Theory of the Anomalous Skin Effect in Normal and Superconducting Metals*

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Chambers' expression for the current density in a normal metal in which the electric field varies over a mean free path is derived from a quantum approach in which use is made of the density matrix in the presence of scattering centers but in the absence of the field. An approximate expression used for the latter is shown to reduce to one derived by Kohn and Luttinger for the case of weak scattering. A general space- and time-varying electromagnetic interaction is treated by first-order perturbation theory. The method is applied to superconductors, and a general expression derived for the kernel of the Pippard integral for fields of arbitrary frequency. The expressions derived can also be used to discuss absorption of electromagnetic radiation in thin superconducting films.

1. INTRODUCTION

FOR calculations of the anomalous skin effect in metals and for related problems, one is interested in the response of conduction electrons to high-frequency fields which vary in amplitude over a mean free path, so that the usual expression for current density in terms of bulk conductivity is invalid. A convenient method, which has been successfully applied to several problems, is to start with an infinite medium, so that boundary effects do not come in. Sources of the field are introduced into the interior, and the response of the electrons to the field is calculated by time-dependent perturbation theory. The total field acting on the electrons, the sum of the external field and that due to the electrons themselves, is taken to be self-consistent.[1,2]

Most earlier calculations of the anomalous skin effect in normal metals have been based on the Boltzmann equation. A particularly simple and elegant derivation is that due to Chambers.[3] A quantum-mechanical derivation similar to the present one has been given by Mattis and Dresselhaus[4] who obtain exactly Chambers' result. Our treatment differs in the way in which scattering of the electrons is introduced but also leads to the same result. A main purpose is to formulate the theory in a way that can readily be extended to superconductors. In the low-frequency limit and in the absence of scattering, our method reduces to the derivation of the Meissner effect as given by Bardeen, Cooper, and Schrieffer.[5]

While in general there are difficulties involved in applying a solution derived for an infinite medium to a finite body of arbitrary shape, there is no problem for the case of most practical importance, that of a plane boundary. Such a solution may also be used to discuss conduction and absorption of radiation in thin films.

There has been considerable interest recently in quantum derivations of conductivity from the density matrix formalism. Kubo[6] has given a formal solution in which it is assumed that scattering is present in the zero-order Hamiltonian. Kohn and Luttinger[7] and also Nakano[8] have treated both scattering and the external field as perturbations, and have shown how Boltzmann's equation appears in a certain approximation. The connection between the quantum formulation and Boltzmann's equation has also been discussed by Lax.[9]

Our treatment follows Kubo and Lax in that we assume that scattering occurs in zero order. We do not use the density matrix formulation directly, but expand to first order in perturbation theory the wave functions appropriate to an applied external field, in terms of those in the absence of the field. This requires some knowledge about the solutions of the wave equation, $\psi_k(\mathbf{r})$, in the presence of the scattering centers but in zero field. What is required for the current is the density matrix for an energy shell; that is, $\langle\psi_k^*(\mathbf{r})\psi_k(\mathbf{r}')\rangle$ averaged over states of the same energy, ϵ_k, and over random distributions of scattering centers. We do not derive an expression for this quantity, but assume a form based on plausible arguments. A similar method was used by one of the authors[2] in a discussion of the effect of a finite mean free path from elastic scattering on the superconducting penetration depth.

We start with the time-dependent Schrödinger equation,

$$H_0\Phi+H_{ex}\Phi=i\hbar\partial\Phi/\partial t, \tag{1.1}$$

in which H_0 is the Hamiltonian in the absence of external fields, but including any scattering which may

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be present, and H_{ex} is the total electromagnetic interaction.

If H_{ex} is expanded in a Fourier series in time, an arbitrary function of space and time may be expressed in the form

$$H_{ex}=e^{st}\sum_{\omega,j}H_\omega(\mathbf{r}_j)e^{i\omega t}. \tag{1.2}$$

The sum over j is over all electrons. Reality requires that $H_{-\omega}=H_\omega^*$. The introduction of s, a small positive parameter to be set equal to zero in the final expressions, insures that the field started from zero in the remote past and sets the direction of time for irreversible phenomena. Mathematically, s determines how the integration over energy denominators is to be carried out.

