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On Gauge Invariance of Noncommutative Chern-Simons Theories

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Abstract

Motivated by possible applications to condensed matter systems, in this paper we construct $U(N)$ noncommutative Chern-Simons (NCCS) action for a disc and for a double-layer geometry, respectively. In both cases, gauge invariance severely constrains the form of the NCCS action. In the first case, it is necessary to introduce a group-valued boson field with a non-local chiral boundary action, whose gauge variation cancels that of the bulk action. In the second case, the coefficient matrix K in the double $U(N)$ NCCS action is restricted to be of the form $K = k \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with integer k. We suggest that this double NCCS theory with $U(1)$ gauge group describes the so-called Halperin (kkk) state in a double-layer quantum Hall system. Possible physical consequences are addressed.

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I. INTRODUCTION

In the past two decades, Chern-Simons (CS) field theory [1–5] in $2+1$ dimensions has received a great deal of attention in theoretical physics. In particular, when a $U(1)$ CS field couples to a matter field, statistics of the latter gets transmuted: The CS coupling automatically produces a CS flux associated with the matter particle. The composite of a particle and CS flux is called an anyon, whose exchange statistics depends on the CS coupling constant. The feat to transmute the exchange statistics in two spatial dimensions turned out to be extremely helpful in understanding the fractional quantum Hall effect and in the early attempt of anyon superconductivity [4,5]. It is also amusing to note that pure CS field theory itself can serve as the low energy effective theory for the quantum Hall systems [6,7]. In this formulation, gauge invariance of the CS field theory plays an important role. For a finite system with an edge, Wen [6] pointed out that a boundary term has to be added to maintain gauge invariance of the system, leading to his chiral Luttinger liquid theory of the quantum Hall edge states. Wen and Zee, and Frölich and Zee [7] also systematically studied the structures of topological fluids in the framework of a multiple $U(1)$ CS theory.

Recently field theories (especially gauge field theory) on a noncommutative space have attracted much interest [8]. Originally, such theory arises from string/M(atrix) theory [9,10], which suggests that space or spacetime noncommutativity should be a general feature of quantum gravity for generic points deep inside the moduli space of M-theory. However, the simplest example of noncommutative space appears in a familiar quantum mechanical problem; namely, a charge in the lowest Landau level (LLL) in a strong magnetic field can be viewed as living in a noncommutative space, because the guiding-center coordinates of the charge are known not to commute[1]. This fact motivates many (include the present authors) to believe that perhaps the most natural field theory formulation for the quantum Hall systems should be a noncommutative one. Indeed, along this route, Susskind [12] recently

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[1]For a pedagogical review, see Ref. [11].

argued that $U(1)$ noncommutative Chern-Simons (NCCS) theory at level k on a plane is equivalent to the Laughlin theory at filling factor $\nu = 1/k$. This scenario can be considered as a projection (or deformation) of the ordinary CS effective field theory formulation [7] of quantum Hall fluids to the LLL which is intrinsically a noncommutative space. For further studies on Susskind's proposal see, e.g., Refs. [13–15].

In an ordinary non-Abelian CS theory the CS coupling (or the level) acquires, due to self interactions, a one-loop shift by the integer N for $SU(N)$ ($N \geq 2$) gauge group. This perturbative result provides an important test for the level quantization and other topological features of non-Abelian CS theory [16]. Since a $U(1)$ NCCS theory is no longer a free theory, it is natural to ask whether the self interactions arising from noncommutativity will also induce a similar one-loop shift in the $U(1)$ NCCS theory. This was confirmed in Ref. [17] by an explicit one-loop calculation: Indeed, the CS coupling (or the level) is *shifted by one for $U(1)$ and by N for $U(N)$ NCCS theory.* This was, to our knowledge, the first evidence for the level quantization in $U(1)$ NCCS theory, which was later discussed by other groups [18,19] using topological arguments. Due to this quantum shift, Susskind's level-k NCCS theory should describe a $\nu = \frac{1}{k+1}$ (rather than $\frac{1}{k}$) Laughlin state. This point has also been emphasized in recent papers [14,20].

Motivated by the possibility that pure NCCS theory provides a low energy effective theory of quantum Hall fluids, in this paper we study how to construct NCCS theory for spatial geometry other than an infinite plane, that corresponds to more realistic geometry for samples used in experimental studies [21]. The first case we will study is a sample of finite size with an edge, say a disc. The second case involves two spatially close but separated quantum Hall fluids in a double quantum well structure. The two layers are coupled to form an interesting system, called a double-layer quantum Hall system, either due to tunneling or due to Coulomb interactions between the layers. We will mainly focus on how to construct a *gauge invariant* NCCS action for these sample configurations.

