PHYSICAL REVIEW D **78**, 065002 (2008)

Twisted supersymmetric invariant formulation of Chern-Simons gauge theory on a lattice

Kazuhiro Nagata[1,*] and Yong-Shi Wu[2,+]

[1]*Department of Physics, Indiana University, Bloomington, Indiana 47405, USA*
[2]*Department of Physics, University of Utah, Salt Lake City, Utah 84112, USA*
(Received 9 April 2008; published 2 September 2008)

We propose a twisted supersymmetric (SUSY) invariant formulation of the Chern-Simons theory on a Euclidean three-dimensional lattice. The SUSY algebra to be realized on the lattice is the $N = 4$ $D = 3$ twisted algebra that was recently proposed by D'Adda *et al.* In order to keep the manifest anti-Hermiticity of the action, we introduce oppositely oriented supercharges. Accordingly, the naive continuum limit of the action formally corresponds to the Landau-gauge fixed version of the Chern-Simons theory with complex gauge group which was originally proposed by Witten. We also show that the resulting action consists of parity even and odd parts with different coefficients.

DOI: 10.1103/PhysRevD.78.065002 PACS numbers: 11.15.Ha, 12.60.Jv

I. INTRODUCTION

Chern-Simons gauge theory is a fundamentally important field theory in both physics and mathematics. The Lagrangian density [1,2] is just the famous Chern-Simons secondary characteristic class [3] for a principal bundle. As a topological field theory, its action can be defined in an odd dimensional spacetime without involving its metric. So mathematically the metric independent physical observables of the theory are topological invariants independent of a spacetime metric [4]. In particular, the partition function of the theory on a compact manifold gives rise to a three-manifold invariant, while the expectation value of Wilson loops gives rise to knot-link invariants, say Jones polynomials [5] in the case with the non-Abelian gauge group $SU(2)$. On the physics side, by now it is well known that Chern-Simons gauge theory can be used as a low energy effective theory to describe a new type of matter, the so-called topological phases, in planar condensed matter systems (or in $2 + 1$ spacetime dimensions), such as the fractional quantum Hall effect [6]. Also, quantum gravity in $(2 + 1)$-dimensional spacetime, which is known to be diffeomorphism invariant, can be formulated as a Chern-Simons theory with the Poincaré group as the gauge group [7]. In recent years, the close relationship between Chern-Simons gauge theory, topological invariants, and topological phases has attracted a lot of attention for developing topological quantum computation [8,9]. The above is just a few examples of the ubiquitousness of the Chern-Simons theory in physical applications. For a recent survey see, e.g., Ref. [10].

Because of the primary importance of the Chern-Simons theory, it is much desirable to put the theory on a lattice for the convenience of computer simulations. However, up to now this task has been achieved with limited success. Previously, lattice formulations of the Chern-Simons theory have been addressed in the context of bosonization or anyonization [11–13] or of topological excitations [14] in a regularized framework.[1] Two major difficulties in formulating the lattice Chern-Simons theory have been identified. One is the problem of an extra zero eigenvalue in the gauge field kernel, which arises from the fact that the gauge kinetic terms involve only first order derivatives. This feature resembles the "doubling problem" for lattice fermions, which is also tightly connected with the Hermiticity issue of the lattice action [17,18]. The other difficulty, in formulating a non-Abelian Chern-Simons theory on a lattice, is related to gauge noninvariance of the action for a non-Abelian theory under large gauge transformations.

In this paper we attack the problem of the lattice formulation of the Chern-Simons theory with a new method. Instead of attempting to directly put the Chern-Simons action on a lattice, we propose to put the gauge-fixed Chern-Simons theory on a Euclidean lattice. We also introduce oppositely oriented component fields in order to ensure the manifest anti-Hermiticity of the lattice action. We are motivated by two observations in the literature. The first observation is an old one [19,20] that there exists a very rich symmetry structure in the Landau-gauge fixed Chern-Simons action; namely, apart from the ordinary BRST symmetry which is the remnant of the original gauge symmetry, there exist more fermionic symmetries of a vector type. In Ref. [21], the set of symmetries together with the anti-BRST–type symmetries are identified as a certain type of twisted supersymmetry (SUSY), which was originally proposed in the context of topological quantum field theory [22]. Since then, the twisted SUSY invariant properties of the Chern-Simons theory in Landau gauge have been studied in more detail concerning its quantum aspects [23] as well as its rich symmetry structure [24]. The

*knagata@indiana.edu
+wu@physics.utah.edu

[1]There are works on simplicial lattices addressing Abelian Chern-Simons theory in terms of a geometric discretization scheme [15] and also Chern-Simons gravity via the Ponzano-Regge model [16].

second observation that inspires us is a recent one, in which the twisted SUSY plays a particularly important role in realizing SUSY on a lattice [25–29]. This is essentially due to the intrinsic relation between twisted fermions and Dirac-Kähler fermions [30]. It is observed that among other recent developments of lattice SUSY [31], the so-called deconstruction formulation of lattice SUSY [32] can also be related to the twisted SUSY framework [33]. Motivated by these recent developments, we naturally anticipate that a lattice formulation of Chern-Simons theory can be given through the lattice realization of the twisted SUSY associated with the Landau-gauge fixed action.

This article is devoted to constructing a Landau-gauge fixed Chern-Simons multiplet directly on a three-dimensional lattice and to proposing a manifestly anti-Hermitian Euclidean lattice action. This paper is organized as follows. In Sec. II, we review the symmetries of the Landau-gauge fixed Chern-Simons action in continuum spacetime. In Sec. III, after giving an overview of the twisted SUSY formulation on a lattice developed in [25] and introducing the twisted $N = 4$ $D = 3$ lattice algebra [27], we proceed to construct a lattice counterpart of the Chern-Simons multiplet. We also introduce oppositely oriented supercharges and component fields in order to realize the manifest (anti-)Hermiticity of the lattice multiplet. In Sec. IV, we construct a lattice version of Landau-gauge fixed Chern-Simons action and show how the twisted SUSY invariance is realized. We further show that the zero-eigenvalue problem does not occur in our formulation owing to manifest anti-Hermiticity of the lattice action. We also discuss about the naive continuum limit and its relation to the Chern-Simons theory with complex gauge group [34]. Section V addresses the parity transformation properties of our lattice action, and Sec. VI summarizes our formulation with some discussions.

II. CHERN-SIMONS IN LANDAU GAUGE

In this section, we review the symmetry aspects of the Chern-Simons action with Landau gauge fixing in the continuum spacetime. Although the original Chern-Simons action is given in a metric independent form, it becomes metric dependent after the gauge-fixing terms are introduced. In this paper, we only consider the Euclidean three-dimensional flat spacetime. The gauge-fixed action is given by

$$S = i\frac{k}{2\pi}\int d^3x \,\mathrm{Tr}\Big[\epsilon_{\mu\nu\rho}\Big(\frac{1}{2}A_\mu\partial_\nu A_\rho + \frac{1}{3}A_\mu A_\nu A_\rho\Big) - b\partial_\mu A_\mu - \bar{c}\partial_\mu D_\mu c\Big], \tag{2.1}$$

where A_μ, b, c, and $\bar{c}$ denote the gauge field, an auxiliary field, the ghost, and the antighost fields, respectively. The coefficient k should be a multiple of integer required by invariance under large gauge transformations. Note the overall purely imaginary factor i in the Euclidean action, because the path integral measure of the topological field theory has to be a pure phase factor. All of the component fields belong to the adjoint representation of the gauge group with the following anti-Hermiticity conditions [21],

$$A_\mu^\dagger = -A_\mu, \qquad b^\dagger = -b, \qquad c^\dagger = -c, \qquad \bar{c}^\dagger = \bar{c}. \tag{2.2}$$

The gauge-fixed action (2.1) is invariant under the BRST transformations which are remnants of the original gauge symmetry,

$$sA_\mu = -D_\mu c, \qquad sc = c^2, \tag{2.3}$$

$$s\bar{c} = b, \qquad sb = 0, \tag{2.4}$$

where the covariant derivative D_μ is defined by $D_\mu c = \partial_\mu c + [A_\mu, c]$. Furthermore, it was pointed out in [19–21] that the action (2.1) is also invariant under additional fermionic transformations including vector-type transformations, $\bar{s}_\mu$, s_μ, and $\bar{s}$, where the index μ runs from 1 to 3. We list their transformation laws for the component fields in Table I. The whole set of eight generators $(s, \bar{s}_\mu, s_\mu, \bar{s})$ is shown to satisfy the following algebra [21]:

$$\{s, \bar{s}_\mu\} \doteq \partial_\mu, \qquad \{s_\mu, \bar{s}_\nu\} \doteq \epsilon_{\mu\nu\rho}\partial_\rho, \tag{2.5}$$

$$\{\bar{s}, s_\mu\} \doteq -\partial_\mu, \qquad \{\text{others}\} = 0. \tag{2.6}$$

Here the dotted equality means that the algebra closes only on shell, namely, up to equations of motion. The anti-Hermiticity conditions for the twisted supercharges can be imposed consistently with those for the component fields (2.2):

$$s^\dagger = -s, \qquad \bar{s}^\dagger = \bar{s}, \qquad s_\mu^\dagger = -s_\mu, \qquad \bar{s}_\mu^\dagger = \bar{s}_\mu, \qquad \partial_\mu^\dagger = -\partial_\mu. \tag{2.7}$$

Since the BRST generator s is supposed to transform as a scalar under the Lorentz transformation, we immediately read off from the algebra (2.5) and (2.6) that the remaining fermionic generators $\bar{s}_\mu$, s_μ, and $\bar{s}$ transform as a vector, another vector, and a scalar, respectively. These transformation properties are identical to the ones in a certain type of twisted SUSY, where the new Lorentz group, which is called the twisted Lorentz group, is defined as the diagonal

TABLE I. Fermionic transformation laws in continuum spacetime.