It is most convenient to choose a gauge such that the vector potential alone represents the applied electromagnetic fields. The portion of the interaction Hamiltonian which is linear in the fields is then

$$H_{ex}=+\frac{e}{mc}\sum_i \mathbf{A}(\mathbf{r}_i,t)\cdot\mathbf{p}_i, \tag{1.3}$$

(we shall consistently neglect terms quadratic in the fields), where $\mathbf{A}$ is subject to the gauge condition

$$\nabla\cdot\mathbf{A}=0, \tag{1.4}$$

which expresses mathematically the physical condition that no external charges be introduced into the sample and that electrical neutrality be maintained throughout. The fields are as usual the derivatives

$$\mathcal{E}=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \quad\text{and}\quad \mathbf{H}=\nabla\times\mathbf{A}. \tag{1.5}$$

We are concerned in the anomalous skin effect only with transverse fields. For these, in the gauge $\nabla\cdot\mathbf{A}=0$, we need to introduce in the perturbation-theory expansion only particle-like excitations of the superconductor. In an arbitrary gauge, collective excitations would have to be included. Further, the expression for the density matrix required for both normal and superconducting metals appears to be simpler in this gauge.

The procedure is to expand Φ in a series of the time-independent eigenstates, φ_j, of H_0:

$$H_0\varphi_j=W_j\varphi_j, \tag{1.6}$$

so that if the unperturbed state is φ_0, then

$$\Phi=\exp(-iW_0t/\hbar)\varphi_0+\sum_{j\neq0}a_j(t)\exp(-iW_jt/\hbar)\varphi_j. \tag{1.7}$$

The expression for the expansion coefficient is

$$a_j(t)=\frac{(j|H_\omega|0)\exp[i(\omega-is)t]}{W_0-W_j-\hbar(\omega-is)}. \tag{1.8}$$

The wave function Φ is then used to calculate the current density.

2. ANOMALOUS SKIN EFFECT IN NORMAL METALS

We take a simple model for which the single-particle states in the absence of scattering are plane waves designated by the wave vector $\mathbf{k}$ and normalized to unit volume. It is assumed that the electrons move independently and the exclusion principle is taken into account only in the statistics. The exact one-particle wave functions, $\psi_\mathbf{k}(\mathbf{r})$, in the presence of elastic scattering are made up in large part of linear combinations of plane waves of approximately the same energy. Here $\mathbf{k}$ is a quantum number which designates the common energy, $\epsilon_\mathbf{k}=\hbar^2k^2/2m$, but is without significance as a wave vector. If the scattering is not too strong, the energy of the states will not be changed much by the presence of the scattering centers. The probability of occupancy of a state $\psi_\mathbf{k}$ of energy $\epsilon_\mathbf{k}$ is given, in the absence of an external field, by the Fermi-Dirac function $f(\epsilon_\mathbf{k})$.

Wave functions, $\Psi_\mathbf{k}$, in the presence of an external field may be expanded in a series of the unperturbed wave functions $\psi_\mathbf{k}$. To the first order in H_ω,

$$\Psi_\mathbf{k}(\mathbf{r},t)=\{\psi_\mathbf{k}(\mathbf{r})+\textstyle\sum_{\mathbf{k}'}a_{\mathbf{k}'\mathbf{k}}(t)\psi_{\mathbf{k}'}(\mathbf{r})\}\times\exp(-i\epsilon_\mathbf{k}t/\hbar), \tag{2.1}$$

where the coefficients are

$$a_{\mathbf{k}'\mathbf{k}}(t)=e^{st}\sum_\omega\frac{(\mathbf{k}'|H_\omega|\mathbf{k})e^{i\omega t}}{\epsilon_\mathbf{k}-\epsilon_{\mathbf{k}'}-\hbar(\omega-is)}, \tag{2.2a}$$

$$a_{\mathbf{k}\mathbf{k}'}^*(t)=-e^{st}\sum_\omega\frac{(\mathbf{k}'|H_\omega|\mathbf{k})e^{i\omega t}}{\epsilon_\mathbf{k}-\epsilon_{\mathbf{k}'}+\hbar(\omega-is)}. \tag{2.2b}$$

In the latter we have made use of $H_\omega^*=H_{-\omega}$.

The expression for the current density is

$$\begin{aligned}\mathbf{j}(\mathbf{r},t)&=-\frac{e\hbar}{2mi}\sum_\mathbf{k}f_\mathbf{k}(\Psi_\mathbf{k}^*\nabla\Psi_\mathbf{k}-\Psi_\mathbf{k}\nabla\Psi_\mathbf{k}^*)\\&\quad-\frac{e^2}{mc}\sum_\mathbf{k}f_\mathbf{k}\mathbf{A}(\mathbf{r},t)|\Psi_\mathbf{k}(\mathbf{r},t)|^2\\&=\frac{-e}{2mi}\sum_{\mathbf{k},\mathbf{k}'}f_\mathbf{k}(\psi_\mathbf{k}^*\nabla\psi_{\mathbf{k}'}a_{\mathbf{k}'\mathbf{k}}+a_{\mathbf{k}'\mathbf{k}}^*\psi_{\mathbf{k}'}^*\nabla\psi_\mathbf{k})\\&\qquad-\text{comp. conj.}-\frac{ne^2}{mc}\mathbf{A}(\mathbf{r},t),\end{aligned} \tag{2.3}$$

where $f_\mathbf{k}$ is the probability that $\mathbf{k}$ is occupied and $n=\sum f_\mathbf{k}\psi_\mathbf{k}^*\psi_\mathbf{k}$ is the density of electrons, assumed constant. We have included in (2.3) only terms to the first order in applied fields and have supposed (as must be true in thermal equilibrium) that the current vanishes in the absence of the field.

