On the Inversion of Certain Nonlinear Systems

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***Abstract*— In this letter, we present some theorems for the exact inversion and the pth-order inversion of a wide class of causal, discrete-time, nonlinear systems. The nonlinear systems we consider are described by the input–output relationship $y(n) = g[x(n)]h[x(n-1), y(n-1)] + f[x(n-1), y(n-1)]$, where $g[\cdot]$, $h[\cdot, \cdot]$, and $f[\cdot, \cdot]$ are causal, discrete-time and nonlinear operators and the inverse function $g^{-1}[\cdot]$ exists. The exact inverse of such systems is given by $z(n) = g^{-1}[\{u(n) - f[z(n-1), u(n-1)]\}/h[z(n-1), u(n-1)]]$. Similarly, when $h[\cdot, \cdot] = 1$, the pth-order inverse is given by $z(n) = g_p^{-1}[u(n) - f[z(n-1), u(n-1)]]$ where $g_p^{-1}[\cdot]$ is the pth-order inverse of $g[\cdot]$.**

***Index Terms*—Inverse systems, nonlinear filters, nonlinear systems.**

I. Introduction

THIS letter presents some results for the exact inversion of nonlinear systems described by the input–output relationship

$$y(n) = g[x(n)]h[x(n-1), y(n-1)] + f[x(n-1), y(n-1)] \tag{1}$$

where $g[\cdot]$, $h[\cdot, \cdot]$, and $f[\cdot, \cdot]$ are causal, discrete-time, and nonlinear operators, and the inverse function $g^{-1}[\cdot]$ exists. We also present expressions for the pth order inverses of systems of the form

$$y(n) = g[x(n)] + f[x(n-1), y(n-1)]. \tag{2}$$

The second result is useful in situations where the exact inverse system does not exist, or is not stable. Even when the exact inverse does not exist, the class of filters in (2) admits efficient realizations of their pth order inverses.

II. The Inverse of Certain Nonlinear Systems

In all of our discussions, we assume causal signals, i.e., all the signals are identically zero for time indices less than zero. The following theorem shows how to evaluate the exact inverse of (1).

Theorem 1: Let $g[\cdot]$, $h[\cdot, \cdot]$ and $f[\cdot, \cdot]$ be causal nonlinear discrete operators and let the inverse operator $g^{-1}[\cdot]$ exist. Then, the exact inverse of the system in (1) is described by the input–output relationship

$$z(n) = g^{-1}\left[\frac{u(n) - f[z(n-1), u(n-1)]}{h[z(n-1), u(n-1)]}\right] \tag{3}$$

where $u(n)$ and $z(n)$ are the input signal and output signal, respectively, of the system.

Proof: We demonstrate first that the system in (3) is the post-inverse of (1), i.e., a cascade interconnection of the system in (1) followed by the system in (3) results in an identity system. We proceed by mathematical induction. Let $x(n)$ and $y(n)$ represent the input and output signals, respectively, of the system in (1). To prove the theorem using induction, we assume that

$$z(n-i) = x(n-i) \qquad \forall\, i > 0. \tag{4}$$

We must now show using (4) that

$$z(n) = x(n) \tag{5}$$

when $u(k) = y(k)$ for $k \leq n$. Now we obtain (6), shown at the bottom of the next page. By substituting $z(n-i) = x(n-i)$ from (4) into (6), it follows in a straightforward manner that $z(n) = x(n)$. We can prove in a similar manner that the system in (3) is also the pre-inverse of the system in (1), i.e., a cascade interconnection of the system in (3) followed by the system in (1) results in an identity system. This completes the proof.

Remark: The inverse of the system in (1) may not exist or may not be stable. For example, if

$$h[z(n-1), u(n-1)] = 0 \tag{7}$$

at any time for some specific input signal, the inverse system of (1) is unstable.

Example 1: We wish to find the inverse of the bilinear system

$$y(n) = x(n) + \sum_{i=1}^{N-1} a_i x(n-i) + \sum_{i=1}^{N-1} b_i y(n-i) + \sum_{i=0}^{N-1}\sum_{j=1}^{N-1} c_{ij} x(n-i) y(n-j). \tag{8}$$

Let us define f, h, and g to be

$$f[x(n-1), y(n-1)] = \sum_{i=1}^{N-1} a_i x(n-i) + \sum_{i=1}^{N-1} b_i y(n-i) + \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} c_{ij} x(n-i) y(n-j) \tag{9}$$

$$h[x(n-1), y(n-1)] = 1 + \sum_{j=1}^{N-1} c_{0j} y(n-j) \tag{10}$$

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and

$$g[x(n)] = x(n) \tag{11}$$

respectively. Then, we can utilize Theorem 1 to find the inverse of the bilinear system to be shown in (12), at the bottom of the page.

A simpler expression can be found for the inverse filter of the system in (2). The following corollary can be immediately derived from Theorem 1.