	s	$\bar{s}_\rho$	s_ρ	$\bar{s}$
c	c^2	$-A_\rho$	0	$-b + \{\bar{c}, c\}$
$\bar{c}$	b	0	A_ρ	$\bar{c}^2$
A_μ	$-D_\mu c$	$-\epsilon_{\rho\mu\nu}\partial_\nu\bar{c}$	$-\epsilon_{\rho\mu\nu}\partial_\nu c$	$-D_\mu\bar{c}$
b	0	$\partial_\rho\bar{c}$	$D_\rho c$	$[\bar{c}, b]$

subgroup of the original Lorentz group and a certain type of internal symmetry group. In the present case with eight supercharges $(s, \bar{s}_\mu, s_\mu, \bar{s})$, the twisted Lorentz group is understood as the diagonal subgroup of $SO(3)_{\rm Lorentz} \times SO(3)_{\rm internal}$ whose covering group is $(SU(2) \times SU(2))_{\rm diag}$. The twisted structure can be explicitly seen from the following combinations of $(s, \bar{s}_\mu, s_\mu, \bar{s})$ into the generators $Q_{\alpha k}$ and $\bar{Q}_{k\alpha}$ with spin index α and internal index k:

$$Q_{\alpha k} = (\mathbf{1}s + \gamma_\mu(-is_\mu))_{\alpha k}, \tag{2.8}$$

$$\bar{Q}_{k\alpha} = (\mathbf{1}\bar{s} + \gamma_\mu(i\bar{s}_\mu))_{k\alpha}, \tag{2.9}$$

where $\mathbf{1}$ represents the unit matrix while γ_μ $(\mu = 1, 2, 3)$ represent three-dimensional gamma matrices which can be taken to be Pauli matrices. One can easily see that s_μ and $\bar{s}_\mu$ transform as vectors if the spin and internal indices are rotated simultaneously. Furthermore, in terms of $Q_{\alpha k}$ and $\bar{Q}_{k\alpha}$, the algebra (2.5) and (2.6) can be reexpressed as

$$\{Q_{\alpha k}, \bar{Q}_{l\beta}\} = 2i\delta_{kl}(\gamma_\mu)_{\alpha\beta}\partial_\mu. \tag{2.10}$$

This clearly shows that the internal symmetry indices k and l can be viewed as the suffices labeling extended SUSY, while α and β remain the ordinary spinor indices. From the above observations it becomes clear that the fermionic symmetries associated with the Landau-gauge fixed Chern-Simons action are essentially connected with a certain type of extended SUSY through the twisting procedure. Following the standard nomenclature in topological field theory [35,36], we refer to the algebra (2.5) and (2.6) as the $N = 4$ $D = 3$ twisted SUSY algebra.[2] A superfield formulation based on the twisted $N = 4$ $D = 3$ SUSY algebra is recently elaborated on in Ref. [28] with a direct application to continuum super Yang-Mills theories in the off-shell regime.

It is important to mention here about parity transformations of the component fields and the supercharges of the Chern-Simons multiplet. Since we are working on a Euclidean three-dimensional spacetime, a parity operation on the spacetime coordinates may be defined by the simultaneous inversion of all the directions,

$$P(x_1, x_2, x_3)P^{-1} = (-x_1, -x_2, -x_3). \tag{2.11}$$

The gauge fields and the derivative operators are supposed to transform as vectors, obeying

$$PA_\mu(x)P^{-1} = -A_\mu(-x), \qquad P\partial_\mu P^{-1} = -\partial_\mu, \tag{2.12}$$

where $-x$ denotes $-x = (-x_1, -x_2, -x_3)$. The parity nature of the supercharges could be determined consistently with the SUSY transformations of the component fields, provided parity is compatible with the SUSY algebra (2.5) and (2.6). Here we assume that the ghost field $c(x)$ transforms as a scalar, namely, $Pc(x)P^{-1} = c(-x)$. We then immediately read off the parity of the supercharges as

$$\begin{aligned} PsP^{-1} &= s, & P\bar{s}_\mu P^{-1} &= -\bar{s}_\mu, \\ Ps_\mu P^{-1} &= s_\mu, & P\bar{s}P^{-1} &= -\bar{s}. \end{aligned} \tag{2.13}$$

The parities of $\bar{c}$ and b are accordingly given by $P\bar{c}(x)P^{-1} = -\bar{c}(-x)$ and $Pb(x)P^{-1} = -b(-x)$. Notice that the entire action (2.1) is parity odd under these assumptions.

III. TWISTED SUSY AND CHERN-SIMONS MULTIPLET ON LATTICE

A. Lattice SUSY algebra

It was recently recognized [27] that the $N = 4$ $D = 3$ twisted SUSY algebra could be realized on a lattice consistently with the lattice Leibniz rule; then it was immediately applied to a twisted super Yang-Mills formulation on a three-dimensional lattice. We first briefly review the lattice formulation of the twisted SUSY proposed in Ref. [25] and then proceed to construct the Chern-Simons multiplet based on the $N = 4$ $D = 3$ twisted SUSY structure on the lattice. Since the lattice spacing is always finite, on a lattice all the derivative operators should be replaced by difference operators:

$$\partial_\mu \rightarrow \Delta_{\pm\mu}, \tag{3.1}$$

where $\pm$ denotes forward and backward differences, respectively. The operation of difference on a function $\Phi(x)$ is defined by the following type of "shifted" commutators,

$$(\Delta_{\pm\mu}\Phi(x)) \equiv \Delta_\pm\Phi(x) - \Phi(x \pm n_\mu)\Delta_{\pm\mu}, \tag{3.2}$$

where n_μ $(\mu = 1, \ldots, r)$ denote the unit vectors in r dimensions, whose component is given by $(n_\mu)_\rho = \delta_{\mu\rho}$. We take the lattice spacing to be unity. The difference operators $\Delta_{\pm\mu}$ are most naturally located on links from x to $x \pm n_\mu$ for a generic value of x, and they take unit values such that the definition (3.2) actually gives the forward and backward differences:

$$\Delta_{\pm\mu} = (\Delta_{\pm\mu})_{x\pm n_\mu, x} = \mp 1. \tag{3.3}$$

Starting from the definition (3.2), one finds that the operation of $\Delta_{\pm\mu}$ on a product of functions $\Phi_1(x)\Phi_2(x)$ gives

$$\begin{aligned} (\Delta_{\pm\mu}\Phi_1(x)\Phi_2(x)) = (\Delta_{\pm\mu}\Phi_1(x))\Phi_2(x) \\ + \Phi_1(x \pm n_\mu)(\Delta_{\pm\mu}\Phi_2(x)), \end{aligned} \tag{3.4}$$

which we refer to as the Leibniz rule on the lattice. The importance of the Leibniz rule has also been recognized in the context of noncommutative differential geometry on a lattice [37]. Since in continuum, SUSY is essentially nothing but the fermionic decomposition of the differential operators ∂_μ, we may then naturally expect that the fermionic decomposition of the difference operators $\Delta_{\pm\mu}$ will

[2] In the early literatures it was referred to as $N = 2$ algebra.

accordingly serve as the starting point of a lattice formulation of SUSY. In order to be compatible with the link nature of difference operators, we introduce a generic lattice supercharge Q_A on a link from x to $x + a_A$:

$$Q_A = (Q_A)_{x+a_A,x}, \tag{3.5}$$

where the a_A denotes a generic vector whose expression is to be determined in the following. The operation of Q_A is again defined as a shifted (anti-)commutator,[3]

$$(Q_A\Phi(x)) \equiv (Q_A)_{x+a_A,x}\Phi(x) - (-1)^{|\Phi|}\Phi(x + a_A) \times (Q_A)_{x+a_A,x}. \tag{3.6}$$

Accordingly, the operation on a product of functions gives

$$(Q_A\Phi_1(x)\Phi_2(x)) = (Q_A\Phi_1(x))\Phi_2(x) + (-1)^{|\Phi_1|}\Phi_1(x + a_A)(Q_A\Phi_2(x)), \tag{3.7}$$

where $|\Phi|$ stands for 0 or 1 for bosonic or fermionic Φ, respectively. The anticommutator of these supercharges may naturally be defined as the successive connections of link operators:

$$\{Q_A, Q_B\}_{x+a_A+a_B,x} \equiv (Q_A)_{x+a_A+a_B,x+a_B}(Q_B)_{x+a_B,x} + (Q_B)_{x+a_A+a_B,x+a_A}(Q_A)_{x+a_A,x}. \tag{3.8}$$

In terms of the above link operators, we can express the generic form of lattice SUSY algebra as

$$\{Q_A, Q_B\} = (\Delta_{\pm\mu})_{x\pm n_\mu,x}, \tag{3.9}$$

provided the following lattice Leibniz rule conditions hold:

$$a_A + a_B = +n_\mu \quad \text{for } \Delta_{+\mu}, \tag{3.10}$$

$$a_A + a_B = -n_\mu \quad \text{for } \Delta_{-\mu}. \tag{3.11}$$

Figures 1 and 2 depict the possible configurations of the general lattice SUSY algebra (3.9) subject to the conditions (3.10) and (3.11), respectively. It is a nontrivial question to ask what type of SUSY algebras satisfy these conditions. As described in [25,26], one can show that the Dirac-Kähler twisted $N = D = 2$ and $N = D = 4$ satisfy the above conditions. Furthermore, it is recently shown in [27] that the twisted $N = 4$ $D = 3$ algebra also satisfies the conditions. We actually find the lattice realization of the algebra (2.5) and (2.6) as

$$\{s, \bar{s}_\mu\} \doteq \Delta_{+\mu}, \qquad \{s_\mu, \bar{s}_\nu\} \doteq \epsilon_{\mu\nu\rho}\Delta_{-\rho}, \tag{3.12}$$

[3]We thank Jourjine for his comment on the shifted (anti-) commutator from the cell-complex cohomological point of view and for letting us know his works [38]. For recent works on algebraic topology in connection with the Dirac-Kähler fermion on a lattice, one may also refer to Ref. [39].