A typical term in the expression for the current

density contains products of the form $\psi_{\mathbf{k}}^*(\mathbf{r})\nabla\psi_{\mathbf{k}'}(\mathbf{r})\times\psi_{\mathbf{k}'}^*(\mathbf{r}')\nabla'\psi_{\mathbf{k}}(\mathbf{r}')$ multiplied by factors dependent on the energies $\epsilon_{\mathbf{k}}$ and $\epsilon_{\mathbf{k}'}$. If we first sum over terms for fixed $\epsilon_{\mathbf{k}}$ and $\epsilon_{\mathbf{k}'}$, we need the average over an energy shell of

$$\rho_\epsilon=\langle\psi_{\mathbf{k}}^*(\mathbf{r})\psi_{\mathbf{k}}(\mathbf{r}')\rangle_{\text{Av}}. \tag{2.4}$$

For plane waves, we average over the directions of the wave vector $\mathbf{k}$ and find

$$\rho_\epsilon=(\sin kR)/kR, \tag{2.5}$$

where $R=|\mathbf{r}-\mathbf{r}'|$ and $k=|\mathbf{k}|$.

For a particular set of scattering centers, ρ_ϵ is a complicated function of both $\mathbf{r}$ and $\mathbf{r}'$. What we want is an average of ρ_ϵ over random distributions of scattering centers, which would be expected to depend only on the distance R between $\mathbf{r}$ and $\mathbf{r}'$. This amounts to the neglect of off-diagonal components of ρ_ϵ in the momentum representation, which are zero only on the average. Arguments given in reference 2 suggest that the appropriate average for elastic scattering describable by a mean free path $l(\epsilon)$ is[10]

$$\rho_\epsilon(R)=\frac{\sin kR}{kR}e^{-R/2l}. \tag{2.6}$$

In a metal, the significant values of k and l are those for the Fermi surface, $k=k_{\text{F}}$.

Some remarks should be made concerning the limitations of (2.6): (1) It may be expected to be valid only if the mean free path is large compared with the interatomic distance, or if $k_{\text{F}}l\gg1$. (2) It should be used only when the charge density is uniform, as would not be the case for an electrostatic potential varying in space. (3) It is not gauge invariant. A change of gauge would multiply the individual wave functions by $\exp[ie\varphi(\mathbf{r})/\hbar c]$ and ρ_ϵ by $\exp\{ie[\varphi(\mathbf{r}')-\varphi(\mathbf{r})]/\hbar c\}$; which in general is not a function of R alone. (4) The expression is an average over a random distribution of scattering centers and so is not a solution of Schrödinger's equation as a true density matrix must be. (5) Isotropic scattering is implied. (6) When it is used to evaluate (2.3), the additional assumption is made that the average of the product of two ρ_ϵ's for different energies is the product of the averages.

From the above discussion, we expect (2.6) to be valid only when the gauge is chosen so that $\text{div}\mathbf{A}=0$. It can be shown (Appendix A) to be correct in the limit of a low density of weak scattering centers, for which l can be computed by conventional scattering theory. It is a simple form which has the right general features to describe scattering for more general cases which would be difficult to treat accurately. A further advantage is that it can be used to discuss superconducting as well as normal metals. A more complete mathematical derivation and justification from first principles would, of course, be highly desirable.

