Bosonization of One-Dimensional Exclusons and Characterization of Luttinger Liquids

Yong-Shi Wu and Yue Yu
Department of Physics, University of Utah, Salt Lake City, Utah 84112
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We achieve a bosonization of one-dimensional ideal gas of particles obeying exclusion statistics λ (so called λ exclusons) at low temperatures, resulting in a new variant of $c = 1$ conformal field theory with compactified radius $R = \sqrt{1/\lambda}$. These ideal excluson gases exactly reproduce the low-T critical properties of Luttinger liquids, so they can be used to characterize the fixed points of the latter. Generalized ideal gases with mutual statistics and nonideal gases with Luttinger-type interactions have also similar low-T behavior, controlled by an effective statistics varying in a fixed-point line.

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Recently a new combinatoric rule for many-body state counting [1,2], which essentially is an abstraction and generalization of Yang-Yang's counting [3,4] in Bethe ansatz solvable models, is shown to be applicable to elementary excitations in a number of exactly solvable models for strongly correlated systems [1,2,4–7] in one, two, and higher dimensions. This has led to the notion of fractional exclusion statistics (FES) [1] and associated generalized ideal gases (GIG) [2]. (For abbreviation, we will call particles or excitations obeying FES *exclusons*.) An important issue to be addressed is the relevance of FES to realistic gapless systems, for which state counting is somewhat obscure due to strong correlations among particles. In this Letter, we suggest that at least *for some strongly correlated systems or non-Fermi-liquids their low-energy or low-temperature fixed point may be described by a GIG associated with FES*, similar to that of Landau-Fermi liquids by ideal Fermi gas [8].

A well-established class of non-Fermi-liquids is the Luttinger liquids in 1D, proposed by Haldane [9]. To produce in this case a testimony to our above suggestion, we will show that the low-T critical properties of Luttinger liquids are exactly reproduced by those of 1D ideal excluson gases (IEG), if one identifies the controlling parameter [9] of the former with the statistics λ of the latter. So *IEG can be used to describe the fixed points of Luttinger liquids*. A main development here is that we have succeeded in bosonizing 1D excluson systems at low T, *à la* Tomonaga [10]. It results in a new variant of conformal field theory (CFT) with central charge $c = 1$ and compactified radius [11] $R = \sqrt{1/\lambda}$. The particle-hole duality between λ and $1/\lambda$ in IEG [4,5] gives rise to a duality between R and $1/R$ in this variant of $c = 1$ CFT.

We have also studied the effects of mutual statistics between different momenta and Luttinger-type (density-density) interactions among exclusons. In both cases, the low-T behavior is controlled by an effective statistics $\lambda_{\rm eff}$ for excitations near the Fermi points, the same way as IEG by λ. In 1D both the momentum independent part of interactions and change in chemical potential μ are *relevant* perturbations [8,12], resulting in a continuous shift in the fixed-point line parametrized by λ.

Consider a GIG of N_0 particles on a ring with size L. Single-particle states are labeled by k_i. The total energy and momentum are given by $E = \sum k_i^2$ and $P = \sum k_i$. We assume [1,2] that in the thermodynamic limit the density $\rho_a(k)$ of available single-particle states is *linearly* dependent on the particle density $\rho(k)$. By definition, the statistics matrix is given by the derivative

$$g(k,k') = -\,\delta\rho_a(k)/\delta\rho(k'). \quad (1)$$

The system is called an IEG of statistics λ (with *no mutual statistics* between different momenta), if $g(k,k') = \lambda\,\delta(k - k')$, or $\rho_a(k) + \lambda\rho(k) = \rho_0(k)$ [4], where $\rho_0(k) \equiv 1/2\pi$ is the bare density of single-particle states. Thus $\lambda = 1$ corresponds to fermions, and $\lambda = 0$ to bosons. The thermodynamics of IEG is shown [2] to be determined by the thermodynamic potential

$$\Omega = -\frac{T}{2\pi}\int_{-\infty}^{\infty} dk\, \ln[1 + w(k,T)^{-1}], \quad (2)$$

with the function $w(k,T) \equiv \rho_a(k)/\rho(k)$ satisfying

$$w(k,T)^{\lambda}[1 + w(k,T)]^{1-\lambda} = e^{(k^2-\mu)/T}. \quad (3)$$

In the ground state, there is a Fermi surface such that $\rho(k) = 1/2\pi\lambda$ for $|k| < k_F$ and $\rho(k) = 0$ for $|k| > k_F$. Then the Fermi momentum is given by $k_F = \pi\lambda\bar{d}_0$, and the ground state energy and momentum by $E_0/L = \pi^2\lambda^2\bar{d}_0^3/3$, $P_0 = 0$, with the average density $\bar{d}_0 = N_0/L$.

