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Minimal conductivity in graphene: interaction corrections and ultraviolet anomaly

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Abstract. - Conductivity of a disorder-free intrinsic graphene is studied to the first order in the long-range Coulomb interaction and is found to be $\sigma = \sigma_0(1 + 0.01g)$, where g is the dimensionless ("fine structure") coupling constant. The calculations are performed using three different methods: i) electron polarization function, ii) Kubo formula for the conductivity, iii) quantum transport equation. Surprisingly, these methods yield *different* results *unless* a proper ultraviolet cut-off procedure is implemented, which requires that the interaction potential in the effective Dirac Hamiltonian is cut-off at small distances (large momenta).

Introduction. – Low-frequency optical conductivity of undoped (intrinsic) graphene free of disorder is known to have a universal value of $\sigma_0 = e^2/4\hbar$ [1–13]. Experimental measurements [14,15], which yielded a value somewhat bigger than the theoretical predictions, motivated the studies of the possible role played by electron-electron interactions. The findings of Ref. [16] that the combined effect of self energy (velocity renormalization) and vertex corrections leads to a suppression of the optical conductivity at low frequencies have been questioned in Refs. [17,18] on the basis of scaling arguments. The latter indicate that the large logarithmic (momentum cut-off dependent) terms in the self-energy and vertex corrections cancel each other. We note that Ref. [16] and Refs. [17,18] agree on this cancellation in the lowest order in electron-electron interaction but differ on whether the higher order terms feature similar cancellation. It appears that the analysis of Ref. [16], though valid in the first order, fails for higher orders, and that the conclusion of the suppression of the conductivity at low frequencies is not valid.

The theory presented in Refs. [17, 18] implies that the low-frequency dependence is properly described by the lowest order correction. Indeed, to the first order in interaction the conductivity is expected to yield, $\sigma/\sigma_0 = 1 + Cg$, where C is some constant, $g = e^2/\kappa v$ is the interaction strength; κ is the dielectric constant of a substrate and v is the electron velocity in graphene. Renormalization group approach for 2D Dirac fermions predicts that the interaction strength g is a running coupling constant that

depends on frequency $g \to \widetilde{g}(\omega)$ [20, 21]. At low frequencies $\widetilde{g}(\omega)$ flows to zero, so that higher order corrections to the electron velocity become progressively negligible and it is sufficient to consider only the first order renormalization of velocity (electric charge is not renormalized): $\widetilde{g}(\omega) = g/[1 + \frac{g}{4}\ln(\mathcal{K}v/\omega)]$, where $\mathcal{K}$ is the momentum cut-off. Combining these expressions gives,

$$\sigma/\sigma_0 = 1 + \frac{Cg}{1 + \frac{g}{4}\ln(\mathcal{K}v/\omega)}, \qquad (1)$$

with the low-frequency behavior of the conductivity being determined by the constant C alone. Calculation of this constant, therefore, becomes an important task. While Ref. [17] did not calculate C, Ref. [18] provided the following value

$$C = \frac{25 - 6\pi}{12} \approx 0.51 . \qquad (2)$$

This result predicts quite a considerable variation of σ with the frequency for typical values of the bare graphene interaction constant g (which can exceed 1).

In the present Letter we test the above prediction (2) by performing a perturbative calculation of the minimal conductivity to the first order in electron-electron interaction using three different methods, based on, a) electron polarization operator, b) Kubo formula for the conductivity, c) kinetic equation. We point out that crucial anomaly, which does not appear in a non-interacting case, occurs for the interaction correction. Three above mentioned methods would give essentially *different* values for the constant
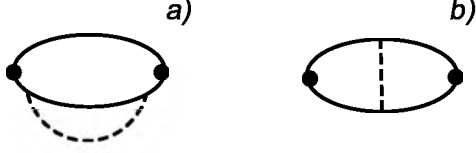
Fig. 1: Self-energy, a), and vertex correction, b), to the conductivity $\sigma(\omega)$ and polarization operator $\Pi(\omega, q)$ in the first order in electron-electron interaction (dashed line). The vertex (black dot) is equal to 1 in the case of the polarization operator and to $ev\hat{\boldsymbol{\sigma}}$ in case of the conductivity. The two quantities are related to each other by the particle conservation condition, Eq. (5).

$\mathcal{C}$ unless some appropriate high-momentum cut-off procedure is implemented. We argue that expression (2) overestimates the interaction correction by almost two orders of magnitude and show that the numerical value of C is

$$C = \frac{19 - 6\pi}{12} \approx 0.01 \ . \tag{3}$$

We will now proceed to demonstrate that the difference between Eqs. (2) and (3) originate from handling of singular integrals at large electron momenta.