With ρ_ϵ defined by (2.3), the expression (1.11) for the current density becomes

$$\mathbf{j}(\mathbf{r},t)=\lim_{s\to0}\sum_\omega e^{i\omega t}$$
$$\times\left\{-\frac{ne^2}{mc}\mathbf{A}_\omega(\mathbf{r})+\frac{e^2\hbar^2}{m^2c}\frac{(4\pi)^2}{(2\pi)^6}\int_0^\infty dk\,k^2\int_0^\infty dk'\,k'^2\right.$$
$$\times\int\frac{d\mathbf{r}'[\mathbf{A}_\omega(\mathbf{r}')\cdot\mathbf{R}]\mathbf{R}}{R^2}\left(\frac{\partial\rho_\epsilon(R)}{\partial R}\right)\left(\frac{\partial\rho_{\epsilon'}(R)}{\partial R}\right)$$
$$\left.\times\left[\frac{f(\epsilon')-f(\epsilon)}{\epsilon-\epsilon'+\hbar(\omega-is)}+\frac{f(\epsilon')-f(\epsilon)}{\epsilon-\epsilon'-\hbar(\omega-is)}\right]\right\}. \tag{2.7}$$

The evaluation of (2.7) for the limit $\omega\to0$ has been given by one of the authors[2] who found that it leads to Landau's expression for the diamagnetism of free electrons. Small corrections which appear, of order $(k_Fl)^{-2}$, are of doubtful validity because of the limitations on (2.6). The conclusion is that, except for possible corrections of this order, scattering does not affect the diamagnetic properties of normal metals.

The integration in (2.7) can be carried out most readily by first performing that with $f(\epsilon)$ over k' and that with $f(\epsilon')$ over k. We need the derivative with respect to R of

$$I_1=\int_0^\infty\left[\frac{1}{\epsilon-\epsilon'+\hbar(\omega-is)}+\frac{1}{\epsilon-\epsilon'-\hbar(\omega-is)}\right]k'^2\rho_{\epsilon'}(R)dk'$$
$$=-\frac{\pi}{2R}\left(\frac{2m}{\hbar^2}\right)\exp(-R/2l)\{\exp[iR(k^2-2\hbar^{-1}m\omega)^{\frac{1}{2}}]$$
$$+\exp[-iR(k^2+2\hbar^{-1}m\omega)^{\frac{1}{2}}]\}, \tag{2.8}$$

provided that $\hbar^2k^2/2m\geq\hbar\omega$. Since most of the contribution from the integral over k is for $\hbar^2k^2/2m\gg\hbar\omega$, we may expand the square roots and keep terms linear in ω as follows:

$$\int_0^\infty f(\epsilon)\frac{\partial\rho_\epsilon}{\partial R}\frac{\partial I_1}{\partial R}k^2dk$$
$$=-\frac{2m\pi}{\hbar^2R^2}e^{-R/l}\int_0^{k_F}dk\exp(-i\omega mR/\hbar k)$$
$$\times\left\{\frac{i\omega mk}{\hbar}\cos^2(kR)+k^2\left(\frac{1}{R}+\frac{1}{2l}\right)\cos(2kR)\right.$$
$$+\tfrac{1}{2}k^3\sin(2kR)\left[1-\frac{1}{k^2}\left(\frac{1}{R}+\frac{1}{2l}\right)\right.$$
$$\left.\left.\times\left(\frac{i\omega m}{\hbar k}+\frac{1}{R}+\frac{1}{2l}\right)\right]\right\}. \tag{2.9}$$

[10] Since the above was written, we have been informed of a derivation by P. R. Weiss and E. Abrahams [Phys. Rev. (to be published)], who show that (2.6) follows approximately from a perturbation expansion. It has also been called to our attention by M. Lax that (2.6) follows from the optical model for treatment of multiple scattering [see M. Lax, Revs. Modern Phys. **23**, 287 (1951), Eqs. (6.35) and (5.29)].

The first term in curly brackets, which gives the part in phase with the electric field and thus the resistive contribution, will be discussed later. The other terms, together with the gauge current proportional to $\mathbf{A}$, give the diamagnetic contribution. Since these terms have $\cos(2kR)$ or $\sin(2kR)$ as factors, they are rapidly oscillating for $kR>1$, and so are important only for R very small, of the order of the interatomic distance. For such small R, we may expand $\mathbf{A}_\omega(\mathbf{r}')$ in a series about $\mathbf{r}$:

$$\mathbf{A}_\omega(\mathbf{r}')=\mathbf{A}_\omega(\mathbf{r})+\mathbf{R}\cdot\nabla\mathbf{A}_\omega(\mathbf{r}) +\tfrac{1}{2}(\mathbf{R}\cdot\nabla)^2\mathbf{A}_\omega(\mathbf{r})+\cdots. \quad (2.10)$$

When the integration over R is carried out, which may be done before the final integration over k, it is found that the constant term, proportional to $\mathbf{A}_\omega(\mathbf{r})$, exactly cancels the gauge current, $-(ne^2/mc)\mathbf{A}_\omega(\mathbf{r})$. The second term averages to zero and the third gives the Landau diamagnetic contribution to the current density.[2]