Now let us examine possible excitations in the IEG. First there are density fluctuations due to particle-hole excitations, i.e., sound waves with velocity $v_s = v_F \equiv 2k_F$ (see below). Moreover, by adding extra M particles to the ground state, one can create particle excitations, and by Galileo boost a persistent current. Our observation is that the velocities of these three classes of elementary excitations in IEG satisfy a fundamental relation that Haldane years ago used to characterize the Luttinger liquid [9]. Indeed, shifting N_0 to $N = N_0 + M$, the change in the ground state energy is $\delta_1 E_0 = \pi(\lambda k_F)M^2$, while a persistent current, created by the boost of the Fermi sea $k \to k + \pi J/L$, leads to the energy shift $\delta_2 E_0 = \pi(k_F/\lambda)J^2$. [Because of periodic boundary conditions, M and J are

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constrained [9] by $M = J\,(\mathrm{mod}\ 2)$.] Therefore the total change in energy and in momentum, due to charge and current excitations, are

$$\delta E_0 = \frac{\pi}{2}(v_N M^2 + v_J J^2), \quad \delta P_0 = \pi\Big(\bar{d}_0 + \frac{M}{L}\Big) J, \tag{4}$$

respectively, with

$$v_N = v_s\lambda, \quad v_J = v_s/\lambda, \quad v_s = \sqrt{v_N v_J}. \tag{5}$$

These coincide with the well-known relations [9] in Luttinger-liquid theory, if we identify λ with Haldane's controlling parameter $\exp(-2\varphi)$. So it is interesting to see whether all critical exponents of Luttinger liquids are also reproduced simply by IEG.

To calculate the exponents in IEG, we need to develop a bosonization approach. Following Yang and Yang [3,13], we introduce the dressed energy $\epsilon(k,T)$ by writing

$$w(k,T) = e^{[\epsilon(k,T)-\mu]/T}. \tag{6}$$

The point is that the grand partition function Z_G, corresponding to the thermodynamic potential (2), is of the form of that for an ideal system of fermions with a complicated, T-dependent energy dispersion given by the dressed energy: $Z_G = \prod_k (1 + e^{[\mu-\epsilon(k,T)]T})$. However, this fermion representation is not very useful, because of the implicit T dependence of the dressed energy. We observe, nevertheless, that in the low-T limit the T dependence of $\epsilon(k,T)$ can be ignored: According to Eqs. (3) and (6), $\epsilon(k,T) = \epsilon(k) + O(e^{-|\epsilon|/T})$, where

$$\epsilon(k) = \begin{cases} (k^2 - k_F^2)/\lambda + k_F^2, & |k| < k_F, \\ k^2, & |k| > k_F. \end{cases} \tag{7}$$

Thus the low-T grand partition function can be obtained from the effective Hamiltonian given by $H_{\rm eff} = \sum_k \epsilon(k)\, c_k^\dagger c_k$, where $c_k^\dagger$ are fermionic creation operators.

Another simplification in the low-T limit is that we need to consider only low-energy excitations near Fermi points $k \sim \pm k_F$, where the left- and right-moving sectors are separable and decoupled: $H_{\rm eff} = H_+ + H_-$, and $H_\pm$ has a linearized energy dispersion

$$\epsilon_\pm(k) = \begin{cases} \pm v_F(k \mp k_F) + k_F^2, & |k| > k_F, \\ \pm v_F(k \mp k_F)/\lambda + k_F^2, & |k| < k_F. \end{cases} \tag{8}$$