The first method to be presented is based on the calculation of electron polarization operator and has an advantage of *being free* from any such singular integrals.

Polarization operator. – Single intrinsic 2D graphene layer is described by the chiral Hamiltonian,

$$H = v \sum_{i\mathbf{p}} \hat{c}_{\mathbf{p}}^{i\dagger} \hat{\boldsymbol{\sigma}} \cdot \mathbf{p} \, \hat{c}_{\mathbf{p}}^{i} + \frac{1}{2} \sum_{ij\mathbf{pkq}} \hat{c}_{\mathbf{p-q}}^{i\dagger} \hat{c}_{\mathbf{k+q}}^{j\dagger} V_{\mathbf{q}} \hat{c}_{\mathbf{k}}^{j} \hat{c}_{\mathbf{p}}^{i}, \tag{4}$$

where "hats" denote operators in a pseudo-spin space ($\hat{\boldsymbol{\sigma}}$ represents the usual set of Pauli matrices), the sum over Latin indices is taken over two nodal points and two (true) spin directions. The interaction potential is $V_{\mathbf{q}} = 2\pi e^2/\kappa q$; we also denote, $\sum_{\mathbf{p}} \equiv \int d^2p/(2\pi)^2$, and set $\hbar = 1$.

First-order interaction corrections to the conductivity are given by the two diagrams shown in Fig. 1, with the vertices denoting the operators of electric current, $ev\hat{\boldsymbol{\sigma}}$. Another possible method to derive the homogeneous optical conductivity is to calculate the corresponding diagrams for the electron *polarization operator* $\Pi(\omega, q)$ and then utilize the particle conservation condition,

$$\sigma(\omega) = \lim_{q \to 0} \frac{ie^2\omega}{q^2} \Pi(\omega, q). \tag{5}$$

The calculation of the polarization operator to the first order in g requires two diagrams [19],

$$\Pi(\omega, q) = 4Tr \sum_{\mathbf{pp'}} \int \frac{d\epsilon d\epsilon'}{(2\pi)^2} \, V_{\mathbf{p-p'}} \Big[2\hat{G}_{\epsilon\mathbf{p}} \hat{G}_{\epsilon'\mathbf{p'}} \hat{G}_{\epsilon\mathbf{p}}$$

$$\times \hat{G}_{\epsilon+\omega,\mathbf{p+q}} + \hat{G}_{\epsilon\mathbf{p}} \hat{G}_{\epsilon'\mathbf{p'}} \hat{G}_{\epsilon'+\omega,\mathbf{p'+q}} \hat{G}_{\epsilon+\omega,\mathbf{p+q}} \Big], \tag{6}$$

where the electron Green's function in the subband representation is

$$\hat{G}_{\epsilon\mathbf{p}} = \frac{1}{2} \sum_{\beta=\pm 1} \frac{1 + \beta\hat{\sigma}_{\mathbf{p}}}{\epsilon - \beta(vp - i\eta)}, \tag{7}$$

with $\hat{\sigma}_{\mathbf{p}} = \hat{\boldsymbol{\sigma}} \cdot \mathbf{n_p}$ being the projection of the pseudo-spin operator onto the direction of the electron momentum $\mathbf{n_p} = \mathbf{p}/p$. Factor 4 in Eq. (6) accounts for the (true) spin and valley degeneracy. Taking energy integrals and performing pseudospin trace operation we obtain for the first term in Eq. (6),

$$\Pi_a(\omega, q) = 2 \sum_{\mathbf{pp'}\beta} V_{\mathbf{p-p'}} \cos\theta_{\mathbf{p,p'}} \frac{1 - \cos\theta_{\mathbf{p,p+q}}}{(\omega + 2\beta vp - i\beta\eta)^2}. \tag{8}$$