A different method[11] is more convenient for evaluation of the resistive contribution. If in place of the gauge current we subtract from the integrand in (2.7) the corresponding expression for $\omega=0$, we get as a factor

$$\tfrac{1}{2}[f(\epsilon')-f(\epsilon)]\left\{\frac{1}{\epsilon-\epsilon'+\hbar(\omega-is)} +\frac{1}{\epsilon-\epsilon'-\hbar(\omega-is)}-\frac{2}{\epsilon-\epsilon'}\right\}, \quad (2.11)$$

which is appreciable only for energies within $\sim\hbar\omega$ or $\sim kT$ of the Fermi surface. Further, because of the antisymmetry of $\{\cdots\}$ in ϵ and ϵ', we may replace $f(\epsilon')-f(\epsilon)$ by $-2f(\epsilon)$ if we are careful to treat ϵ and ϵ' symmetrically in the evaluation of semiconvergent integrals. For this purpose, it is convenient to let ϵ be the energy measured from the Fermi surface, to introduce the convergence factor

$$\frac{a^2}{\epsilon^2+\epsilon'^2+a^2}, \quad (2.12)$$

and then take the limit as $a\to\infty$.

Near the Fermi surface, we may approximate the factors from the density matrices as follows:

$$\frac{\partial\rho_\epsilon}{\partial R}\frac{\partial\rho_{\epsilon'}}{\partial R}=\frac{e^{-R/l}}{kk'R^4}\left[kR\cos kR-\left(1+\frac{R}{2l}\right)\sin kR\right] \times\left[k'R\cos k'R-\left(1+\frac{R}{2l}\right)\sin k'R\right] \simeq\frac{e^{-R/l}}{2R^2}\cos[(k-k')R]=\frac{e^{-R/l}}{2R^2}\cos[\alpha(\epsilon-\epsilon')], \quad (2.13)$$

[11] This is similar to that used for the calculation of the Meissner effect (reference 5).

where $\alpha=(dk/d\epsilon)_F R$. We have omitted rapidly oscillating terms and neglected terms of order $(1/k_F R)^2$ or $(1/k_F l)^2$. Further, we replace integrations over k by integrations over ϵ as follows:

$$k^2dk=2\pi^2N(0)d\epsilon, \quad (2.14)$$

where $N(0)$ is the density of states of one spin in energy at the Fermi surface:

$$2\pi^2N(0)=k_F^2(dk/d\epsilon)_F. \quad (2.15)$$

We then need to evaluate the integral

$$I_2=-\lim_{\substack{a\to\infty\\ s\to 0}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{a^2f(\epsilon)\cos[\alpha(\epsilon-\epsilon')]}{\epsilon^2+\epsilon'^2+a^2} \times\left\{\frac{1}{\epsilon-\epsilon'+\hbar(\omega-is)}+\frac{1}{\epsilon-\epsilon'-\hbar(\omega-is)}\right\}d\epsilon d\epsilon'. \quad (2.16)$$

The integration over ϵ' can be carried out by contour integration. The contribution from poles at $\epsilon'=\pm i(\epsilon^2+a^2)^{\frac{1}{2}}$ vanish in the limit $a\to\infty$. Poles at $\epsilon'=\epsilon\pm\hbar(\omega-is)$ give

$$I_2=-\lim_{a\to\infty}\pi i\int_{-\infty}^{\infty}f(\epsilon)\exp(-i\alpha\hbar\omega) \times\left\{\frac{a^2}{a^2+\epsilon^2+(\epsilon+\hbar\omega)^2}-\frac{a^2}{a^2+\epsilon^2+(\epsilon-\hbar\omega)^2}\right\}d\epsilon \quad (2.17)$$

$$=-\pi i\lim_{a\to\infty}\exp(-i\alpha\hbar\omega)\int_{-\infty}^{\infty}\frac{a^2[f(\epsilon)-f(\epsilon+\hbar\omega)]}{a^2+\epsilon^2+(\epsilon+\hbar\omega)^2}d\epsilon =-\pi i\hbar\omega e^{-R\omega/v_0}, \quad (2.18)$$

where we have introduced the velocity at the Fermi surface, $v_0=\hbar^{-1}(d\epsilon/dk)$. The third term, $2/(\epsilon-\epsilon')$, in (2.11) gives a vanishing contribution for finite R, but serves to cancel the δ function at $R=0$ from the first two terms.

