Chiral Bosonization and Local Lorentz-Invariant Actions for Chiral Bosons

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We formulate a geometrical theory of self-dual scalar fields in two dimensions with a local Lorentz-invariant action. Path-integral quantization is used to derive chiral bosonization by demonstrating equivalence of correlation functions of currents and energy-momentum tensors in our theory with those for Weyl fermions.

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Chiral bosons in two dimensions are the basic building blocks of string theory[1] and are essential to the bosonic formulation of the world-sheet field theory of the heterotic superstring.[2] However, as in the case of self-dual fields in other numbers of space-time dimensions, a manifestly local Lorentz-invariant description of their quantum mechanics is notoriously difficult to formulate.[3] The chiral constraint $\partial_-\phi(x)\approx 0$ is second class and is not easily imposed at the quantum level—the constraint has quantum fluctuations and only expectation values of the fields are chiral. If enforced with a Lagrange multiplier, the multiplier becomes a dynamical field. A second-order constraint suggested by Siegel,[3] $[\partial_-\phi(x)]^2\approx 0$, is first class at the classical level but its algebra has an anomaly and at the quantum level it is a second-class constraint and again Lagrange multipliers fail to decouple. Recently proposed solutions to this problem[4,5] are only partially satisfactory—additional anomaly-canceling terms can be added to the action of Abelian chiral bosons[4]—however, for the non-Abelian case the best one can do is to arrange the number of degrees of freedom so as to cancel the anomaly.[5] Coupling two-dimensional gravity to these models remains problematic.[6] Manifestly covariant Becchi-Rouet-Stora-Tyutin quantization of a single chiral scalar has been argued to be inconsistent.[7] Of course, at the Hamiltonian level, chiral bosons form a consistent Poincaré-invariant and unitary field theory which is equivalent to a theory of Weyl fermions.[8,9] However, to couple to external two-dimensional gravity or to ensure Lorentz invariance in coupling to external gauge fields it is desirable to begin with a manifestly covariant Lagrangean formalism. In this Letter we show that chiral bosonization, namely the requirement that correlation functions of currents and energy-momentum tensors as well as gravitational and gauge anomalies of chiral bosons to match those of Weyl fermions, provides a principle which defines the chiral Bose theory at the quantum level.

Bosonization is essential to the solution of two-dimensional field theories. There have recently been several efforts to extend bosonization to chiral (Weyl) fermions on arbitrary two-dimensional Riemann surfaces.[8] These work in Euclidean space and take advantage of the complex structure of the Reimann surface to define chiral fields as holomorphic operators. In this Letter we take an alternative approach. We work in a 2D space-time where the metric has Minkowski signature and define chirality with respect to the local tangent spaces. To construct our model at the classical level consider a scalar coupled to a two-dimensional background metric:

$$S_B=\int d^2x\,\tfrac{1}{2}[-|g(x)|]^{1/2}g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x). \tag{1}$$

We decompose the metric into a product of *Zweibein* fields contracted with the Minkowski metric, $g_{\mu\nu}(x)=e_\mu^a(x)\eta_{ab}e_\nu^b(x)$. We consider the light-cone frame components of the *Zweibein*,[10] $e_\mu^\pm(x)$, set the positive light-cone components flat and orthonormal, $e_\mu^+=\delta_\mu^+$, and treat the other components as dynamical degrees of freedom. We obtain the action

$$S_{\text{ch}}=\int d^2x[\partial_+\phi(x)\partial_-\phi(x)-\lambda_{++}(x)\partial_-\phi(x)\partial_-\phi(x)], \tag{2}$$

where $\lambda_{++}(x)=e_+^-(x)/e_-^-(x)$ is a Lagrange multiplier which enforces the chiral constraint $[\partial_-\phi(x)]^2\approx 0$. This is the action proposed by Siegel.[3] The classical Poisson-bracket algebra of the constraints closes to form the one-dimensional diffeomorphism algebra corresponding to reparametrizations of x^-. Indeed, under an infinitesimal coordinate transformation $\delta_f x^\mu=f^\mu(x)$ the *Zweibein* transforms like a vector field, $\delta_f e_\mu^a(x)=f^\nu(x)\partial_\nu e_\mu^a(x)+\partial_\mu f^\nu(x)e_\nu^a(x)$. This preserves $e_\mu^+(x)=\delta_\mu^+$ when $f^+(x)=0$. Under the remaining symmetry we have