In the expression (8) we kept only terms that lead to the lowest order contribution in the external momentum, $\Pi_a(\omega, q) \propto q^2$, which are necessary for the calculation of the homogeneous conductivity. Using Eq. (5) we obtain the corresponding contribution,

$$\sigma_a(\omega) = ie^2\omega \int \sum_{\mathbf{pp'}} V_{\mathbf{p-p'}} \cos\theta_{\mathbf{p,p'}} \frac{\omega^2 + 4v^2p^2}{p^2(\omega^2 - 4v^2p^2)^2}, \tag{9}$$

where the frequency is presumed to have a positive infinitesimal imaginary part. The second term in Eq. (6) is evaluated similarly,

$$\Pi_b(\omega, q) = -2 \sum_{\mathbf{pp'}} \left(\frac{\omega^2(\mathbf{n_p} - \mathbf{n_{p+q}}) \cdot (\mathbf{n_{p'}} - \mathbf{n_{p'+q}})}{(\omega^2 - 4v^2p^2)(\omega^2 - 4v^2p'^2)} \right.$$

$$\left. + \frac{4v^2pp' \sin\theta_{\mathbf{p,p+q}} \sin\theta_{\mathbf{p',p'+q}}}{(\omega^2 - 4v^2p^2)(\omega^2 - 4v^2p'^2)} \right) V_{\mathbf{p-p'}}. \tag{10}$$

Expanding to the quadratic order in q we obtain the vertex correction,

$$\sigma_b(\omega) = -ie^2\omega \sum_{\mathbf{pp'}} V_{\mathbf{p-p'}} \frac{\frac{\omega^2}{pp'} \cos^2\theta_{\mathbf{p,p'}} + 4v^2 \cos\theta_{\mathbf{p,p'}}}{(\omega^2 - 4v^2p^2)(\omega^2 - 4v^2p'^2)}. \tag{11}$$

To the first order in interaction the conductivity is given by the sum $\sigma = \sigma_0 + \sigma_a + \sigma_b$. The second term here, given by Eq. (9), contains a strong divergence at $p = \omega/2v$. This divergence, however, is simply a consequence of the renormalization of the electron velocity by electron-electron interactions. To make the integrals regular we note that *both* the zeroth-order term [19] and σ_a can be written as

$$\sigma_0 + \sigma_a = 2ie^2\omega \sum_{\mathbf{p}} \frac{v_p}{p(\omega^2 - 4v_p^2p^2)}, \tag{12}$$

with $v_p = v + \frac{1}{2p} \sum_{\mathbf{p'}} V_{\mathbf{p-p'}} \cos\theta_{\mathbf{p,p'}} = v[1 + \frac{g}{4} \ln(\mathcal{K}/p)]$ being the renormalized velocity (where $\mathcal{K}$ is the upper momentum cut-off). Indeed, expanding the integrand to the first order in the interaction one recovers Eq. (9). Note that the value of v_p coincides with the electron velocity found from the perturbation expansion for the electron

Green's function [20]. The integral in Eq. (12) is regular. Calculating the real part of Eq. (12) we obtain

$$\sigma_0 + \sigma_a' = \sigma_0 \left(1 + \frac{g}{4}\right). \tag{13}$$

Note that the interaction correction in Eq. (13) is due to the *curvature* of electron spectrum.

Calculation of the real part of Eq. (11) can be reduced to the following dimensionless integral ($x = 2vp/\omega$),

$$\sigma_b' = -\sigma_0 \ g \int_0^\pi \frac{d\theta}{\pi} \int_0^\infty \frac{dx \cos\theta(x + \cos\theta)}{(1 - x^2)\sqrt{x^2 + 1 - 2x\cos\theta}}, \tag{14}$$

where the integral is taken in the principal value sense. Using the identity

$$\int_0^\infty \frac{dx(x + \cos\theta)}{(1 - x^2)\sqrt{x^2 + 1 - 2x\cos\theta}} = \frac{\sin^2(\theta/2)}{\cos(\theta/2)} \ln\left[\tan(\theta/4)\right], \tag{15}$$

and integrating over the angle θ we obtain,

$$\sigma_b' = \sigma_0 \ g \frac{8 - 3\pi}{6}. \tag{16}$$

Combining Eqs. (13) and (16) we finally arrive at Eq. (3). The vertex correction is negative and nearly cancels the self energy correction. The frequency independence of Eqs. (13) and (16) is analogous to the independence of the non-interacting conductivity σ_0.

