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Eigenfunction localization in dilute lattices of various dimensionalities

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As a fraction of bonds p is removed from a lattice, we find a threshold p_Q above which all eigenstates are nonpropagating (Anderson localized). This occurs well before the classical percolation threshold. For example, in two dimensions where the classical threshold is $p_c = \frac{1}{2}$, we find $p_Q \approx 0$.

Anderson introduced the concept of disorder-induced localization of eigenstates of a wave equation.¹ Subsequently, Licciardello and Thouless² and Abrahams, Anderson, Licciardello, and Ramakrishnan³ have given reasons for all such eigenstates to be localized in two dimensions ($d=2$) at arbitrarily small values of the random potential V . Such localization has long been known to be a feature of one dimension,⁴ whereas in four or more dimensions, the disorder parameter $|V|$ must exceed a critical magnitude V_c before eigenstates become localized. The critical magnitude V_c depends, of course, on the number d of dimensions.

Now, a number of questions concerning localization in $d=2$ or 3 have not been definitively answered. For example, of the most recent various studies of the random potential model, we have on the one hand those of Soukoulis and Economou⁵ and the renormalization-group (RG) analysis of Domany and Sarker⁶ which confirm the predictions of Refs. 2 and 3, finding $V_c=0$ in $d=2$. While on the other hand, we find the accurate numerical analysis of Stein and Krey⁷ and a variant RG analysis by Lee⁸ which contradict Refs. 2 and 3, by obtaining a sizable V_c (of the order of one-sixth the bandwidth) for $d=2$. Considerations of the relative merits of the various calculations aside, it is entirely possible that in $d=2$ (and perhaps $d=3$) the behavior is "nonuniversal," hence sensitive to the precise method of solution as well as to the nature of the disorder. We believe this to be the case for $d=2$, as

we apply the very methods of Stein and Krey⁷ to a variant model, and obtain results quite different from theirs.

Our work centers on the random-bond-dilution model, well known in connection with classical percolation, diffusion, and conductivity,⁹ having a host of applications to solids, liquids, gases, lasers, etc. It is desired to obtain the effects on eigenstates of removal, at random, of a fraction p of the bonds.¹⁰ One would not expect any extended states to persist once p exceeds the classical percolation threshold p_c , and indeed, we find that the quantum threshold p_Q is lower. With an estimated uncertainty of ± 0.05 , our results indicate $p_Q \approx 0, 0.63, \text{ and } 0.77$ in $d=2, 3, \text{ and } 4$ dimensions, respectively. This may be compared with $p_c \approx 0.50, 0.75, \text{ and } 0.80$ in $d=2, 3, \text{ and } 4$ dimensions.¹¹ For the trivial case $d=1$ where $p_c=0$, we also obtain $p_Q=0$ and have nothing to add to the known results.⁴

We base our analysis on a numerically efficient "trick," the tridiagonalization of a Hamiltonian which has interactions limited to nearest neighbors on a generalized cubic lattice [lattice sites at $R_i = a(n, m, \dots)$ where n, m, \dots are integers]. Tridiagonalization was first applied to a related problem when Haydock¹² used it to study the mobility edge on a Cayley tree lattice, which is in some sense equivalent to infinite d . Recently, this method was chosen by Stein and Krey⁷ for their previously noted analysis, and by Mattis and Raghavan¹³ for a rederivation of Wigner's famous semicircular density

of states for a totally random matrix. Tridiagonalization requires far fewer steps than diagonalization, and often yields greater insights, as we shall see. We refer the interested reader to a rapidly evolving literature on the subject of tridiagonalization,¹⁴ and limit explanatory remarks here to the few details necessary to describe the model. A fraction p of nearest-neighbor bonds is removed from the lattice; the remaining bonds have magnitude $\frac{1}{2}$. Let us introduce the parameter ϵ_{ij} which is $[1,0]$ depending on whether the bond is [present, absent] and write the Hamiltonian as

$$H = -\frac{1}{2} \sum_{\langle ij \rangle} \epsilon_{ij} (c_i^\dagger c_j + \text{H.c.}) . \quad (1)$$

The distributions of ϵ_{ij} is “frozen in.” We then consider an eigenstate propagating out from some initial site R_i . This connects to the shell of nearest neighbors through the extant bonds, thence to the next shell, etc. The (real) matrix element b_n connecting the n th shell to the $(n+1)$ st is obtained after normalizing the operators at each shell. Our symmetric, tridiagonal matrix consists of the b_n above and below the main diagonal, and zero everywhere else.

As we vary the initial site R_i or vary the statistical sample for a given value of p , we find the b_n 's change but asymptotic ($n \rightarrow \infty$) properties do not. These properties will be the main focus of discussion in this paper. Our analysis is based on sample of well over 20 000 sites in $d=2, 3, 4$ dimensions and a large number of numerical experiments. The experiments could be run quickly, as tridiagonalization is a very rapid and efficient procedure¹⁴ compared with other numerical procedures.

In $d=4$ (and $d=3$) at small dilution, the matrix elements b_n rapidly approach an asymptotic value b_∞ which we plot as a function of p in Fig. 1. Beyond a certain critical dilution, which is denoted p_Q and indicated by a star in the figure, oscillations persist out to the largest attainable shell index n and, presumably, b_∞ ceases to exist. However, when this occurs, we find that the amplitude of the oscillations—the *noise* in b_n —does approach an asymptotic, stable value. Figure 1 shows the dependence of b_∞ on p in the range, $p < p_Q$, for which an asymptotic value is obtained. Significantly, such a value is never obtained in $d=2$, however large the sample we have used and however large the shell index (up to $n=150$), whereas the “noise” parameter is always asymptotically stable in $d=2$. Thus, the appropriate curve for $d=2$ shrinks to a point at $p=0$, $b_\infty=2$.