We note the "refractions" at $k = \pm k_F$. In spite of this peculiarity, we have succeeded in bosonizing the effective Hamiltonian as follows. The density fluctuation operator at $k \sim k_F$ is constructed as follows:

$$\rho_q^{(+)} = \sum_{k>k_F} :c_{k+q}^\dagger c_k: + \sum_{k<k_F-\lambda q} :c_{k+\lambda q}^\dagger c_k: + \sum_{k_F-\lambda q<k<k_F} :c_{(k-k_F)/\lambda+k_F+q}^\dagger c_k: \tag{9}$$

for $q > 0$. A similar density operator $\rho_q^{(-)}$ can also be defined for $k \sim -k_F$. Within the Tomonaga approximation [14], in which commutators are taken to be their ground state expectation value, we obtain

$$[\rho_q^{(\pm)}, \rho_q^{(\pm)\dagger}] = \frac{\lambda L}{2\pi} q\, \delta_{q,q'}, \ [H_\pm, \rho_q^{(\pm)}] = \pm v_F q \rho_q, \tag{10}$$

which describe 1D free phonons with the sound velocity $v_s = v_F$. Introducing normalized boson annihilation operators $b_q = \sqrt{2\pi/\lambda q L}\,\rho_q^{(+)}$, $\tilde{b}_q = \sqrt{2\pi/\lambda q L}\,\rho_q^{(-)\dagger}$, the bosonized Hamiltonian satisfying (10) is given by

$$H_B = v_s\left\{\sum_{q>0} q(b_q^\dagger b_q + \tilde{b}_q^\dagger \tilde{b}_q) + \frac{1}{2}\frac{\pi}{L} \times \left[\lambda M^2 + \frac{1}{\lambda} J^2\right]\right\}. \tag{11}$$

The construction of the bosonized momentum operator is a bit more tricky. Each term in Eq. (9) should carry same momentum q, therefore the fermion created by $c_k^\dagger$ carries a dressed momentum p, which is related to k by

$$p(k) = \begin{cases} k - k_F + (k_F/\lambda), & k > k_F, \\ k/\lambda, & |k| < k_F, \\ k + k_F - (k_F/\lambda), & k < -k_F. \end{cases} \tag{12}$$

In terms of this variable, the linearized dressed energy $\epsilon(p)$ is of a simple form: $\epsilon_\pm(p) = \pm v_s(p \mp p_F) + \mu$, with $p_F = k_F/\lambda$. The bosonized total momentum operator, corresponding to the fermionized $P = \sum_k p(k)\, c_k^\dagger c_k$, is

$$P = \sum_{q>0} q(b_q^\dagger b_q - \tilde{b}_q^\dagger \tilde{b}_q) + \pi(\bar{d}_0 + M/L) J. \tag{13}$$

In the coordinate-space formulation, the normalized density field $\rho(x)$ is given by $\rho(x) = \rho_R(x) + \rho_L(x)$:

$$\rho_R(x) = \frac{M}{2L} + \sum_{q>0} \sqrt{q/2\pi L\lambda}\,(e^{iqx} b_q + e^{-iqx} b_q^\dagger), \tag{14}$$

and $\rho_L(x)$ is similarly constructed from $\tilde{b}_q$ and $\tilde{b}_q^\dagger$. The boson field $\phi(x)$, which is conjugated to $\rho(x)$ and satisfies $[\phi(x), \rho(x')] = i\delta(x - x')$, is $\phi(x) = \phi_R(x) + \phi_L(x)$ with

$$\phi_R(x) = \frac{\phi_0}{2} + \frac{\pi J x}{2L} + i \sum_{q>0} \sqrt{\pi\lambda/2qL} \times (e^{iqx} b_q - e^{-iqx} b_q^\dagger),$$

and a similar $\phi_L(x)$. Here M and J are operators with integer eigenvalues, and ϕ_0 is an angular variable conjugated to M: $[\phi_0, M] = i$. The Hamiltonian (11) becomes

$$H_B = \frac{v_s}{2\pi}\int_0^L dx\{\Pi(x)^2 + [\partial_x X(x)]^2\}, \tag{15}$$

where $\Pi(x) = \pi\lambda^{1/2}\rho(x)$ and $X(x) = \lambda^{-1/2}\phi(x)$. With $X(x,t) = e^{iHt} X(x) e^{-iHt}$, the Lagrangian density reads

$$\mathcal{L} = (v_s/2\pi)\, \partial_\alpha X(x,t)\, \partial^\alpha X(x,t). \tag{16}$$

We recognize that $\mathcal{L}$ is the Lagrangian of a $c = 1$ CFT [11]. Since ϕ_0 is an angular variable, there is a hidden invariance in the theory under $\phi \to \phi + 2\pi$. The field

