavenger species in a web food model incorporating time delay and prey harvesting is formulated mathematically. Boundednes of all solutions of the model carried out. The existence as well as stability analysis of all possible positive equilibrium points are discussed. Also, we proved that under certain time delay, our model exhibits a subcritical Hopfbifurcation. Furthermore, to confirm our analytical finding, we studied numerical simulation for the model.

Keyword: Scavengers, Predator- prey model, Stability analysis, Hopf bifurcation.

1. Introduction

After the pioneering works of Lotka and Voltera[1],[18] for per-predator interactions, predator prey models have been important in mathematical ecology and were studied extensively [3],[16]. In the last decades the classical model of Lotka and Voltera is modified by many researchers [7]. In 1963, M. Rosenzweig and R. MacArthur, considered a modification model of lotka voltera model; they replaced the exponential growth by the logistic growth because of limit source for prey and they replaced the functional responses of Hollying type I by functional responses of Hollying type II which has been presented by Hollying [8], [9]. Also, many researchers considered the prey predator model incorporated by the effect of spreading disease on species [2],[6],[10-12], and [19], time delay [13], prey refuge [14-15], and herding behavior of prey [17]. On the other hand, in many articles, there is an extension of simple prey predator model to food chain model [9], and food web [6]. However, till now, there is no mathematical model for the receiving benefit from prey predator interaction by a scavenger. The scavenger species usually consume animals that have either died of natural causes or been killed by another. Therefore, in this paper we model the benefit dynamics of a scavenger from natural death of prey and a parts of bodies of predated prey that is not eaten by predators. Our model incorporates time delay and harvesting factor on prey. Our work is structured as follows: in section two, we have discussed the details of the assumption in the model, the significance of parameters used in it and proof of its boundedness. In section three, all possible positive equilibrium points and criteria for stability are discussed, and Hopf bifurcation is also studied. In section four, we study numerical simulation for our analytical finding using Euler 'method with help of MATLAB. Finally, in section five, we discussed significance of the analytical result and the numerical simulation.

2. Model formulation

Consider an ecological model consisting of a prey, predator and scavenger species, with the following assumption.

1. The prey species increases due to birth with birth rate b and the prey species decreases due to natural death with death rate d_1, intraspecific competition with completion rate c_1, harevesting with harvesting rate h and predation by predators.
2. Increasing the predator species density is dependent on predation of the prey species according to the Holing type II functional responses, and the predator species decreases due to natural death with death rate d_2 and the intraspecific competition with competition rate c_2 .
3. Increasing the Scavenger species density dependent on eating death prey and a part of bodies of predated prey that does not eaten by predators and they decreases due to natural death with death rate d_3 and intra specific competition with competition rate c_3.
4. At time $t, x(t)$, $y(t)$ and $z(t)$ are prey density, predator species and scavenger species repectivly

Then the dynamics of such model can be represented in the following set of nonlinear differential equations

$$\begin{aligned}
\frac{dx}{dt} &= bx - c_1x^2 - (d_1 + h)x - \frac{\alpha xy}{1 + \alpha T_h x} \\
\frac{dy}{dt} &= \frac{e_1\alpha x(t-\tau_1)y}{1+\alpha T_h x(t-\tau_1)} - c_2y^2 - d_2y \qquad (1) \\
\frac{dz}{dt} &= e_2d_1x(t-\tau_2)z + \frac{e_3\alpha x(t-\tau_2)y(t-\tau_2)}{1+\alpha T_h x(t-\tau_2)}z - c_3z^2 - d_3z
\end{aligned}$$

Here, all the parameters are assumed to be positive. Moreover, the parameter α repersented the predation rate; α is the search efficiency of the predator to prey T_h is the average handling time for each prey; τ_1 is the time-lag from predation of prey to convert it in to predator biomass; τ_2 is the time- lag from eating the dead prey or predated prey by predator to converted it in to scavenger biomass; e_1 is the biomass conversation rate of prey population to predator population; e_2 and e_3 are biomass conversation rate of natural death of prey and predated prey by predator. obviously, the right side of the system (1) are continuous and have continuous partial derivatives on the state space R_+^3, therfor they are Lipschizian function on $R_+^3$and then the solution of the system (1) which initiate in non-negative octant are positive and uniformly bounded as shown in the following theorems.

Theorem 1. All solution of the system (1) that initiate with positive values are positive and uniformly bounded if the following condition holds.

$$d_3 < 2 * min\left\{\left(\frac{e_2d_1(b-d_1-h)}{c_1}\right), \frac{e_3(e_1-T_hd_2)}{c_2T_h^2}\right\} \qquad (2)$$

Proof. It is easy to prove that all solution that initiate with positive initial value, remain positive, therefore the proof is omitted. From the first equation of the system (1), it is obtained that

$$\frac{dx}{dt} \le (b - d_1 - h)x - c_1 x^2$$

It is clear from the condition given by in (2), $b - d_1 - h > 0$.

