# Convexity in Random Resistor Networks*

Kenneth Golden†

**Abstract.** The bulk conductivity $\sigma^*(p)$ of the bond lattice in $\mathbb{Z}^d$ is considered, where the conductivity of the bonds is either 1 with probability $p$ or $\epsilon \geq 0$ with probability $1-p$. Rigorous and non-rigorous results demonstrating convexity of $\sigma^*(p)$ near the percolation threshold $p_c$ are presented. For $\epsilon = 0$, a certain transformation on graphs which drives the system to $p_c$ is found to be "convexity improving". This analysis leads us to propose upper and lower bounds on the conductivity critical exponent $t$ in terms of some percolation exponents. These bounds become tighter with increasing dimension and coincide in $d = 6$, where mean field behavior is believed to hold.

**1. Introduction.** Disordered conductors are encountered frequently in science and engineering. Examples include inhomogeneous gold films on glass, impure semiconductors, suspensions of particles in a fluid, composite materials, and even sea ice. Of particular practical and theoretical interest are those systems which undergo an insulator/conductor transition as some parameter is varied. For example, sea ice is composed of salty water, or brine inclusions (conductor) embedded in a pure ice matrix (insulator). When the sea ice is cold, the brine pockets occupy only a small volume fraction and the sea ice behaves as an insulator. However, when the sea ice is warmer, the brine pockets tend to coalesce and form a conducting matrix, so that the sea ice behaves as a conductor.

In the late 1960's Ziman [1], Eggarter and Cohen [**2, 3**] and others suggested that random resistor networks based on the percolation model [4] provide a good

*Supported in part by NSF and AFOSR through NSF Grant DMS - 8801673.
† Department of Mathematics, Princeton University, Princeton, NJ 08544 USA

description of disordered conductors. In particular, consider the bulk conductivity $\sigma^*(p)$ of the bond lattice in $\mathbf{Z}^d$, where the conductivity of the bonds is either 1 with probability $p$, or $\epsilon \geq 0$ with probability $1-p$. When $\epsilon = 0$, we can view the conductivity 0 bonds as vacant. In this case, the associated percolation problem concerns $P_\infty(p)$, the probability that the origin is connected to an infinite cluster of occupied (conductivity 1) bonds. For $p$ below some critical probability $p_c$, called the percolation threshold, $P_\infty(p) = 0$, while for $p > p_c$, $P_\infty(p) > 0$. The bulk conductivity $\sigma^*(p)$ has a similar behavior, with $\sigma^*(p) = 0$ for $p < p_c$, and $\sigma^*(p) > 0$ for $p > p_c$, although there is apparently no simple relation between $\sigma^*(p)$ and $P_\infty(p)$ [**5**]. As $p \to p_c^+$, it is believed that $\sigma^*(p)$ exhibits critical scaling, $\sigma^*(p) \sim (p-p_c)^t$, where $t$ is called the conductivity critical exponent [**6**].

Since their introduction, random resistor networks have been widely studied in the physics literature [**7, 8**]. Given their central place in the theory of disordered conductors, it is surprising that there has been little rigorous analysis of random resistor networks. The main exceptions are the works of Grimmett and Kesten [**9**] (see also ref. [**10**]) and Chayes and Chayes [**11, 12**]. One of the principal contributions of these works is to establish the coincidence of the conduction and percolation thresholds for $d = 2$, where $p_c = \frac{1}{2}$ [**9**], as well as for higher dimensions [**11**]. In addition, the Chayes [**11**] obtain bounds on $t$ (assuming it exists) in terms of some percolation exponents. In $d = 2$ it has also been established, using arguments that can be made rigorous, that $t \geq 1$ [**13, 14,** see also ref. **12**]. Furthermore, it is rigorously known that $\sigma^*(p)$ is continuous at $p_c = \frac{1}{2}$ in $d = 2$, which is proven via the continuity of $P_\infty(p)$ at $p_c = \frac{1}{2}$ [**15**] and the bound $\sigma^*(p) \leq pdP_\infty^2(p)$ [**11**]. In higher dimensions it is certainly believed that $\sigma^*(p)$ is continuous at $p_c$, but this has not been rigorously proven yet.

The purpose of the present work is to introduce a new approach to the random resistor problem based on convexity, which has played an important role in many problems of statistical physics, but has apparently all but been ignored in the present context. As Straley [**16**] remarks, $\sigma^*(p)$ is " necessarily positive but has no convexity property, with the consequence that no rigorous exponent inequalities can be proved." Nevertheless, casual inspection of numerical simulations [**6, 17-20**] of the graph of $\sigma^*(p)$ for bond or site models in $d \geq 2$ suggests convexity in $p$, at least near the percolation threshold $p_c$. (In the site problem, vertices of $\mathbf{Z}^d$, along with all $2d$ attached bonds, are removed (when $\epsilon = 0$) at random with probability $1-p$.) Given the broad and enduring interest in these models, we believe that it is important to investigate this convexity, which appears to be a general feature of the conductivity of lattices near $p_c$.

The principal results discussed in this paper are as follows. First we observe directly that $\sigma^*(p)$ cannot be convex for all $p$ when $\epsilon = 0$, and present numerical results outlining the regimes of $\epsilon$ and $p$ for which $\sigma^*(p)$ is convex. Next, rigorous results for $\epsilon > 0$ are obtained, the main one being that $\sigma^*(p)$ for the $d = 2$ bond problem is convex near $p_c = \frac{1}{2}$ for every $\epsilon > 0$. The proof is based on Keller's Interchange Theorem, which holds for certain continuum systems, so that our result holds for them as well. Finally the $\epsilon = 0$ case is considered, and a general physical argument explaining convexity near $p_c$ is offered. Our physical argument is supported by a rigorous result asserting that a certain transformation which maps the $p = 1$ lattice to a well-known model of the conducting backbone near $p_c$, namely the node-link model, is "convexity improving". Further analysis leads us to propose upper and lower bounds on $t$, in terms of some percolation exponents, which become tighter with increasing dimension, and coincide for $d = 6$, where mean field behavior is believed to hold, with $t = 3$.

**2. Formulation.** We formulate the bond conductivity problem for an arbitrary graph. Let $G$ be a finite graph consisting of $N$ bonds $\{b_i\}$ and $N'$ vertices $\{x_i\}$. Assigned to $G$ are $N$ independent random variables $c_i$, $1 \le i \le N$, the bond conductivities, which take the values 1 with probability $p$ and $\epsilon \ge 0$ with probability $q = 1 - p$. Distinguish two vertices, say $x_1 = x$ and $x_{N'} = y$, and connect them to a battery which keeps the voltage drop between them equal to 1. The effective conductivity $\tilde{\sigma}(\omega)$ of the network for any realization $\omega$ of the bond conductivities is just the total current $i(\omega)$ that flows through the network, which is obtained via Kirchoff's laws. We define $\sigma(p) =< \tilde{\sigma}(\omega) >$, where the expectation $< \cdot >$ is over all $2^N$ realizations. For example, a two bond network has

$$\sigma(p) = p^2\tilde{\sigma}(1,1) + pq\big(\tilde{\sigma}(1,\epsilon) + \tilde{\sigma}(\epsilon,1)\big) + q^2\tilde{\sigma}(\epsilon,\epsilon) \ ,$$

where $\tilde{\sigma}(1,1) = \tilde{\sigma}(\omega)$ with $\omega = (1,1)$, and so on. For $N$ bonds, $\sigma(p)$ is an $N$th order homogeneous polynomial in $p$ and $q$,

$$\begin{aligned} \sigma(p) &= \sum_{k=0}^{N} \alpha_k p^{N-k} q^k \\ \alpha_k &= \sum_{\omega^k \in \Omega^k} \tilde{\sigma}(\omega^k), \quad q = 1 - p, \end{aligned} \tag{2.1}$$

where $\Omega^k = \big\{\omega^k = (\omega_1, \ldots, \omega_N) | \omega_\ell = \epsilon \text{ for exactly k of the } \omega_\ell\text{'s}\big\}$.