X is thus said to be "compactified" on a circle, with a radius that is determined by the exclusion statistics:

$$X \sim X + 2\pi R, \quad R^2 = 1/\lambda. \tag{17}$$

States $V[X]|0\rangle$ or operators $V[X]$ are *allowed* only if they respect this invariance, so quantum numbers of quasiparticles are strongly constrained. The particle-hole duality [4,5], i.e., $\lambda \leftrightarrow 1/\lambda$ and $M \leftrightarrow J$, in Eq. (11) is just a duality $R \leftrightarrow 1/R$. Moreover, the partition function of IEG (in the low-T limit) can be rewritten as $Z = \mathrm{Tr}_{\mathcal{H}}[q^{L_0^R}\bar{q}^{L_0^L}]$, where $q = e^{iv_s/T}$. The zeroth generators of the Virasoro algebra are $L_0^{R,L} = v_s^{-1}H_{R,L}^{(b)} + (\pi/4L)[JR \mp M/R]^2$. The constraint $M = J$ (mod2) makes our variant of $c = 1$ CFT have an unusual spectrum and duality relation [11].

We also see that the Hamiltonian (15) agrees with Haldane's harmonic fluid description of Luttinger liquids [15]. Thus the critical properties of IEG reproduce those of Luttinger liquids. Or one may say that IEG can be used to characterize the fixed points of Luttinger liquids.

Using bosonization techniques, the computation of correlation functions and critical exponents in IEG is similar to that for Luttinger liquids [9,15]. A careful construction for the allowed boson field with charge 1 leads to

$$\Psi_B^\dagger(x,t) = \rho(x)^{1/2} \sum_{m=-\infty}^{\infty} e^{iO_m} : e^{i(\lambda^{1/2}+2m/\lambda^{1/2})X_R(x\ )} : \times : e^{i(\lambda^{1/2}-2m/\lambda^{1/2})X_L(x,)} :, \tag{18}$$

where the Hermitian, constant-valued operators O_m satisfy $[O_m, O_{m'}] = i\pi(m - m')$ [16]. The excluson operator reads $\Psi_\lambda^\dagger(x) =: \Psi_B^\dagger(x)e^{i\lambda^{1/2}[X_R(x)-X_L(x)]}:$, obeying $\Psi_\lambda^\dagger(x)\Psi_\lambda^\dagger(x') - e^{i\pi\lambda \mathrm{sgn}(x-x')}\Psi_\lambda^\dagger(x')\Psi_\lambda^\dagger(x) = 0$ for $x \neq x'$. The multisector density operator for exclusons is

$$\hat{\rho}(x) = \Psi_\lambda^\dagger(x)\Psi_\lambda(x) = \Psi_B^\dagger(x)\Psi_B(x) = \rho(x)\sum_m : \exp\{i2m[X_R(x) - X_L(x)]/\lambda^{1/2}\} : . \tag{19}$$

Dynamical correlation functions can be easily calculated:

$$\langle\hat{\rho}(x,t)\hat{\rho}(0,0)\rangle \approx \bar{d}_0^2\left[1 + \frac{1}{(2\pi\bar{d}_0)^2\lambda}\left(\frac{1}{x_+^2} + \frac{1}{x_-^2}\right) + \sum_{m=1}^{\infty} A_m \frac{1}{[x_+x_-]^{m^2/\lambda}}\cos(2\pi\bar{d}_0 mx)\right],$$

$$G(x,t;\lambda) \equiv \langle\Psi_\lambda^\dagger(x,t)\Psi_\lambda(0,0)\rangle \approx \bar{d}_0 \sum_{m=-\infty}^{\infty} B_m \frac{1}{x_-^{(m+\lambda)^2/\lambda}}\frac{1}{x_+^{m^2/\lambda}} e^{i[2\pi(m+\lambda/2)\bar{d}_0 x + \mu t]},$$

with A_m and B_m regularization-dependent constants. These correlation functions coincide with the asymptotic ones [6] in the Calogero-Sutherland model.