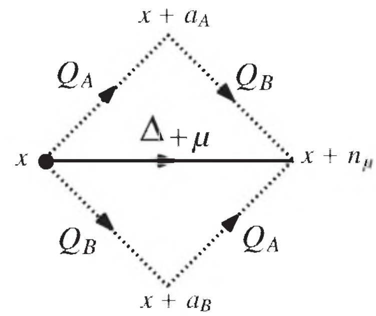

FIG. 1. Lattice SUSY algebra subject to the condition (3.10).

$$\{\bar{s}, s_\mu\} \doteq -\Delta_{+\mu}, \qquad \{\text{others}\} = 0, \tag{3.13}$$

where μ, ν, $\rho = 1, 2, 3$ and the link anticommutators in the left-hand side are understood. The corresponding Leibniz rule conditions on the choice of a_A can be expressed as

$$a + \bar{a}_\mu = +n_\mu, \qquad a_\mu + \bar{a}_\nu = -|\epsilon_{\mu\nu\rho}|n_\rho, \qquad \bar{a} + a_\mu = +n_\mu, \tag{3.14}$$

which are satisfied by the following general solutions:

$$a = (\text{arbitrary}), \qquad \bar{a}_\mu = +n_\mu - a, \tag{3.15}$$

$$a_\mu = -\sum_{\lambda\neq\mu} n_\lambda + a, \qquad \bar{a} = +\sum_{\lambda=1}^{3} n_\lambda - a. \tag{3.16}$$

Note that there is a one-vector arbitrariness in the choice of a_A, which eventually governs the resulting lattice configuration of the model. We will come back to this point when we construct the lattice Chern-Simons action. Notice also that the total sum of all the shift parameters vanishes despite the one-vector arbitrariness:

$$\sum a_A = a + \bar{a}_1 + \bar{a}_2 + \bar{a}_3 + a_1 + a_2 + a_3 + \bar{a} = 0. \tag{3.17}$$

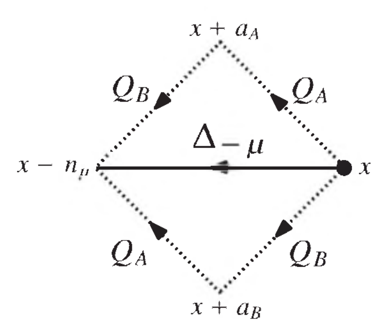

FIG. 2. Lattice SUSY algebra subject to the condition (3.11).

TABLE II. Link assignment of the fields and supercharges for a generic value of x. Note that the shift parameters $(a, \bar{a}_\mu, a_\mu, \bar{a})$ are subject to (3.15) and (3.16).

	c	$\bar{c}$	A_μ	b	s	$\bar{s}_\mu$	s_μ	$\bar{s}$
Link	$(c)_{x+a,x}$	$(\bar{c})_{x+\bar{a},x}$	$(A_\mu)_{x+n_\mu,x}$	$(b)_{x+\sum n,x}$	$(s)_{x+a,x}$	$(\bar{s}_\mu)_{x+\bar{a}_\mu,x}$	$(s_\mu)_{x+a_\mu,x}$	$(\bar{s})_{x+\bar{a},x}$.

TABLE III. Twisted SUSY transformation laws on the lattice. The link attributes of the products and (anti-)commutators are understood.

	s	$\bar{s}_\rho$	s_ρ	$\bar{s}$
c	c^2	$-A_\rho$	0	$-b + \{\bar{c}, c\}$
$\bar{c}$	b	0	A_ρ	$\bar{c}^2$
A_μ	$-[D_{+\mu}, c]$	$-\epsilon_{\rho\mu\nu}[\Delta_{-\nu}, \bar{c}]$	$-\epsilon_{\rho\mu\nu}[\Delta_{-\nu}, c]$	$-[D_{+\mu}, \bar{c}]$
b	0	$[\Delta_{+\rho}, \bar{c}]$	$[D_{+\rho}, c]$	$[\bar{c}, b]$

B. Twisted SUSY Chern-Simons multiplet on the lattice

The lattice implementation of the twisted SUSY transformation laws is possible only with an appropriate link assignment for each component field. For example, the transformation law $sc = c^2$ requires that the ghost field c should be located on a generic link from x to $x + a$ in order to be consistent with the link assignment of s which is also from x to $x + a$. With this link assignment, the corresponding lattice transformation law can be expressed as

$$(sc)_{x+2a,x} = (c)_{x+2a,x+a}(c)_{x+a,x}. \tag{3.18}$$

By studying all the twisted SUSY transformation laws in a similar way, one finds that the link attributes can be consistently assigned for all the component fields. Tables II and III summarize the link attributes of the component fields and their twisted SUSY transformation laws. In Table II and in the following, the symbol $\sum n$ represents the abbreviation $\sum_{\lambda=1}^{3} n_\lambda$. In Table III, all the field products and (anti-)commutators should be understood as link products and link (anti-)commutators, with the link indices suppressed for simplicity. $D_{+\mu}$ denotes the covariant derivative with forward difference, $D_{+\mu} \equiv \Delta_{+\mu} + A_\mu$.

Notice that the gauge fields are associated only with the forward difference and not with the backward difference. The absence of the backward covariant derivative implies that the (anti-)Hermiticity cannot be maintained if only one lattice multiplet $(A_\mu, b, c, \bar{c})$ is considered. One obvious way to maintain the (anti-)Hermiticity on the lattice is to introduce the oppositely oriented multiplet associated with a set of oppositely oriented supercharges. From now on, we slightly change the notations and denote the set of supercharges introduced in the above as $s_A^+ = (s^+, \bar{s}_\mu^+, s_\mu^+, \bar{s}^+)$. Then we introduce an additional set of oppositely oriented supercharges and denote them by $s_A^- = (s^-, \bar{s}_\mu^-, s_\mu^-, \bar{s}^-)$. The SUSY algebra is assumed to be

$$\{s^+, \bar{s}_\mu^+\} \doteq \Delta_{+\mu}, \qquad \{s_\mu^+, \bar{s}_\nu^+\} \doteq \epsilon_{\mu\nu\rho}\Delta_{-\rho}, \qquad \{\bar{s}^+, s_\mu^+\} \doteq -\Delta_{+\mu}, \tag{3.19}$$

$$\{s^-, \bar{s}_\mu^-\} \doteq \Delta_{-\mu}, \qquad \{s_\mu^-, \bar{s}_\mu^-\} \doteq \epsilon_{\mu\nu\rho}\Delta_{+\rho}, \qquad \{\bar{s}^-, s_\mu^-\} \doteq -\Delta_{-\mu}, \tag{3.20}$$

with other anticommutators of the supercharges vanishing: $\{\text{others}\} = 0$. We anticipate the on-shell closure of the algebra and express them with dotted equalities. We have assumed that the mixing sector of the algebra is just zero: $\{s_A^+, s_B^-\} = 0$. The Hermitian conjugation of the lattice supercharges and difference operators are defined as

$$(s^+)^\dagger = -s^-, \qquad (\bar{s}^+)^\dagger = \bar{s}^-, \tag{3.21}$$

$$(s_\mu^+)^\dagger = -s_\mu^-, \qquad (\bar{s}_\mu^+)^\dagger = \bar{s}_\mu^-, \tag{3.22}$$

$$(\Delta_{+\mu})^\dagger = -\Delta_{-\mu}, \qquad (\Delta_{-\mu})^\dagger = -\Delta_{+\mu}. \tag{3.23}$$

We assign the supercharges s_A^+ and s_A^- to be located on the same links but with mutually opposite orientation, namely, $(s_A^+)_{x+a_A,x}$ and $(s_A^-)_{x,x+a_A}$, respectively, as summarized in

TABLE IV. Link properties of oppositely oriented supercharges and component fields.

	s^+	$\bar{s}_\mu^+$	s_μ^+	$\bar{s}^+$	s^-	$\bar{s}_\mu^-$	s_μ^-	$\bar{s}^-$
Link	$(s^+)_{x+a,x}$	$(\bar{s}_\mu^+)_{x+\bar{a}_\mu,x}$	$(s_\mu^+)_{x+a_\mu,x}$	$(\bar{s}^+)_{x+\bar{a},x}$	$(s^-)_{x,x+a}$	$(\bar{s}_\mu^-)_{x,x+\bar{a}_\mu}$	$(s_\mu^-)_{x,x+a_\mu}$	$(\bar{s}^-)_{x,x+\bar{a}}$

	c^+	$\bar{c}^+$	$A_{+\mu}$	b^+	c^-	$\bar{c}^-$	$A_{-\mu}$	b^-
Link	$(c^+)_{x+a,x}$	$(\bar{c}^+)_{x+\bar{a},x}$	$(A_{+\mu})_{x+n_\mu,x}$	$(b^+)_{x+\sum n,x}$	$(c^-)_{x,x+a}$	$(\bar{c}^-)_{x,x+\bar{a}}$	$(A_{-\mu})_{x,x+n_\mu}$	$(b^-)_{x,x+\sum n}$

TABLE V. Twisted SUSY transformation laws on the lattice. The upper and lower signs show the transformation laws of $(c^+, \bar{c}^+, A_{+\mu}, b^+)$ under $(s^+, \bar{s}^+_\mu, s_{+\mu}, \bar{s})$ and $(c^+, \bar{c}^+, A_{+\mu}, b^+)$ under $(s^+, \bar{s}^+_\mu, s_{+\mu}, \bar{s})$, respectively. The link attributes of the products and (anti-)commutators are understood.