Therefore, we get $\lim_{t\to\infty} Sup(x(t)) \le \frac{b-d_1-h}{c_1}$, and from the second equation of the system (1), it is obtained that

$$\frac{dy}{dt} \le \left(\frac{e_1}{T_h} - d_2\right) y - c_2 y^2$$

the condition given by in (2), guarantees that $e_1 - T_h d_2 > 0$,

So, we get $\lim_{t\to\infty} Sup(y(t)) \le \frac{e_1 - T_h d_2}{c_2 T_h}$. Finaly from third equation of the system (1), we have

$$\frac{dz}{dt} = \left(\frac{e_2 d_1 (b - d_1 - h)}{c_1} + \frac{e_3(e_1 - T_h d_2)}{c_2 T_h^2} - d_3\right) z - c_3 z^2$$

Again from the condition given by in (2), we get $\frac{e_2 d_1 (b-d_1-h)}{c_1} + \frac{e_3(e_1 - T_h d_2)}{c_2 T_h^2} > d_3$.

So,

$$\lim_{t\to\infty} Sup(z(t)) \le \frac{e_2 d_1 (b - d_1 - h)}{c_1 c_3} + \frac{e_3(e_1 - T_h d_2)}{c_2 c_3 T_h^2} - \frac{d_3}{c_3}$$

And hence the theorem.

3. Stability analysis and Hopf bifurcation

The system (1) has at most five positive equilibrium points, namely $E_1 = (0,0,0)$, $E_2 = (\bar{x}, 0,0)$, $E_3 = (\bar{x}, 0, \bar{z})$, $E_4 = (\tilde{x}, \tilde{y}, 0)$ and $E_5 = (\tilde{x}, \tilde{y}, \tilde{z})$.

Where

$\bar{x} = \frac{b-(d_1+h)}{c_1}$, $\bar{z} = \frac{1}{c_3}\left(e_1 d_1 \frac{b-(d_1+h)}{c_1} - d_3\right)$, $\tilde{y} = \frac{1}{c_2}\left(\frac{e_1 \alpha \tilde{x}}{1+\alpha T_h \tilde{x}} - d_2\right)$,

$$\tilde{z} = \frac{1}{c_3}\left(e_2 d_1 \tilde{x} + \frac{e_3 \alpha \tilde{x} \tilde{y}}{1 + \alpha T_h \tilde{x}} - d_3\right)$$

And $\tilde{x}$ is the positive solution of the following equation.

$$A_1 x^3 + A_2 x^2 + A_3 x + A_4 = 0 \qquad (3)$$

With $A_1 = c_1 c_2 \alpha^2 T_h$, $A_2 = 2c_1 c_2 T_h - c_2 \alpha^2 T_h^2 (b - d_1 - h)$,
$A_3 = c_1 c_2 + e_1 \alpha^2 + \alpha^2 d_2 T_h - 2c_2 \alpha T_h (b - d_1 - h)$ and $A_4 = c_2(b - d_1 - h) - \alpha d_2$

Therefore, the trivial equilibrium point is always exist, while the axial equilibrium exists if the following condition holds.

$$b > (d_1 + h) \qquad (4)$$

The predator free equilibrium point exist if in addition condition given by equation (4), the following condition holds.

$$e_1 d_1 \frac{b-(d_1+h)}{c_1} > d_3 \qquad (5)$$

The scavenger free equilibrium point exist if the equation (3) has unique positive root and the following condition holds

$\frac{e_1\alpha\tilde{x}}{1+\alpha T_h\tilde{x}} > d_2$ (6)

Finally, the positive equilibrium exist if in addition to existence condition of free scavenger equilibrium point the following condition hold

$e_2 d_1\tilde{x} + \frac{e_3\alpha\tilde{x}\tilde{y}}{1+\alpha T_h\tilde{x}} - d_3$ (7)

Now to study the stability behavior of an equilibrium for the system (1), we linearize the system (1) using the transformations $u_1(t) = x(t) - x_0$, $u_2(t) = y(t) - y_0$ and $u_3(t) = z(t) - z_0$ where (x_0, y_0, z_0) is an equilibrium point of the system (1).

