Holography and Noncommutative Yang-Mills Theory

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In this Letter a recently proposed gravity dual of noncommutative Yang-Mills theory is derived from the relations between closed string moduli and open string moduli recently suggested by Seiberg and Witten. The only new input one needs is a simple form of the running string tension as a function of energy. This derivation provides convincing evidence that string theory integrates with the holographical principle and demonstrates a direct link between noncommutative Yang-Mills theory and holography.

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By now it becomes clear that any consistent theory that unifies quantum mechanics and general relativity requires dramatically new ideas beyond what we have been familiar with. Two such ideas, *the holographic principle* [1] and *noncommutative geometry* [2], have recently attracted increasing attention in the string theory community. The holographic principle [1], originally motivated by the area dependence of black hole entropy, asserts that all information on a quantum theory of gravity in a volume is encoded in the boundary surface of the volume. Though this principle seems to conflict with our intuition from local quantum field theory, string theory as a promising candidate of quantum gravity is believed [3] to integrate with it. Indeed the Maldacena conjecture [4], motivated by the D-brane models of black hole in string theory, is nothing but an embodiment [5] of the holographic principle: There is an equivalence or correspondence between supergravity (or closed string theory) on an anti–de Sitter space (AdS), say of five dimensions, and a supersymmetric Yang-Mills gauge theory on its four-dimensional boundary.

In a parallel development, Yang-Mills theory on a space with noncommutative coordinates [6], which we will call noncommutative Yang-Mills theory (NCYM), has been found to arise naturally in string theory, first in the multi-D-brane description [7], then in matrix theory [8,9] or string theory [10] compactifications with nonvanishing antisymmetric tensor background, and most recently in a special limit that decouples closed string contributions from the open string description for coincident D-branes with a constant rank-2 antisymmetric tensor B-background (see [11], and references therein). Right now it is the last case that is the focus of attention. A rather thorough discussion of the aspects of NCYM from the open string versus closed string perspectives has been given in the work of Seiberg and Witten [11] which, among other things, also clarifies several puzzles previously encountered in NCYM, including the one raised by one of us [12]. Moreover, the supposed-to-be gravity duals of NCYM's in the decoupling limit, which generalize the usual Maldacena conjecture without B-background, were also constructed [13,14].

One might think that NCYM is relevant only in nongeneric situations, such as the above-mentioned decoupling limit in which the B-background is brought to infinity, so it cannot shed much light on deep issues such as holography and quantum nature of spacetime. In this Letter we show that this is not true. With an observation made on a direct link between the NCYM and its gravity dual, we will try to argue for the opposite: Switching on a B-background allows one to probe the nature of holography with NCYM and will probably lead to uncovering more, previously unsuspected links between a large-N theory and its closed string dual.

One of the central observations in [11] is that the natural moduli to use in open string theory with ends on a set of N coincident D-branes in the presence of a constant B field are different from those defined for closed strings. The effective action for the system is more elegantly written if one uses the open string metric G_{ij} and an antisymmetric tensor θ^{ij}, and their relation to the closed string metric g_{ij} and the antisymmetric tensor field B_{ij} is

$$G_{ij} = g_{ij} - (\alpha')^2 (Bg^{-1}B)_{ij}\,, \qquad \theta^{ij} = 2\pi\alpha' \left(\frac{1}{g + \alpha' B}\right)_A^{ij}, \tag{1}$$

where the subscript A indicates the antisymmetric part. Our normalization of the B field differs from that in [11] by a factor of 2π. Seiberg and Witten noted that in the limit $\alpha' \to 0$ and $g_{ij} \to 0$ (assuming B_{ij} is nondegenerate), it is possible to keep G_{ij} and θ^{ij} fixed with a fixed B_{ij}. The tree level effective action surviving this limit is the noncommutative Yang-Mills action, with a star product of functions defined using θ^{ij}:

$$f * g(x) = e^{i/2\theta^{ij}\partial_i^x \partial_j^y} f(x) g(y)|_{y=x}\,. \tag{2}$$

This is equivalent to the noncommutativity: $x^i * x^j - x^j * x^i = i\theta^{ij}$ [15].