	$s^\pm$	$\bar{s}^\pm_\rho$	$s^\pm_\rho$	$\bar{s}^\pm$
$c^\pm$	$(c^\pm)^2$	$-A_{\pm\rho}$	0	$-b^\pm + \{\bar{c}^\pm, c^\pm\}$
$\bar{c}^\pm$	$b^\pm$	0	$A_{\pm\rho}$	$(\bar{c}^\pm)^2$
$A_{\pm\mu}$	$-[D_{\pm\mu}, c^\pm]$	$-\epsilon_{\rho\mu\nu}[\Delta_{\mp\nu}, \bar{c}^\pm]$	$-\epsilon_{\rho\mu\nu}[\Delta_{\mp\nu}, c^\pm]$	$-[D_{\pm\mu}, \bar{c}^\pm]$
$b^\pm$	0	$[\Delta_{\pm\rho}, \bar{c}^\pm]$	$[D_{\pm\rho}, c^\pm]$	$[\bar{c}^\pm, b^\pm]$

Table IV. Correspondingly, we introduce oppositely oriented lattice Chern-Simons multiplets $(c^+, \bar{c}^+, A_{+\mu}, b^+)$ and $(c^-, \bar{c}^-, A_{-\mu}, b^-)$, together with the following Hermitian conjugation conditions,

$$(c^+)^\dagger = -c^-, \qquad (\bar{c}^+)^\dagger = \bar{c}^-, \tag{3.24}$$

$$(A_{+\mu})^\dagger = -A_{-\mu}, \qquad (b^+)^\dagger = -b^-. \tag{3.25}$$

The link attributes of the multiplets and their SUSY transformation laws are given in Tables IV and V, respectively. The covariant differences in Table V, $D_{+\mu}$ and $D_{-\mu}$, are defined as $D_{\pm\mu} \equiv \Delta_{\pm\mu} + A_{\pm\mu}$, which obey the obvious Hermitian conjugation relations, $D^\dagger_{\pm\mu} = -D_{\mp\mu}$. We again assume that the SUSY transformation between the different sectors be trivial, namely,

$$[s^+_A, \varphi^-\} = [s_{\bar{A}}, \varphi^+\} = 0, \tag{3.26}$$

where φ^+ and φ^- denote any component of $(c^+, \bar{c}^+, A_{+\mu}, b^+)$ and $(c^-, \bar{c}^-, A_{-\mu}, b^-)$, respectively. Note that although the number of total supercharges is doubled in the present (anti-)Hermitian formulation, the lattice Leibniz rule requirements associated with the algebra (3.19) and (3.20) remain unchanged and are expressed as (3.14). The generic solutions are still given by (3.15) and (3.16).

IV. LATTICE CHERN-SIMONS ACTION

In terms of the two oppositely oriented multiplets, the anti-Hermitian, Landau-gauge fixed Chern-Simons action on a three-dimensional lattice is given by

$$S^{\rm tot} = k^+S^+ + k^-S^-, \tag{4.1}$$

with

$$\begin{aligned} S^+ = \frac{i}{4\pi}\sum_x \mathrm{Tr}\Big[&\frac{1}{2}\epsilon_{\mu\nu\rho}(A_{+\mu})_{x+\sum n, x+n_\nu+n_\rho} \\ &\times [\Delta_{+\nu}, A_{+\rho}]_{x+n_\nu+n_\rho, x} \\ &+ \frac{1}{3}\epsilon_{\mu\nu\rho}(A_{+\mu}A_{+\nu}A_{+\rho})_{x+\sum n, x} \\ &- (b^+)_{x+\sum n, x}[\Delta_{-\mu}, A_{+\mu}]_{x,x} \\ &- (\bar{c}^+)_{x+\sum n, x+a}[\Delta_{-\mu}, [D_{+\mu}, c^+]]_{x+a,x}\Big], \end{aligned} \tag{4.2}$$

$$\begin{aligned} S^- = \frac{i}{4\pi}\sum_x \mathrm{Tr}\Big[&\frac{1}{2}\epsilon_{\mu\nu\rho}(A_{-\mu})_{x-\sum n, x-n_\nu-n_\rho} \\ &\times [\Delta_{-\nu}, A_{-\rho}]_{x-n_\nu-n_\rho, x} \\ &+ \frac{1}{3}\epsilon_{\mu\nu\rho}(A_{-\mu}A_{-\nu}A_{-\rho})_{x-\sum n, x} \\ &- (b^-)_{x-\sum n, x}[\Delta_{+\mu}, A_{-\mu}]_{x,x} \\ &- (\bar{c}^-)_{x-\sum n, x-a}[\Delta_{+\mu}, [D_{-\mu}, c^-]]_{x-a,x}\Big], \end{aligned} \tag{4.3}$$

where k^+ and k^- denote complex parameters related to each other by complex conjugation, $(k^\pm)^* = k^\mp$. The summation over x in (4.2) and (4.3) covers all the integer sites of a three-dimensional regular lattice, anticipating the fact that the a needs to be integer vectors. The anti-Hermiticity of the total action is manifest.

A. Twisted SUSY invariance

Before showing the SUSY invariance of the lattice action (4.1), (4.2), and (4.3), we would like to make the following remarks. First, in order to ensure the SUSY invariance of the action, one needs to take care of the ordering of the link fields. The notion of proper ordering in lattice SUSY formulations has been addressed in Ref. [27]. Here in the lattice Chern-Simons action, the proper ordering is nothing but the geometrically connected ordering; namely, each term of S^+ or S^- consists of factors on connected links. Furthermore, all the terms in S^+ and S^- connect x to $x + \sum n$ and x to $x - \sum n$, respectively, through a sequence of links. The homogeneous connecting property is a direct consequence of the link component fields consistently allocated with the $N = 4$ $D = 3$ twisted

SUSY transformation laws on the lattice. Figure 3 depicts the configuration of the component fields per unit cell in the case of $a = -\sum n$. The second remark is that the $N = 4$ $D = 3$ twisted SUSY invariance of the action (4.1) is intrinsically related to the one-vector arbitrariness (3.15) and (3.16) in the solutions for the lattice Leibniz rule conditions. Since the twisted SUSY variations satisfy Eq. (3.26), the only nontrivial variations come from either $s_A^+ S^+$ or $s_A^- S^-$, whose link attributes are given by $(x + \sum n + a_A, x)$ and $(x - \sum n - a_A, x)$, respectively. One observes here that if one takes $a_A = -\sum n$, then the twisted SUSY variation of the action is reduced to that for closed loops.

The twisted SUSY invariance of the action can be explicitly verified by exploiting the above remarks. For example, the s^+ variation of the second term in (4.2) gives

$$\begin{aligned} s^+S^+|_{\text{2nd term}} = \frac{i}{4\pi}\sum_x \text{Tr}\Big[&\frac{1}{3}\epsilon_{\mu\nu\rho}((s^+A_{+\mu})A_{+\nu}A_{+\rho})_{x+\sum n+a,x} \\ &+\frac{1}{3}\epsilon_{\mu\nu\rho}(A_{+\mu}(s^+A_{+\nu})A_{+\rho})_{x+\sum n+a,x} \\ &+\frac{1}{3}\epsilon_{\mu\nu\rho}(A_{+\mu}A_{+\nu}(s^+A_{+\rho}))_{x+\sum n+a,x}\Big], \end{aligned} \tag{4.4}$$

whose link attribute is given by $(x + \sum n + a, x)$. If we take $a = -\sum n$, then each term above is reduced to connected links forming a closed loop. After using the cyclic property of trace under the summation over x and the s^+ transformation law of $A_{+\mu}$, $s^+A_{+\mu} = -[D_{+\mu}, c^+]$, one obtains

$$\begin{aligned} s^+S^+|_{\text{2nd term}} &= -\frac{i}{4\pi}\sum_x \text{Tr}\epsilon_{\mu\nu\rho}([D_{+\mu}, c^+]A_{+\nu}A_{+\rho})_{x,x} \\ &= -\frac{i}{4\pi}\sum_x \text{Tr}\Big[\epsilon_{\mu\nu\rho}([\Delta_{+\mu}, c^+]A_{+\nu}A_{+\rho})_{x,x} \\ &\quad + \epsilon_{\mu\nu\rho}([A_{+\mu}, c^+]A_{+\nu}A_{+\rho})_{x,x}\Big] \\ &= -\frac{i}{4\pi}\sum_x \text{Tr}\epsilon_{\mu\nu\rho}([\Delta_{+\mu}, c^+]A_{+\nu}A_{+\rho})_{x,x}, \end{aligned} \tag{4.5}$$

where from the first to the second line, we just inserted the expression of forward covariant differences, $D_{+\mu} = \Delta_{+\mu} + A_{+\mu}$, while from the second to the third equality, we used the trace property and antisymmetric property of $\epsilon_{\mu\nu\rho}$ to cancel out the second term. Figure 4 depicts the typical configuration of component fields in the s^+ transformed action with the particular choice of $a = -\sum n$. The operation of s^+ plays the role to close the loop. Performing the same procedure for the other terms in (4.2), one can explicitly show that s^+ variations of S^+ give the total difference terms which are vanishing under the summation over x. Furthermore, $s^-S^- = 0$ can also be shown explicitly with the choice of $a = -\sum n$. In a similar manner, we may verify the invariance of the total action (4.1) with respect to each supercharge of $(s^{\pm}, \bar{s}^{\pm}_{\mu}, s^{\pm}_{\mu}, \bar{s}^{\pm})$ under an appropriate choice of a_A:

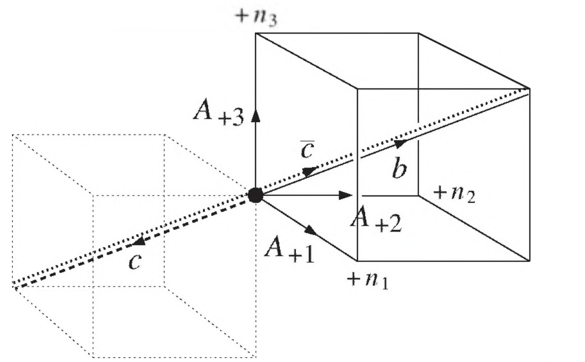

FIG. 3. Configurations of the component fields $(A_{+\mu}, c, \bar{c}, b)$ in the action S^+ for $a = -\sum n$: All the edges of each unit cell are occupied by $A_{+\mu}$.

$$s^{\pm}S^{\text{tot}} = 0, \quad \text{for } a = -\sum n, \tag{4.6}$$

$$\bar{s}^{\pm}_{\mu}S^{\text{tot}} = 0, \quad \text{for } \bar{a}_{\mu} = -\sum n \quad (\mu = 1, 2, 3), \tag{4.7}$$

$$s^{\pm}_{\mu}S^{\text{tot}} = 0, \quad \text{for } a_{\mu} = -\sum n \quad (\mu = 1, 2, 3), \tag{4.8}$$

$$\bar{s}^{\pm}S^{\text{tot}} = 0, \quad \text{for } \bar{a} = -\sum n. \tag{4.9}$$

Notice again that the one-vector arbitrariness associated with the lattice algebra (3.19) and (3.20) has played a fundamental role in the natural realization of the invariance under the full lattice SUSY algebra.