Now $\frac{du}{dt} = Au(t) + Bu(t-\tau_1) + Cu(t-\tau_1)$

Where $u(t) = (u_1(t), u_2(t), u_3(t))^T$,

$$A = \begin{pmatrix} b - 2c_1x_0 - (d_1+h) - \frac{\alpha y_0}{(1+\alpha T_h x_0)^2} & -\frac{\alpha x_0}{1+\alpha T_h x_0} & 0 \\ 0 & \frac{e_1\alpha x_0}{1+\alpha T_h x_0} - 2c_2y_0 - d_2 & 0 \\ 0 & 0 & e_2d_1x_0 + \frac{e_3\alpha x_0 y_0}{1+\alpha T_h x_0} - 2c_3z_0 - d \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ \frac{e_1\alpha y_0}{(1+\alpha T_h x_0)^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ And } C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_2d_1z_0 - \frac{e_3\alpha z_0 y_0}{(1+\alpha T_h x_0)^2} & -\frac{e_3\alpha x_0 z_0}{1+\alpha T_h x_0} & 0 \end{pmatrix}.$$

We look for solution of system (1) in the form$u(t) = pe^{\lambda t}$, $0 \neq p \in R^3$. This gives the following characteristic equation:

$(\lambda - a_1)(\lambda^2 + a_2\lambda + a_3) = 0$ (8)

Where

$$a_1 = e_2d_1x_0 - \frac{e_3\alpha x_0 y_0}{1+\alpha T_h x_0} - 2c_3z_0 - d_3$$
$$a_2 = 2c_1x_0 + (d_1+h) + +2c_2y_0 + d_2 + \frac{\alpha y_0}{(1+\alpha T_h x_0)^2} - \frac{e_1\alpha x_0}{1+\alpha T_h x_0} - b$$
$$a_3 = \left(b - 2c_1x_0 - (d_1+h) - \frac{\alpha y_0}{(1+\alpha T_h x_0)^2}\right)\left(\frac{e_1\alpha x_0}{1+\alpha T_h x_0} - 2c_2y_0 - d_2\right) - \left(\frac{e_1\alpha^2 x_0 y_0}{(1+\alpha T_h x_0)^3}\right)e^{-\lambda\tau_1}$$

At the trivial equilibrium $E_1 = (0,0,0)$, the characteristic equation (8) has the following roots

$\lambda_{11} = (b - (d_1) + h)$, $\lambda_{12} = -d_2$ and $\lambda_{13} = -d_3$, so $E_1 = (0,0,0)$ is asymptotically stable if the following condition holds.

$b < h + d_1$ (9)

At the axial equilibrium $E_2 = (\bar{x}, 0,0)$ the characteristic equation (8).has the following roots

$\lambda_{21} = -c_1\bar{x},\ \lambda_{22} = \frac{e_1\alpha\bar{x}}{1+\alpha T_h\bar{x}} - d_2, \lambda_{23} = e_2 d_1\bar{x} - d_3$

So, in addition to existence condition of $E_2 = (\bar{x}, 0,0)$, if the following conditions holds then it becomes asyptotically stable

$$d_2 > \frac{e_1\alpha\bar{x}}{1+\alpha T_h\bar{x}} \tag{10}$$

$$d_3 > e_2 d_1\bar{x} \tag{11}$$

The root of characteristic equation at the predator free equilibrium point $E_3 = (\bar{x}, 0, \bar{z})$ are

$\lambda_{31} = -c_1\bar{x}, \lambda_{32} = \frac{e_1\alpha\bar{x}}{1+\alpha T_h\bar{x}} - d_2, \lambda_{33} = -c_3\bar{z}$ if in addition to its existence condition, the following condition holds

$$d_2 > \frac{e_1\alpha\bar{x}}{1+\alpha T_h\bar{x}} \tag{12}$$

Then $E_3 = (\bar{x}, 0, \bar{z})$ is asymptotically stable

Now we have the following theorem regarding stability switch at $E_4 = (\tilde{x}, \tilde{y}, 0)$ and $E_5 = (\tilde{x}, \tilde{y}, \tilde{z})$.

Theorem 2. Suppose E_4 exists with the following condition

$$b < h + d_1 + 2c_1\tilde{x} + \frac{\alpha\tilde{y}}{(1+\alpha T_h\tilde{x})^2} \tag{13}$$

and $b_1^2 < b_3$ (14)

Then

1. $E_5 = (\tilde{x}, \tilde{y}, \tilde{z})$ is locally asymptotically stable for all $\tau_1 \in [0, \tau_1^*]$ and exhibits a supercritical Hopf bifuracation near E_5 for $\tau_1 = \tau_1^*$
2. $E_5 = (\tilde{x}, \tilde{y}, 0)$ is locally asymptotically stable for all $\tau_1 \in [0, \tau_1^*]$ and exhibits a supercritical Hopf bifuracation near E_5 for $\tau_1 = \tau_1^*$ if in addition to given conditions (13,14) the following condition holds
$$d_3 > e_3 d_1\tilde{x} + \frac{e_3\alpha\tilde{x}\tilde{y}}{1+\alpha h\tilde{x}}\,. \tag{15}$$

Where τ_1^* , b_1 *and* b_3 are given in the proof.