$$\delta_f\lambda_{++}(x)=f^-(x)\partial_-\lambda_{++}(x)-\partial_-f^-(x)\lambda_{++}(x)+\partial_+f^-(x),\quad \delta_f\phi(x)=f^-(x)\partial_-\phi(x). \tag{3}$$

This reparametrization invariance coincides with Siegel's gauge symmetry.[3] There is also trivial invariance under the local Lorentz transformation $e_\mu^-(x) \to \Lambda^{-1}(x) e_\mu^-(x)$ since the action depends only on the ratio $\lambda_{++}(x)$. This geometrical picture of the chiral constraint indicates a natural coupling of chiral bosons to a background metric, i.e., leaves $e_\mu^+(x)$ an external classical field. The model is

$$S_{\rm ch} = \int d^2x \, \tfrac{1}{2} \, |e(x)| \, E_a^\mu(x) \eta^{ab} E_b^\nu(x) \partial_\mu \phi(x) \partial_\nu \phi(x), \tag{4}$$

where $E_a^\mu(x) = [e_\mu^a(x)]^{-1}$, $\phi(x)$ and $e_\mu^-(x)$ are dynamical fields, and $e_\mu^+(x)$ is the external gravitational field. This poses a classical field theory of chiral scalars with the novel feature that they couple to only one set of chiral components of a background *Zweibein*. At the classical level this coupling has both general coordinate and local Lorentz invariance.

We expect that, as in the case of ordinary bosons and Dirac fermions, there is a correspondence at the quantum level between the dynamics of Weyl fermions and chiral bosons. Indeed, in Minkowski space Weyl fermions also couple to one chiral component of the *Zweibein*. To see this, consider the Weyl action,

$$S_F = i\sqrt{2} \int d^2x \, |e(x)| \, \psi_L^\dagger(x) E_-^\mu(x) [\partial_\mu + \omega_\mu(x)] \psi_L(x),$$

where $\omega_\mu(x)$ is the spin connection. Using the identity $|e(x)| E_a^\mu = -\epsilon^{\mu\nu}\epsilon_{ab} e_\nu^b(x)$,[10] we get

$$S_F = -i\sqrt{2} \int d^2x \, \psi_L^\dagger(x) [e_\mu^+(x) \epsilon^{\mu\nu} \partial_\nu - \tfrac{1}{2} \epsilon^{\mu\nu} \partial_\mu e_\nu^+(x)] \psi_L(x). \tag{5}$$

At the quantum level, the Weyl fermion and chiral boson theories are equivalent if they have identical correlation functions of their energy-momentum tensors, i.e., when the effective actions

$$iS_{\rm ch}^{\rm eff}[e_\mu^+] = \ln \int de_\mu^-(x) d\phi(x) \exp(iS_B[e_\mu^+, e_\mu^-, \phi] + iS_{\rm ct}[e_\mu^+, e_\mu^-]), \tag{6}$$

$$iS_F^{\rm eff}[e_\mu^+] = \ln \int d\psi_L(x) d\psi_L^\dagger(x) \exp(iS_F[e_\mu^+, \psi_L, \psi_L^\dagger]), \tag{7}$$

coincide. In (6) we add local counterterms $S_{\rm ct}$, which are necessary to cancel the anomaly of the Siegel symmetry and to adjust the effective action to match the fermion determinant.