Corollary 1: Let $g[\cdot]$ and $f[\cdot,\cdot]$ be causal nonlinear discrete operators and let the inverse operator $g^{-1}[\cdot]$ exist. Then, the causal discrete nonlinear system described in (2) has the inverse system whose input–output relationship is given by

$$z(n) = g^{-1}[u(n) - f[z(n-1), u(n-1)]]. \tag{13}$$

Example 2: The inverse of the bilinear system

$$\begin{aligned} y(n) = {} & x(n) + \sum_{i=1}^{N-1} a_i x(n-i) + \sum_{i=1}^{N-1} b_i y(n-i) \\ & + \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} c_{ij} x(n-i) y(n-j) \end{aligned} \tag{14}$$

is the bilinear system

$$\begin{aligned} z(n) = {} & u(n) - \sum_{i=1}^{N-1} b_i u(n-i) - \sum_{i=1}^{N-1} a_i z(n-i) \\ & - \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} c_{ij} z(n-i) u(n-j). \end{aligned} \tag{15}$$

Note that the double summation in (14) is slightly different from the double summation in (8), and this difference contributes to the simpler inverse system in (15).

III. pTH ORDER INVERSES

Not all nonlinear systems possess an inverse and many nonlinear systems admit an inverse only for a certain subset of input signals. For these reasons, Schetzen developed the theory of the pth-order inverse of a nonlinear system whose input–output relation can be represented using a Volterra series expansion [7], [8]. The pth-order inverse of a nonlinear system H is defined in [7] and [8] as the pth-order system which, connected in cascade with H, results in a system whose linear kernel is the identity system and whose Volterra kernels from the second up to the pth-order are zero. A pth-order system is one in which all the Volterra kernels of order greater than p are zero. The definition of the pth-order inverse was relaxed in [6] by allowing the inverse system to possess nonzero Volterra operators of order greater than p. These operators do not affect the first p Volterra operators of the cascade system. This relaxed definition of the pth-order inverse was employed in [6] to derive simpler and computationally more efficient expressions for the inverse system. However, because of the presence of higher order components, the definition of the pth-order inverse in [6] does not result in a unique inverse system.

The following theorem presents an efficient method of computing a pth order inverse of the system in (2). Note that this system is a special case of the system in (1) when $h[\cdot,\cdot] = 1$.

Theorem 2: Let $g[\cdot]$ and $f[\cdot,\cdot]$ be causal, discrete-time nonlinear operators with convergent Volterra series expansions with respect to all the arguments. Moreover, let the pth order inverse $g_p^{-1}[\cdot]$ of the system $g[\cdot]$ exist. Then a pth order inverse of the causal, discrete-time nonlinear system described in (2) is given by the following input–output relationship:

$$z(n) = g_p^{-1}[u(n) - f[z(n-1), u(n-1)]]. \tag{16}$$

Proof: As was the case for the Theorem 1, we first show that the system in (16) is the pth-order post-inverse of the system in (2). Using the same variables as in the derivation of Theorem 1, we express $z(n)$ as

$$\begin{aligned} z(n) &= g_p^{-1}[y(n) - f[z(n-1), y(n-1)]] \\ &= g_p^{-1}[g[x(n)] + f[x(n-1), y(n-1)] \\ &\quad - f[z(n-1), y(n-1)]]. \end{aligned} \tag{17}$$

We proceed by mathematical induction. We assume that, for any i greater than zero, the output $z(n-i)$ differs from $x(n-i)$ only by $T_p(n-i)$, a term whose Volterra series expansion in $x(n)$ contains only kernels of order larger than p, i.e.,

$$z(n-i) = x(n-i) + T_p(n-i) \qquad \forall\, i > 0. \tag{18}$$

We have to prove that the Volterra series expansion of $z(n) - x(n)$ have zero kernels of order up to p. Since $f[\cdot,\cdot]$ admits a convergent Volterra series expansion, we have from (18) that the Volterra series expansion of the difference $f[x(n-1), y(n-1)] - f[z(n-1), y(n-1)]$ contains only kernels of order greater than p, i.e.,

$$f[x(n-1), y(n-1)] - f[z(n-1), y(n-1)] = 0 + T'_p(n) \tag{19}$$

$$\begin{aligned} z(n) &= g^{-1}\left[\frac{y(n) - f[z(n-1), y(n-1)]}{h[z(n-1), y(n-1)]}\right] \\ &= g^{-1}\left[\frac{g[x(n)]h[x(n-1), y(n-1)] + f[x(n-1), y(n-1)] - f[z(n-1), y(n-1)]}{h[z(n-1), y(n-1)]}\right] \end{aligned} \tag{6}$$