Proof.

1. At $E_5 = (\tilde{x}, \tilde{y}, \tilde{z})$ the characteristic equation become
$$(\lambda + c_3\tilde{z})\left(\lambda^2 + b_1\lambda + b_2 + b_3 e^{-\lambda\tau_1}\right) = 0$$

(16)

$b_1 = c_2\tilde{y} + h + d_1 + 2c_1\tilde{x} + \frac{\alpha\tilde{y}}{(1+\alpha T_h\tilde{x})^2} - b$, $b_2 = c_2\tilde{y}\left(h + d_1 + 2c_1\tilde{x} + \frac{\alpha\tilde{y}}{(1+\alpha T_h\tilde{x})^2} - b\right)$

And $b_3 = \frac{e_1\alpha^2\tilde{x}\tilde{y}}{(1+\alpha T_h)^3}$, so the root in the z-direction in negative and other roots in x-, y-direction satisfies the following equation

$$\lambda^2 + b_1\lambda + b_2 + b_3 e^{-\lambda\tau_1} = 0 \tag{17}$$

Now if $\tau_1 \in [0, \tau_1^*[$ and the conditions(13,14) holds then the root in the x-, y-directions are negative and hence $E_5 = (\tilde{x}, \tilde{y}, \tilde{z})$ is locally asymptotically stable. For accurance of Hopf bifurcation

Suppose $\lambda = \omega + i\varpi$ is the root of the equation, we get

$$\omega^2 - \varpi^2 + b_1\omega + b_2 + b_3\cos(\tau_1\varpi)\,e^{-\omega\tau_1} = 0 \tag{18}$$

$$2\omega\varpi + b_1\varpi - b_3\sin(\tau_1\varpi)e^{-\omega\tau_1} = 0 \tag{19}$$

The equation 17, should have imaginary roots for stability change of $E_5 = (\tilde{x}, \tilde{y}, \tilde{z})$, hence, to obtain the stability criterion, we set $\omega = 0$ in equation (17) and get

$$\varpi^2 - b_2 = b_3 \cos(\tau_1 \varpi) \quad (20)$$

$$b_1 \varpi = b_3 \sin(\tau_1 \varpi) \quad (21)$$

Squaring and adding these two equation, we get the τ_1 eliminated equation

$$\varpi^4 + (a_1^2 - 2a_2)\varpi^2 + b_1^2 - b_3 = 0 \quad (22)$$

It is obvious that the equation always has one and only positive root ϖ_0^2 due to condition 14.

Now the time delay at which the equation has imaginary roots, $\pm\varpi$ is $\tau_1^* = \frac{1}{\varpi_0} \arccos\left(\frac{\varpi_0^2 - a_2}{a_3}\right)$

Now $\omega(\tau_1^*) = 0$ and $\varpi(\tau_1^*) = \varpi_0$, so, if we take differentiation for both equation with respect to τ_1, we get $\left[\frac{d\omega}{d\tau_1}\right]_{\tau_1 = \tau_1^*} = \frac{\varpi_0^2(b_1^2 + 2\varpi_0^2)}{b_1^2 + 4\varpi_0^2} > 0$

And hence the proof 1, is complete

2. If At $E_5 = (\tilde{x}, \tilde{y}, 0)$ the charastrictic equation become

$$\left(\lambda - \left(e_2 d_1 x_0 + \frac{e_3 \alpha x_0 y_0}{1 + \alpha T_h x_0} - d_3\right)\right)\left(\lambda^2 + b_1 \lambda + b_2 + b_3 e^{-\lambda \tau_1}\right) = 0 \quad (23)$$

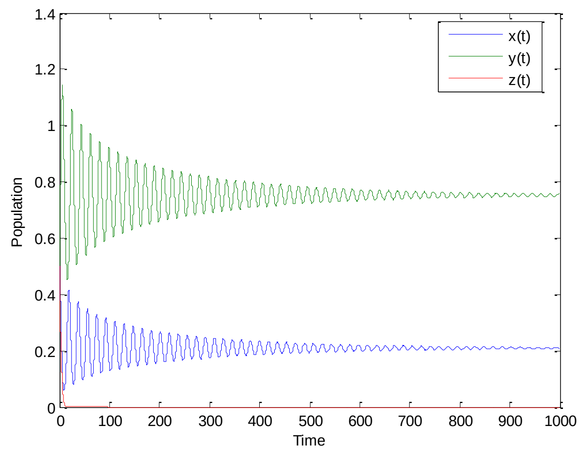
$b_1 = c_2 \tilde{y} + h + d_1 + 2c_1 \tilde{x} + \frac{\alpha \tilde{y}}{(1 + \alpha T_h \tilde{x})^2} - b$, $b_2 = c_2 \tilde{y}\left(h + d_1 + 2c_1 \tilde{x} + \frac{\alpha \tilde{y}}{(1 + \alpha T_h \tilde{x})^2} - b\right)$

So, the root in the z-direction is negative if the condition holds and the other two roots in x-,y-direction are the same as in the first proof part, and hence the proof is complete.