The cases of most interest are when $G$ is a square, cubic, or hypercubic lattice. Then, with $d = 2$ for simplicity, we take an $L \times L$ sample of the lattice and attach a perfectly conducting bus bar to each of two opposite edges of the sample. This can be accomplished [9] in the above language by attaching to each vertex of these opposing edges a perfectly conducting bond. All of these bonds from one edge meet at a new vertex $x$ and all the bonds from the other edge meet at a new vertex $y$. Then $x$ and $y$ are connected again with the unit battery. Random bond conductivities are assigned only to the bonds in the original $L \times L$ sample. Let $\sigma_L(p)$ denote (2.1) for the effective conductivity measured between $x$ and $y$. Then for $d \ge 1$, the bulk conductivity $\sigma_L^*(p)$ is defined as

$$\sigma_L^*(p) = L^{2-d}\sigma_L(p) \ . \tag{2.2}$$

For $\epsilon > 0$, Künnemann [21] proved that the infinite volume limit

$$\sigma^*(p) = \lim_{L\to\infty} \sigma_L^*(p) \tag{2.3}$$

exists. For $\epsilon = 0$, the existence of (2.3) has still not been proven [9-11]. In what follows we shall assume that this limit exists. It should be remarked that since for $\epsilon > 0$, $\sigma^*(p,\epsilon) = \lim_{L\to\infty} \sigma_L^*(p,\epsilon)$ exists and is monotonically decreasing in $\epsilon$, $\lim_{\epsilon\to 0} \sigma^*(p,\epsilon)$ exists, and provides a reasonable definition for the bulk conductivity in the $\epsilon = 0$ case. The unsolved problem of the existence of (2.3) directly for $\epsilon = 0$ then boils down to whether or not the $\epsilon \to 0$ and $L \to \infty$ limits can be interchanged.

**3. Regimes of Convexity.** We now make an observation which directly indicates that $\sigma^*(p)$ cannot generally be a convex function of $p$. For the $d = 2$ bond problem with $\epsilon = 0$, it is known [7, 19] (although not rigorously) that $\frac{d\sigma^*}{dp}|_{p=1} = 2$. Now, a straight line with slope 2 at $\sigma^*(1) = 1$ intersects the $p$-axis at $p = \frac{1}{2}$. If $\sigma^*(p)$ is convex for all $p$, then either $p_c < \frac{1}{2}$ or the graph of $\sigma^*(p)$ is the above straight line, which is the effective medium solution for this problem. Since it is rigorously known [9] that $p_c = \frac{1}{2}$ for $d = 2$, the only way $\sigma^*(p)$ could be convex for all $p$ is if effective medium theory gives the correct solution for all $p$, which gives a critical exponent of $t = 1$ and contradicts practically every numerical simulation of this problem. From inspection of the simulations in references [6] or [19], as $p$ increases from $\frac{1}{2}$ to 1, what apparently happens is that $\sigma^*(p)$ starts off convex at $p = \frac{1}{2}$, but eventually the curve "turns over", i.e. becomes slightly concave, which allows it to have the correct slope of 2 at $p = 1$. We note that this effect is subtle, because away from the critical regime ($p \approx \frac{1}{2}$), the graph of $\sigma^*(p)$ looks nearly linear, where effective medium theory is believed to provide a very good approximation. In fact, an expression for $\frac{d^2\sigma^*}{dp^2}|_{p=1}$ is found in reference [19] and is numerically evaluated, with a result of about $-.21$, which supports the accuracy of effective medium theory near $p = 1$. The calculations of $\frac{d^2\sigma^*}{dp^2}|_{p=1}$ are somewhat involved, and it is not at all obvious before numerical evaluation that the result will be negative, hence the need for our above direct argument.

Another direct way of seeing that convexity is not a general principle is the following finite network pointed out to the author by P. Doyle. Let the network consist of three circuit elements in series, the first consists of two bonds in parallel, the second (and middle) is a single bond, and the third is again two bonds in parallel. When $\epsilon = 0$, an elementary calculation shows $\frac{d^2\sigma}{dp^2}|_{p=1} < 0$.

In order to build intuition about convexity in resistor networks, we have examined $\frac{d^2\sigma}{dp^2}$ for a variety of finite graphs. The calculations are based on an exact formula [22, 23] for the effective conductivity $\tilde{\sigma}(\omega)$ of a conductor network based on a graph

$G$ for any realization of the bond conductivities $c_i$,

$$\tilde{\sigma}(\omega) = \frac{\sum_T \prod_{b_i \in T} c_i}{\sum_{T_{xy}} \prod_{b_i \in T_{xy}} c_i} , \tag{3.1}$$

where the sum in the numerator is over all spanning trees $T$ in $G$, and the sum in the denominator is over all spanning trees $T_{xy}$ in $G$ with the vertices $x$ and $y$ identified. The numerator and denominator are computed via the determinants of the appropriate adjacency matrices. The graphs we have considered include the square lattice, triangular lattice, trees (Bethe lattice), ladders, Wheatstone bridge, and others. We have considered bond and site problems, and various ranges of $\epsilon \geq 0$. Most of the networks we have looked at have been rather small, the largest being a $15 \times 15$ sample of the square lattice.

To summarize our results, away from $p = 1$ all networks we have considered have $\sigma(p)$ convex for any $\epsilon \geq 0$, and for both bond and site problems. Near $p = 1$, however, bond problems typically are <u>not</u> convex when $\epsilon$ is small enough, while site problems typically are convex for all $\epsilon \geq 0$. For a typical bond problem, if we start with $\epsilon$ close to 1 and allow it to shrink, $\frac{d^2\sigma}{dp^2}|_{p=1}$ is positive until, say $\epsilon \approx 0.1$, below which it becomes and stays negative all the way down to $\epsilon = 0$. This concavity near $p = 1$ for $\epsilon = 0$ is, of course, consistent with the results in [**19**]. Furthermore, the convexity near $p = 1$ for site problems is consistent with the results of [**20**]. We see then for the site problem, $\sigma^*(p)$ for the square lattice with $\epsilon = 0$ appears to be convex for all $p$. If indeed this is the case, then one can obtain a bound on $p_c$ using the result [**17, 20**] for $d = 2$ that $\frac{d\sigma^*}{dp}|_{p=1} = \pi$. The bound would be obtained by drawing a straight line of slope $\pi$ through the point $p = 1$, $\sigma^*(p) = 1$, and noting that $p_c$ is less than the intercept of this line, which leads to

$$p_c < 1 - \frac{1}{\pi} . \tag{3.2}$$

For the $d = 2$ site problem, $p_c$ is believed [**17**] to be about .59, whereas $1 - 1/\pi \approx .68$.

**4. Convexity when $\epsilon > 0$.** In this section we directly consider the infinite volume limit $\sigma^*(p)$ for the bond problem in dimensions $d \geq 2$ with $\epsilon > 0$. In order to prove our principal results, we shall need some smoothness in $p$ of $\sigma^*(p)$. The analysis for this is based on an integral representation for $\sigma^*(p)$ which was proved for two component stationary random media in [**24**] (see also [**25**]). The formulation there is in the continuum, but applies in the present context by replacing the continuum equations for the electric and current fields with their discrete analogs, i.e., Kirchoff's laws. We repeat here only the relevant features.

Let $s = 1/(1-\epsilon)$. We shall consider $s$ to be a complex variable. It can be shown that $\sigma^*(p, s)$ is analytic everywhere in the $s$-plane except for the interval $[0, 1]$. Furthermore, $\sigma^*(p, s)$ maps the upper half plane to the upper half plane, i.e., Im $\sigma^*(p,s) > 0$ when Im $s > 0$. As a consequence of these analytic properties, $\sigma^*(p,s)$ has the following integral representation,

$$1 - \sigma^*(p,s) = \int_0^1 \frac{d\mu(x)}{s-x} , \tag{4.1}$$

where $\mu$ is a positive Borel measure on $[0,1]$ which depends on $p$. Notice that this representation separates the dependence of $\sigma^*(p,s)$ on $s$ from its dependence on $p$. (In fact, (4.1) applies even when $\epsilon = 0$.) The dependence of $\mu$ on $p$ is most easily obtained through its moments, as follows. For $|s| > 1$, (4.1) can be expanded about a homogeneous medium ($s = \infty$ or $\epsilon = 1$), yielding

$$1 - \sigma^*(p,s) = \frac{\mu_0(p)}{s} + \frac{\mu_1(p)}{s^2} + \frac{\mu_2(p)}{s^3} + \dots , \tag{4.2}$$

$$\mu_n = \int_0^1 x^n d\mu(x) . \tag{4.3}$$

By equating (4.2) to a similar expansion of a resolvent representation for $\sigma^*$, one can obtain a formula for $\mu_n(p)$ in terms of the iterates of a self adjoint operator on $L^2$ ($\Omega$ = set of realizations of the bond conductivities) involving the Green's function of the discrete Laplacian. Because the bond conductivities are independent, these moments can be computed in principle (see, e.g., [**26**]), but they become very complicated. The first two are

$$\begin{aligned} \mu_0(p) &= 1 - p \\ \mu_1(p) &= \frac{p(1-p)}{d} . \end{aligned} \tag{4.4}$$

In general, $\mu_n(p)$ is an $(n+1)$-order polynomial in $p$.