Kubo formula for conductivity. – An advantage of deriving minimal conductivity from the polarization operator originates from the fact that large logarithmic contributions *do not appear* in different terms in this formalism. On the other hand one could begin with a straightforward application of the Kubo formula, which as we will see, does not offer such a simplification. As a result one has to deal with logarithmic contributions which ultimately cancel. The starting expression is the expression for the optical conductivity

$$\widetilde{\sigma}(\omega) = \frac{K(\omega) - K(0)}{\omega}, \tag{17}$$

via the current-current correlation function, which is given in the zeroth order by

$$K_0(\omega) = 4e^2v^2\mathrm{Tr}\sum_{\mathbf{p}} \int \frac{d\epsilon}{2\pi} \ \sigma_x \hat{G}_{\epsilon\mathbf{p}}\hat{\sigma}_x\hat{G}_{\epsilon+\omega,\mathbf{p}}, \tag{18}$$

and in the first order by

$$K_a(\omega) = 8ie^2v^2\mathrm{Tr}\sum_{\mathbf{pp}'} \int \frac{d\epsilon d\epsilon'}{(2\pi)^2} \ V_{\mathbf{p}-\mathbf{p}'}$$
$$\times \hat{\sigma}_x \hat{G}_{\epsilon\mathbf{p}}\hat{G}_{\epsilon'\mathbf{p}'}\hat{\sigma}_x\hat{G}_{\epsilon\mathbf{p}}\hat{G}_{\epsilon+\omega,\mathbf{p}}, \tag{19}$$

$$K_b(\omega) = 4ie^2v^2\mathrm{Tr}\sum_{\mathbf{pp}'} \int \frac{d\epsilon d\epsilon'}{(2\pi)^2} \ V_{\mathbf{p}-\mathbf{p}'}$$
$$\times \hat{\sigma}_x \hat{G}_{\epsilon\mathbf{p}}\hat{G}_{\epsilon'\mathbf{p}'}\hat{\sigma}_x\hat{G}_{\epsilon'+\omega,\mathbf{p}'}\hat{G}_{\epsilon+\omega,\mathbf{p}}, \tag{20}$$

here the subscripts a, b denote the contributions from the self energy and vertex diagrams, Fig. 1. Note, however, that the corresponding contributions into the conductivity, which we denote here by $\widetilde{\sigma}_a$ and $\widetilde{\sigma}_b$, do not satisfy the condition (5) *term by term*. However their *sum has to obey it*, $\widetilde{\sigma}_a + \widetilde{\sigma}_b = \sigma_a + \sigma_b$.

Calculation of Eqs. (18-20) is similar to the above derivation for the polarization operator.

$$\widetilde{\sigma}_a(\omega) = ie^2\omega \sum_{\mathbf{pp}'} V_{\mathbf{p}-\mathbf{p}'} \cos\theta \frac{12v^2p^2 - \omega^2}{p^2(\omega^2 - 4v^2p^2)^2}, \tag{21}$$

$$\widetilde{\sigma}_b(\omega) = ie^2\omega \sum_{\mathbf{pp}'} V_{\mathbf{p}-\mathbf{p}'} \cos\theta \frac{\frac{1}{pp'}(\omega^2 - 8v^2p^2)\cos\theta - 4v^2}{(\omega^2 - 4v^2p^2)(\omega^2 - 4v^2p'^2)}, \tag{22}$$

where we omit the subscripts in $\cos\theta_{\mathbf{p},\mathbf{p}'}$. The expressions (21,22) are to be contrasted with Eqs. (9) and (11). The obvious distinction arises from the fact that the integrals in Eqs. (21) and (22) are logarithmically divergent though these divergencies ultimately cancel in their sum, $\widetilde{\sigma}(\omega) = \widetilde{\sigma}_a(\omega) + \widetilde{\sigma}_b(\omega)$. However, a more striking observation can be made if one calculates the difference of the two expressions,

$$\sigma(\omega) - \widetilde{\sigma}(\omega) = I, \tag{23}$$

where

$$I = 2ie^2\omega \sum_{\mathbf{pp}'} V_{\mathbf{p}-\mathbf{p}'} \cos\theta \frac{p' - p\cos\theta}{p^2p'(\omega^2 - 4v^2p^2)}. \tag{24}$$

Before addressing the issue of a numerical value of I let us briefly describe the third method for the calculation of the conductivity.

Kinetic equation. – The kinetic equation to the lowest order in electron-electron interaction and in the presence of electric filed has the form [16]

$$\frac{\partial \hat{f}_{\mathbf{p}}}{\partial t} + ivp[\hat{\sigma}_{\mathbf{p}}, \hat{f}_{\mathbf{p}}] + e\mathbf{E} \cdot \frac{\partial \hat{f}_{\mathbf{p}}}{\partial \mathbf{p}} = i\sum_{\mathbf{p}'} V_{\mathbf{p}-\mathbf{p}'}[\hat{f}_{\mathbf{p}'}, \hat{f}_{\mathbf{p}}], \tag{25}$$