4. Numerical simulation

In this section the Hopf bifurcation of the system is investigated numerically. We have performed the simulation using mat lab it is observed that they have good agreement with our analytical finding. For the parameters given by equation (24), we have r $\cong 1.71$. The solution of the system (1) at $\tau_1 = 1.7 < \tau_1^*$ plotted in Fig.1 and at $\tau_1 = 1.8 > \tau_1^*$ plotted in figure (2).

$$\begin{aligned} &a = 1.99, c_1 = 1.3, \alpha = 1.2, T_h = 0.41, d_1 = 0.5, h = 0.4 \\ &e_1 = 0.825, d_2 = 0.15, c_2 = 0.05 \\ &e_2 = 0.98, e_3 = 0.16, c_3 = 0.1 \; and \; d_3 = 0.5 \end{aligned} \quad (24)$$



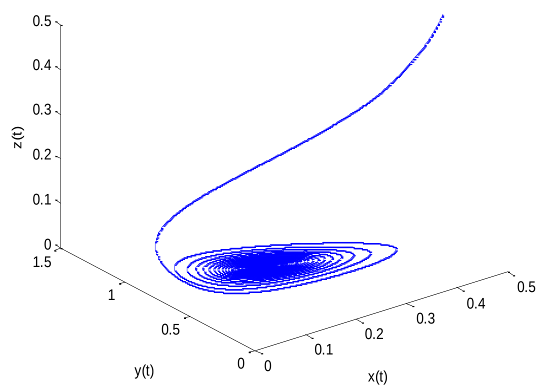
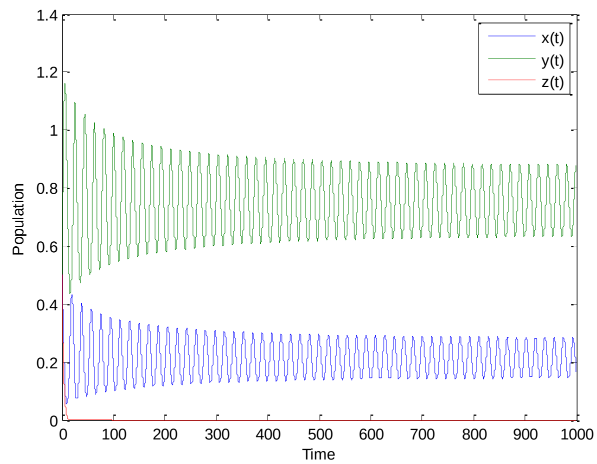


Fig.(1) Both the time series and phase portrait of system (1) for the data set with $\tau_1 = 1.7$, showing that the scavenger free equilibrium point $E_4 = (0.2075,0.7576,0)$ is locally asymptotically stable.



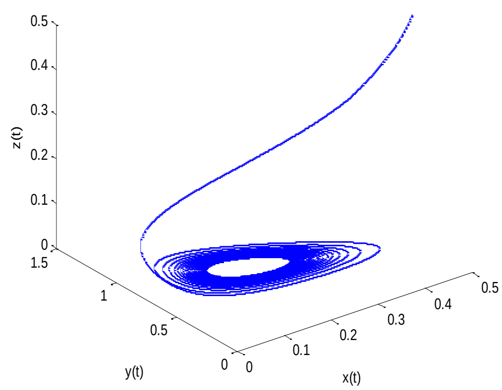


Fig.(2) For date set given by equation (24) with $\tau_1 = 1.8$ both the time series and phase portrait of system (1) showing periodic orbit near $E_4 = (0.2075, 0.7576, 0)$

It is obvious that the parameters used in Fig. (1) and Fig. (2) satisfies the condition confirm the second part of the theorem, which confirm our analytical result.