Keeping in accordance with the above invariance of the lattice Chern-Simons action, one may define the twisted SUSY variations δ_A for the component fields as follows:

$$\delta_A(\varphi)_{x+a_\varphi,x} = \overleftarrow{T}_{a_A}\eta_A(s_A\varphi)_{x+a_\varphi+a_A,x} \quad \text{(no sum)}, \tag{4.10}$$

where $(\varphi)_{x+a_\varphi,x}$ denotes any of the component fields $(c^{\pm}, \bar{c}^{\pm}, A_{\pm\mu}, b^{\pm})$. $\overleftarrow{T}_{a_A}$ represents a shift operator acting

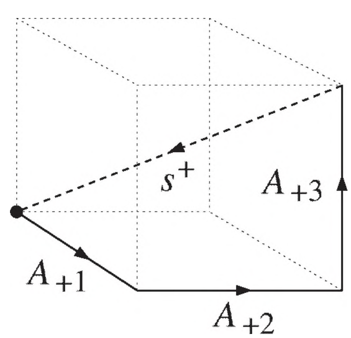

FIG. 4. A typical configuration in the transformed action s^+S^+.

on the functions from the right, $f(x)\overleftarrow{T}_{a_A} = f(x + a_A)$, while η_A represents a constant Grassmann parameter. One can verify the invariance of the total action (4.1) under the above componentwise twisted SUSY variations,

$$S^{\rm tot}[\varphi + \delta_A\varphi] - S^{\rm tot}[\varphi] = 0, \quad \text{for } a_A = -\sum n, \tag{4.11}$$

where φ represents collectively the set of all component fields that appear in the total action. The existence of the shift operator $\overleftarrow{T}_{a_A}$ in the componentwise SUSY variations (4.10) and the notion of the proper ordering in the lattice action seems to imply that the entire lattice SUSY formulation could be embedded in a certain noncommutative (super)space framework, which will be addressed in the future development.

B. Kernels

Another important feature of the lattice Chern-Simons action (4.1), (4.2), and (4.3) is that the kinetic terms of the lattice gauge fields can be expressed in terms of the one-sided difference version of the Fröhlich-Marchetti kernels [11],

$$S^{+}|_{\rm 1st\ term} = \frac{i}{8\pi}\sum_{xy}\mathrm{Tr}(A_{+\mu})_{x+n_\mu,x}\hat{K}_{\mu\nu}(x-y)(A_{+\nu})_{y+n_\nu,y}, \tag{4.12}$$

$$S^{-}|_{\rm 1st\ term} = \frac{i}{8\pi}\sum_{xy}\mathrm{Tr}(A_{-\mu})_{x,x+n_\mu}K_{\mu\nu}(x-y)(A_{-\nu})_{y,y+n_\nu}, \tag{4.13}$$

where the kernels $K(x-y)$ and $\hat{K}(x-y)$ are given by [17]

$$K_{\mu\nu}(x-y) = \vec{T}_{n_\mu}\epsilon_{\mu\rho\nu}\partial_{+\rho}\delta_{xy}, \tag{4.14}$$

$$\hat{K}_{\mu\nu}(x-y) = \vec{T}_{-n_\nu}\epsilon_{\mu\rho\nu}\partial_{-\rho}\delta_{xy}, \tag{4.15}$$

with $\partial_{+\mu}f(x) = f(x+n_\mu) - f(x)$, $\partial_{-\mu}f(x) = f(x) - f(x-n_\mu)$, $\vec{T}_{n_\mu}f(x) = f(x+n_\mu)$, and $\vec{T}_{-n_\mu}f(x) = f(x-n_\mu)$. Since the gauge fields are located on links, the analysis in the momentum space superficially depends on where to pick up their representatives in the configuration space. Fourier transformation of the link gauge fields is given by

$$(A_{+\mu})_{x+n_\mu,x} = \int_B \frac{d^3p}{(2\pi)^3}e^{-ip\cdot(x+\alpha n_\mu)}A_{+\mu}(p), \tag{4.16}$$

$$(A_{-\mu})_{x,x+n_\mu} = \int_B \frac{d^3p}{(2\pi)^3}e^{-ip\cdot(x+\alpha n_\mu)}A_{-\mu}(p), \tag{4.17}$$

where the constant α parametrizes the representative points of the gauge fields. Namely, $\alpha = 0, \frac{1}{2}$, and 1 correspond to the initial point, midpoint, and ending point prescriptions, respectively. B denotes the Brillouin zone: $B = \{p_\mu | -\pi \le p_\mu \le \pi, \mu = 1, 2, 3\}$. $A_{+\mu}(p)$ and $A_{-\mu}(p)$ are related to each other by the complex conjugation, $A_{\pm\mu}(p)^\dagger = -A_{\mp\mu}(-p)$, in order to satisfy the conjugation relation of the gauge fields in the configuration space. Momentum space representation of the kernels (4.14) and (4.15) is accordingly given by

$$K^{(\alpha)}_{\mu\nu}(p) = -2i\epsilon_{\mu\rho\nu}e^{-i((1-\alpha)p_\mu+\alpha p_\nu+(1/2)p_\rho)}\sin\frac{p_\rho}{2}, \tag{4.18}$$

$$\hat{K}^{(\alpha)}_{\mu\nu}(p) = -2i\epsilon_{\mu\rho\nu}e^{+i(\alpha p_\mu+(1-\alpha)p_\nu+(1/2)p_\rho)}\sin\frac{p_\rho}{2}. \tag{4.19}$$

Although the form of the kernels is explicitly dependent on the parameter α, their eigenvalues should be independent of α. In fact, one may easily verify that the eigenvalues of K are given by $\lambda(p) = 0$,

$$\pm 2e^{-(i/2)\sum_{\mu=1}^{3}p_\mu}\sqrt{\sum_{\mu=1}^{3}\sin^2\frac{p_\mu}{2}}.$$

Likewise the eigenvalues of $\hat{K}$ are given by $\hat{\lambda}(p) = 0$,

$$\pm 2e^{+(i/2)\sum_{\mu=1}^{3}p_\mu}\sqrt{\sum_{\mu=1}^{3}\sin^2\frac{p_\mu}{2}},$$

the complex conjugate of $\lambda(p)$. The zero eigenvalue, which arises from the original gauge invariance of the action, should be cured by the gauge-fixing terms. It is important to notice that they do not have any other extra zero eigenvalues, which implies that both (4.18) and (4.19) could serve as the invertible kernels after the gauge-fixing terms are properly taken into account. Notice again that the eigenvalues always come in complex conjugated pairs, ensuring the anti-Hermiticity of the entire formulation. These features are direct consequences of the use of two sets of oppositely oriented component fields on the lattice.

C. Naive continuum limit

The naive continuum limit of the total action is taken by replacing the difference operators by differential operators,

$$\Delta_{\pm\mu} \to \partial_\mu. \tag{4.20}$$

The Hermitian conjugation property of $\Delta_{\pm\mu}$ is accordingly reduced into the anti-Hermiticity of ∂_μ,

$$(\Delta_{\pm\mu})^\dagger = -\Delta_{\mp\mu} \to (\partial_\mu)^\dagger = -\partial_\mu. \tag{4.21}$$

The Hermitian conjugation properties, Eqs. (3.24) and (3.25), of the component fields are supposed to be retained in the continuum. The component fields in the continuum limit are accordingly given by

$$\begin{aligned}(A_{+\mu})_{x+n_\mu,x} &\to A^+_\mu(x) \equiv A_\mu(x) + iB_\mu(x),\\ (A_{-\mu})_{x,x+n_\mu} &\to A^-_\mu(x) \equiv A_\mu(x) - iB_\mu(x),\end{aligned} \tag{4.22}$$

$$\begin{aligned}(c^+)_{x+a,x} &\to c^+(x) \equiv c(x) + id(x),\\ (c^-)_{x,x+a} &\to c^-(x) \equiv c(x) - id(x),\end{aligned} \tag{4.23}$$

$$\begin{aligned}(\bar c^+)_{x+\bar a,x} &\to \bar c^+(x) \equiv \bar c(x) + i\bar d(x),\\ (\bar c^-)_{x,x+\bar a} &\to \bar c^-(x) \equiv \bar c(x) - i\bar d(x),\end{aligned} \tag{4.24}$$

$$\begin{aligned}(b^+)_{x+\sum n,x} &\to b^+(x) \equiv b(x) + ih(x),\\ (b^-)_{x,x+\sum n} &\to b^-(x) \equiv b(x) - ih(x).\end{aligned} \tag{4.25}$$

Here (A_μ, B_μ, b, h), (c, d), and $(\bar c, \bar d)$ denote bosonic anti-Hermitian fields, Grassmann odd anti-Hermitian fields, and Grassmann odd Hermitian fields, respectively. Note that the two possible orientations of the lattice component fields can naturally be interpreted as the complex structure of the gauge group. In terms of the above expansions, the entire action (4.1), (4.2), and (4.3) can be expressed in the continuum limit as

$$\begin{aligned}S^{\rm tot}_{\rm cont} &= k^+S^+_{\rm cont} + k^-S^-_{\rm cont}\\ &= \frac{i}{2\pi}u\int d^3x{\rm Tr}\Big[\frac{1}{2}\epsilon_{\mu\nu\rho}(A_\mu\partial_\nu A_\rho - B_\mu\partial_\nu B_\rho) + \frac{1}{3}\epsilon_{\mu\nu\rho}(A_\mu A_\nu A_\rho - 3A_\mu B_\nu B_\rho) - b\partial_\mu A_\mu + h\partial_\mu B_\mu\\ &\quad - \bar c\partial_\mu(D_\mu c - [B_\mu, d]) + \bar d\partial_\mu(D_\mu d + [B_\mu, c])\Big] - \frac{i}{2\pi}v\int d^3x{\rm Tr}\Big[\frac{1}{2}\epsilon_{\mu\nu\rho}(A_\mu\partial_\nu B_\rho + B_\mu\partial_\nu A_\rho)\\ &\quad + \frac{1}{3}\epsilon_{\mu\nu\rho}(3A_\mu A_\nu B_\rho - B_\mu B_\nu B_\rho) - b\partial_\mu B_\mu - h\partial_\mu A_\mu - \bar d\partial_\mu(D_\mu c - [B_\mu, d]) - \bar c\partial_\mu(D_\mu d + [B_\mu, c])\Big],\end{aligned} \tag{4.26}$$

where the constants u and v are the real and imaginary parts of the complex parameters $k^\pm = u \pm iv$. The covariant derivative D_μ is again defined by $D_\mu c = \partial_\mu c + [A_\mu, c]$. The action (4.26) can be regarded as the Landau-gauge fixed version of the Chern-Simons action with complex gauge group originally proposed in Ref. [34]. Obviously, if one takes $B_\mu = d = \bar d = h = 0$, the entire action (4.26) is reduced into the expression (2.1) with the coefficient $u = k$. In the general case, according to Ref. [34], the parameter u must always be quantized to be

TABLE VI. Twisted SUSY transformation laws in the naive continuum limit for the expanded component fields $(c, d, \bar c, \bar d, A_\mu, B_\mu, b, h)$.