where $\hat{f}_{\mathbf{p}}$ is the 2×2 matrix distribution function. The second term in the left-hand side represents the rate of change of the electron distribution function during its precession in the momentum-dependent "pseudo-Zeeman" field. The third term is the usual drift in the momentum space caused by external electric field. Finally, the right-hand side account for the exchange electron-electron interaction (Hartree contribution being zero by virtue of electric neutrality and spatial homogeneity). Given the solution of kinetic equation (25) one can find electric current and optical conductivity from $\mathbf{j} = \sigma(\omega)\mathbf{E} = 4ev\mathrm{Tr}\sum_{\mathbf{p}} \hat{f}_{\mathbf{p}}\hat{\boldsymbol{\sigma}}$. The detailed solution of Eq. (25) to the first order in $\mathbf{E}$ and $V_{\mathbf{p}-\mathbf{p}'}$ was found in Ref. [16]. For the conductivity it yields,

$$\sigma_{kin}(\omega) = 8ie^2\omega \sum_{\mathbf{pp}'} V_{\mathbf{p}-\mathbf{p}'} \cos\theta \frac{v^2}{(\omega^2 - 4v^2p^2)^2}$$

$$-4ie^2\omega \sum_{\mathbf{pp'}} V_{\mathbf{p}-\mathbf{p'}} \frac{v^2 p \cos^2\theta + v^2 p' \cos\theta}{p'(\omega^2 - 4v^2 p^2)(\omega^2 - 4v^2 p'^2)}. \quad (26)$$

This expression is different from both $\sigma(\omega)$ and $\widetilde{\sigma}(\omega)$. Interestingly,

$$\sigma(\omega) - \sigma_{kin}(\omega) = I/2. \quad (27)$$

Discussion. — Three different values obtained from the polarization operator, $\sigma(\omega)$, Kubo formula, $\widetilde{\sigma}(\omega)$, and kinetic equation, $\sigma_{kin}(\omega)$, respectively, indicate an inconsistency of the theory of interacting two-dimensional Dirac fermions *unless $I = 0$.* We will now demonstrate that the conclusion of whether $I = 0$ or $I \neq 0$ depends on the way the ultraviolet cut-off is imposed in the calculation of a singular integral over $\mathbf{p'}$ in Eq. (24).

(i) *Hard cut-off.* Let us first assume that the divergent momentum integral is extended only to $p' \leq \mathcal{K}$. By noting that interaction potential depends only on $s = (\mathbf{p} - \mathbf{p'})^2$ and that $p' - p\cos\theta = \frac{1}{2}\partial s/\partial p'$ we then obtain for the latter integral in case when $V(s) = 2\pi e^2/\sqrt{s}$,

$$\sum_{\mathbf{p'}} V_{\mathbf{p}-\mathbf{p'}} \cos\theta \frac{p - p\cos\theta}{p'} = \frac{e^2}{2} \int_0^{2\pi} \frac{d\theta}{2\pi} \cos\theta \int_0^{\mathcal{K}} \frac{dp'}{\sqrt{s}} \frac{\partial s}{\partial p'}$$

$$= e^2 \int_0^{2\pi} \frac{d\theta}{2\pi} \cos\theta \sqrt{s(\mathcal{K}, p)}. \quad (28)$$

Expanding $\sqrt{s(\mathcal{K}, p)} \approx \mathcal{K} - p\cos\theta$ for large values of $\mathcal{K}$, we observe that the integral here is cut-off independent and equals $-e^2 p/2$. It is now straightforward to verify that equation (24) gives $I = -\sigma_0 g/2$. Such value of I yields Eq. (2) reproducing the result of Ref. [18], and precisely accounts for the difference between Eq. (2) and our result (3). However, as shown above, such an ultraviolet cut-off yields *three different* values of the conductivity depending on which method is being used and is therefore *unphysical.*

(ii) *Soft cut-off.* The anomaly encountered in the expression (24) is specific for $V_q \propto q^{-1}$ behavior of the interaction potential. For *any* faster decay of interaction at large momenta the integral I *vanishes.* Let us demonstrate this point by assuming

$$V(q) = \frac{2\pi e^2}{q} e^{-q/\mathcal{K}}, \quad \mathcal{K} \to \infty. \quad (29)$$

Calculation similar to the preceding one gives,

$$\lim_{\mathcal{K}\to\infty} \frac{e^2}{2} \int_0^{2\pi} \frac{d\theta}{2\pi} \cos\theta \int_0^\infty dp' \frac{e^{-\sqrt{s}/\mathcal{K}}}{\sqrt{s}} \frac{\partial s}{\partial p'} = 0, \quad (30)$$

so that $I = 0$ and all three methods yield *the same* value (3). Similar conclusion will be reached if one assumes $V(q) \propto q^{-1-\eta}$ and subsequently takes the limit $\eta \to 0$.