We are now ready to state

LEMMA 4.1: *($d \geq 1$ bond problem) For every $\epsilon > 0$, there exists an open neighborhood $V_\epsilon$ in the complex $p$-plane such that $[0,1] \subset V_\epsilon$ and $\sigma^*(p)$ is analytic in $V_\epsilon$.*

PROOF: Fix $s = 1/(1-\epsilon) > 1$. The idea is to produce a neighborhood containing $[0,1]$ in which (4.2) converges uniformly. Since for $p \in [0,1]$, $\mu_0(p) = 1 - p$ and $\mu_n(p) \geq \mu_{n+1}(p)$ for all $n$ (via (4.3)),

$$\mu_n(p) \leq 1, \quad p \in [0,1] . \tag{4.5}$$

Now we must extend what we can of (4.5) into the complex plane. Consider $W = \{p \in \mathbb{C} | p \notin [0,1]\}$. Conformally map $W$ onto the unit disk $D$ in the $z$-plane, so that $p = \infty$ gets mapped to $z = 0$, and $[0,1]$ gets mapped to the unit circle $|z| = 1$. Let $m = n+1$. Since $\mu_n(p)$ is an $m$th order polynomial in $p$, $\mu_n(z)$ has at worst an $m$th order pole at $z = 0$. Thus $z^m \mu_n(z)$ is analytic in $D$. Since $|\mu_n(z)| \leq 1$ for $|z| = 1$, by the maximum modulus principle,

$$|\mu_n(z)| \leq \frac{1}{|z|^m}\ , \quad z \in D\ . \tag{4.6}$$

For any small $\delta' > 0$, there is a small $\delta > \delta' > 0$ such that in the annulus $A_{\delta'}$ defined by $1 \geq |z| > 1 - \delta'$

$$|\mu_n(z)| \leq (1+\delta)^m\ , \quad z \in A_{\delta'}\ . \tag{4.7}$$

For our given $s > 1$ (or $\epsilon > 0$), we can choose $\delta$ and $\delta'$ such that

$$|\mu_n(p)| \leq (1+\delta)^m < s^m\ , \quad p \in V_\epsilon\ , \tag{4.8}$$

where $V_\epsilon$ conformally maps to $A_{\delta'}$. Then (4.2) converges uniformly in $V_\epsilon$, which proves the lemma.

The conformal mapping trick used to obtain (4.8) arose from a wonderful conversation with C. McMullen and C. Simpson, and the author gratefully acknowledges this.

**Remark.** Lemma 4.1 and its proof hold for a large class of continuum systems as well, namely infinitely interchangeable media, which have recently been introduced by O. Bruno [**27**]. This class is a generalization of Miller's cell materials [**28**], where all of space is divided up into cells, such as spheres of all sizes, and then the conductivity of each cell is a random variable (independent from the others) taking two (or more) values with probability $p$ and $1-p$. While the integral representation (4.1) holds in great generality, along with (4.5), what is needed to make the proof go through is that the $\mu_n(p)$ are polynomials in $p$. The proof of this fact for infinitely interchangeable media is contained in [**29**] (along with rigorous upper and lower bounds on $\sigma^*(p)$ for the $d = 2$ bond problem with $\epsilon > 0$). We also note that Lemma 4.1 presumably does not hold for all composite media. For example, $\sigma^*(p)$ for a periodic array of spheres of volume fraction $p$ embedded in a host material is believed to be analytic at $p = 0$ only in the variable $p^{\frac{1}{3}}$, so that $\sigma^*(p)$ has a branch cut there (see, e.g., [**30**]).

We now return to convexity. In the previous section it was found that convexity of $\sigma^*(p,\epsilon)$ appears to be lost only when $\epsilon$ becomes small enough. The following result provides some basis for this observation.

Proposition 4.1: *($d \geq 1$ bond problem) For $\epsilon$ sufficiently close to 1, $\sigma^*(p)$ is convex for all $p \in [0,1]$.*

Proof: From (4.2) and (4.4)

$$\sigma^*(p,s) = 1 - \left(\frac{1-p}{s} + \frac{p(1-p)/d}{s^2} + \ldots\right)\ . \tag{4.9}$$

By Lemma 4.1,

$$\frac{d^2\sigma^*}{dp^2} = \frac{2/d}{s^2} + O\left(\frac{1}{s^3}\right) , \tag{4.10}$$

which is positive for all $p \in [0,1]$ when $s$ is sufficiently large.

In the previous section it was observed that convexity can only be lost near $p = 1$. At the other end, $p = 0$, convexity can never be lost, as shown in

PROPOSITION 4.2: *For any finite graph $G$ with $\epsilon \geq 0$,*

$$\left.\frac{d^2\sigma}{dp^2}\right|_{p=0} \geq 0 . \tag{4.11}$$

PROOF: The proof is elementary and follows directly from (5.9).

Finally we come to the principal result of this section.

THEOREM 4.1: *($d = 2$ bond problem) For every $\epsilon > 0$, there exists an open neighborhood $\mathcal{U}_\epsilon \subset [0,1]$ containing $p_c = \frac{1}{2}$ such that $\sigma^*(p)$ is convex on $\mathcal{U}_\epsilon$.*

PROOF: The proof is based on Keller's Interchange Theorem

$$\sigma^*(\sigma_1, \sigma_2)\sigma^*(\sigma_2, \sigma_1) = \sigma_1\sigma_2 , \tag{4.12}$$

where $\sigma^*(\sigma_1, \sigma_2)$ is the bulk conductivity of a statistically isotropic, two-component stationary random medium in $d = 2$ with component conductivities $\sigma_1$ and $\sigma_2$, and $\sigma^*(\sigma_2, \sigma_1)$ is the bulk conductivity with $\sigma_1$ and $\sigma_2$ interchanged [**31-33**]. For the $d = 2$ bond lattice with $\sigma_1 = 1$ in proportion $p$ and $\sigma_2 = \epsilon$ in proportion $1 - p$, (4.12) is also known as a duality relation [**16**], and is written as

$$\sigma^*(p)\sigma^*(1-p) = \epsilon , \tag{4.13}$$

so that at $p = \frac{1}{2}$

$$\sigma^*(\tfrac{1}{2}) = \sqrt{\epsilon} . \tag{4.14}$$

In order to prove the theorem, we compute the second derivative of $\sigma^*(p)$ at $p = \frac{1}{2}$,

$$\frac{d^2\sigma^*}{dp^2}|_{p=\frac{1}{2}} = \lim_{h\to 0} \frac{\sigma^*(\frac{1}{2}+h) - 2\sigma^*(\frac{1}{2}) + \sigma^*(\frac{1}{2}-h)}{h^2} \tag{4.15}$$

$$= \lim_{h\to 0} \frac{\sigma^*(\frac{1}{2}+h) - 2\sqrt{\epsilon} + \epsilon/\sigma^*(\frac{1}{2}+h)}{h^2}\,, \tag{4.16}$$

where (4.16) is obtained via (4.13) and (4.14). To compute the limit, we need some of the smoothness of $\sigma^*(p)$ provided by Lemma 4.1,

$$\sigma^*(\tfrac{1}{2}+h) = \sqrt{\epsilon} + \alpha h + O(h^2)\ , \tag{4.17}$$

where $\alpha > 0$ (when $0 < \epsilon < 1$). Inserting (4.17) into (4.16) and taking the limit, we obtain

$$\left.\frac{d^2\sigma^*}{dp^2}\right|_{p=\frac{1}{2}} = \frac{\alpha^2}{\sqrt{\epsilon}} > 0\ . \tag{4.18}$$

Again using Lemma 4.1, $\frac{d^2\sigma^*}{dp^2}$ is a continuous function of $p$, so that it is positive in a neighborhood $\mathcal{U}_\epsilon$ of $p_c = \frac{1}{2}$, which proves the theorem.