	$s^{(1)}$	$\bar s^{(1)}_\rho$	$s^{(1)}_\rho$	$\bar s^{(1)}$
c	$c^2 - d^2$	$-A_\rho$	0	$-b + \{\bar c, c\} - \{\bar d, d\}$
d	$\{c, d\}$	$-B_\rho$	0	$-h + \{\bar c, d\} + \{\bar d, c\}$
$\bar c$	b	0	A_ρ	$\bar c^2 - \bar d^2$
$\bar d$	h	0	B_ρ	$\{\bar c, \bar d\}$
A_μ	$-D_\mu c + [B_\mu, d]$	$-\epsilon_{\rho\mu\nu}\partial_\nu\bar c$	$-\epsilon_{\rho\mu\nu}\partial_\nu c$	$-D_\mu\bar c + [B_\mu, \bar d]$
B_μ	$-D_\mu d - [B_\mu, c]$	$-\epsilon_{\rho\mu\nu}\partial_\nu\bar d$	$-\epsilon_{\rho\mu\nu}\partial_\nu d$	$-D_\mu\bar d - [B_\mu, \bar c]$
b	0	$\partial_\rho\bar c$	$D_\rho c - [B_\rho, d]$	$[\bar c, b] - [\bar d, h]$
h	0	$\partial_\rho\bar d$	$D_\rho d + [B_\rho, c]$	$[\bar c, h] + [\bar d, b]$

	$s^{(2)}$	$\bar s^{(2)}_\rho$	$s^{(2)}_\rho$	$\bar s^{(2)}$
c	$\{c, d\}$	B_ρ	0	$h - \{\bar c, d\} - \{\bar d, c\}$
d	$-c^2 + d^2$	$-A_\rho$	0	$-b + \{\bar c, c\} - \{\bar d, d\}$
$\bar c$	h	0	B_ρ	$-\{\bar c, \bar d\}$
$\bar d$	$-b$	0	$-A_\rho$	$\bar c^2 - \bar d^2$
A_μ	$-D_\mu d - [B_\mu, c]$	$\epsilon_{\rho\mu\nu}\partial_\nu\bar d$	$-\epsilon_{\rho\mu\nu}\partial_\nu d$	$D_\mu\bar d + [B_\mu, \bar c]$
B_μ	$D_\mu c - [B_\mu, d]$	$-\epsilon_{\rho\mu\nu}\partial_\nu\bar c$	$\epsilon_{\rho\mu\nu}\partial_\nu c$	$-D_\mu\bar c + [B_\mu, \bar d]$
b	0	$-\partial_\rho\bar d$	$D_\rho d + [B_\rho, c]$	$-[\bar c, h] - [\bar d, b]$
h	0	$\partial_\rho\bar c$	$-D_\rho c + [B_\rho, d]$	$[\bar c, b] - [\bar d, h]$

an integer k if the Tr is normalized correctly, while there is no quantization condition for the real parameter v.

The lattice supercharges $(s^{\pm}, \bar{s}^{\pm}_{\mu}, s^{\pm}_{\mu}, \bar{s}^{\pm})$ may also be expanded as

$$s^{\pm} = \tfrac{1}{2}(s^{(1)} \pm is^{(2)}), \qquad \bar{s}^{\pm} = \tfrac{1}{2}(\bar{s}^{(1)} \mp i\bar{s}^{(2)}), \tag{4.27}$$

$$\bar{s}^{\pm}_{\mu} = \tfrac{1}{2}(\bar{s}^{(1)}_{\mu} \mp i\bar{s}^{(2)}_{\mu}), \qquad s^{\pm}_{\mu} = \tfrac{1}{2}(s^{(1)}_{\mu} \pm is^{(2)}_{\mu}), \tag{4.28}$$

with which the naive continuum limit of the lattice SUSY algebra (3.19) and (3.20) is given by

$$\begin{aligned} \{s^{(i)}, \bar{s}^{(j)}_{\mu}\} &\doteq (\delta^{ij} \pm i\epsilon^{ij})\partial_{\mu}, \\ \{s^{(i)}_{\mu}, \bar{s}^{(j)}_{\mu}\} &\doteq (\delta^{ij} \pm i\epsilon^{ij})\epsilon_{\mu\nu\rho}\partial_{\rho}, \end{aligned} \tag{4.29}$$

$$\{\bar{s}^{(i)}, s^{(j)}_{\mu}\} \doteq -(\delta^{ij} \mp i\epsilon^{ij})\partial_{\mu}, \qquad \{\text{others}\} = 0, \tag{4.30}$$

for the continuum-limit multiplet $\varphi^{\pm} = (c^{\pm}, \bar{c}^{\pm}, A^{\pm}_{\mu}, b^{\pm})$, respectively. The suffixes i, j take 1 or 2, and $\epsilon^{12} = -\epsilon^{21} = 1$. The SUSY transformation laws in terms of $(s^{(i)}, \bar{s}^{(i)}_{\mu}, s^{(i)}_{\mu}, \bar{s}^{(i)})$ for the expanded component fields $(A_{\mu}, B_{\mu}, c, d, \bar{c}, \bar{d}, b, h)$ are summarized in Table VI. It is straightforward to verify that the action with the coefficient u and the action with v in (4.26) are separately invariant under the twisted SUSY transformations $(s^{(i)}, \bar{s}^{(i)}_{\mu}, s^{(i)}_{\mu}, \bar{s}^{(i)})$.

V. TRANSFORMATION PROPERTIES UNDER PARITY

The properties under parity transformation are an important issue for continuum Chern-Simons theory. In this section we address this issue for our twisted SUSY Chern-Simons action on a lattice. We first recall that on a Euclidean three-dimensional lattice or spacetime, parity may be defined by the simultaneous inversion of all coordinates (2.11). Since the gauge fields $A_{\pm\mu}$ are located on links $(A_{\pm\mu})_{x\pm n_{\mu},x}$ and the parity also flips the link orientations, one may naturally define the parity operation P for $A_{\pm\mu}$ on the lattice by

$$P(A_{+\mu})_{x+n_{\mu},x}P^{-1} = -(A_{-\mu})_{-x-n_{\mu},-x}, \tag{5.1}$$

where $-x$ denotes $(-x_1, -x_2, -x_3)$. The difference operators are also located on links so that their parity transformation law is

$$P(\Delta_{+\mu})_{x+n_{\mu},x}P^{-1} = -(\Delta_{-\mu})_{-x-n_{\mu},-x}, \tag{5.2}$$

which is also consistent with the fact that $\Delta_{\pm\mu}$ actually take the unit values in the link commutators [see (3.3)]. As for the gauge-fixing component fields and the supercharges, we define

$$\begin{aligned} P(c^{+})_{x+a,x}P^{-1} &= +(c^{-})_{-x-a,-x}, \\ P(\bar{c}^{+})_{x+\bar{a},x}P^{-1} &= -(\bar{c}^{-})_{-x-\bar{a},-x}, \end{aligned} \tag{5.3}$$

$$P(b^{+})_{x+\sum n,x}P^{-1} = -(b^{-})_{-x-\sum n,-x}, \tag{5.4}$$

$$\begin{aligned} P(s^{+})_{x+a,x}P^{-1} &= +(s^{-})_{-x-a,-x}, \\ P(\bar{s}^{+})_{x+\bar{a},x}P^{-1} &= -(\bar{s}^{-})_{-x-\bar{a},-x}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} P(\bar{s}_{\mu})_{x+\bar{a}_{\mu},x}P^{-1} &= -(\bar{s}_{\mu})_{-x-\bar{a}_{\mu},-x}, \\ P(s^{+}_{\mu})_{x+a_{\mu},x}P^{-1} &= +(s^{-}_{\mu})_{-x-a_{\mu},-x}. \end{aligned} \tag{5.6}$$

In the following we will see two interesting features resulting from these definitions. One is regarding the parity of the lattice Chern-Simons action. The other one is the parity property in the continuum limit.