The open string coupling constant G_s, proportional to the Yang-Mills coupling $g_{\rm YM}^2$, is also different from the

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closed string coupling constant. The relation between the two is

$$G_s = g_s\left(\frac{\det G}{\det(g + \alpha' B)}\right)^{1/2}. \tag{3}$$

It is easy to see that in the "double scaling limit" when $\alpha' \to 0$, $g \to 0$, keeping fixed the open string coupling G_s (say for D3-branes), g_s must be taken to zero too. Thus closed strings decouple from the open string sector described by the NCYM. We will see that this perfectly matches with the closed string dual description of the NCYM.

Using the conventions of [14], the Neveu-Schwarz (NS) fields in the gravity dual proposed for D3-branes with a constant B_{23} are [13,14]

$$\begin{aligned} ds_{str}^2 &= R^2u^2\left[(-dt^2 + dx_1^2) + \frac{1}{1 + a^4u^4}(dx_2^2 + dx_3^2)\right] \\ &\quad + R^2\frac{du^2}{u^2} + R^2 d\Omega_5^2, \\ B_{23} &= B\frac{a^4u^4}{1 + a^4u^4}, \\ e^{2\phi} &= g^2\frac{1}{1 + a^4u^4}, \end{aligned} \tag{4}$$

where we have set $\alpha' = 1$. The constant B is the value of B_{23} at the boundary $u = \infty$, and the constant g is the closed string coupling in the infrared $u = 0$. Here $R^4 = 4\pi gN$, and the parameter a is given by

$$Ba^2 = R^2. \tag{5}$$

In addition to the usual $C^{(4)}$ induced by the presence of D3-branes, there is also an induced $C^{(2)}$ field. Its presence is quite natural for D3-branes. Recall that a constant B_{23} on the branes can be replaced by a constant magnetic field. Performing S-duality transformation, this field becomes the electric field $E = F_{01}$. This electric field is defined using the dual quanta, thus it is equivalent to a C_{01}. The u-dependent C_{01} is given in [14]:

$$C_{01} = \frac{a^2R^2}{g}u^4. \tag{6}$$

It is natural to interpret the fields appearing in the gravity dual (4) as closed string moduli. Note that apart from the u^2 factor, there is an additional factor $1/(1 + a^4u^4)$ in the closed string metric on the plane (x_2, x_3). Thus if one is to hold the geometry on the plane (t, x_1) fixed, then the geometry on the plane (x_2, x_3) shrinks when the boundary is approached. By the UV/IR relation [16], this means that the closed string metric shrinks to zero in the UV limit from the open string perspective. The following is our central observation. We identify the UV limit as the double scaling limit of [11], thus that g_{ij} shrinks in the UV limit is quite natural. In this limit, α' must also approach zero. This is just right in the AdS conformal field theory (AdS/CFT) correspondence [4]. Note that there is an overall factor R^2u^2 in (4) for the 4D geometry along D3-branes. This redshift factor can be interpreted as the effective string tension

$$\alpha'_{\text{eff}} = \frac{1}{R^2u^2}. \tag{7}$$

Therefore α'_{eff} also approaches zero in the UV limit. The manner in which it approaches zero compared to g_{ij} agrees with the limit taken in [11]. Note that here we differ from the philosophy of [14] in which α' itself is taken to zero, while we have set it to be 1.

Now we are ready to derive the NS fields in the closed string dual (4) by applying formulas (1) and (3). The way in which Seiberg and Witten derived these formulas is valid if we treat strings as effective strings at a fixed energy scale when loop effects are included. Thus we can take these formulas as giving relations among the open string moduli and the closed string moduli at a fixed cutoff $E = u$. Bigatti and Susskind argued [17] that in the large-N limit, the effective action of NCYM can be obtained by replacing the usual product in the effective action of $\mathcal{N} = 4$ SYM by the star product. This, in particular, implies that there is no renormalization for the open string metric, the Yang-Mills coupling constant G_s, and the noncommutative moduli θ^{ij}. *Now with α' replaced by α'_{eff} at a fixed energy scale, the closed string moduli are renormalized.* Because of the rotational symmetry on the (x_2, x_3) plane, we introduce ansatz