**Remark 1.** It is interesting to note that $\frac{d\sigma^*}{dp^2}|_{p=\frac{1}{2}}$ depends only on $\frac{d\sigma^*}{dp}|_{p=\frac{1}{2}}$ and $\sigma^*(\frac{1}{2})$.

**Remark 2.** Our argument, while very simple, gives no control on the size of $\mathcal{U}_\epsilon$ as $\epsilon \to 0$, so that one cannot rigorously conclude from it alone that convexity is preserved in the limit. As stated before, however, it is known that the conductivity exponent for $d = 2$ satisfies $t \geq 1$, which indicates convexity near $p_c = \frac{1}{2}$ in the $\epsilon \to 0$ limit. In view of our result one can view the fact that $t \geq 1$ in $d = 2$ as arising from duality.

**Remark 3.** As noted above, Keller's Theorem holds in great generality, in particular, for infinitely interchangeable media in the continuum, as does Lemma 4.1. Thus Theorem 4.1 holds for infinitely interchangeable media as well. Information about the shape of $\sigma^*(p)$ in the insulator/conductor transition regime for small $\epsilon > 0$ is provided by the theorem only for those systems whose percolation threshold is known to occur at $p_c = \frac{1}{2}$. For percolating random systems in the continuum, apparently little is known about this. However, there is a periodic system which has an insulator/conductor transition occurring at $p_c = \frac{1}{2}$ and to which Theorem 4.1 presumably applies. This system is a variant of the one considered in a classical problem dating back to Maxwell [**34**] and Rayleigh [**35**], namely, a periodic array of perfectly conducting spheres occupying a volume fraction $p$ of a host medium of unit conductivity. The variation that we consider here in $d = 2$ has been studied in [**36**] and [**37**], and is described as follows. At each integer pair $(i, j) \in \mathbf{Z}^2$ place a small square prism so that the center of the prism is at $(i, j)$ and its vertices lie on the line segments joining $(i, j)$ to its four nearest neighbors. Let the prisms have conductivity 1 and occupy a volume fraction $p$, and fill the rest of space with a medium of conductivity $\epsilon$, with $0 < \epsilon << 1$. As the prisms grow in size, $p$ increases. When $p = \frac{1}{2}$, the corners of the prisms touch, and it is around $p = \frac{1}{2}$ that the conducting transition occurs (when $\epsilon = 0$, $p_c = \frac{1}{2}$). Keller's Theorem applies to this problem, and presumably $\sigma^*(p)$ is analytic in $p$ when $\epsilon > 0$, at least near $p = \frac{1}{2}$, in which case Theorem 4.1 applies as well, which says that $\sigma^*(p)$ is convex in the transition regime when $\epsilon > 0$. This finding is consistent with Figure 4 in [**36**]. The interesting question here is what happens as $\epsilon \to 0$. Is the convexity near $p = \frac{1}{2}$ a finite $\epsilon$ effect which vanishes as $\epsilon \to 0$, or does it persist as $\epsilon \to 0$. In the $\epsilon = 0$ limit, convexity is <u>not</u> a general feature of these types of systems near the transition. For a related problem in $d = 2$ involving circles (with $p_c \neq \frac{1}{2}$) rather than squares, the critical exponent for conductivity is $\frac{1}{2}$ [**38**], indicating concavity rather than convexity at threshold when $\epsilon = 0$.

**5. Convexity when $\epsilon = 0$ and Conjectured Bounds on $t$.** We begin this section by proposing a physical argument which we believe explains observed convexity of $\sigma^*(p)$ in bond and site lattice problems with $\epsilon = 0$ and $d \geq 2$ for $p$ near $p_c$, $p > p_c$. Before giving the argument, we must introduce the notion of correlation length, which for $p > p_c$ is somewhat more delicate than for $p < p_c$. For the infinite bond lattice in $d \geq 2$ with a fraction $p$ of occupied bonds, let

$$\tau^f(0, x) = \text{Prob}_p\{0 \text{ and } x \text{ belong to the same } \underline{\text{finite}} \text{ cluster of occupied bonds}\} . \tag{5.1}$$

Then the correlation length can be defined (see [**39**]) by

$$\frac{1}{\xi(p)} = \lim_{x \to \infty} -\frac{1}{|x|} \log \tau^f(0, x) , \tag{5.2}$$

where the limit is taken as $x$ moves out to infinity in a fixed direction. This limit was proved to exist for $p > p_c$ in [**39**]. We shall assume that $\xi$ diverges with exponent $\nu$,

$$\xi(p) \sim (p - p_c)^{-\nu}, \quad p \to p_c^+ . \tag{5.3}$$

For simplicity we formulate the argument for the $d = 2$ bond problem. Fix $p > p_c$ and consider an $L \times L$ sample of the square lattice with $L >> \xi(p)$. Let us view convexity as a decrease in $\frac{d\sigma^*}{dp}$ as $p$ decreases, where for convenience we have dropped the $L$ subscript in (2.2). To accomplish a decrease in $\frac{d\sigma^*}{dp}$, remove one occupied bond $b_1$ from the sample, and let $\delta_1\sigma$ be the expected drop in the conductivity as we average over all possible removals. It should be remarked that the occupied bonds can be divided into two types, the "backbone" and the "dangling" bonds. Backbone bonds have current flowing through them and dangling bonds do not – they are dead ends for the current. Consequently, a decrease in the conductivity can only be obtained by removing a backbone bond; the removal of a dangling bond does not contribute to $\delta_1\sigma$. Now, the removal of a backbone bond can create many new dangling bonds, for example if it breaks a "string" of connected bonds. Presumably, the probability of creating dangling bonds many correlation lengths away from $b_1$ is exponentially small. Consider then the removal of a second occupied bond $b_2$. If $b_2$ is far away (with respect to $\xi$) from $b_1$, then the contribution from such bonds to the expected drop $\delta_2\sigma$ in the conductivity will be essentially $\delta_1\sigma$. However, when $b_2$ is close to $b_1$, the corresponding average contribution to $\delta_2\sigma$ will be less than $\delta_1\sigma$, due to the increased density of dangling bonds around $b_1$, the removal of which contributes nothing to $\delta_2\sigma$. Thus $\delta_2\sigma < \delta_1\sigma$, which is equivalent to convexity. Far away from $p_c$ the creation of dangling bonds will be a minor effect. For example, at $p = 1$ the removal of a single bond cannot create any dangling bonds. However as $p \to p_c^+$, $\xi$ diverges, the backbone becomes more "stringy", and this effect will be more pronounced.