We thus arrive at Chambers' expression[3] for the current density:

$$\mathbf{j}(\mathbf{r},t)=-\frac{e^2\hbar^2}{m^2c}\sum_\omega N(0)^2(i\pi\hbar\omega)\int\frac{\mathbf{R}(\mathbf{R}\cdot\mathbf{A})e^{-R/l}e^{-iR\omega/v_0}}{R^4}d\tau' =\frac{e^2N(0)v_0}{2\pi}\int\frac{\mathbf{R}[\mathbf{R}\cdot\mathcal{E}(r',\,t-R/v_0)]}{R^4}e^{-R/l}d\tau'. \quad (2.19)$$

Note that the electric field, $\mathcal{E}$, is evaluated at the retarded time, $t-R/v_0$. When the field is constant over a mean free path, (2.19) reduces to Ohm's law, with conductivity σ given by

$$\sigma=\tfrac{2}{3}e^2N(0)v_0l. \quad (2.20)$$

In the earlier derivation,[4] scattering was introduced by use of a phenomenological relaxation time, τ. Instead of taking the limit $s\to 0$, one sets $s=1/\tau$ and

uses the plane wave expression (2.5) for the density matrix. This procedure also gives (2.19) with $l=v_0\tau$.

3. ANOMALOUS SKIN EFFECT IN SUPERCONDUCTORS

Expressions for the current density in superconductors which apply at microwave or infrared frequencies may be obtained by extensions of the method used in Secs. 1 and 2 and of the method used in the low-frequency limit ($\omega=0$) by Bardeen, Cooper, and Schrieffer[5] to discuss the Meissner effect. We find that $L(\epsilon,\epsilon')$ used in (5.15) of BCS[5] is to be replaced by

$$L(\omega,\epsilon,\epsilon')=\frac{1}{4}\left(1+\frac{\epsilon\epsilon'+\epsilon_0^2}{EE'}\right)\times\left(\frac{f'-f}{E-E'-\hbar(\omega-is)}+\frac{f'-f}{E-E'+\hbar(\omega-is)}\right)+\frac{1}{4}\left(1-\frac{\epsilon\epsilon'+\epsilon_0^2}{EE'}\right)\times\left(\frac{1-f-f'}{E+E'-\hbar(\omega-is)}+\frac{1-f-f'}{E+E'+\hbar(\omega-is)}\right),\quad(3.1)$$

which may also be written in the form

$$L(\omega,\epsilon,\epsilon')=-\tfrac{1}{2}(1-2f)\left\{\frac{E+\hbar(\omega-is)+[(\epsilon_0^2+\epsilon\epsilon')/E]}{E'^2-[E+\hbar(\omega-is)]^2}+\frac{E-\hbar(\omega-is)+[(\epsilon_0^2+\epsilon\epsilon')/E]}{E'^2-[E-\hbar(\omega-is)]^2}\right\},\quad(3.2)$$

in which use has been made of the symmetry in ϵ and ϵ'. The second form is most convenient for subsequent integration over ϵ'. We have followed the convention of BCS to take $E=+(\epsilon^2+\epsilon_0^2)^{\frac{1}{2}}$ and define $f=f(E)$, $f'=f(E')$, where f is the usual Fermi-Dirac function. Note that (2.11) of Sec. 2 corresponds to the difference $L(\omega,\epsilon,\epsilon')-L(0,\epsilon,\epsilon')$ for normal metals ($\epsilon_0=0$). The apparent difference in form arises from the fact that ϵ takes on both positive and negative values, while E is always positive.

Scattering can be introduced as in Sec. 2. If ψ_k is a given wave function in the presence of scattering, another wave function of the same energy is $\psi_{-k}\equiv\psi_k^*$. Unless ψ_k is real (except for a possible constant phase factor), these can be chosen to be orthogonal. If the paired states are $(\mathbf{k}\uparrow,-\mathbf{k}\downarrow)$, as for plane waves, the same density matrix over an energy shell, (2.4), is required for evaluation of superconducting as for normal metals. The net effect of scattering is then to introduce an extra factor, $e^{-R/l}$, into the kernel of the integral for the current density.

The integration for the current density can be carried out as in Sec. 2 by introduction of the convergence factor (2.12) and contour integration. The expression for the kernel in the Pippard expression for the current density,

$$\mathbf{j}(\mathbf{r},t)=\sum_\omega\frac{e^2N(0)v_0}{2\pi^2\hbar c}\times\int\frac{\mathbf{R}[\mathbf{R}\cdot\mathbf{A}_\omega(r')]I(\omega,R,T)e^{-R/l}dr'}{R^4},\quad(3.3)$$

is

$$I(\omega,R,T)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{L(\omega,\epsilon,\epsilon')-\frac{f(\epsilon)-f(\epsilon')}{\epsilon'-\epsilon}\right\}\times\cos[\alpha(\epsilon-\epsilon')]d\epsilon d\epsilon',\quad(3.4)$$