As for the parity transformation of the action, it is easy to see that the definitions (5.1), (5.2), (5.3), and (5.4) interchange the two oppositely oriented parts of the action, S^{+} given by Eq. (4.2) and S^{-} given by Eq. (4.3):

$$\begin{aligned} PS^{+}P^{-1} &= \frac{i}{4\pi}P\sum_x \mathrm{Tr}\Big[\frac{1}{2}\epsilon_{\mu\nu\rho}(A_{+\mu})_{x+\sum n,x+n_{\nu}+n_{\rho}}[\Delta_{+\nu}, A_{+\rho}]_{x+n_{\nu}+n_{\rho},x} + \cdots\Big]P^{-1} \\ &= -\frac{i}{4\pi}\sum_x \mathrm{Tr}\Big[\frac{1}{2}\epsilon_{\mu\nu\rho}(A_{-\mu})_{-x-\sum n,-x-n_{\nu}-n_{\rho}}[\Delta_{-\nu}, A_{-\rho}]_{-x-n_{\rho}-n_{\rho},-x} + \cdots\Big] \\ &= -\frac{i}{4\pi}\sum_x \mathrm{Tr}\Big[\frac{1}{2}\epsilon_{\mu\nu\rho}(A_{-\mu})_{x-\sum n,x-n_{\nu}-n_{\rho}}[\Delta_{-\nu}, A_{-\rho}]_{x-n_{\rho}-n_{\rho},x} + \cdots\Big] = -S^{-}. \end{aligned} \tag{5.7}$$

Here from the second line to the third, we have replaced $x \to -x$. Likewise, we also have $PS^{-}P^{-1} = -S^{+}$. We thus have the parity transformation for the total action S^{tot} (4.1) as

$$PS^{\mathrm{tot}}P^{-1} = -k^{+}S^{-} - k^{-}S^{+}, \tag{5.8}$$

which implies that the total action is not an eigenstate of the parity defined by (5.1), (5.2), (5.3), and (5.4). Writing the complex parameters $k^{\pm}$ as $k^{\pm} = u \pm iv$, we actually have

$$S^{\mathrm{tot}} = u(S^{+} + S^{-}) + iv(S^{+} - S^{-}), \tag{5.9}$$

$$PS^{\mathrm{tot}}P^{-1} = -u(S^{+} + S^{-}) + iv(S^{+} - S^{-}). \tag{5.10}$$

Now it becomes clear that the total action is a sum of a parity even part with the coefficient u and a parity odd part with the coefficient iv.

$$S^{\text{tot}}|_{v=0}\text{: parity odd,} \qquad S^{\text{tot}}|_{u=0}\text{: parity even.} \tag{5.11}$$

One can understand the mixed behavior of the total action under parity more clearly by examining the parity behavior of the component fields in the continuum limit. In fact, by considering the continuum limit (4.22) of the lattice parity operation (5.1), one obtains

$$PA_\mu(x)P^{-1} = -A_\mu(-x), \qquad PB_\mu(x)P^{-1} = +B_\mu(-x), \tag{5.12}$$

which imply that the $A_\mu(x)$ is an ordinary vector, while the $B_\mu(x)$ is a pseudovector. By considering the continuum limit of the relations (5.3), (5.4), (5.5), and (5.6), one also obtains the parity behavior of the other component fields and the supercharges as listed in Table VII. In the language of forms, the complex gauge fields $A^{\pm}_\mu$ may be regarded as complex combinations of a one-form A and a two-form B,

$$A^{\pm}_\mu dx_\mu = A \pm i * B, \tag{5.13}$$

where $A = A_\mu dx_\mu$ and $B = \frac{1}{2}B_{\mu\nu}dx_\mu \wedge dx_\nu$. The symbol $*$ denotes the Hodge star operation. Likewise, the continuum limits of the gauge-fixing component fields $(c^{\pm}, \bar{c}^{\pm}, b^{\pm})$ are divided into the complex combinations of 0-forms and 3-forms. It is interesting to note that our anti-Hermitian lattice formulation together with the twisted SUSY structure actually involves all possible simplicial forms in the three-dimensional spacetime.

The mixed behavior under parity of the continuum action (4.26) is now clearly understood. One can easily see from Table VII that part of the action with the coefficient u is actually parity odd, just like the ordinary Chern-Simons action for a single gauge field, while part of the action with the coefficient v is parity even. The manifestly anti-Hermitian formulation on the lattice thus eventually leads to a unified picture of even and odd parity Chern-Simons theory. It is worthwhile to mention that the parity even part of the continuum action (4.26) shares the same parity behavior as the so-called "dumbbell" Chern-Simons action addressed in [13], where vector and pseudovector gauge fields are introduced as the lattice objects dual to each other. We also note that the parity even part of the continuum action (4.26) shares the same parity behavior with the so-called "doubled" Chern-Simons theory discussed in [8], though the action is actually not the same.

VI. SUMMARY AND DISCUSSIONS

We have constructed the Landau-gauge fixed Chern-Simons theory on a three-dimensional regular lattice. The $N = 4$ $D = 3$ twisted SUSY associated with the Chern-Simons action in Landau gauge has played a crucial role as the guiding principle in the present lattice construction. The one-vector arbitrariness associated with the $N = 4$ $D = 3$ lattice algebra is shown to play an important role in maintaining the twisted SUSY invariance of the lattice action. In order to ensure the manifest anti-Hermiticity on the lattice, we have introduced two sets of oppositely oriented component fields attached to every possible link. Owing to this "doubling" of the lattice component fields, the gauge kernels are shown to be free from the extra zero-eigenvalue problem. We have also addressed the transformation properties under parity of the fields involved in our construction. It was pointed out that a natural definition of parity on the lattice involves component fields of opposite parity. Parity invariance then puts a constraint between the coefficients in front of the actions for the oppositely oriented component fields.

It is important to ask whether one can recover the appropriate $N = 4$ $D = 3$ twisted SUSY Chern-Simons theory in the continuum limit. In particular, whether the continuum rotational symmetry and the entire $N = 4$ $D = 3$ twisted SUSY invariance are restored in the continuum limit is an important issue, worth further study. Discussing these aspects requires a careful examination of possible quantum corrections on the lattice. Here we would like to point out an important correlation between the rotational symmetry and the twisted SUSY invariance of the lattice action (4.6), (4.7), (4.8), and (4.9). Since in our formulation we respect only part of the entire set of SUSY generators, not only the continuous rotational symmetry but also the discrete rotational symmetry (for the square lattice) are broken on the lattice. However, as one can see in Table IV and Fig. 3, the lattice action with the parameter choice $a = -\sum n$, which corresponds to the invariance (4.6), has a symmetry subgroup with a single 3-fold rotation axis, $\mathcal{C}_3 = (E, C_3, C_3^2)$, of the octahedral group $\mathcal{O}$. This is because all the gauge-fixing component fields $(c^{\pm}, \bar{c}^{\pm}, b^{\pm})$ are located on the diagonal link parallel to $a = -\sum n$, while the gauge fields $A_{\pm\mu}$ are located on the regular edges. The lattice action with $\bar{a} = -\sum n$, which corresponds to the invariance (4.9), also has the same symmetry. Figure 5 shows the projected field configurations normal to $a = -\sum n$, where the 3-fold rotational symmetry is manifest. It is interesting to notice that the gauge-fixing component fields $(c, \bar{c}, b)$ and the supercharge s are projected onto a point, which

TABLE VII. Behavior under parity of the component fields and supercharges in the continuum limit.

	A_μ	B_μ	c	d	$\bar{c}$	$\bar{d}$	b	h	$s^{(1)}$	$s^{(2)}$	$\bar{s}^{(1)}_\mu$	$\bar{s}^{(2)}_\mu$	$s^{(1)}_\mu$	$s^{(2)}_\mu$	$\bar{s}^{(1)}$	$\bar{s}^{(2)}$	∂_μ
Parity	$-$	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$-$

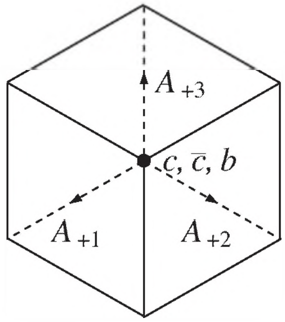

FIG. 5. Field configurations of Fig. 3 on a projected plane normal to $a = -\sum n$: All the edges are occupied by $A_{+\mu}$.

corresponds to the fact that these component fields and the supercharge should behave as (pseudo-)scalars in the continuum limit.

It should be stressed again that our lattice action is a gauge-fixed one. Namely, it is not invariant under gauge symmetry. Instead, it has $N = 4$ $D = 3$ twisted supersymmetry whose scalar transformation corresponds to the BRST transformation associated with the Landau-gauge fixed Chern-Simons theory. Although further study is needed to clarify whether the entire $N = 4$ $D = 3$ twisted SUSY invariance can be properly restored in the continuum limit, the following three important features of our formulation may be explored to argue for the gauge invariance in the continuum limit: (1) The Landau gauge-fixed action (2.1) enables us to make use of the $N = 4$ $D = 3$ twisted SUSY structure in building the lattice action; (2) the remnant of the gauge symmetry in the original Chern-Simons action has turned into the scalar part of the $N = 4$ $D = 3$ twisted SUSY; (3) the *infinitesimal* BRST transformations are preserved on the lattice. Therefore, at least formally, in our gauge-fixed formulation there is no need to be concerned about large gauge transformations, which would be far more difficult to realize directly on the lattice.

It is also important to ask whether the lattice formulation presented in this paper could really serve as a useful regularization scheme; namely, whether the quantum aspects such as the Chern-Simons coefficient renormalization [40] could be calculated in this framework. We should also address the possibility that the entire lattice SUSY description presented in this paper could be formulated more rigidly in terms of a certain noncommutative (super)space formalism. The work is in progress.

Another interesting question for possible applications in physics is whether there exists a real or model system in condensed matter physics that has a topological phase described by the Chern-Simons action with complex gauge group.

ACKNOWLEDGMENTS

K. N. would like to thank N. Kawamoto for discussions and comments. K. N. was supported by the Department of Energy U.S. Government, Grant No. FG02-91ER 40661. Y. S. W. was supported in part by the U.S. NSF through Grant No. PHY-0457018.