$$g_{ij} = f(u)\delta_{ij}, \qquad B_{ij} = h(u)\epsilon_{ij}. \tag{8}$$

The first equation in (1) yields

$$\delta_{ij} = \delta_{ij}\big(f + h^2f^{-1}/(R^4u^4)\big), \tag{9}$$

or

$$f^2 + \frac{h^2}{R^4u^4} = f. \tag{10}$$

The second equation in (1) leads to

$$2\pi\frac{1}{R^2u^2}\left(f^2 + \frac{h^2}{R^4u^4}\right)^{-1}\frac{h}{R^2u^2}\epsilon_{ij} = 2\pi\frac{a^2}{R^2}\epsilon_{ij}, \tag{11}$$

where we used the fact that θ^{ij} is not renormalized and is given by $(2\pi/B)\epsilon_{ij}$, and $B = R^2/a^2$. This second equation is just

$$h = a^2R^2u^4\left(f^2 + \frac{h^2}{R^4u^4}\right). \tag{12}$$

Combined with Eq. (10) we have $h = a^2R^2u^4f$, and substituting this into Eq. (10) we find

$$f(u) = \frac{1}{1 + a^4u^4}. \tag{13}$$

This is precisely what appeared in (4) which is obtained as a solution to classical equations of motion in closed string theory. With $h = a^2R^2u^4f$ we find

$$h(u) = \frac{a^2R^2u^4}{1 + a^4u^4} = B\frac{a^4u^4}{1 + a^4u^4}, \tag{14}$$

also agreeing with (4),

Substitute the above solution into (3) with the identification $G_s = g$, and the energy dependent closed string coupling is also solved

$$g_s = g[\det(g + \alpha'_{\rm eff}B)]^{1/2} = g(1 + a^4u^4)^{-1/2}. \quad (15)$$

Again, this agrees with (4). The closed string coupling becomes weaker and weaker in the UV limit. Although this UV asymptotic freedom appears in the closed string dual, it does not mean that there is asymptotic freedom in NCYM, as we have seen that the Yang-Mills coupling is not renormalized in the large-N limit.

Having shown that the rather *ad hoc* looking closed string background is naturally a solution to (1) and (3), one still has room to doubt whether this is a coincidence. To check that our procedure is indeed a correct one, we turn to the case when both B_{01} and B_{23} are turned on. The solution is also found in [13,14]. We use the Euclidean signature for all coordinates. In such a case, both the geometry of (t, x_1) and the one of (x_2, x_3) shrink at the boundary in the similar manner. Let a be defined as before, and a' related to B_{01} in the same way as a is related to B_{23}. We need not repeat the above steps in deriving the metric and the B field, since these fields are block diagonalized, and so we have similar results. The closed string coupling is given by, in this case,

$$\begin{aligned} g_s &= g[\det(g + \alpha'_{\rm eff}B)]^{1/2} \\ &= g(1 + a^4u^4)^{-1/2}(1 + a'^4u^4)^{-1/2}, \end{aligned} \quad (16)$$

where the determinant is taken of the matrix including all components. This result agrees with the classical solution in [13,14]. Other parts of the closed string metric cannot be reproduced so simply. Because of the result of Bigatti and Susskind, and the unbroken R-symmetry SO(6), it must be identical to that in the AdS case without B field background.

We see that the relations among the closed string moduli and the open string moduli contain much more than we could have imagined. With the input $\alpha'_{\rm eff}$, they determine the closed string dual of NCYM. We venture to conjecture that this is a quite general fact. It can be applied, for instance, to other Dp-brane cases with constant B fields, and perhaps other backgrounds. Also, we expect that $1/N$ corrections will at least renormalize the Yang-Mills coupling. The relation (3) likely holds in this case, thus the closed string coupling must be renormalized by the $1/N$ effects.

We now apply our procedure to Dp-branes with constant B field. Switching on B_{23} only, the solution in this case is given in [14]. This solution is similar to (4). For the shrinking metric, we replace $1/(1 + a^4u^4)$ by $1/[1 + (au)^{7-p}]$, where a is determined by $B^2a^{7-p} = R^{7-p}$. The overall factor $(Ru)^2$ in (4) is replaced by $(Ru)^{(7-p)/2}$. The u-dependent B field is

$$B_{23} = B\frac{(au)^{7-p}}{1 + (au)^{7-p}}, \quad (17)$$

and the dilaton field is

$$e^{2\phi} = g^2u^{(7-p)(p-3)/2}\frac{1}{1 + (au)^{7-p}}. \quad (18)$$

To obtain this solution from (1), we need to use $\alpha'_{\rm eff} = (Ru)^{(p-7)/2}$. Introducing the same ansatz for g_{ij} and B_{ij} as before, the two equations in (1) combine to yield

$$h = (aR)^{(7-p)/2}u^{7-p}f. \quad (19)$$

Substitute this into either equation in (1) and we obtain

$$f = \frac{1}{1 + (au)^{7-p}}. \quad (20)$$

This together with (19) results in the correct answer for the B field.