To begin, let us consider the analogous and more familiar case of chiral coupling to external gauge fields. The determinant of the Dirac operator coupled to an external U(1) vector gauge field is

$$\ln \det \gamma^\mu(\partial_\mu - iA_\mu) = -\frac{i}{2\pi} \int d^2x (\partial_- A_+ - \partial_+ A_-) \frac{1}{\partial_+ \partial_-} (\partial_- A_+ - \partial_+ A_-) + \operatorname{tr} \ln \partial_+ \partial_-. \tag{8}$$

Since the massless Dirac operator is the direct sum of left- and right-handed Weyl operators which couple to each of the light-cone components of the vector potential separately, we expect that its determinant factorizes, i.e.,

$$\ln \det \gamma^0 \gamma^\mu (\partial_\mu - iA_\mu) = \ln \det(\partial_+ - iA_+) + \ln \det(\partial_- - iA_-) + (\text{local counterterms}). \tag{9}$$

Inspection of (8) confirms this with the local counterterm $(i/\pi)\int A_+ A_-$ necessary to decouple the left-moving and right-moving parts and yields the Weyl fermion determinant

$$\ln \det(\partial_- - iA_-) = \frac{i}{2\pi} \int A_- \frac{\partial_+}{\partial_-} A_- + \operatorname{tr} \ln \partial_-.$$

The vector coupling of the gauge field bosons is

$$\begin{aligned} iS_B^{\rm eff}[A_+, A_-] &= \ln \int d\phi(x) \exp \left[i \int d^2x [\partial_+ \phi \partial_- \phi + (2\pi)^{-1/2} A_+ \partial_- \phi - (2\pi)^{-1/2} A_- \partial_+ \phi] \right] \\ &= \frac{i}{2\pi} \int d^2x \left| A_+ \frac{\partial_-}{\partial_+} A_+ + A_- \frac{\partial_+}{\partial_-} A_- - 2A_- A_+ \right| - \tfrac{1}{2} \operatorname{tr} \ln \partial_+ \partial_-, \end{aligned} \tag{10}$$

and we could define a theory of bosons with a chiral coupling by adding the local counterterm $(i/\pi)\int A_+ A_-$ and integrating over the unwanted component $A_+(x)$:

$$\begin{aligned} iS_{\rm ch}^{\rm eff}[A_-] &= \ln \int d\phi(x) dA_+(x) \exp \left[i \int d^2x [\partial_+ \phi \partial_- \phi - (2\pi)^{-1/2} A_- \partial_+ \phi + (2\pi)^{-1/2} A_+ \partial_- \phi + \pi^{-1} A_+ A_-] \right] \\ &= \frac{i}{2\pi} \int d^2x \, A_- \frac{\partial_+}{\partial_-} A_- - \operatorname{tr} \ln \partial_-. \end{aligned} \tag{11}$$

This effective action yields the gauge anomaly of a Weyl fermion. Note that the normalization of the path integral indicates that this model has the (negative) degrees of freedom of a Weyl fermion, i.e., the normalization $\det\partial_-^{-1}$ corresponds to twice the degrees of freedom of a chiral Bose field which would have $\det\partial_-^{-1/2}$ or $\det\partial_+^{-1/2}$. Our theory corresponds to a bosonized Wyel fermion which has two Majorana-Weyl degrees of freedom.

For external gravitational fields we revert to the second-order constraint and the formalism of (6) and (7). The Dirac operator is the direct sum of Weyl operators which couple to $e_\mu^+(x)$ and $e_\mu^-(x)$ separately,

$$|e|\gamma^0\gamma^a E_a^\mu(\partial_\mu+\omega_\mu\sigma) = -\sqrt{2}\begin{pmatrix} e_\mu^+\epsilon^{\mu\nu}\partial_\nu - \frac{1}{2}\epsilon^{\mu\nu}\partial_\mu e_\nu^+ & 0 \\ 0 & -e_\mu^-\epsilon^{\mu\nu}\partial_\nu + \frac{1}{2}\epsilon^{\mu\nu}\partial_\mu e_\nu^- \end{pmatrix}, \tag{12}$$

where σ is the spin matrix and the Dirac determinant must factorize,

$$\ln\det|e|\gamma^0\gamma^a E_a^\mu(\partial_\mu+\omega_\mu\sigma) = \Gamma^+[e_\mu^+] + \Gamma^-[e_\mu^-] + (\text{local counterterms}). \tag{13}$$