where α is again $R/\hbar v_0$. The integral in the limit $s\to 0$ is

$$\begin{aligned}I(\omega,R,T)=&-\pi i\int_{\epsilon_0-\hbar\omega}^{\epsilon_0}[1-2f(E+\hbar\omega)]\\&\times[g(E)\cos(\alpha\epsilon_2)-i\sin(\alpha\epsilon_2)]e^{i\alpha\epsilon_1}dE\\&-\pi i\int_{\epsilon_0}^{\infty}\{[1-2f(E+\hbar\omega)]\\&\times[g(E)\cos(\alpha\epsilon_2)-i\sin(\alpha\epsilon_2)]e^{i\alpha\epsilon_1}-[1-2f(E)]\\&\times[g(E)\cos(\alpha\epsilon_1)+i\sin(\alpha\epsilon_1)]e^{-i\alpha\epsilon_2}\}dE,\end{aligned}\quad(3.5)$$

where ϵ_1 and ϵ_2 are the Bloch energies corresponding to E and $E+\hbar\omega$, respectively:

$$\epsilon_1=(E^2-\epsilon_0^2)^{\frac{1}{2}},\quad \epsilon_2=[(E+\hbar\omega)^2-\epsilon_0^2]^{\frac{1}{2}};\quad(3.6)$$

and

$$g(E)=(E^2+\epsilon_0^2+\hbar\omega E)/\epsilon_1\epsilon_2.\quad(3.7)$$

For a negative argument, $-x$, $(-x)^{\frac{1}{2}}=ix^{\frac{1}{2}}$. The negative sign of the square root is to be taken when $\hbar\omega-E>\epsilon_0$. The expression reduces to the appropriate value for the normal state, $-\pi i\hbar\omega e^{-iR\omega/v_0}$, in the limit $\epsilon_0\to 0$.

In the extreme anomalous limit, for which the penetration of field is small compared with the coherence distance $\xi_0\sim v_0/\pi\epsilon_0(0)$, we may set $\alpha=R/\hbar v_0=0$. It is then convenient to introduce, following Glover and Tinkham,[12] a complex conductivity $\sigma=\sigma_1-i\sigma_2$ for the superconducting state. We then have, for the ratio of the superconducting to normal conductivity,

$$\frac{\sigma_1-i\sigma_2}{\sigma_N}=\frac{I(\omega,0,T)}{-\pi i\hbar\omega}.\quad(3.8)$$

Expressions for σ_1 and σ_2 are

$$\frac{\sigma_1}{\sigma_N}=\frac{2}{\hbar\omega}\int_{\epsilon_0}^{\infty}[f(E)-f(E+\hbar\omega)]g(E)dE+\frac{1}{\hbar\omega}\int_{\epsilon_0-\hbar\omega}^{-\epsilon_0}[1-2f(E+\hbar\omega)]g(E)dE,\quad(3.9)$$

$$\frac{\sigma_2}{\sigma_N}=\frac{1}{\hbar\omega}\int_{\epsilon_0-\hbar\omega,-\epsilon_0}^{\epsilon_0}\frac{[1-2f(E+\hbar\omega)](E^2+\epsilon_0^2+\hbar\omega E)}{(\epsilon_0^2-E^2)^{\frac{1}{2}}[(E+\hbar\omega)^2-\epsilon_0^2]^{\frac{1}{2}}}.\quad(3.10)$$

[12] R. E. Glover, III, and M. Tinkham, Phys. Rev. **108**, 243 (1957).

Note that (3.9) is the same as the expression for the ratio of absorption for superconducting to normal metals for an interaction which follows case II of BCS.[5] The second term of (3.9) does not appear unless $\hbar\omega>2\epsilon_0$, in which case the lower limit of the integral in (3.10) is $-\epsilon_0$ instead of $\epsilon_0-\hbar\omega$. Signs of the square roots are such that $g(E)$ is positive in both integrals of (3.9).

Numerical integration is required for $T>0$, but the integrals can be carried out explicitly in terms of complete elliptic integrals for $T=0$. When we set $f(E)=0$, the first integral of (3.9) vanishes. There is absorption only for $\hbar\omega>2\epsilon_0$, in which case

$$\frac{\sigma_1}{\sigma_N}=\left(1+\frac{2\epsilon_0}{\hbar\omega}\right)E(k)-2\left(\frac{2\epsilon_0}{\hbar\omega}\right)K(k), \quad (\hbar\omega>2\epsilon_0). \tag{3.11}$$

