VECTOR QUANTIZATION OF IMAGES USING THE L_∞ DISTORTION MEASURE

V. John Mathews

Department of Electrical Engineering
University of Utah
Salt Lake City, Utah 84112, USA

ABSTRACT

This paper considers vector quantization of signals using the L_∞ distortion measure. The key contribution is a result that allows one to characterize the centroid of a set of vectors for the L_∞ distortion measure. A method similar to the LBG algorithm for designing codebooks has been developed and tested. The paper also discusses the design of vector quantizers employing the L_∞ distortion measure in an application in which the occurances of quantization errors with larger magnitudes than a pre-selected threshold must be minimized.

1. INTRODUCTION

In many applications of image compression using vector quantization, it is necessary to limit the maximum distortion introduced in the image during the coding process. An example involves image compression using perceptual threshold functions [1]. Perceptual threshold functions for images define the amount of distortion that can be introduced into the images without being detected by human observers [2], [3], [4]. An image compression system that constrains the quantization errors to below the perceptual threshold values is ***perceptually lossless***. One way in which we can constrain the maximum value of the quantization errors in a vector quantizer system is to employ the L_∞ distortion measure in which the distance between two K-dimensional vectors X and Y is defined as

$$\| X - Y \|_\infty = \max_{i \in [1,K]} | x_i - y_i |, \qquad (1)$$

where x_i and y_i denote the ith samples of X and Y, respectively. In addition to being useful in perceptual coding of images, vector quantization using L_∞ norm has the advantage of computational simplicity. Note that the selection of the nearest neighbor using the L_∞ distortion measure do not require any multiplications. This characteristic of vector quantizer systems employing the L_∞ distortion measure is also shared by vector quantizers that use the L_1 distortion measure [5].

The objective of this paper is to discuss an approach for developing codebooks for vector quantizers employing the L_∞ distortion measure. The key contribution of this paper is a result that allows one to characterize the centroid of a set of vectors for the L_∞ distortion measure. A method similar to the Linde-Buzo-Gray (LBG) algorithm [6] for designing codebooks for this distortion measure is developed using our result. The technique is then extended to the situations in which occurrences of quantization errors larger than some pre-selected threshold should be minimized. The extension of this approach to applications involving perceptual threshold functions is straightforward, but not considered in this paper. Even though several researchers have used the L_∞ distortion measure in image compression problems, this author is not aware of any published results for designing codebooks that minimize such distortion measures.

The rest of the paper is organized as follows: The next section presents the result that characterizes the centroid of a set of vectors for the L_∞ distortion measure. The method for calculating the centroid is also discussed in this Section. The design of codebooks for systems that attempt to limit the occurrences of quantization errors larger than some threshold values is described in Section 3. Section 4 contains experimental results. The concluding remarks are made in Section 5.

2. CENTROID CALCULATION

The information necessary for designing the codebooks using variations of the LBG algorithm is the means for calculating the centroid that minimizes the average L_∞ distortion for a set of vectors. The following theorem is key to our development.

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Theorem 1: Let $X_1, X_2, \cdots, X_N$ be a set of K-dimensional vectors. Let C be the centroid of the vectors that minimizes the average L_∞ distortion between C and the vectors in the above set. Assume that the centroid satisfies the property that the L_∞ distortion between C and any vector in the set is uniquely determined by one of the K indices. Then the ith element of C satisfies the property that it is the median of the ith samples of all the vectors in the set such that

$$\| X_l - C \|_\infty = | x_{l,i} - c_i |,$$

where $x_{l,i}$ and c_i are the ith elements of X_l and C, respectively.

Remark. This theorem provides us with a property of the centroid of a set of vectors when the L_∞ distortion measure is employed. However, it does not provide a direct method for evaluating the centroid itself.

Proof of Theorem 1: Recall that the L_∞ distortion between two vectors can be defined as the limiting case of the L_p distortion when $p \to \infty$, *i.e.*,

$$\| X - Y \|_\infty = \lim_{p\to\infty} \left\{ \sum_{i=1}^{K} | x_i - y_i |^p \right\}^{1/p}. \quad (2)$$