Relation (3) determines

$$g_s^2 = G_s^2 f. \quad (21)$$

To agree with (18) we must have $G_s^2 = g^2u^{(7-p)(p-3)/2}$. This just means that the open string coupling runs in the same way as in the case when there is no B field. This fact certainly agrees with the result of [17] in the large-N limit.

Our derivation of the closed string dual from NCYM is based on the idea that at the dynamical level, namely, when the quantum effects in NCYM are included by renormalization down to fixed energy scale $E = u$, the relations among the open string moduli and closed string moduli in view of effective strings should be the one derived at the tree level, as done in [11]. Although this idea may find its root in some previous known facts, such as the Fischler-Susskind mechanism and Polyakov's introduction of Liouville field to mimic quantum effects in QCD, we have not found a similar precise statement in the literature.

To complete our derivation of the closed string dual in the D3-brane case, we still need to explain the effective string tension, namely, $\alpha'_{\rm eff} = 1/(R^2u^2)$. As we explained, this relation is natural in the supergravity side of the original AdS/CFT correspondence. We are yet to understand it directly in the gauge theory. If we measure energy with time t in the gauge theory with metric component $g_{00} = -1$, we identify $E = u$. In SYM without noncommutative parameters, this is the only scale, thus $\alpha'_{\rm eff} \sim 1/E^2 = 1/u^2$. Yet we have to attribute the coefficient $1/R^2$ to quantum effects. Back to NCYM, there are two scales, one of which is determined by θ, another energy scale. According to the result of [17], the large-N string tension is the same as in ordinary SYM, thus $\alpha'_{\rm eff} = 1/(Ru)^2$ in NCYM too.

We have left out the derivation of the metric du^2/u^2 in the induced dimension. This together with other parts of the metric induces an anomaly term in the special conformal transformation. This term is computed in SYM in [18] at the one loop level, providing a satisfactory interpretation of the full AdS metric. Although NCYM is not conformally invariant, we expect that a similar calculation to that in [18] can be done and thus justifies du^2/u^2.

The fact that the geometry on (x_2, x_3) shrinks toward the boundary when $B_{23} \neq 0$ has been a source of confusion. Our scheme clarifies this issue. The geometry in the supergravity dual is not to be confused with the geometry in NCYM. The latter remains fixed at all energy scales, while the geometry felt by closed strings becomes degenerate in the UV limit, as it ought to be according to [11].

There remain quite a few puzzles to be understood within our scheme. We need to understand why and how the Gubser-Klebanov-Polyakov-Witten prescription [5] works, and why the calculation of the force between a pair of heavy quark and antiquark requires a different prescription [14]. Among other things, it can be shown that there does not exist a nontrivial geodesic connecting two points on the boundary and separated in x_2. This already indicates interesting behavior of correlation functions at the two point function level. We leave these issues for future study.

In conclusion, we have observed that the moduli in the gravity duals of NCYM should be understood as closed string moduli, and they can be reproduced from the Seiberg-Witten relations (1) between open and closed string moduli, once the string tension in these relations is understood as a running one with simple energy dependence as given in (7). This observation not only helps clarify a few puzzles in understanding the gravity dual of NCYM but also, more importantly, demonstrates a simple and direct connection between NCYM and the holographic principle, either of which is believed to play a role in the ultimate theoretical structure for quantum gravity. In particular, the existence of such a link between NCYM and holography seems to indicate the fundamental significance of noncommutativity in space or spacetime. (For another perspective pointing to the same direction, see [19].) To search for an appropriate geometric framework for the nonperturbative formulation of string/M theory, further exploring the connections between these two themes should be helpful. For instance, it would be interesting to consider further deformation of NCYM, to see what it will lead to as the holographic image on boundary for a closed string dual in bulk.

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