An Efficient Algorithm for Joint Estimation of Differential Time Delays and Frequency Offsets†

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ABSTRACT

This paper introduces an efficient algorithm that jointly estimates differential time delays and frequency offsets between two signals. The approach is a two-step procedure. First, the differential frequency offsets are estimated from measurement of the autocorrelation functions of the received and transmitted signals. The time delays are estimated from estimates of the higher-order statistics of the two signals involved. The major advantage of the new approach is its remarkably reduced computational complexity over traditional approaches. The experimental results indicate that the algorithm performs better than the traditional methods in most cases of interest in spite of its reduced computational complexity.

I. INTRODUCTION

Estimation of differential time delays (DTDs) and differential frequency offsets (DFOs) between two signals generated by the same source is important in many applications. Consider two signals represented in the following form

$$S_t(n) = X(n) + W_1(n)$$
$$S_r(n) = \sum_{m=1}^{M} a_m X(n-D_m)\, e^{-j(\omega_m n + \theta_m)} + W_2(n) \quad . \tag{1}$$

Here X(n) is the source signal and is assumed to be at least locally stationary in the derivations, and $W_1(n)$ and $W_2(n)$ are zero mean, white, additive noise signals uncorrelated with each other and with the source signals. M is the number of the targets, $\{a_m, m=1,2,\ldots, M\}$ are the complex attenuation factors and $\{\theta_m, m=1,2,\ldots, M\}$ are the initial phase shifts. θ_m and θ_k are assumed to be uncorrelated with each other for $m \neq k$. Furthermore, it is assumed that each θ_m is uniformly distributed in the range $[\pi, -\pi)$. D_m and ω_m are the time delay and frequency offset parameters, respectively, that are associated with the m^{th} target.

Our objective is to estimate the differential time delays D_m and differential frequency offset ω_m between the "transmitted signal" $S_t(n)$ and the "received signal" $S_r(n)$ from a time-limited segment of the signals.

Traditional approaches to the joint estimation of DTDs and DFOs between the transmitted and received signals involve use of the complex cross-ambiguity function of the two signals [1,5,7]. The complex cross-ambiguity function of $S_t(n)$ and $S_r(n)$ is defined as [6]

$$A(\tau,\omega) = \sum_{n=-\infty}^{+\infty} S_t(n)\, S_r^*(n+\tau)\, e^{-j\omega n} \quad , \tag{2}$$

where * denotes complex conjugation, and τ, and ω are the time lag and frequency offset variables, respectively. It is well known that $E\{A(\tau,\omega)\}$ peaks at the true values of the (time delay, frequency offset) pairs and therefore one can estimate the DTD and DFO parameters by finding the values of τ and ω for which the magnitude of the complex cross-ambiguity function of the two waveforms peaks.

Even though the above approach is conceptually simple, it is computationally extremely complex. We will present an approach that is computationally much more efficient and whose performance is superior to the traditional approach.

II. ALGORITHM DERIVATION

A. Estimation of Differential Frequency Offsets

Consider the autocorrelation function of the received signal defined as

$$R_{rr}(\tau) = E\{\, S_r(n)\, S_r^*(n-\tau)\,\} \; . \tag{3}$$

Substituting the expression for $S_r(n)$ given in Eq. 1 into Eq. 3 and making use of the fact that the random phase shifts are uncorrelated, and that the source and noise signals are also mutually uncorrelated, $R_{rr}(\tau)$ reduces to the form

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$$R_{rr}(\tau) = \sum_{m=1}^{M} |a_m|^2 R_{XX}(\tau) e^{-j\omega_m\tau} + \sigma_2^2\delta(\tau), \quad (4)$$

where $R_{XX}(\tau)$ is the autocorrelation function of the source signal $X(n)$ at lag τ, σ_2^2 is the variance of the white noise sequence $W_2(n)$ that corrupts the received signal, and $\delta(\tau)$ is the Dirac delta function. Note that $R_{XX}(\tau)$ is related to $R_{tt}(\tau)$ as

$$R_{tt}(\tau) = R_{xx}(\tau) + \sigma_1^2\delta(\tau), \quad (5)$$

where σ_1^2 is the variance of white noise sequence $W_1(n)$ that corrupts the source signal $X(n)$, and $R_{tt}(\tau)$ is the autocorrelation function of the transmitted signal.

Let $\eta(\tau)$ denote the ratio of the autocorrelation functions of the received and transmitted signals. Dividing Eq. 4 by Eq. 5, we get

$$\eta(\tau) = \sum_{m=1}^{M} |a_m|^2 e^{j\omega_m\tau} + \left\{ \frac{\sigma_2^2 - \sum_{m=1}^{M} |a_m|^2 \sigma_1^2}{R_{tt}(0)} \right\} \delta(\tau). \quad (6)$$

The right-hand side of Eq. 6 has the same functional form as the expression for the autocorrelation function of M complex sinusoidal signals in the presence of additive white noise. Recognition of this fact immediately reduces the problem of the frequency offsets estimation to one of estimating the frequencies of M complex sinusoids embedded in additive, white noise. There are several available methods that can be used to estimate the frequencies of sine waves in additive white noise, including many high-resolution algorithms. We used the root-MUSIC algorithm [2] for estimating the frequency offsets.