In order to make the intuition in the above argument more quantitative, we define the following quantities. Let $G$ be any graph (we have in mind an $L \times L$ sample of the square lattice) with $N$ bonds having conductivities 1 or $\epsilon \geq 0$, and let $\omega^k$ be as in (2.1). Now define

$$\delta\tilde{\sigma}(\omega^k) = \sum_{i=1}^{N-k} \left[\tilde{\sigma}(\omega_i^k(1)) - \tilde{\sigma}(\omega_i^k(\epsilon))\right], \tag{5.4}$$

where given $\omega^k$, $i$ runs over the $N-k$ bonds which have conductivity 1, $\omega_i^k(1) = \omega^k$, and $\omega_i^k(\epsilon)$ is the same realization but with the $i$th bond conductivity changed to $\epsilon$. Similarly, let

$$\delta^2\tilde{\sigma}(\omega^k) = \sum_{\substack{i,j=1 \\ i\neq j}}^{N-k} \left[\tilde{\sigma}(\omega_{ij}^k(1,1)) + \tilde{\sigma}(\omega_{ij}^k(\epsilon,\epsilon)) - \tilde{\sigma}(\omega_{ij}^k(1,\epsilon)) - \tilde{\sigma}(\omega_{ij}^k(\epsilon,1))\right], \tag{5.5}$$

where, given $\omega^k$, $i$ and $j$ run over the $N-k$ bonds which have conductivity 1, $\omega_{ij}^k(1,1) = \omega^k$, $\omega_{ij}^k(\epsilon,\epsilon)$ is $\omega^k$ but with the $i$th and $j$th bond conductivities changed to $\epsilon$, $\omega_{ij}^k(1,\epsilon)$ is $\omega^k$ but with the $j$th bond conductivity changed to $\epsilon$, and similarly for $\omega_{ij}^k(\epsilon,1)$ with $i$ instead of $j$. The expressions in (5.4) and (5.5) represent discrete first and second derivatives of the conductivity with respect to $p$. To make this connection more precise, we define

$$\beta_k = \sum_{\omega^k\in\Omega^k} \delta\tilde{\sigma}(\omega^k) \tag{5.6}$$

$$\gamma_k = \sum_{\omega^k \in \Omega^k} \delta^2 \tilde{\sigma}(\omega^k) \ . \tag{5.7}$$

Then the exact relation is contained in

LEMMA 5.1: *Let $G$ be any $N$-bond graph with bond conductivities 1 and $\epsilon \geq 0$. Then*

$$\frac{d\sigma}{dp} = \sum_{k=0}^{N-1} \beta_k p^{N-1-k} q^k \tag{5.8}$$

$$\frac{d^2\sigma}{dp^2} = \sum_{k=0}^{N-2} \gamma_k p^{N-2-k} q^k \ . \tag{5.9}$$

PROOF: Differentiate $\sigma(p)$ in (2.1) keeping in mind that since $q = 1 - p$, $\frac{d}{dp} = -\frac{d}{dq}$. Then we have

$$\begin{aligned}
\frac{d\sigma}{dp} &= \sum_{k=0}^{N} \alpha_k \left[ (N-k) p^{N-k-1} q^k - k p^{N-k} q^{k-1} \right] \\
&= \sum_{k=0}^{N-1} p^{N-k-1} q^k \left[ (N-k)\alpha_k - (k+1)\alpha_{k+1} \right] ,
\end{aligned}$$

which can be written as (5.8). Taking one more derivative, we have

$$\begin{aligned}
\frac{d^2\sigma}{dp^2} &= \sum_{k=0}^{N} \alpha_k \Big[ (N-k)(N-k-1) p^{N-k-2} q^k \\
&\qquad - 2(N-k) k p^{N-k-1} q^{k-1} + k(k-1) p^{N-k} q^{k-2} \Big] \\
&= \sum_{k=0}^{N-2} p^{N-2-k} q^k \Big[ (N-k)(N-k-1)\alpha_k \\
&\qquad - 2(N-k-1)(k+1)\alpha_{k+1} + (k+2)(k+1)\alpha_{k+2} \Big] ,
\end{aligned}$$

which can be written as (5.9).

When the number of bonds in $G$ goes to infinity, the appropriate limits of $\beta_k$ and $\gamma_k$ yield exact formulas for the first and second derivatives of $\sigma(p)$. When $G$ is a hypercubic sample of the lattice with side $L$, then $N \sim dL^d$. In this case we consider $\sigma^*(p)$ in (2.3), assuming that the infinite volume limit exists. Then we have

**LEMMA 5.2:** *For the lattice in $d \geq 2$ with $\epsilon \geq 0$ and fixed $p$,*

$$\lim_{k,L\to\infty} L^{2-d}\beta_k/\binom{N}{k} = p\frac{d\sigma^*}{dp} \tag{5.10}$$

$$\lim_{k,L\to\infty} L^{2-d}\gamma_k/\binom{N}{k} = p^2\frac{d^2\sigma^*}{dp^2}\ , \tag{5.11}$$

*where the simultaneous limits of $k$ and $L \to \infty$ are taken so that*

$$\lim_{k,L\to\infty}\frac{k}{N} = \lim_{k,L\to\infty}\frac{k}{dL^d} = q = 1-p\ .$$

**PROOF:** Equation (5.8) can be written as

$$p\frac{d\sigma}{dp} = \sum_{k=0}^{N-1}\beta_k p^{N-k}q^k\ . \tag{5.12}$$

Note that in (5.6) there are $\binom{N}{k}$ terms $\delta\tilde{\sigma}(\omega^k)$. When $N$ is large, the weight of the binomial distribution is concentrated on values of $k$ such that $k/N$ is nearly $q = 1-p$. Appropriately scaling $\beta_k$ with $L^{2-d}$ yields the result (5.10), and similarly for (5.11). See [40] for more details about this type of argument.

It is important at this juncture to point out the implications of Lemmas 5.1 and 5.2 for our analysis of $\frac{d^2\sigma^*}{dp^2}$. For simplicity, let us consider $d = 2$ (with $\epsilon = 0$) so that $L^{2-d} = 1$ in (5.11), which can be written as (after dropping the "$*$" notation)

$$p^2\frac{d^2\sigma}{dp^2} = <\delta^2\tilde{\sigma}(\omega^p)>_p\ , \tag{5.13}$$

where $\omega^p$ is a configuration of the bond lattice with a fraction $p$ of the bonds occupied (which can be viewed as a random graph), and $<\cdot>_p$ denotes averaging over such configurations. Since $\delta^2\tilde{\sigma}(\omega^p)$ involves all pairs of bonds in $\omega^p$, presumably an ergodic theory argument shows that it suffices in (5.13) to consider a single "typical" configuration $B(p)$ of the bond lattice with a fraction $p$ of occupied bonds (i.e., $B(p)$ belongs to a set of full Bernoulli measure in the standard percolation problem). Then (5.13) can be written as

$$p^2\frac{d^2\sigma}{dp^2} = \delta^2\sigma(B(p))\ , \tag{5.14}$$

where we have now dropped the "$\sim$" notation. Also note that, given any graph $G$ with bond conductivities 1 and 0 and conductivity function $\sigma(p)$,

$$\left.\frac{d^2\sigma}{dp^2}\right|_{p=1} = \delta^2\sigma(G)\ . \tag{5.15}$$

Then we see via (5.14) that $p^2\frac{d^2\sigma}{dp^2}(p)$ for the lattice can be computed by finding $\frac{d^2\widehat{\sigma}}{dp^2}|_{p=1}$, where $\widehat{\sigma}$ is the conductivity function for $G = B(p)$. Similar considerations hold for $\frac{d\sigma}{dp}$, with the analog of (5.15) being

$$\left.\frac{d\sigma}{dp}\right|_{p=1} = \delta\sigma(G) \ . \tag{5.16}$$

We now return to the physical argument given above. The principal objection that one can raise to this argument is that the first removal $b_1$ can set up a situation where the second removal results in a larger drop in the conductivity than did the first. For example, consider a configuration of connected bonds in the shape of a Y, where the current flows in through the single leg and out through the two arms. For simplicity let each "limb" of the Y be composed of 1 bond of conductivity 1, with $b_1$ the leg and $b_2$ and $b_3$ the arms. Consider now $\delta^2\sigma(Y) = 2(\delta_{12}+\delta_{23}+\delta_{13})$, where the $\delta_{ij}$ are the summands in (5.5). Elementary calculation shows that $\delta_{12} = \delta_{13} = +\frac{1}{6}$ and $\delta_{23} = \frac{-1}{3}$, so that $\delta^2\sigma(Y) = 0$. The important point to note is that $(2,3)$ is a pair of bonds in parallel, while $(1,2)$ and $(1,3)$ are pairs in series. The pair $(2,3)$ is an example of the objection raised. When $b_2$ is removed, the current can still flow through $b_3$, and the drop in conductivity is minimal. When $b_3$ is subsequently removed, the effect is to cut off all current flow, with the result that the net contribution of the pair to $\delta^2\sigma(Y)$ is negative. However, when $b_1$ is removed first, the flow is stopped immediately, so that $b_2$ and $b_3$ are dangling bonds whose subsequent removal does not affect the conductivity, which is the principal reason why $\delta_{12}$ and $\delta_{13}$ are positive.