[1] W. Siegel, Phys. Lett. **84B**, 193 (1979).
[2] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982); **185**, 406 (E) (1988).
[3] S. S. Chern and J. H. Simons, Ann. Math. **99**, 48 (1974).
[4] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[5] V. F. R. Jones, Bull. Am. Math. Soc. **12**, 103 (1985).
[6] X. G. Wen, Adv. Phys. **44**, 405 (1995).
[7] E. Witten, Nucl. Phys. **B311**, 46 (1988).
[8] M. Freedman, C. Nayak, K. Shtengel, K. Walker, and Z. H. Wang, Ann. Phys. (N.Y.) **310**, 428 (2004).
[9] S. Das Sarma, M. Freedman, C. Nayak, S. H. Simon, and A. Stern, arXiv:0707.1889 [Rev. Mod. Phys. (to be published)].
[10] R. Jackiw, in *Proceedings of the XXIII International Conference of Differential Geometric Methods in Theoretical Physics* (Chern Institute of Mathematics, Tianjin, China, 2005), pp. 20–26.
[11] J. Fröhlich and P. A. Marchetti, Commun. Math. Phys. **121**, 177 (1989).
[12] M. Luscher, Nucl. Phys. **B326**, 557 (1989); V. F. Muller, Z. Phys. C **47**, 301 (1990).
[13] R. Kantor and L. Susskind, Nucl. Phys. **B366**, 533 (1991); D. Eliezer and G. W. Semenoff, Ann. Phys. (N.Y.) **217**, 66 (1992); Phys. Lett. B **286**, 118 (1992); **266**, 375 (1991); D. Eliezer, G. W. Semenoff, and S. S. C. Wu, Mod. Phys. Lett. A **7**, 513 (1992).
[14] M. C. Diamantini, P. Sodano, and C. A. Trugenberger, Phys. Rev. Lett. **75**, 3517 (1995); **71**, 1969 (1993).
[15] S. Sen, S. Sen, J. C. Sexton, and D. H. Adams, Phys. Rev. E **61**, 3174 (2000).
[16] N. Kawamoto, H. B. Nielsen, and N. Sato, Nucl. Phys. **B555**, 629 (1999); N. Kawamoto, N. Sato, and Y. Uchida, Nucl. Phys. **B574**, 809 (2000).
[17] F. Berruto, M. C. Diamantini, and P. Sodano, Phys. Lett. B **487**, 366 (2000).
[18] C. D. Fosco and A. Lopez, Phys. Rev. D **64**, 025017 (2001); W. Bietenholz, J. Nishimura, and P. Sodano, Nucl. Phys. B, Proc. Suppl. **119**, 935 (2003); W. Bietenholz and P. Sodano, arXiv:hep-lat/0305006.
[19] D. Birmingham, M. Rakowski, and G. Thompson, Nucl. Phys. **B329**, 83 (1990).
[20] D. Birmingham and M. Rakowski, Mod. Phys. Lett. A **4**, 1753 (1989).

[21] F. Delduc, F. Gieres, and S. P. Sorella, Phys. Lett. B **225**, 367 (1989).
[22] E. Witten, Commun. Math. Phys. **117**, 353 (1988).
[23] F. Delduc, C. Lucchesi, O. Piguet, and S. P. Sorella, Nucl. Phys. **B346**, 313 (1990); N. Dorey, Phys. Lett. B **246**, 87 (1990); D. Daniel and N. Dorey, Phys. Lett. B **246**, 82 (1990).
[24] P. H. Damgaard and V. O. Rivelles, Phys. Lett. B **245**, 48 (1990); E. Guadagnini, N. Maggiore, and S. P. Sorella, Phys. Lett. B **247**, 543 (1990); D. Birmingham and M. Rakowski, Phys. Lett. B **269**, 103 (1991); C. Lucchesi and O. Piguet, Nucl. Phys. **B381**, 281 (1992).
[25] A. D'Adda, I. Kanamori, N. Kawamoto, and K. Nagata, Nucl. Phys. **B707**, 100 (2005); Nucl. Phys. B, Proc. Suppl. **140**, 754 (2005); **140**, 757 (2005).
[26] A. D'Adda, I. Kanamori, N. Kawamoto, and K. Nagata, Phys. Lett. B **633**, 645 (2006).
[27] A. D'Adda, I. Kanamori, N. Kawamoto, and K. Nagata, Nucl. Phys. **B798**, 168 (2008); Proc. Sci., LAT2007 (2007) 271 [arXiv:0709.0722].
[28] K. Nagata, J. High Energy Phys. 01 (2008) 041.
[29] S. Catterall and S. Karamov, Phys. Rev. D **65**, 094501 (2002); **68**, 014503 (2003); S. Catterall, J. High Energy Phys. 05 (2003) 038; S. Catterall and S. Ghadab, J. High Energy Phys. 05 (2004) 044; 10 (2006) 063; S. Catterall, J. High Energy Phys. 11 (2004) 006; 06 (2005) 027; 03 (2006) 032; 04 (2007) 015; S. Catterall and T. Wiseman, J. High Energy Phys. 12 (2007) 104; Proc. Sci., LAT2007 (2007) 051 [arXiv:0709.3497]; S. Catterall and A. Joseph, arXiv:0712.3074; M. Hanada, J. Nishimura, and S. Takeuchi, Phys. Rev. Lett. **99**, 161602 (2007); K. N. Anagnostopoulos, M. Hanada, J. Nishimura, and S. Takeuchi, Phys. Rev. Lett. **100**, 021601 (2008); Proc. Sci., LAT2007 (2007) 059 [arXiv:0801.4205]; F. Sugino, J. High Energy Phys. 01 (2004) 015; 03 (2004) 067; 01 (2005) 016; Phys. Lett. B **635**, 218 (2006); I. Kanamori, F. Sugino, and H. Suzuki, arXiv:0711.2132; Phys. Rev. D **77**, 091502 (2008); K. Ohta and T. Takimi, Prog. Theor. Phys. **117**, 317 (2007); F. Bruckmann and M. de Kok, Phys. Rev. D **73**, 074511 (2006); F. Bruckmann, S. Catterall, and M. de Kok, Phys. Rev. D **75**, 045016 (2007); S. Arianos, A. D'Adda, N. Kawamoto, and J. Saito, Proc. Sci., LAT2007 (2007) 259 [arXiv:0710.0487].
[30] N. Kawamoto and T. Tsukioka, Phys. Rev. D **61**, 105009 (2000); J. Kato, N. Kawamoto, and Y. Uchida, Int. J. Mod. Phys. A **19**, 2149 (2004); J. Kato, N. Kawamoto, and A. Miyake, Nucl. Phys. **B721**, 229 (2005).
[31] W. Bietenholz, Mod. Phys. Lett. A **14**, 51 (1999); K. Fujikawa and M. Ishibashi, Nucl. Phys. **B622**, 115 (2002); Phys. Lett. B **528**, 295 (2002); Y. Kikukawa and Y. Nakayama, Phys. Rev. D **66**, 094508 (2002); K. Fujikawa, Phys. Rev. D **66**, 074510 (2002); Nucl. Phys. **B636**, 80 (2002); M. Bonini and A. Feo, J. High Energy Phys. 09 (2004) 011; Phys. Rev. D **71**, 114512 (2005); J. W. Elliott and G. D. Moore, J. High Energy Phys. 11 (2005) 010; 11 (2007) 067; K. Itoh, M. Kato, H. Sawanaka, H. So, and N. Ukita, J. High Energy Phys. 02 (2003) 033; Prog. Theor. Phys. **108**, 363 (2002); A. Feo, Nucl. Phys. B, Proc. Suppl. **119**, 198 (2003); arXiv: hep-lat/0311037; Mod. Phys. Lett. A **19**, 2387 (2004); H. Suzuki and Y. Taniguchi, J. High Energy Phys. 10 (2005) 082; H. Suzuki, J. High Energy Phys. 09 (2007) 052; H. Fukaya, I. Kanamori, H. Suzuki, and T. Takimi, Proc. Sci., LAT2007 (2007) 264 [arXiv:0709.4076]; Y. Kikukawa and H. Suzuki, J. High Energy Phys. 02 (2005) 012; M. Harada and S. Pinsky, Phys. Rev. D **71**, 065013 (2005).
[32] D. B. Kaplan, E. Katz, and M. Ünsal, J. High Energy Phys. 05 (2003) 037; A. G. Cohen, D. B. Kaplan, E. Katz, and M. Ünsal, J. High Energy Phys. 08 (2003) 024; 12 (2003) 031; J. Nishimura, S. J. Rey, and F. Sugino, J. High Energy Phys. 02 (2003) 032; J. Giedt, E. Poppitz, and M. Rozali, J. High Energy Phys. 03 (2003) 035; J. Giedt, Nucl. Phys. **B668**, 138 (2003); **B674**, 259 (2003); Int. J. Mod. Phys. A **21**, 3039 (2006); D. B. Kaplan and M. Unsal, J. High Energy Phys. 09 (2005) 042; M. Unsal, J. High Energy Phys. 11 (2005) 013; 04 (2006) 002; T. Onogi and T. Takimi, Phys. Rev. D **72**, 074504 (2005); M. G. Endres and D. B. Kaplan, J. High Energy Phys. 10 (2006) 076; P. H. Damgaard and S. Matsuura, J. High Energy Phys. 07 (2007) 051; Phys. Lett. B **661**, 52 (2008); S. Matsuura, J. High Energy Phys. 12 (2007) 048.
[33] M. Ünsal, J. High Energy Phys. 10 (2006) 089; T. Takimi, J. High Energy Phys. 07 (2007) 010; P. H. Damgaard and S. Matsuura, J. High Energy Phys. 08 (2007) 087; 09 (2007) 097; S. Catterall, J. High Energy Phys. 01 (2008) 048.
[34] E. Witten, Commun. Math. Phys. **137**, 29 (1991).
[35] M. Blau and G. Thompson, Nucl. Phys. **B492**, 545 (1997).
[36] B. Geyer and D. Mulsch, Nucl. Phys. **B616**, 476 (2001).
[37] I. Kanamori and N. Kawamoto, Int. J. Mod. Phys. A **19**, 695 (2004); Nucl. Phys. B, Proc. Suppl. **129**, 877 (2004).
[38] A. N. Jourjine, Phys. Rev. D **34**, 3058 (1986); **35**, 2983 (1987).
[39] V. de Beauce, Proc. Sci., LAT2005 (2006) 276 [arXiv:hep-lat/0510028]; V. de Beauce and S. Sen, arXiv:hep-th/0610065.
[40] E. Guadagnini, M. Martellini, and M. Mintchev, Phys. Lett. B **227**, 111 (1989); L. Alvarez-Gaume, J. M. F. Labastida, and A. V. Ramallo, Nucl. Phys. **B334**, 103 (1990); W. Chen, G. W. Semenoff, and Y. S. Wu, Mod. Phys. Lett. A **5**, 1833 (1990); Phys. Rev. D **44**, R1625 (1991); G. Giavarini, C. P. Martin, and F. Ruiz Ruiz, Phys. Lett. B **314**, 328 (1993); **332**, 345 (1994); M. Asorey, F. Falceto, J. L. Lopez, and G. Luzon, Phys. Rev. D **49**, 5377 (1994); Nucl. Phys. **B429**, 344 (1994); K. i. Nittoh and T. Ebihara, Mod. Phys. Lett. A **13**, 2231 (1998).