B. Estimation of Differential Time Delays

Let the estimated values of the frequency offsets be $\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_M$. We will use these estimates in our approach for estimating the unknown time delays. Let us define the product signals $Y_p(n)$ as

$$Y_p(n) = S_t(n) S_r^*(n-p) \quad (7)$$

for several integer values of p. Consider the autocorrelation function of $Y_p(n)$ evaluated as

$$R_{y_p y_p}(\tau) = E\{Y_p(n) Y_p^*(n-\tau)\}. \quad (8)$$

It is easy to see that

$$R_{y_p y_p}(\tau) = \sum_{m=1}^{M} |a_m|^2 E\{ X(n)X^*(n-D_m-p)X^*(n-\tau)X(n-D_m-p-\tau) \} e^{j\omega_m\tau}$$

$$\text{for } \tau \geq 1. \quad (9)$$

Let

$$Z_\tau(n) = X(n)X(n-\tau). \quad (10)$$

The fourth-order correlation function in Eq. 9 can be expressed as an autocorrelation function of $Z_\tau(n)$. That is,

$$R_{Z_\tau Z_\tau}(p+D_m) = E\{Z_\tau(n) Z_\tau^*(n-p-D_m)\}$$

$$= E\{X(n)X^*(n-\tau)X^*(n-p-D_m)X(n-\tau-p-D_m)\}. \quad (11)$$

This implies that we can rewrite (9) as

$$R_{y_p y_p}(\tau) \equiv \sum_{m=1}^{M} |a_m|^2 R_{Z_\tau Z_\tau}(p+D_m) e^{j\omega_m\tau}; \ \tau \geq 1. \quad (12)$$

Now, we can easily estimate the time delays by realizing the fact that the autocorrelation function $R_{Z_\tau Z_\tau}(p+D_m)$ attains its peak value at the time lag $p = -D_m$, $m=1,2,\ldots$ M. One can solve for $|a_m|^2 R_{Z_\tau Z_\tau}(p+D_m)$ from (12) if one knows $R_{y_p y_p}(\tau)$. $R_{y_p y_p}(\tau)$ can be estimated from the measured data.

III. EXPERIMENTAL RESULTS

Several experiments were conducted using a narrowband and a broadband versions of linear-FM waveforms as the source signal. In the experiments, the statistical expectations were replaced with the corresponding time averages. It is not very difficult to see that the results in Section II hold at least approximately even when the statistical autocorrelation functions are replaced with deterministic autocorrelations defined using time averages. The source signal has the functional form given by [3]

$$X(n) = a(n) e^{j\pi b n^2}, \quad (13)$$

where a(n) is the envelope of the signal and b is a constant that is designated as the slope of the instantaneous frequency of the FM waveform. Note that if a(n) is confined to a time duration T, the instantaneous frequency of the waveform will sweep over the bandwidth of bT radians/sample. The narrowband and broadband source signals were created by selecting b to be 10^{-5} and 5×10^{-4}, respectively. The estimates of the time delays assumed that the unknown time delays were integer multiples of the sampling period in all cases. While this is not a realistic assumption, it considerably reduced the complexity of the simulations. In practice one will have to interpolate between samples of the crosscorrelation estimates to find the peak values. All the results presented are averages of fifty independent experiments. All the experiments made use of 4096 data samples.

Two cases were considered. In the first case, there was only one unknown (DFO, DTD) set. The unknown DFO was 1 radian per sample and the unknown time delay was 5 samples. In the second example, there were two unknown (DFO, DTD) pairs given by (1 radian/sample, 5 time units) and (1.2 radian/sample, 50 time units). The attenuation factor associated with the components of the received signal corresponding to both (time delay, frequency offset) pairs was one. Table 1 presents the mean and mean-squared deviation from the actual value of the frequency offset estimates obtained using our approach and the direct method that involves ambiguity function calculations for the first case. The corresponding results for time delay estimates are shown in Table 2. Tables 3 and 4 display the results involving multiple (DFO, DTD) sets.

The results show that the new method of frequency offset estimation performs better than the direct method for both narrowband and broadband source signals in all SNR environments. Using the broadband source signal results in poorer performance than using the narrowband signal in all experiments. This is to be expected since computation of $\eta(\tau)$ in equation (6) is more noisy when the broadband signal is used than when the narrowband signal is used. This is so because the autocorrelation function of the broadband signal decays rapidly and the estimation noise may dominate the estimation of $R_{tt}(\tau)$ that appears in the denominator of (6) especially for large values of τ.