For an arbitrary graph $G$, there is no particular reason why the positive $\delta_{ij}$'s should outweigh the negative ones. However, for graphs that are sufficiently stringy, positive contributions from series pairs in a given string should tip the balance to a net positive $\delta^2\sigma(G) > 0$. For example, if we replace each bond in $Y$ above with 2 bonds in series, and call the new graph $Y_2$, then $\delta^2\sigma(Y_2) > 0$. The reason is that the three series pairs in the leg and two arms give new, positive contributions to $\delta^2\sigma(Y_2)$ which were not present in $\delta^2\sigma(Y)$. Such considerations led us to the following

THEOREM 5.1: *Let $G$ be any finite connected graph with bond conductivities 1 or 0, and let $S_nG$ be a new graph formed by replacing each bond of $G$ with $n$ bonds in series. Then*

$$\delta^2\sigma(S_nG) = n\delta^2\sigma(G) + (n-1)\delta\sigma(G) \tag{5.17}$$

$$\delta\sigma(G) \geq 0 \ . \tag{5.18}$$

Before we prove the theorem, its principal consequence is given in

COROLLARY 5.1: *If $G$ satisfies*

$$|\delta^2\sigma(G)| < \delta\sigma(G) \ , \tag{5.19}$$

*then there exists a positive constant $C$ such that*

$$\delta^2\sigma(S_nG) \sim Cn\,, \quad C > 0\,,\ n \to \infty\,. \tag{5.20}$$

PROOF 1 OF THEOREM 5.1: In order to calculate $\delta^2\sigma(S_nG)$, we must consider all pairs of bonds $(i,j)$, $i \neq j$, in the new graph $S_nG$. Let the pairs in the original graph be labeled by $(i',j')$, $i' \neq j'$. Now, the pairs in $S_nG$ are of two types, those which arise from $i' \neq j'$ and those which arise from one original bond $i'$. Denoting the summands in (5.5) again as $\delta_{ij}$, for those $(i,j)$ pairs of the first type, we have

$$\delta_{ij} = \frac{1}{n}\delta_{i'j'}\,, \tag{5.21}$$

where the factor of $\frac{1}{n}$ appears because

$$\sigma(S_nG) = \frac{1}{n}\sigma(G)\,, \tag{5.22}$$

since the conductivity of $n$ bonds in series is $\frac{1}{n}$. For pairs of the second type,

$$\delta_{ij} = \frac{1}{n}\left(\sigma_{i'}(1) - \sigma_{i'}(0)\right)\,, \tag{5.23}$$

where $\sigma_{i'}(1) = \sigma(G)$ and $\sigma_{i'}(0)$ is $\sigma$ of $G$ without bond $i'$. Note that $\delta_{ij}$ in (5.23) is always non-negative. For each $(i',j')$ pair in $G$, $i' \neq j'$, there are $n^2$ pairs of the first type in $S_nG$. For each $i'$ in $G$, there are $n(n-1)$ pairs of the second type in $S_nG$. Thus

$$\delta^2\sigma(S_nG) = \frac{1}{n}\left(n^2\delta^2\sigma(G) + n(n-1)\delta\sigma(G)\right)\,, \tag{5.24}$$

which yields (5.17).

**Remark.** $S_nG$ is composed of a graph $G$ whose elements are strings of $n$ bonds in series. Removal of any one of these bonds converts all the other bonds in that string into dangling bonds. Their subsequent removal has no effect on the conductivity, and this effect is the source of the positive term (5.23), which becomes the second term in (5.17).

PROOF 2 OF THEOREM 5.1: Let the conductivity function of $S_nG$ be denoted by $\widehat{\sigma}(p)$, and that of $G$ be denoted by $\sigma(p)$. Then

$$\widehat{\sigma}(p) = \frac{1}{n}\sigma(p^n)\,. \tag{5.25}$$

Differentiating both sides of (5.25) twice with respect to $p$ and setting $p = 1$ yields

$$\left.\frac{d^2\widehat{\sigma}}{dp^2}\right|_{p=1} = n\left.\frac{d^2\sigma}{dp^2}\right|_{p=1} + (n-1)\left.\frac{d\sigma}{dp}\right|_{p=1}\,, \tag{5.26}$$

which is equivalent to (5.17).

Proof 2 was a joint observation with S. Goldstein, which was made subsequent to the original Proof 1. We chose to include Proof 1 as well because it shows explicitly the convexity improving effect of creating dangling bonds.

We are interested in applying Theorem 5.1 and its corollary to the lattice. Henceforth let $G_L$ be a square (or cubic or hypercubic) sample of side $L$ of the square (or cubic or hypercubic) lattice in $d \geq 2$ with $p = 1$. (Think of $G_L$ as standing for "grid".) Let $G$ be the infinite volume limit of $G_L$. Furthermore, let

$$\delta\sigma^*(G) = \lim_{L\to\infty} L^{2-d}\delta\sigma_L(G_L) \tag{5.27}$$

$$\delta^2\sigma^*(G) = \lim_{L\to\infty} L^{2-d}\delta^2\sigma_L(G_L) \ , \tag{5.28}$$

where on the right hand sides of (5.27) and (5.28) we have simply added an $L$ subscript to the notation used in (5.15) and (5.16). In order for Corollary 5.1 to apply to $G_L$ and $G$, condition (5.19) (appropriately scaled in $L$) must be satisfied. As stated before, for $d = 2$,

$$\delta\sigma^*(G) = \frac{d\sigma^*}{dp}\Big|_{p=1} = 2 \tag{5.29}$$

$$\delta^2\sigma^*(G) = \frac{d^2\sigma^*}{dp^2}\Big|_{p=1} \approx -0.21 \ , \tag{5.30}$$

so that condition (5.19) is presumably satisfied for both $G$ and $G_L$ with $L$ large, although we have not rigorously proven such a statement. In higher dimensions [**7**]

$$\delta\sigma^*(G) = \frac{d\sigma^*}{dp}\Big|_{p=1} = \frac{d}{d-1} \ , \tag{5.31}$$

while numerical simulation of $\sigma^*(p)$ in $d = 3$ [**6**] and analytical solution of $\sigma^*(p)$ for the Bethe lattice, supposedly representing large $d$ [**41, 42**], are practically linear near $p = 1$, so that $\delta^2\sigma^*(G)$ is also small, as in $d = 2$. We thus state an unproven

CONJECTURE 5.1: *Condition (5.19) is satisfied by the bulk conductivity $\sigma^*$ for the square (or cubic or hypercubic) lattice in $d \geq 2$.*

Let us now describe the picture we have in mind. We begin with $G_L$, and apply $S_n$ to it, for some large $n$. The result, $S_nG_L$, can be thought of as a super lattice or grid with side of length $nL$, composed of "strings" or "macrolinks" connecting the old vertices or "nodes" of $G_L$. This super lattice is closely connected to the so-called "node-link" model of the backbone of the infinite cluster for $p$ near $p_c$, $p > p_c$, proposed independently by Skal and Shlovskii [**43**] and de Gennes [**44**]. In this model, the length $n$ of the strings connecting the nodes is greater than the distance between the nodes, reflecting the observation in actual clusters that "strings" are not straight but tend to wander around. The distance between the nodes is assumed to be on the order of a correlation length $\xi$. The geometrical parameters $n$ and $\xi$ in the node-link model are related to $p$ via the scaling relations as $p \to p_c^+$,

$$\xi(p) \sim (p - p_c)^{-\nu} \tag{5.32}$$

$$n(p) \sim (p - p_c)^{-\zeta} . \tag{5.33}$$

The Chayes [12] have rigorously shown that $\zeta \geq \nu$, and also that $\zeta \geq \min\{1, \frac{\nu}{\nu'}\}$, for an appropriately defined $\zeta$, where $\nu'$ is the correlation length exponent as $p$ approaches $p_c$ from below. It is believed, however, that $\nu' = \nu$, and we shall assume that $\zeta \geq 1$. Under these assumptions, an easy calculation leads to the following expression for the conductivity critical exponent,

$$t = (d-2)\nu + \zeta . \tag{5.34}$$

We can make the correspondence between our super lattice $S_nG_L$ and the node-link model exact by allowing our strings to wander as well, and stipulating that the length of a side of $S_nG_L$ is $\xi L$, rather than $nL$. This variation does not alter the conductivity or its derivatives, but only the way the graph is situated in space. It should be remarked that in the node-link model we have generated via $S_nG_L$, four strings meet at each node in $d = 2$. Apparently, though, it is much more common in actual percolation clusters to observe three fold meetings (D. Fisher, private communication). This can be taken into account in our model by letting $G_L$ be a sample of the hexagonal lattice instead of the square lattice. Presumably (5.19) still holds for the hexagonal lattice. Similar considerations apply in higher dimensions as well.