$$z(n) = \frac{u(n) - \sum_{i=1}^{N-1} b_i u(n-i) - \sum_{i=1}^{N-1} a_i z(n-i) - \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} c_{ij} z(n-i) u(n-j)}{1 + \sum_{j=1}^{N-1} c_{0j} y(n-j)} \tag{12}$$

where the Volterra kernels of $T'_p(n)$ up to order p are zero. Substituting (19) in (17), we get

$$z(n) = g_p^{-1}[g[x(n)] + T'_p(n)]. \tag{20}$$

The pth-order inverse of the operator $g[\cdot]$ derived in [7] is given by a pth-order truncated Volterra series whose kernels depend only on the first p kernels of the Volterra series expansion of $g[\cdot]$. The pth-order inverse derived in [6] may have Volterra kernels of order greater than p. However, the inverse still has a Volterra series expansion with finite order of nonlinearity, and it depends only on the first p kernels of the Volterra series expansion of $g[\cdot]$. Consequently, it immediately follows from (20) that

$$z(n) = x(n) + T_p(n) \tag{21}$$

and that the system in (16) is the pth-order post-inverse of the system in (2). We can prove in a similar manner that it is also a pre-inverse of the system in (2).

Remark: Due to the rational structure of the system in (3), a similar expression for the pth-order inverse of the system in (1) does not exist.

Example 3: We wish to derive a pth-order inverse for the second-order Volterra filter given by the following expression:

$$y(n) = \sum_{i=0}^{N-1} a_i x(n-i) + \sum_{i=0}^{N-1}\sum_{j=i}^{N-1} \cdot b_{ij} x(n-i)x(n-j). \tag{22}$$

Let

$$g[x(n)] = a_o x(n) + x(n)\sum_{j=0}^{N-1} b_{0j} x(n-j) \tag{23}$$

and

$$f[x(n-1)] = \sum_{i=1}^{N-1} a_i x(n-i) + \sum_{i=1}^{N-1}\sum_{j=i}^{N-1} \cdot b_{ij} x(n-i)x(n-j). \tag{24}$$

According to Theorem 2, a pth-order inverse of (22), is

$$z(n) = g_p^{-1}\left[u(n) - \sum_{i=1}^{N-1} a_i z(n-i) - \sum_{i=1}^{N-1}\sum_{j=i}^{N-1} b_{ij} z(n-i)z(n-j)g\right]. \tag{25}$$

The pth-order inverse $g_p^{-1}[\cdot]$ can be computed iteratively as in [6] and is given by

$$g_p^{-1}[u(n)] = -g_1^{-1}[q_p[g_{p-1}^{-1}[u(n)]] - u(n)] \tag{26}$$

where $g_1^{-1}[\cdot]$ is the inverse of the first Volterra operator of $g[\cdot]$ (i.e., a_0^{-1}, in our case) and $q_p[\cdot]$ is the truncated Volterra series expansion of the system $g[\cdot]$ that contains only the second- through pth-order Volterra kernels.

While it is possible to compute the pth-order inverse of the system of (22) as in [6], using the structure in [6] for inverting a smaller subsystem and then using Theorem 2, as we have done here, is a more efficient procedure in most situations.

The computational cost for the evaluation of (25) is $2(N-1) + (N-1)N/2 + (N+2)(p-1)$ multiplications per time instant. The corresponding computational cost for directly computing the pth-order inverse of (22) as in (26) is $N + [2N + N(N+1)/2](p-1)$ multiplications per time instant. Implementing (25) has a computational cost of $O(N^2 + pN)$ multiplications per time instant, while for the method in [6] the computational cost is $O(N^2 p)$. The methodology suggested by Theorem 2 is more efficient for evaluating the pth order inverse of a Volterra filter of order q when p is greater than q. On the other hand, when $p < q$, only the first p Volterra operators are significant for the evaluation of the pth-order inverse. In this situation, both methods of inversion require almost the same number of multiplications per sample.

IV. Concluding Remarks

This letter presented expressions for the exact inverse and the pth order inverse of a wide class of discrete-time nonlinear systems. This class includes most causal polynomial systems with finite order as well as many nonlinear filters with nonpolynomial input–output relationships. In particular, Theorem 1 allows the inversion of all recursive polynomial systems whose dependence on the input sample $x(n)$ can be characterized using an invertible component $g[x(n)]$. The computational cost of the exact inverse filter coincides with the cost of implementing the direct system and the operator $g^{-1}[\cdot]$. Theorem 2 applies to all recursive polynomial systems with the same characteristic as described above, as well as many other nonlinear systems. In this case also, the cost of implementing the inverse filter is that of implementing the direct system and the pth-order inverse of $g[\cdot]$. All the inverse filters presented in this paper are recursive and, therefore, may possess poor stability properties. Consequently, the stability of such systems must be tested after the inversion of the filter. Stability of recursive nonlinear systems is a topic of active research. Some useful stability results for recursive polynomial filters can be found in [1]–[5] and [9].

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