The results for time delay estimation show that our approach for DTD estimation perform as well as the direct ambiguity function method for both narrowband and broadband source signals in high SNR environments. However, at low SNR, the time delay estimates of the new approaches are somewhat poorer when compared with the ambiguity function method for narrowband situation. Again, note that using the broadband source signal results in poorer performance than using the narrowband signal in all time delay estimation experiments. A large number of additional experiments involving a variety of situations have been done and documented in [4].

V. CONCLUDING REMARKS

In this paper, we presented a new and efficient approach for estimating differential time delays and differential frequency offsets. The major advantage of the new approach is its remarkably reduced computational complexity. Besides, the experimental results indicate that the new method is capable of estimating the frequency offsets more accurately than the traditional approaches, especially when the signal-to-noise ratio is very poor and when a narrowband source signal is used. The performance of the time delay estimator is slightly worse, but comparable to the traditional schemes for good signal to noise ratios. These aspects of the new approach should make it very useful and attractive in practical applications.

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		Narrow Band		Broad Band	
		Ambi-guity	New Method	Ambi-guity	New Method
20 dB	μ	1.004	1.000	1.007	0.999
	MSD	1.875E-5	1.000E-8	4.436E-5	1.904E-5
10 dB	μ	0.994	1.000	1.002	0.998
	MSD	3.352E-5	2.560E-8	9.526E-5	3.920E-6
0 dB	μ	0.982	0.998	1.021	0.997
	MSD	3.258E-4	2.341E-6	4.452E-4	8.880E-6
-10dB	μ	1.032	1.003	0.960	0.994
	MSD	9.998E-4	6.656E-6	1.583E-3	3.745E-5

Table 1.•The mean and mean squared deviation of the frequency offset estimates in the first example. The actual frequency offset was 1.0 radian per sample.

		Narrow Band		Broad Band	
		Ambi-guity	Higher Order	Ambi-guity	Higher Order
20 dB	μ	5.000	5.000	5.000	5.000
	MSD	0.0	0.0	0.0	0.0
10 dB	μ	5.000	5.000	5.000	5.000
	MSD	0.0	0.0	0.0	0.0
0 dB	μ	5.000	5.000	5.000	5.020
	MSD	0.0	0.0	0.0	4.000E-4
-10 dB	μ	5.000	5.060	4.740	4.940
	MSD	0.0	3.600E-3	6.760E-2	3.600E-3

Table 2. The mean and mean squared deviation of the time delay estimates in the first example. The actual time delay was 5 time units.

		Ambiguity		New Approach	
20 dB	μ	1.003	1.209	1.001	1.201
	MSD	9.923E-6	7.709E-5	2.132E-6	1.254E-6
10 dB	μ	1.006	1.213	1.002	1.202
	MSD	3.956E-5	1.563E-4	2.856E-6	5.617E-6
0 dB	μ	1.009	1.170	1.004	1.203
	MSD	8.354E-5	8.928E-4	1.552E-5	6.970E-6
-10 dB	μ	1.012	1.251	1.011	1.204
	MSD	1.553E-4	2.621E-3	1.272E-4	1.246E-5

(a)

		Ambiguity		New Approach	
20 dB	μ	5.000	50.000	5.000	50.000
	MSD	0.0	0.0	0.0	0.0
10 dB	μ	5.000	50.000	5.000	50.000
	MSD	0.0	0.0	0.0	0.0
0 dB	μ	5.000	50.000	5.080	49.900
	MSD	0.0	0.0	3.600E-3	1.000E-2
-10 dB	μ	5.120	36.580	5.180	49.240
	MSD	1.440E-2	11.695	3.240E-2	5.776E-1

(b)

Table 3. The mean and mean squared deviation of the (a) frequency offsets and (b) time delays estimates in the second example when the narrow band source signal was used.

		Ambiguity		New Approach	
20 dB	μ	1.009	1.239	1.004	1.198
	MSD	7.709E-5	1.557E-3	1.927E-5	2.856E-6
10 dB	μ	0.988	1.247	1.010	1.208
	MSD	1.390E-4	2.173E-3	1.098E-4	7.089E-5
0 dB	μ	0.963	1.308	0.967	1.170
	MSD	1.359E-3	1.167E-2	1.069E-3	8.952E-4
-10 dB	μ	1.069	1.314	1.060	1.262
	MSD	4.821E-3	1.309E-2	3.649E-3	3.839E-3

(a)

		Ambiguity		New Approach	
20 dB	μ	5.000	50.000	4.940	50.080
	MSD	0.0	0.0	3.600E-3	4.000E-4
10 dB	μ	5.200	50.120	5.140	50.020
	MSD	4.000E-2	1.440E-2	1.960E-2	6.400E-3
0 dB	μ	15.120	52.440	5.240	51.820
	MSD	102.414	5.954	5.760E-2	3.312
-10 dB	μ	-13.000	24.920	5.660	44.640
	MSD	324.000	629.004	4.356E-1	28.730

(b)

Table 4. The mean and mean squared deviation of the (a) frequency offsets and (b) time delays estimates in the second example when the broad band source signal was used.