To get the Weyl determinant we use the result of bosonization of Dirac fermions in a gravitational background field[11]:

$$\begin{aligned}\ln\det|e|\gamma^0\gamma^a E_a^\mu(\partial_\mu+\omega_\mu\sigma) &= \ln\int d\phi(x)\exp(iS_B[\phi,e_\mu^a]) + (\text{local counterterms})\\ &= \frac{i}{96\pi}\int d^2x\,|e|\,R\frac{1}{\nabla^2}R + (\text{local counterterms}),\end{aligned} \tag{14}$$

from which, using (13), we identify the effective actions for the left-moving and right-moving Weyl fermions by inspection:

$$\Gamma^+[e_\mu^+] = \frac{i}{48\pi}\int d^2x\,\partial_+^2\lambda_{--}\frac{1}{\partial_+(\partial_- - \lambda_{--}\partial_+)}\partial_+^2\lambda_{--}, \tag{15a}$$

$$\Gamma^-[e_\mu^-] = \frac{i}{48\pi}\int d^2x\,\partial_-^2\lambda_{++}\frac{1}{\partial_-(\partial_+ - \lambda_{++}\partial_-)}\partial_-^2\lambda_{++}. \tag{15b}$$

Here Γ^+ and Γ^- depend only on the ratios $\lambda_{--}=e_-^+/e_+^+$ and $\lambda_{++}=e_+^-/e_-^-$, respectively, and are therefore invariant under local Lorentz transformations $e_\mu^+(x)\to\Lambda(x)e_\mu^+(x)$, $e_\mu^-(x)\to\Lambda^{-1}(x)e_\mu^-(x)$. They are not invariant under general coordinate transformations and local counterterms can be added so that their variation reproduces the consistent gravitational anomalies for right-moving and left-moving fermions. We also obtain the local counterterms, $S_{\rm ct}=\Gamma^++\Gamma^- - (1/96\pi)\int|e|R\nabla^{-2}R$, which must be added to the classical action in (6) to decouple the left-moving and right-moving sectors:

$$\begin{aligned}S_{\rm ct}[e_\mu^a] = -\frac{1}{48\pi}\int d^2x\Big\{&\partial_-\lambda_{++}\frac{\lambda_{--}}{1-\lambda_{++}\lambda_{--}}\partial_-\lambda_{++} + \partial_+\lambda_{--}\frac{\lambda_{++}}{1-\lambda_{++}\lambda_{--}}\partial_+\lambda_{--}\\ &-\partial_+\lambda_{--}\frac{2}{1-\lambda_{++}\lambda_{--}}\partial_-\lambda_{++} + \tfrac{1}{2}\ln(e_+^+e_-^-)|e|\nabla^2\ln(e_+^+e_-^-)\\ &-\frac{2}{1-\lambda_{++}\lambda_{--}}[\partial_+\lambda_{--}(\partial_+ - \lambda_{++}\partial_-) + \partial_-\lambda_{++}(\partial_- - \lambda_{--}\partial_+)]\ln(e_+^+e_-^-)\Big\}.\end{aligned} \tag{16}$$

Using (6), (13), and (14) we compute the effective action for chiral bosons:

$$\int de_\mu^-(x)d\phi(x)\exp(iS_B[\phi,e_\mu^a]+iS_{\rm ct}[e_\mu^a]) = \exp(i\Gamma^+[e_\mu^+])\int de_\mu^-(x)\exp(i\Gamma^-[e_\mu^-]) \tag{17}$$

and $S_{\rm ch}^{\rm eff}[e_\mu^+]=\Gamma^+[e_\mu^+]$ which coincides with that for Weyl fermions. This completes the proof that correlation functions for energy-momentum tensors in the present formulation of chiral bosons coincide with those for complex Weyl fermions.