An expression for σ_2 valid for all ω is

$$\frac{\sigma_2}{\sigma_N}=\frac{1}{2}\left\{\left(\frac{2\epsilon_0}{\hbar\omega}+1\right)E(k')+\left(\frac{2\epsilon_0}{\hbar\omega}-1\right)K(k')\right\}, \tag{3.12}$$

where the arguments of the elliptic integrals E and K are

$$k=|2\epsilon_0-\hbar\omega|/|2\epsilon_0+\hbar\omega|, \quad k'=(1-k^2)^{\frac{1}{2}}. \tag{3.13}$$

The ratio of the surface impedance in the extreme anomalous limit in the superconducting state, $Z_{\infty s}$, to that in the normal state, $Z_{\infty n}$, is given by

$$\frac{Z_{\infty s}}{Z_{\infty n}}=\left(\frac{\sigma_1-i\sigma_2}{\sigma_N}\right)^{\frac{1}{3}}. \tag{3.14}$$

These expressions should be useful not only for interpretation of measurements on the anomalous skin effect, but also for absorption in thin superconducting films. Comparisons which have been made with data of Glover and Tinkham[12] on absorption of microwave and far-infrared radiation in thin superconducting films show good agreement between theory and experiment.[13]

APPENDIX A

We wish to show here the connection between the expression we have used for the density matrix over an energy shell and a corresponding expression derived by Kohn and Luttinger[7] by a perturbation theory expansion. If the probability of occupancy of a state of energy ϵ' is $\xi(\epsilon')$, the over-all density matrix is

$$\rho=\sum_{k'} f(\epsilon')\frac{\sin(k'R)}{k'R}e^{-R/2l}, \tag{A.1}$$

where in general the mean free path l may be a function of ϵ'.

In the perturbation expansion, it is assumed that the unperturbed states are plane waves and the density matrix is expressed in the momentum representation. When averaged over random positions of the scattering centers, the density matrix is diagonal. The expansion (C.1) of KL gives to terms of second order

$$\rho_{kk}=f(\epsilon)+\sum_{k'}\langle|H_{kk'}|^2\rangle \times\frac{f(\epsilon')-f(\epsilon)+(\epsilon-\epsilon')(df/d\epsilon)}{(\epsilon'-\epsilon)^2}, \tag{A.2}$$

where $\langle|H_{kk'}|^2\rangle$ is an average of the scattering matrix element over random distributions. The mean free path, l, is given in terms of the relaxation time, τ, by

$$v/l=1/\tau=2\pi(dk/d\epsilon)\langle|H_{kk'}|^2\rangle N(\epsilon), \quad \text{for} \quad \epsilon'\simeq\epsilon, \tag{A.3}$$

where $N(\epsilon)$ is the density of states in energy.

The expression (A.1) is diagonal in the momentum representation and the diagonal component is given by the Fourier transform of (A.1):

$$\rho_{kk}=\sum_{k'} f(\epsilon')(4\pi/kk')\int_0^\infty \sin kR\,\sin k'R\,e^{-R/2l}dR$$

$$=\sum_{k'}\frac{\pi}{kk'l}\left\{\frac{1}{(k-k')^2+(2l)^{-2}}-\frac{1}{(k+k')^2+(2l)^{-2}}\right\}. \tag{A.4}$$

If we change variables from k and k' to ϵ and ϵ' and the sum over k' to an integral over $\epsilon'=\hbar^2k'^2/2m$, we find after some reduction

$$\rho_{kk}=\frac{1}{\pi}\int\frac{f(\epsilon')(\hbar/2\tau')d\epsilon'}{[\epsilon-\epsilon'+(\hbar^2/8ml^2)]^2+(\hbar/2\tau')^2}, \tag{A.5}$$

where $\tau'=lv'=ml/\hbar k'$.

We may neglect $(\hbar^2/8ml^2)$ in comparison with ϵ in the denominator, since we are assuming that $kl\gg1$. For weak scattering, $\hbar/\tau$ is likewise small, so that the major contribution to the integral is for ϵ' close to ϵ. The singularity at $\epsilon=\epsilon'$ gives just $f(\epsilon)$. If we add terms to $f(\epsilon')$ to eliminate this singularity and write $f(\epsilon)$ as a separate term, we find, with neglect of small terms in the denominator,

$$\rho_{kk}=f(\epsilon)+\frac{\hbar}{2\pi}\int\frac{f(\epsilon')-f(\epsilon)+(\epsilon-\epsilon')(df/d\epsilon)}{(\epsilon-\epsilon')^2}\frac{d\epsilon'}{\tau'}. \tag{A.6}$$

The equivalence to the perturbation-theory result follows when $1/\tau'$ is replaced by (A.3).

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[13] See A. T. Forrester, Phys. Rev. **110**, 769 (1958); M. Tinkham and R. E. Glover, III, Phys. Rev. **110**, 771 (1958).