Our objective is to select C such that

$$\begin{aligned} J_\infty &= \textstyle\sum_{l=1}^{N} \| X_l - C \|_\infty \\ &= \textstyle\sum_{l=1}^{N} \lim_{p\to\infty} \left\{ \sum_{i=1}^{K} | x_{l,i} - c_i |^p \right\}^{1/p} \end{aligned} \quad (3)$$

is minimized over all choices of C. The approach we will pursue is to differentiate

$$J_p = \sum_{l=1}^{N} \left\{ \sum_{i=1}^{K} | x_{l,i} - c_i |^p \right\}^{1/p} \quad (4)$$

with respect to each c_m, find the limit of the derivatives as $p \to \infty$, and then set the resulting equations to zero to solve for the centroid vector. The differentiation of J_p gives

$$\frac{\partial}{\partial c_m} J_p = -\sum_{l=1}^{N} \sum_{i=1}^{K} \left(\frac{| (x_{l,i} - c_i) |^p}{| (x_{l,m} - c_m) |^p} \right)^{\frac{1-p}{p}} \cdot \operatorname{sign}\{x_{l,m} - c_m\}, \quad (5)$$

where sign(.) denotes the sign of the function within the parenthesis. Let us consider the quantity

$$\sum_{i=1}^{K} \left(\frac{| (x_{l,i} - c_i) |^p}{| (x_{l,m} - c_m) |^p} \right)^{\frac{1-p}{p}} \operatorname{sign}\{x_{l,m} - c_m\} \quad (6)$$

as $p \to \infty$. One can show that as p approaches ∞, the above term approaches one if $| x_{l,m} - c_m |$ is larger than the magnitudes of all other entries of the difference vector $X_l - C$. When $| x_{l,m} - c_m |$ is smaller than the magnitudes of at least one of the other entries of $X_l - C$, (6) approaches 0 as p approaches ∞. Since the theorem assumes that there is always a single index that results in the maximum magnitude of $| x_{l,i} - c_i |$, we only need to consider the two cases discussed above. On the basis of the above discussion, we can express the partial derivative of J_∞ as

$$\frac{\partial}{\partial c_m} J_\infty = -\sum_{A_m} \operatorname{sign}\{x_{l,m} - c_m\}, \quad (7)$$

where A_m denotes the set of values of l such that $| x_{l,m} - c_m |$ is equal to $\| X_l - C \|_\infty$. The value of c_m that makes the above expression zero is indeed the median of the set $\{x_{l,m} : l \in A_m\}$. This completes the proof of the theorem.

Even though the theorem does not provide us with a direct method of calculating the centroid of a set of vectors, it can be used to iteratively find the centroid. It is tempting to devise a procedure in which we initialize the centroid calculations using an arbitrary vector, and then at each iteration, (i) partition the input vector set into K non-overlapping sets such that the mth set contains all the vectors for which the mth element determined the L_∞ distortion between each vector and the current estimate of the centroid, and (ii) replace the mth element of the current estimate of the centroid with the median of the mth elements of all the vectors in the mth subset in the partition. However, such a method will not, in general, decrease the average distortion between the estimated centroid vector and the input vectors at each iteration. The approach presented here is similar in spirit as the technique suggested above. However, we make very gradual changes by changing the current estimate of the centroid by a small amount in the direction of the vector containing the medians as described above. The algorithm is as follows:

Initialize the centroid estimate with an arbitrary estimate $\hat{C}_0$. Let $\hat{C}_i$ be the estimate of the centroid after the ith iteration. At the ith iteration, compute the L_∞ distortion between each vector X_l in the input set and $\hat{C}_{i-1}$. Partition the input vector set into K non-overlapping subsets so that all input vectors for which the mth element determined the L_∞ distortion between $\hat{C}_{i-1}$ and the vectors are clustered into the mth subset. Let $\hat{c}_{i-1,m}$ denote the mth element of $\hat{C}_{i-1}$ and let $w_{i,m}$ represent the median of the mth elements of the vectors in the mth cluster. Then $\hat{c}_{i-1,m}$ is updated during the

ith iteration as

$$\hat{c}_{i,m} = \hat{c}_{i-1,m} + \mu(w_{i,m} - \hat{c}_{i-1,m}), \qquad (8)$$

where μ is a small positive constant. Experiments with images represented using eight-bit integer pixels have shown that values of μ in the range 0.01 to 0.05 result in very good convergence characteristics. The iterations can be terminated using an appropriate stopping criterion.

3. AN EXTENSION

We now consider a situation in which it is desired to minimize the occurrences of quantization errors with magnitudes larger than a certain threshold. There are several applications in which such a criterion is useful. Image coding systems that employ perceptual threshold models require that the distortion introduced by the quantization process at any location in the image is below the predicted value of the perceptual threshold function at that location. Another application involves residual vector quantizers in which the residual vectors are coded only when the distortion in the vector exceeds certain magnitude [7]. By limiting the number of coded vectors with distortions above this threshold value at any stage, one can reduce the number of vectors that must be coded at subsequent stages.

In this Section, we consider only the case of a constant threshold, even though in many applications, a space-varying threshold function is desirable. The extension of our results to the case involving space-varying threshold is relatively straightforward.