Now for the same date set with $d_3 = 0.1$, the system (1) for $\tau_1 = 1.7$ plotted in fig.(4). And for $\tau_1 = 1.8$ is plotted in fig.(4)

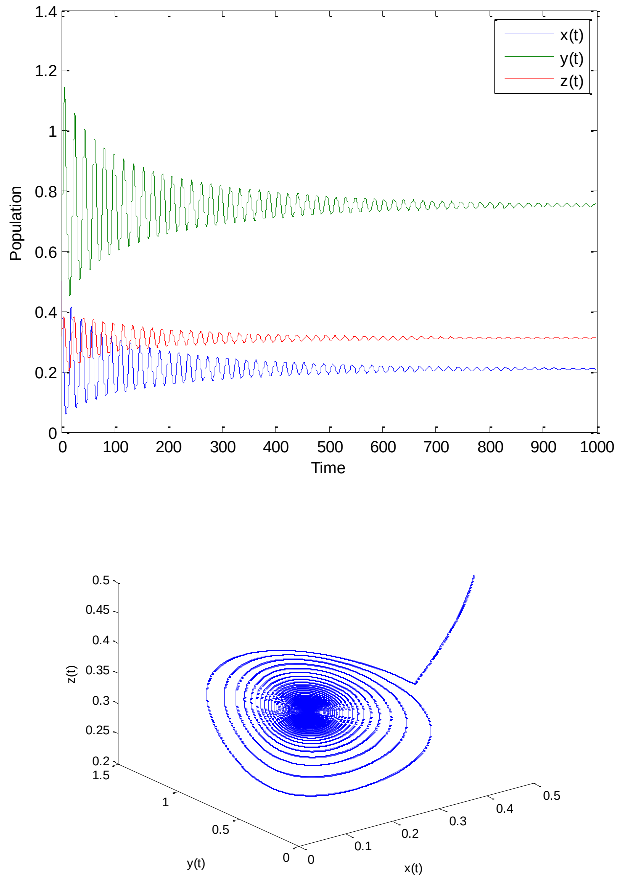
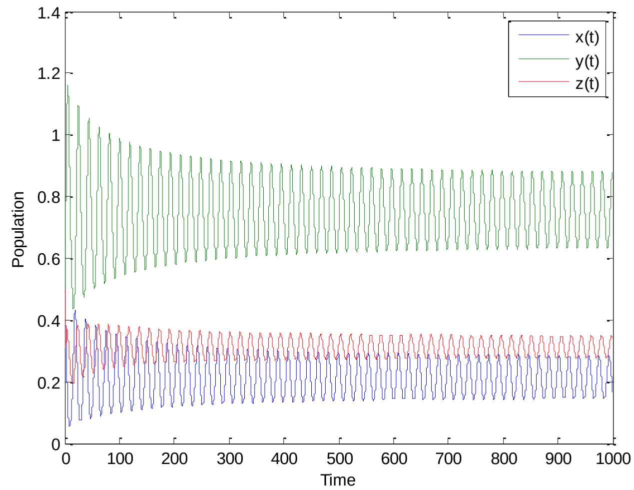


Fig.(3) Both the time series and phase portrait of system (1) for the data set given by equation (24) with $d_1 = 0.1$ and $\tau_1 = 1.7$, showing that the positive equilibrium point $E_5 = (0.2075, 0.7576, 0.3125)$ is locally asymptotically stable.



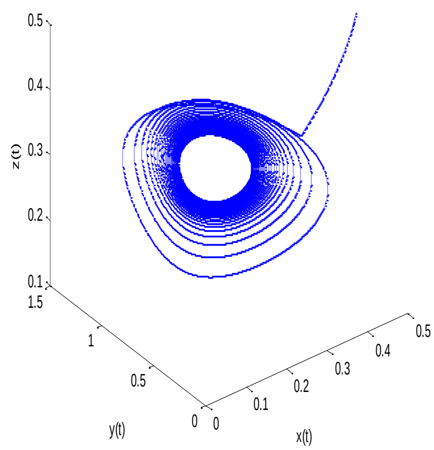


Fig. (4) For date set by equation (24) with $d_3 = 0.1$ *and* $\tau_1 = 1.8$ both the time series and phase portrait of system (1) showing periodic orbit near $E_5 = (0.2075, 0.7576, 0.3125)$.

It is easy to verify that the parameters used in fig.(3) and fig.(4) satisfies the condition of first part of theorem which confirms our analytical result.