The significance of the noisy asymptotic behavior is to be found in the nature of eigenfunctions in one dimension. Tridiagonalization is tantamount to the extraction of the one-dimensional manifold of states having nonvanishing amplitudes at the chosen initial site R_i . The eigenstates of the tridiagonal form are eigenstates of the total Hamiltonian. As the tridiago-

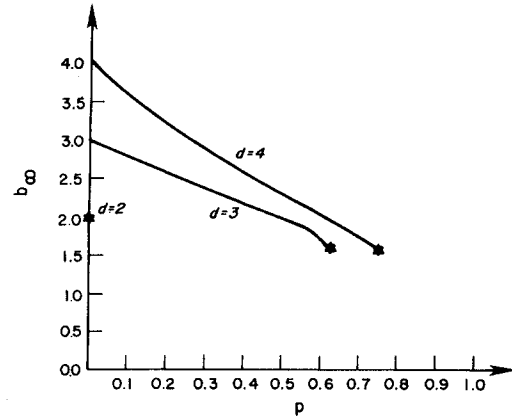


FIG. 1. Asymptotic b_∞ as a function of dilution p in the d -dimensional cubic lattices. For $p > p_Q$ (indicated by a star) the matrix elements b_n do not approach an asymptotic value b_∞ , but continue to oscillate indefinitely. This occurs immediately for $d=1$ (not shown) and for $d=2$ (indicated by a star).

nal form is isomorphic to a semi-infinite linear chain, all the well-known theorems of one dimension⁴ apply to it and, in particular, that which states that if $|b_{n+1} - b_n|$ does not vanish at large n , all the eigenstates are localized. The converse, however, is not true. For even if the b_n approach an asymptotic value, some or all the eigenstates could be localized. Thus, the threshold we shall calculate will be an upper bound.

Figure 2, drawn from three dimensions, is an example which clearly shows the difference between the smooth approach to an asymptotic value at low dilution, and the noisy behavior for $p > p_Q$. It is in-

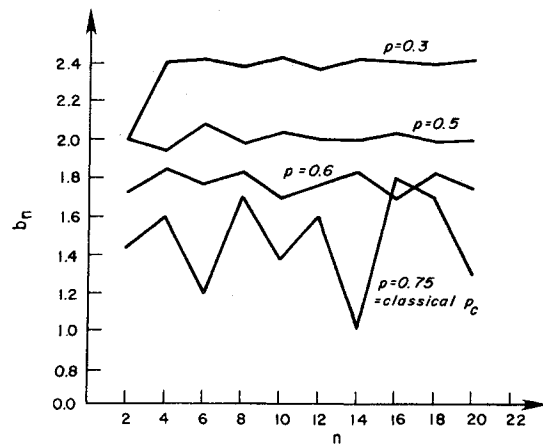


FIG. 2. Typical behavior of b_n as a function of shell index n . This example, in $d=3$ dimensions, shows the approach to asymptotic constancy at small dilution, the increasing “noise” with increasing dilution, and the lack of asymptotic constancy in b_n (tantamount to localization of all the eigenstates) at $p=0.75$.

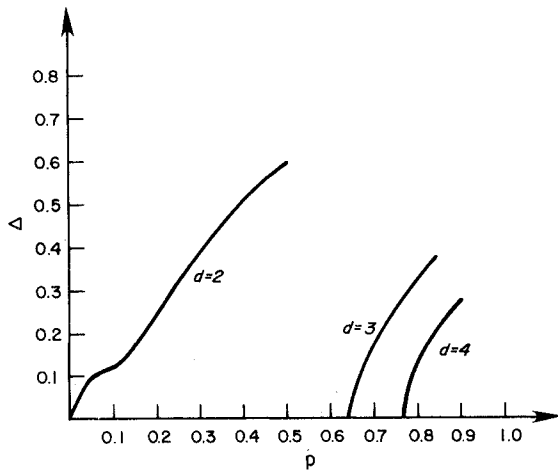


FIG. 3. Plot of the numerically computed parameter $\Delta(p)$ Eq. (2), for various d . Our data indicate p_Q (the value at which Δ vanishes) to be 0, 0.63, and 0.77 for $d=2, 3, 4$ as compared with classical $p_c=0.50, 0.75$, and 0.80. Thus for $d=2$ (as well as $d=1$), eigenstates are localized except when $p=0$.

interesting that the point $p=0.75$ which illustrates the noisy behavior is in fact the classical percolation threshold in $d=3$; we find the onset of noisy behavior to have occurred at $p_Q=0.63$ (not illustrated) for this case.

Finally, Figure 3 shows the noise parameter Δ we

have chosen to calculate defined as follows:

$$\Delta \equiv \lim_{n \rightarrow \infty} \left| \frac{b_{n+1} - b_n}{b_n} \right|. \quad (2)$$

Error bars (omitted for clarity) would show an uncertainty in Δ of not more than ± 0.05 . The “knee” seen in the $d=2$ figure at small p (the noise does not decrease rapidly as $p \rightarrow 0$) falls within this error bar, and could be an artifact. Nevertheless, it is clear that for $d=2$ the critical p_Q for localization is very much less than the classical $p_c = \frac{1}{2}$ and is either zero or very close to it. In $d=3, 4$ we also find that p_Q is substantially below p_c . Clearly, the phenomenon of quantum percolation differs from the classical version and is deserving of further study.

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