The single-hole state, i.e., $\Psi_{1/\lambda}^\dagger|0\rangle = \Psi_\lambda(\lambda \to 1/\lambda)|0\rangle$, with charge $-1/\lambda$ alone is not allowed. The minimum allowed multihole state is given by $\Psi_{1/\lambda}^\dagger(x_1)\cdots\Psi_{1/\lambda}^\dagger(x_p)|0\rangle$ if $\lambda = p/q$ is rational. One may obtain, e.g.,

$$\langle[\Psi_{1/\lambda}^\dagger(x,t)]^p[\Psi_{1/\lambda}(0,0)]^p\rangle \sim [G(x,t;1/\lambda)]^p. \tag{20}$$

A more interesting allowed operator is what creates q particle excitations accompanied by p hole excitations: $\hat{n}(x,t) = [\Psi_\lambda^\dagger(x,t)]^q[\Psi_{1/\lambda}^\dagger(x,t)]^p$. We note the similarity of this operator to Read's order parameter [17] for fractional quantum Hall fluids (in bulk). Its correlation function can be calculated by using Wick's theorem:

$$\langle\hat{n}(x,t)\hat{n}(0,0)\rangle \sim [G(x,t;\lambda)]^q[G(x,t;1/\lambda)]^p. \tag{21}$$

If the contribution from the $m = 0$ sector dominates, then one gets $\langle\hat{n}(x,t)\hat{n}(0,0)\rangle \sim (x - v_s t)^{-(p+q)}$.

Now we turn to discussing the effects of mutual statistics. Consider a GIG with the statistical matrix (1) given by $g(k - k') = \delta(k - k') + \Phi(k - k')$. Here $\Phi(k) = \Phi(-k)$ is a smooth function and stands for mutual statistics between particles with different momenta, in contrast to IEG, for which $\Phi(k)$ is a δ function. The thermodynamic properties of GIG is also given by Eq. (2), but now $w(k,T)$ satisfies instead an integral equation [2,4] which, in terms of the dressed energy (6), is of the form

$$\epsilon(k,T) = \epsilon_0(k) + T\int\frac{dk'}{2\pi}\Phi(k - k') \times \ln(1 + e^{-\epsilon(k',T)/T}),$$

where $\epsilon_0(k) \equiv k^2$, and we have shifted the dressed energy by chemical potential μ. At $T = 0$, the Fermi momentum k_F is determined by $\epsilon(\pm k_F) = 0$. Introduce

$$(\alpha\beta)[-k_F,k_F] \equiv \int_{-k_F}^{k_F}\frac{dk}{2\pi}\alpha(k)\beta(k), \tag{22}$$

$$(\Phi\alpha)(k;-k_F,k_F] \equiv \int_{-k_F}^{k_F}\frac{dk'}{2\pi}\Phi(k - k')\alpha(k'). \tag{23}$$

Then both $\rho(k)$ and $\epsilon(k)$ in the ground state satisfy an integral equation like

$$\alpha(k) = \alpha_0(k) - (\Phi\alpha)(k;-k_F,k_F]. \tag{24}$$

The dressed momentum $p(k)$ is related to $\rho(k)$ by $p'(k) = 2\pi\rho(k)$ and $p(k) = -p(-k)$. The ground state energy is given by $E_0/L = (\epsilon_0\rho)[-k_F,k_F] = (\epsilon\rho_0)[-k_F,k_F]$. These equations are of the same form as those in the thermodynamic Bethe ansatz [3], hence the Luttinger-liquid relation $v_s = \sqrt{v_N v_J}$ remains true [18]: The charge velocity is $v_N = v_s z(k_F)^{-2}$, where the dressed charge $z(k)$ is given [18] by the solution to the integral equation $z(k) = 1 - (\Phi z)(k;-k_F,k_F]$. This relation can be easily derived from the definitions $v_N = L\partial\mu/\partial N_0$ and $z(k) = -\delta\epsilon(k)/\delta\mu$. To create a persistent current, let us boost the Fermi sea by $\pm k_F \to \pm k_F + \Delta$, where $\Delta =$

$z(k_F)/L\rho(k_F)$. Then the total energy of the state with the persistent current is

$$E_\Delta/L = (\epsilon_\Delta \rho_0)[-k_F + \Delta, k_F + \Delta], \quad (25)$$

where $\epsilon_\Delta(k) = \epsilon_0(k) - (\Phi\epsilon_\Delta)(k; -k_F + \Delta, k_F + \Delta]$. Thus, $E_\Delta - E_0 = L\Delta^2 \epsilon'(k_F)\rho(k_F) = (2\pi/L)v_s z(k_F)^2$. This verifies $v_J = v_s z(k_F)^2$. In view of Eq. (5), at low energies GIG looks like IEG with