We will modify the objective function for designing the codebook as follows to achieve the desired result. Given a training sequence $\{X_1, X_2, \cdots, X_L\}$, we desire to choose M codevectors $C_1, C_2, \cdots, C_M$ such that

$$J_\infty = \sum_{l=1}^{L} d_\infty(X_l, \hat{X}_l) \qquad (9)$$

is minimized, where $\hat{X}_l$ is the code vector closest to X_l and $d_\infty(X, Y)$ represents a measure of the distortion between the vectors X and Y defined as

$$d_\infty(X,Y) = \begin{cases} 0; \max_{i\in[1,K]} \mid x_i - y_i \mid < \tau \\ \max_{i\in[1,K]} \mid x_i - y_i \mid - \tau; \text{otherwise.} \end{cases} \qquad (10)$$

In the above equation, τ is a pre-selected threshold. A similar modification of the Euclidean cost function was used in [8]. Ideally, we would like to constrain all the distortions to below this threshold. In situations where this is not possible, the system would attempt to design the codebook such that the occurrences and magnitudes of the distortions larger than τ are kept at the minimum possible level.

The following theorem characterizes the centroid of a set of vectors for the distortion measure in (10).

Theorem 2: Let $X_1, X_2, \cdots, X_N$ and C be as in Theorem 1 with the exception that the distortion measure is as defined in (10). Then the ith element of C satisfies the property that it is the median of the ith samples of all the vectors such that $\| X_l - C \|_\infty > \tau$ and $\| X_l - C \|_\infty = \mid x_{l,i} - c_i \mid$.

The proof is a straightforward extension of the proof of Theorem 1, and is therefore, omitted here. It should be clear at this stage that the codebook design algorithm of the previous stage can be easily modified to suit the situation considered here.

4. EXPERIMENTAL RESULTS

A large number of experiments were conducted to assess the usefulness of the techniques described in the previous sections. The results of one such experiment is described here. For this experiment, we compared the performances of a predictive vector quantizer (PVQ) system [9, 10] that uses the L_∞ distortion measure with that of another PVQ system that attempts to keep the quantization errors below some threshold. The PVQ system uses the same predictor structure as described in [5]. One key advantage of this predictor structure is that it can be implemented without multiplications. Consequently, the complete system can be realized using multiplication-free architectures.

Our system employed vectors of size 16 pixels formed from blocks of size 4x4 pixels each. A codebook containing 32 vectors were designed using a training sequence obtained by generating the prediction error sequences for three different monochrome images of size 512x512 pixels with 8 bits per pixel resolution. The centroid calculations used a step size (μ) value of 0.05 in this experiment. We also created a different codebook of the same size for a vector quantizer system that attempts to minimize the occurrences of quantization errors with magnitudes larger than 25.

Figure 4 displays the cumulative distribution functions of the quantization errors estimated by computing the histogram of the error sequences when a fourth image that did not belong to the training set was quantized using the two codebooks. The curve marked 's00' corresponds to the system with no constraints imposed on the error sequences and the curve marked 's25' corresponds to the case for which the system attempted to reduce the occurrences of errors larger than 25 in magnitude. It can be clearly seen that the latter system

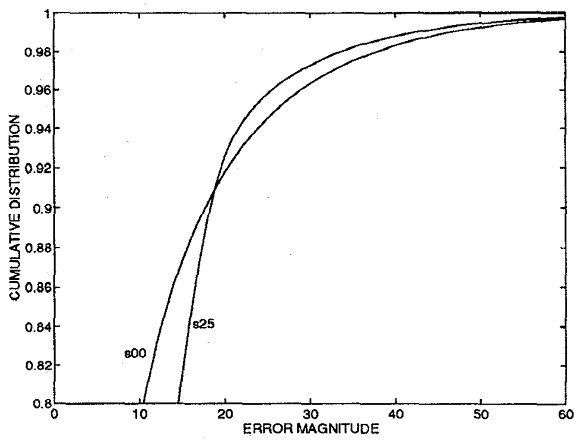

Figure 1: Cumulative distribution of error magnitudes in the experiment.

succeeded in reducing the errors larger than 25 in this experiment. In fact, the number of pixels coded with error magnitudes larger than 25 was approximately 25 percent fewer for this case than the system that attempted to minimize the L_∞ distortion directly. All the experiments that we conducted produced similar results.

5. CONCLUDING REMARKS

This paper presented a method for designing codebooks for vector quantizers that employ the L_∞ distortion measure. The ideas were extended to an application in which it was desired to minimize the occurrences and magnitudes of the quantization errors larger than some pre-selected threshold. A large number of experiments have been conducted, and the results indicate that the technique that attempts to constrain the distortions was successful in reducing the occurrences of error magnitudes larger than the threshold.

A considerable advantage the methods discussed in this paper have is their computational simplicity. The vector quantizer system can be implemented without using any multiplications. The techniques presented in this paper are also useful in image compression systems that utilize models of the perceptual threshold function. The author of this paper is at present studying these and other applications of vector quantizers that employ the L_∞ distortion measure.

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