Now the system (1) solved numerically using the following date set, see fig.(1), fig.(6) .

$$\begin{aligned} &a = 0.89, c_1 = 1.3, \alpha = 1.2, T_h = 4.1, d_1 = 0.5, h = 0.4 \\ &e_1 = 0.7, d_2 = 0.7, c_2 = 0.5 \\ &e_2 = 0.58, e_3 = 0.16, c_3 = 0.5 \text{ and } d_3 = 0.5 \end{aligned} \tag{25}$$

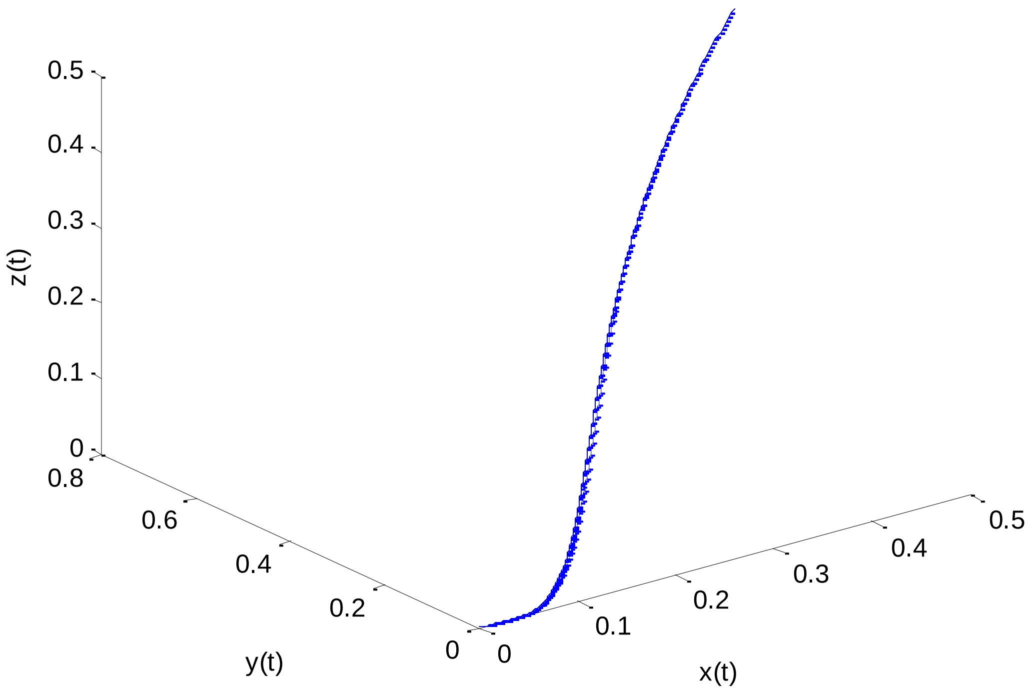


Fig.(5) The phase portrait showing that the solution for all τ_1=1, 2, 20 with parameters given by equation (25), approaches the trivial equilibrium point.

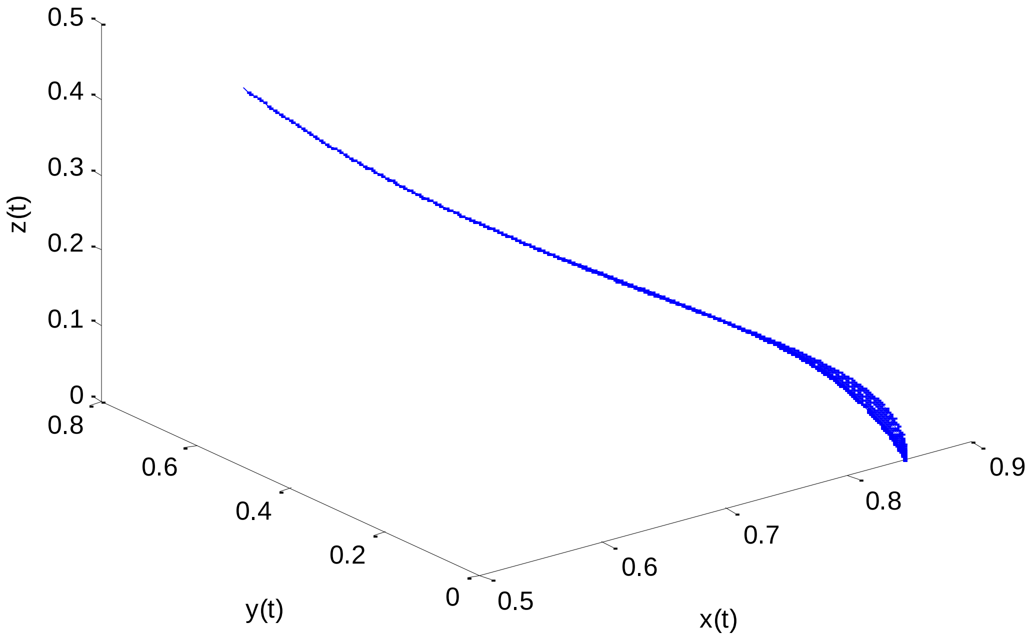


Fig.(6) The phase portrait showing that the system (1) for all τ_1=1, 2, 20 with parameters given by equation (25) with $a = 2$, has locally asymptotically stable axial equilibrium point $E_2 = (0.8462,0,0)$.