$$\lambda_{\text{eff}} = z(k_F)^{-2}. \quad (26)$$

It can be shown that it is the effective statistics (26) that controls the low-T critical properties of GIG, as λ does for IEG. Linearization near the Fermi points and bosonization of the low-energy effective Hamiltonian go the same way as before for IEG. The only difference now is that the linearized dispersion for dressed energy $\epsilon_\pm(k) = \pm\epsilon'(k_F)(k \mp k_F) + \mu = \pm v_s[p(k) \mp p_F] + \mu$ is smooth at $k \sim \pm k_F$. So bosonization is standard and the bosonized Hamiltonian is the same as Eq. (11) for IEG, only with λ replaced by λ_{eff}. An (allowed) $\Psi^\dagger_{\lambda_{\text{eff}}}$ describes the particle excitations near the Fermi surface with both anyon and exlcusion statistics being λ_{eff}. In this sense, one may say that the effect of mutual statistics is to renormalize the statistics.

Here we remark that in IEG $\Phi(k,k') = (\lambda - 1)\,\delta(k - k')$ is not smooth, so the dressed charge has a jump at k_F: $z(k_F^+) = 1$ and $z(k_F^-) = \lambda^{-1}$ for $k_F^\mp = k_F \pm 0^+$. The general Luttinger-liquid relation is of the form

$$v_N = v_s[z(k_F^+)z(k_F^-)]^{-1}, \quad v_J = v_s z(k_F^+)z(k_F^-). \quad (27)$$

Finally, we examine nonideal gases, e.g., with general Luttinger-type density-density interactions

$$H_I = \frac{\pi}{L}\sum_{q\geq 0}[U_q(\rho_q\rho_q^\dagger + \tilde\rho_q\tilde\rho_q^\dagger) + V_q(\rho_q\tilde\rho_q^\dagger + \tilde\rho_q\rho_q^\dagger)], \quad (28)$$

which is added to the Hamiltonian describing a GIG. After bosonization, the total Hamiltonian remains bilinear in densities, so it is trivial to diagonalize it using the Bogoliubov transformation. The diagonalized Hamiltonian is again of the harmonic-fluid form (11) with b_q and $\tilde b_q$ replaced by corresponding operators for Bogoliubov quasiparticles, and with the velocities renormalized: $v_s \to \tilde v_s = |(v_s + U_0)^2 - V_0^2|^{1/2}$, $v_N \to \tilde v_N = \tilde v_s e^{-2\tilde\varphi_0}$, and $v_J \to \tilde v_J = \tilde v_s e^{2\tilde\varphi_0}$. Thus the Luttinger-liquid relation (5) survives, with λ_{eff} of GIG renormalized to

$$\tilde\lambda_{\text{eff}} = e^{-2\tilde\varphi_0}, \quad \tanh(2\tilde\varphi_0) = \frac{v_J - v_N - 2V_0}{v_J + v_N + 2U_0}. \quad (29)$$

Note that the new fixed point depends both on the position of the Fermi points and on the interaction parameters U_0 and V_0, leading to "nonuniversal" exponents.

In passing we observe several additional implications of this work: (1) Our bosonization and operator derivation of CFT at low energies or in low-T limit can be applied to Bethe ansatz solvable models, including the long-range (e.g., Calogero-Sutherland) one. (2) Here we have only considered one-species cases, i.e., with excitations having no internal quantum numbers such as spin. Our bosonization and characterization of Luttinger liquids are generalizable to GIG with multispecies, presumably with the effective statistics matrix related to the dressed charge matrix. (3) The chiral current algebra in Eq. (10) with $\lambda = 1/m$ coincides with that derived by Wen [19] for edge states in $\nu = 1/m$ (m odd) fractional quantum Hall fluids. So these edge states and their chiral Luttinger-liquid fixed points can be described in terms of chiral IEG.

In conclusion, we have shown that 1D IEG without mutual statistics can exactly reproduce the low-energy and low-T properties of (one-component) Luttinger liquids. Moreover, mutual statistics and Luttinger-type interactions in a GIG only shift the value of λ_{eff}. Thus the essence of Luttinger liquids is to have an IEG obeying FES as their fixed point. It is conceivable that some strongly correlated systems, exhibiting non-Fermi-liquid behavior, in two or higher dimensions can also be characterized as having a GIG with appropriate statistics matrix as their low-energy or low-temperature fixed point.

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