To make sense of the right-hand side of (17) it is necessary to define the integration measure $de_\mu^-(x)$. A 2D metric can be parametrized by a conformal transformation $\delta g_{\mu\nu}=\rho g_{\mu\nu}$ and a diffeomorphism $\delta g_{\mu\nu}=\nabla_\mu f_\nu+\nabla_\nu f_\mu$. It was shown by Polyakov[12] in the context of his string theory that the measure for the integration over two-metrics decomposes as

$$dg_{\mu\nu}(x) = d\rho(x)df^+(x)df^-(x)\det\nabla_1^+\det\nabla_1^-,$$

where $\nabla_\alpha^\pm=\epsilon^{\mu\nu}e_\mu^\pm\partial_\nu - \alpha\epsilon^{\mu\nu}\partial_\mu e_\nu^\pm$ and $f^\pm$ parametrize the diffeomorphisms and $\det\nabla_1^\pm$ are Faddeev-Popov determinants. The integration over *Zweibeine* can be parametrized similarly and defined as

$$de_\mu^a(x) = d\Lambda^+(x)d\Lambda^-(x)df^+(x)df^-(x)\det\nabla_1^+\det\nabla_1^-,$$

where $\Lambda^{\pm}(x)=\frac{1}{2}[\rho(x)\pm\ln\Lambda(x)]$ are combinations of local Lorentz and conformal transformations. Since ∇_1^+ and ∇_1^- depend only on e_μ^+ and e_μ^-, respectively, the *zweibein* integration measure factorizes into measures for left-moving and right-moving degrees of freedom and we identify $de_\mu^-(x)=d\Lambda^-(x)df^-(x)\det\nabla_1^-$. Up to local counterterms, the Faddeev-Popov determinant is fixed by the conformal anomaly of the ghost variables. Its essential nonlocal part is given by[12] $\ln\det\nabla_1^-=-26i\Gamma^-[e_\mu^-]$ and the functional integral on the right-hand side of (16) is

$$N=\int d\Lambda^-(x)df^-(x)\exp\{(1-26)i\Gamma^-[e_\mu^-]\}.$$

Since Γ^- is invariant under local Lorentz transformations this contains an unrestricted integration over $\Lambda^+(x)$ and is therefore proportional to the volume of the local Lorentz group. By adding local counterterms to Γ^- which are proportional to

$$-2\partial_-^2\lambda_{++}\ln e_-^-+\ln e_-^-(\partial_+\partial_--\partial_-\lambda_{++}\partial_-)\ln e_-^-$$

we regain invariance under the diffeomorphism (Siegel gauge) transformation (3) (but no longer Lorentz invariance) and the integration is proportional to the volume of the Siegel gauge group:

$$N=(\text{vol}.\delta_f)\int d\Lambda^-(x)\exp\left\{i\frac{(1-26)}{48\pi}\int d^2x\Lambda^-\partial_+\partial_-\Lambda^-\right\}.$$

If we had used our formalism for D chiral bosons instead of 1 we would replace the $(1-26)$ in the integrand $(D-26)$ and for the critical number $D=26$ the integral would be proportional to the volumes of both the local Lorentz and Seigel gauge groups. The mechanism we have used for canceling the Siegel anomaly by introducing the conformal field $\ln e_-^\pm$ is similar to Polyakov's[12] method for canceling the conformal anomaly in string theory away from critical dimensions and we see that the conformal degrees of freedom decouple only for critical D.

It is straightforward to extend the present analysis to consider both gauge currents and gravity. It is also straightforward to extend these considerations to chiral non-Abelian bosonization where our method for canceling the Siegel anomaly yields a consistent theory of non-Abelian chiral bosons away from critical dimensions,[13] and we expect that our results will be useful in constructing bosonic formulations of superstring theories on group manifolds and for 2D supergravity. A canonical analysis to determine the spectrum of the model would be interesting. How to bosonize 2D Weyl-Majorana fermions remains unknown. The effective actions $\Gamma^{\pm}$ correspond to the sums of all Feynman diagrams with one Weyl fermion loop and arbitrary numbers of external gravitons. As in the case of a non-Abelian chiral couping to gauge fields it is usually presented in the form of a Wess-Zumino action which is local but refers to one extra dimension. Here we have obtained the exact nonlocal chiral effective action in two dimensions.

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