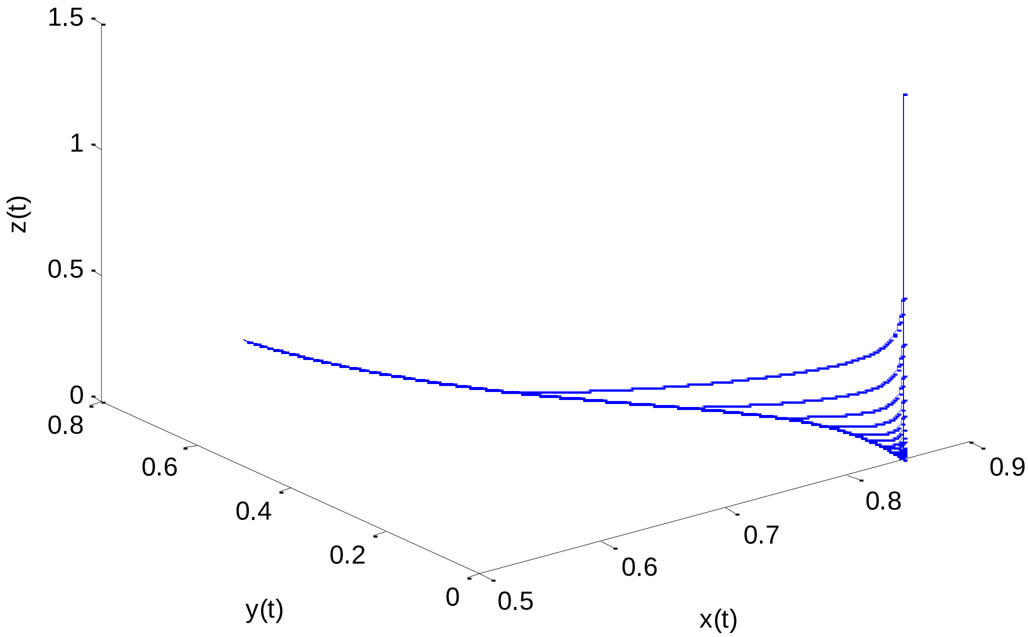


Fig.(6) The phase portrait showing that the system (1) for all τ_1=1, 2, 20 with parameters given by equation (25) with $a = 2\ and\ c_3 = d_3 = 0.1$, has locally asymptotically stable predator free equilibrium point $E_3 = (0.8462,1.4538)$

It is clear that the parameters used in Fig.(5) Satisfies the stability condition for trivial equilibrium point which confirm our result, when we increase the birth rate to $a = 2$, The system (1), approaches the axial equilibrium point because the used parameters in Fig.(6) satisfies the stability condition of axial equilibrium point. Finally when we decrease the value c_3 and d_3 to $c_3 = d_3 = 0.1$ and fixing $a = 1$ the system (1) approaches the predator free equilibrium point this confirm our analytical result for stability condition for free predator free equilibrium point. Note that in each Fig.(5),(6),(7) different time delay values $\tau_1 = 1, 2, 20$ used. It is obvious it does not affect the stability as we shown analytically.

5. Conclusion and discussion

In this paper, we considered an ecological model for dynamics of three species: prey, predator and scavengers, incorporating harvesting factor on prey species and time delay. We have seen that time delay for conservation biomass of dead prey naturally and predated prey by predators, to a scavenger does not affect the dynamics of the model. However, time delay for conservation biomass of prey to predators is effective as shown in theorem (2). In our study for system (1), we have given all locally stability conditions for each of the equilibrium points. Also, numerically, we supported our result as follows: the figures (Fig.(1), Fig.(2), Fig.(3), Fig.(4)) confirm our analytical finding in theorem (2).

Fig.(5), Fig.(6) and Fig.(7) confirm our analytical finding for locally asymptotically stability condition for trivial, axial and predator free equilibrium points, respectively.

Conflict of Interests.

There are non-conflicts of interest .

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الخلاصة

في هذا البحث قد تم رياضيا صياغة ديناميكية أنواع صائد الفرائس في نموذج شبكة الغذاء بدمج التأخير الزمني و حصاد الفريسة. ان حدود جميع حلول النموذج قد تم تنفيذها. تمت مناقشة الوجود وكذلك تحليل الأستقرار لجميع احتمالات نقاط الأتزان الموجبة. كذلك في ظل تأخير زمني معين أثبتنا أن نموذجنا. قد أظهر تشعب هوبف دون الحرج. وعلاوة على ذلك فاننا قد درسنا المحاكاة العددية للنموذج لتأكيد اكتشافنا التحليلي.

الكليمة الدالة: صائد الفرائس, نموذج المفترس ، تحليل الثبات ، تشعب هوبف.