Apparently it is now generally accepted [**14, 12**] that the node-link model is an oversimplification of the backbone structure, particularly in low dimension, especially $d = 2$. Stanley [**45**] has suggested that a more accurate representation of the backbone is provided by a "node-link-blob" model. In this model, nodes separated by a distance $\xi$ are connected by strings or links (of singly connected bonds) and blobs (of multiply connected bonds). One can visualize the connection between nodes as a segment of a necklace of beads on a string where there is some distance between the beads. Also, a node may actually be a blob. These blobs have a self-similar structure, i.e., they have a node-link-blob structure themselves.

We now wish to explore the consequences of Theorem 5.1 and its corollary, under the assumption of Conjecture 5.1 The case of $d = 2$ is considered first. Theorem 5.1 yields

$$\delta^2\sigma(S_nG_L) = n\delta^2\sigma(G_L) + (n-1)\delta\sigma(G_L) . \tag{5.35}$$

Since $L^{2-d} = 1$ for $d = 2$, we can directly take the infinite volume limit of (5.35) (assuming it exists) to obtain

$$\delta^2\sigma^*(S_nG) = n\delta^2\sigma^*(G) + (n-1)\delta\sigma^*(G) . \tag{5.36}$$

Under Conjecture 5.1,

$$\delta^2\sigma^*(S_nG) \sim C_2 n\,, \quad n \to \infty\,, C_2 > 0\,. \tag{5.37}$$

The upshot of (5.37) is that, if one assumes that the backbone of the infinite cluster behaves like the node-link model, then not only is $\frac{d^2\sigma^*}{dp^2}$ positive as $p \to p_c^+$, indicating convexity, but it diverges to $+\infty$. Presumably a similar result can be obtained for a good approximation of a node-link-blob model by taking an appropriate $G$ with a sufficient number of levels of self similarity in each blob (and such that (5.19) holds). Such considerations lead us to

CONJECTURE 5.2: *($d = 2$ bond problem)*

$$\frac{d^2\sigma^*}{dp^2} \to +\infty \quad \text{as } p \to p_c^+\,. \tag{5.38}$$

*Consequently, $\sigma^*(p)$ is convex in $(p_c, p_c + a)$, for some small $a$, and*

$$1 \leq t < 2\,. \tag{5.39}$$

The first inequality in (5.39), as already mentioned, has been rigorously established, but is explained here via convexity of $\sigma^*$ near $p_c = \frac{1}{2}$. The second inequality $t < 2$ comes from (5.38) and

$$\frac{d^2\sigma^*}{dp^2} \sim (p - p_c)^{t-2}\,. \tag{5.40}$$

A tighter upper bound on $t$ is provided by the node-link relation for $t$ in (5.34) for $d = 2$,

$$t \leq \zeta\,, \tag{5.41}$$

where a reasonable numerical estimate for $\zeta$ in $d = 2$ is about 1.35 [**46**]. A rigorous argument yielding (5.41) was shown to the author by H. Kesten. What this bound amounts to is that the conductivity of the node link backbone is smaller than the conductivity of the actual backbone, i.e.,

$$\sigma^*(S_nG) \leq \sigma^*\big(B(p)\big)\,, \tag{5.42}$$

where $B(p)$ is a typical backbone configuration at volume fraction $p$, and $n$ is given by (5.33). Inequality (5.42) is physically reasonable if we imagine $B(p)$ to be constructed from $S_nG$ by adding bonds, which increases the conductivity (see also [**12**]).

We now apply Theorem 5.1 to higher dimensions, with $d = 3$ first. Dividing (5.35) by the length $\xi L$ of a side of $S_nG_L$ yields

$$\frac{\delta^2\sigma(S_nG_L)}{\xi L} = \frac{n\delta^2\sigma(G_L)}{\xi L} + \frac{(n-1)\delta\sigma(G_L)}{\xi L}\,. \tag{5.43}$$

Taking the limit as $L \to \infty$ gives

$$\delta^2\sigma^*(S_nG) = \frac{n}{\xi}\delta^2\sigma^*(G) + \frac{n-1}{\xi}\delta\sigma^*(G) \ . \tag{5.44}$$

Under Conjecture 5.1,

$$\delta^2\sigma^*(S_nG) \sim C_3\frac{n}{\xi} \ , \quad n\,,\xi \to \infty\,, C_3 > 0 \ . \tag{5.45}$$

Now, as we will discuss below, we have rigorous evidence that

$$\delta^2\sigma^*(S_nG) \geq \delta^2\sigma^*(B(p)) \ , \tag{5.46}$$

which, using (5.45), (5.32) and (5.33), becomes

$$\frac{d^2\sigma^*}{dp^2} \sim (p-p_c)^{t-2} \leq (p-p_c)^{\nu-\zeta} \ , \tag{5.47}$$

giving a lower bound on $t$. Repeating the same procedure, but dividing by the appropriate power of $\xi L$ in (5.43), and considering the analog of (5.42) in higher dimensions, leads us to

CONJECTURE 5.3: *($3 \leq d \leq 6$ bond problem)*

$$(d-2)\nu + (2-\zeta) \leq t \leq (d-2)\nu + \zeta, \quad \zeta \geq 1 \ . \tag{5.48}$$

These bounds become tighter with increasing dimension and converge in $d = 6$ where it is believed that $\nu = \frac{1}{2}$ and $\zeta = 1$, with $t = 3$. Using numerical values for $\zeta$ and $\nu$ in dimensions 2 through 5 [**46, 47, 8**], we have plotted in Figure 1 the proposed bounds (5.48), as well as the bound $1 \leq t \leq \zeta$ in $d = 2$. In our choices of values for $\zeta$, we have chosen the largest reasonable ones, which make the bounds the widest. The best current numerical estimate for $t$ in $d = 2$ appears to be $t = 1.303^{+0.004}_{-0.014}$ [**48**], while in $d = 3$ the situation is not as well established. In [**49**], a "relatively well established" value of $\approx 1.9$ is quoted, while very recent numerical estimates of C. Lobb indicate that $t$ in $d = 3$ is "very close to 2" [C. Lobb, private communication]. These values fall within the proposed bounds. The only recent numerical estimates of $t$ for $d = 4$ and 5 known to the author are those in [**46**] and [**47**], where $t$ is computed via (5.34), which is just the upper bound in (5.48).

We now explain (5.46), which says that $\delta^2\sigma^*$ for the node-link model is greater than $\delta^2\sigma^*$ for an actual backbone configuration. The intuitive reason is fairly simple. The node-link model is composed purely of strings, and pairs of bonds from within a given string give purely positive contributions to $\delta^2\sigma^*$. However, the actual backbone is composed of strings <u>and</u> regions of multiply connected bonds. Imagine a blob composed of a piece of the lattice $G$ at $p = 1$ (a "full" blob). Pairs of bonds from within the blob will give a small negative contribution to $\delta^2\sigma^*$. Even if the blob has some string structure within it, the net contribution to $\delta^2\sigma^*$ will still be less than if there were in its place a pure string with length of the same order as the size of the blob. Such statements can be proved under certain conditions, which provides a rigorous basis for (5.46) and forms the content of the informally stated

THEOREM 5.2: *Let $S(n)$ be a graph composed of $n$ bonds in series, i.e., $S(n)$ is a string of length $n$. Let $T(n)$ be a "necklace" composed from $S(n)$, i.e., replace some sections of $S(n)$ with blobs, which can be pieces of the lattice $G$ at $p = 1$, or pieces of $S_m G$, with $m$ sufficiently small compared to the length of the section of $S(n)$ the blob replaced. Then*

$$\delta^2 \sigma(S(n)) \geq \delta^2 \sigma(T(n)) \ . \tag{5.49}$$

The proof of this theorem follows along the lines of Proof 1 of Theorem 5.1, but will be omitted here.