Having established that the hard cut-off utilized in Ref. [18] in the course of Kubo calculations actually results in different and hence inconsistent results when other methods are used, it is time now to discuss the origin of this inconsistency. Terminating momentum integrals at some value $p = \mathcal{K}$ means in fact an essential modification of electron spectrum at large momenta that effectively excludes these states from possible virtual processes. Such a procedure, though not necessarily incorrect, can be made self-consistent *only* if it is accompanied by the appropriate change in the operators of electric current. Otherwise, the Ward identity, which ensures particle conservation, is violated. This is why the polarization function method, which does not involve current vertices in the course of calculations, gives results (in the form of convergent integrals) *independent* of the cut-off procedure. On the other hand both the Kubo formalism and kinetic equation do involve current operators explicitly and thus *fail* if the hard cut-off is implemented without a proper modification of current vertices.

To the contrary, the soft cut-off procedure presented in this Letter does not require modifications of the electron spectra (Green's functions) nor of the electric current vertices. It is thus self-consistent and quite naturally yields *identical* values for the conductivity irrespective of the method used.

Conclusion. — We have calculated the first order interaction correction to the conductivity of intrinsic graphene. Within the Kubo and kinetic equation formalisms the self-energy and vertex corrections contain large logarithmic frequency-dependent terms which ultimately cancel each other. Within the more convenient approach based on the calculation of the polarization operator, such terms do not appear at all. Such a simplification originates from a simpler scalar vertex in the case of a polarization operator.

Nevertheless, the three methods discussed in the present Letter result in different, and hence, unphysical values for the interaction correction unless the large-momentum cut-off is imposed in the form of Eq. (29), or similar. In that case all methods yield the same value given by Eq. (3).

To summarize, the calculations presented above indicate that the effects of electron-electron interactions lead to finite though numerically very small corrections to the minimal conductivity. Finally, the calculations of the present Letter are performed in the limit of zero temperature and their validity implies that $\hbar\omega \gg k_B T$.

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Appendix

After publication of our paper in Europhys. Lett. **83**, 17005 (2008), a preprint by I.F. Herbut, V. Juricic, O. Vafek, and M.J. Case, **"Comment on "Minimal conductivity in graphene: Interaction corrections and ultraviolet anomaly" by Mishchenko E. G."**, appeared in arXiv:0809.0725. It was previously under consideration for publication in the Europhysics Letters. Below we present our reply:

The Comment argues against the procedure implemented above, which leads to Eq. (3), and advocates dimensional regularization scheme in support of the value, Eq. (2), obtained in Ref. [18]. Yet, the Comment fails to offer a consistent resolution of the issue. Indeed, following our suggestion to utilize the charge conservation law, Eq. (5), the authors of the Comment analyzed the derivation of the interaction corrections from the polarization operator and reported that the dimensional regularization yielded, $C = (11 - 3\pi)/6 \approx 0.26$, the value *different* from their Eq. (2). (Note that this value coincides with σ_{kin} given by Eq. (27) of the present paper when I is calculated with the help of the hard cut-off.) Addressing this discrepancy, the authors of the Comment conclude only that, "The origin of this non-uniqueness is unclear at the moment, but we suspect that it may be the non-gauge invariant contribution to the conductivity which is not properly treated within the density polarization approach."

Citing some unidentified contribution does not add clarity to the discussion. The equivalence between the Kubo and the density polarization approaches in the calculation of the homogeneous conductivity is *ensured by the charge conservation law* (the Ward identity). It is surprising that the authors end the discussion with the above statement and do not even attempt to find out what happens to the charge conservation in their calculations. It is thus difficult to conclude that the authors of the Comment were able "to clarify and correct some of the statements made" in our paper. If anything, the credibility of their result, Eq. (2), is even more questionable as it is now clear that this result is based on the scheme that yields values which vary depending on the method used.

Interestingly, recent measurements of dynamic conductivity [22] show $\sigma = \sigma_0(1.01 \pm 0.04)$ over visible frequencies range and thus point towards smaller values of interaction corrections.

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$$\sigma_0 = 2ie^2\omega \sum_{\mathbf{p}} \frac{v}{p\left(\omega^2 - 4v^2p^2\right)},$$

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