We close by remarking that Straley [**13**] has proposed a (non-rigorous) lower bound on $t$ that is better than the one in (5.48), namely, $t \geq (d-2)\nu + 1$. However, our analysis leading to the lower bound in (5.48) has some interesting consequences for the behavior of $\frac{d^2\sigma^*}{dp^2}$ as $p \to p_c^+$, particularly in $d \geq 3$, which is discussed in [**50**].

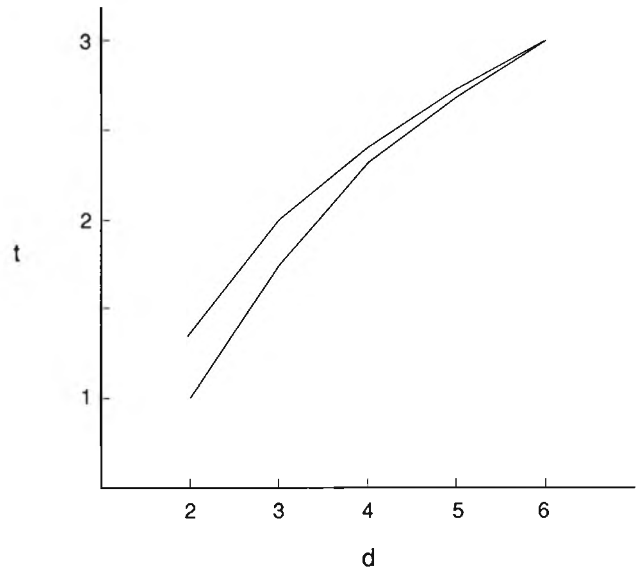


Figure 1. Proposed upper and lower bounds on the conductivity exponent $t$ in terms of the percolation exponents $\nu$ and $\zeta$. Numerical values for $\nu$ and $\zeta$ are used to evaluate the bounds in dimensions $d = 2, 3, 4, 5,$ and $6$. Straight lines have been drawn between these points.

**Acknowledgements.** It is a pleasure to thank a number of people for enjoyable and enlightening discussions during the course of the work: O. Bruno, J. and L. Chayes, D. Fisher, S. Goldstein, H. Kesten, C. McMullen, G. Papanicolaou, C. Simpson, and S. Varadhan. Special thanks is owed to P. Doyle for many long and very useful discussions which clarified certain matters in this work. The author is also very grateful to N. Wilcox for persevering through the difficulties of typing a camera ready copy.

## REFERENCES


1. J.M. Ziman, J. Phys. C, 1 (1968), p. 1532.
2. T.P. Eggarter and M.H. Cohen, Phys. Rev. Lett., 25 (1970), p. 807.
3. T.P. Eggarter and M.H. Cohen, Phys. Rev. Lett., 27 (1971), p. 129.
4. S.R. Broadbent and J.M. Hammersly, Proc. Cambridge Philos. Soc., 53 (1957), p. 629.
5. B.J. Last and D.J. Thouless, Phys. Rev. Lett., 27 (1971), p. 1719.
6. S. Kirkpatrick, Phys. Rev. Lett., 27 (1971), p. 1722.
7. S. Kirkpatrick, Rev. Mod. Phys., 45 (1973), p. 574.
8. D. Stauffer, Phys. Rep., 54 (1979), p. 1.
9. G. Grimmett and H. Kesten, Z. Wahr., 66 (1984), p. 335.
10. H. Kesten, Percolation Theory for Mathematicians, Birkhaüser, Boston, 1982.
11. J.T. Chayes and L. Chayes, Comm. Math. Phys., 105 (1986), p, 133.
12. J.T. Chayes and L. Chayes, Comm. Math. Phys., 113 (1987), p. 27.
13. J.P. Straley, Phys. Rev. C, 15 (1982), p. 2333.
14. A. Coniglio, J. Phys. A, 15 (1982), p. 3829.
15. L. Russo, Z. Wahr., 56 (1981), p. 229.
16. J.P. Straley, Phys. Rev. B, 15 (1977), p. 5733.
17. B.P. Watson and P.L. Leath, Phys. Rev. B, 9 (1974), p. 4893.
18. A.B. Harris and S. Kirkpatrick, Phys. Rev. B, 16 (1977), p. 542.
19. M.H. Ernst, P.F.J. van Velthoven, and Th. M. Nieuwenhuizen, J. Phys. A, 20 (1987), p. 949.
20. Th. M. Nieuwenhuizen, P.F.J. van Velthoven, and M.H. Ernst, Phys. Rev. Lett. 57 (1986), p. 2477.
21. R. Künnemann, Effective conductivity on a lattice as a limit of box conductivities, in Ph.D. Thesis, U. Of Heidelberg, 1983.
22. S.J. Mason and H.J. Zimmermann, Electronic Circuits, Signals, and Systems, John Wiley and Sons, New York, 1960.
23. C. Thomassen, Resistances and currents in infinite electrical networks, to appear in J. Comb. Th. B.
24. K. Golden and G. Papanicolaou, Comm. Math. Phys., 90 (1983), p. 473.
25. D.J. Bergman, Phys. Rep., C43 (1978), p.377.
26. D.J. Bergman and Y. Kantor, J. Phys. C, 14 (1981), p. 3365.
27. O. Bruno, The Effective Conductivity of an Infinitely Interchangeable Mixture, Ph.D. Thesis, New York University, 1989.
28. M.N. Miller, J. Math. Phys., 10 (1969), p. 1988.
29. O. Bruno and K. Golden, Interchangeability and bounds for the effective conductivity of the square lattice, in preparation.
30. A.S. Sangani and A. Acrivos, Proc. R. Soc. Lond. A, 386 (1983), p. 263.
31. J.B. Keller, J. Math. Phys., 5 (1964), p. 548.
32. A.M. Dykhne, Sov. Phys. JETP, 32 (1971), p. 63.

33. W. Kohler and G. Papanicolaou, Bounds for the effective conductivity of random media, in Lecture Notes in Physics Vol. 154, Springer Verlag, New York, 1982, p. 111.
34. J. C. Maxwell, A Treatise on Electricity and Magnetism, Clarendon Press, Oxford, 1892.
35. Lord Rayleigh, Phil. Mag., 34 (1892), p. 481.
36. G. W. Milton, R. C. McPhedran, and D. R. McKenzie, Appl. Phys., 25 (1981), p. 23.
37. R. Fogelholm and G. Grimvall, J. Phys. C, 16 (1983), p. 1077.
38. J. B. Keller, J. Appl. Phys., 34 (1963), p. 991.
39. J. T. Chayes, L. Chayes, G. R. Grimmett, H. Kesten, and R. H. Schonmann, The correlation length for the high density phase of Bernoulli percolation, to appear in Ann. Prob.
40. J. Lamperti, Probability, W. A. Benjamin, Inc., Reading, 1966.
41. R. B. Stinchcombe, J. Phys. C, 6 (1973), p. L1.
42. J. P. Straley, J. Phys. C, 10 (1977), p. 3009.
43. A. S. Skal and B. I. Shlovskii, Sov. Phys.- Semicond., 8 (1975), p. 1029.
44. P. G. de Gennes, J. Physique Lett., 37 (1976), p. L1.
45. H. E. Stanley, J. Phys. A, 10 (1977), p. L211.
46. J. Adler, J. Phys. A, 18 (1985), p. 307.
47. R. Fisch and A. B. Harris, Phys. Rev. B, 18 (1978), p. 416.
48. D. J. Frank and C. J. Lobb, Phys. Rev. B, 37 (1988), p. 302.
49. S. Feng, B. I. Halperin, and P. N. Sen, Phys. Rev. B, 35 (1987), p. 197.
50. K. Golden, Convexity and exponent inequalities for conduction near percolation, in preparation.