In the rest of this section, we establish our notations and conventions. A $(2 + 1)$-dimensional noncommutative spacetime has coordinates satisfying

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \qquad \mu, \nu = 0, 1, 2, \tag{1}$$

where $\theta^{\mu\nu}$ are anti-symmetric and real parameters of dimension length squared. To realize the commutators (1), we follow Moyal's deformation techniques, adopting a representation in which the coordinates x^μ are commuting as usual, but the product of any two functions of x^μ is deformed to the star product:

$$(f * g)(x) = \exp\left[\frac{i}{2}\theta^{\mu\nu}\partial_\mu^f\partial_\nu^g\right]f(x)g(x). \tag{2}$$

Then the commutators (1) are interpreted as the bracket with respect to the star product:

$$[x^\mu, x^\nu] \equiv x^\mu * x^\nu - x^\nu * x^\mu. \tag{3}$$

The advantage of this representation is that we can use the usual differentiation and integration, and in particular we can *define the boundary of noncommutative space or spacetime in the usual way.* Of course, in this case we have to pay attention to the non-locality arising in the theory on the boundary due to coordinate noncommutativity in the bulk. For applications to a system in the LLL, one considers

only the spatial noncommutativity: $\theta^{01} = \theta^{02} = 0$ and $[x^1, x^2] = i\theta$.

The action for a pure $U(N)$ CS theory on this spacetime reads

$$\begin{aligned} I_{CS} &= -\frac{ik}{4\pi}\int_{\mathcal{M}} d^3x \varepsilon^{\mu\nu\lambda}\mathrm{Tr}(A_\mu * \partial_\nu A_\lambda + \frac{2}{3}A_\mu * A_\nu * A_\lambda), \\ &= -\frac{ik}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}(AdA + \frac{2}{3}A^3). \end{aligned} \tag{4}$$

Here the dynamical fields are the gauge potential $A^\mu = A_\mu^a T^a$, with T^a the generators of the gauge group $G = U(N)$, normalized to $\mathrm{Tr}(T^a T^b) = -\delta^{ab}/2$, with $T^0 = i/\sqrt{2N}$ for the $U(1)$ sector. Moreover, k is the CS coupling, or the level parameter, and $\varepsilon^{\mu\nu\lambda}$ the totally anti-symmetric tensor with $\varepsilon^{012} = 1$. The integral is taken over a three dimensional spacetime $\mathcal{M}$, which will be specified later. In the second line of eq. (4), we introduced the gauge potential one-form $A = A_\mu dx^\mu$. We also *suppress both the star product symbol and the wedge product symbol* in the forms. They are understood as

$$A^3 = A \wedge A \wedge A = A_\mu * A_\nu * A_\rho dx^\mu \wedge dx^\nu \wedge dx^\rho. \tag{5}$$

In our convention, the field strength two-form is given by $F = dA + A^2$.

It is well-known that in the LLL the transverse plane, where the electrons live, becomes noncommutative. However, this does not forbid us to think that in a real sample of Hall bar or corbino geometry, there are electronic states that live on the edge of the sample. We emphasize that our adoption of the Moyal representation is crucial for providing a legitimate definition of a noncommutative space with edge or boundary. Indeed, in this representation, though the product in the function algebra is deformed, coordinates are still the ordinary ones, so the boundary of a space with finite geometry can be easily identified and dealt with as usual. (If we had realized the coordinates as operators, it would be hard to identify the boundary of a noncommutative space). Of course, we have to be careful about the product of two functions on the boundary which becomes non-local due to noncommutativity in the bulk.

The paper is organized as follows. In the Section II, we discuss the noncommutative edge theory, described by the boundary terms in the complete action of a $U(1)$ NCCS theory, for a quantum Hall fluid of disc geometry. In Section III, we discuss the multiple $U(1)$ NCCS theory for a double-layer quantum Hall system. In both cases the central issue is gauge invariance which, as we will see, severely constrains the form of the NCCS actions. Finally, conclusions and remarks are given in the Section IV.

II. EDGE THEORY OF A QUANTUM HALL FLUID

In this section, we study the NCCS on a cylindrical spacetime $\mathcal{M} = D \times R$, where D stands for a disc, R the time line. Though the main applications would be the case with $U(1)$ gauge group, it is not hard to first present the more general case with $U(N)$ gauge group. Under a gauge transformation given by the map g, $g : \mathcal{M} \to U(N)$, the gauge potential A and field strength F change to

$$A^g = gAg^{-1} + gdg^{-1}; \qquad F^g = gFg^{-1}. \tag{6}$$

Note that the star product is understood hereafter and g^{-1} is defined by $g*g^{-1} = g^{-1}*g = 1$. Correspondingly, the NCCS action (4) transforms into

$$I_{cs}(A^g) = I_{cs}(A) + \frac{ik}{4\pi}\int_{D\times R}\mathrm{Tr}d(dg^{-1}gA) + \frac{ik}{12\pi}\int_{D\times R}\mathrm{Tr}(gdg^{-1})^3. \tag{7}$$

Usually $\mathcal{M}$ is assumed to have no boundary, then the second term vanishes. The third (the Wess-Zumino-Witten) term in eq. (7) would be a topological term, which does not vanish for a topologically nontrivial (or large) gauge transformation. Gauge invariance of the path integral measure in the NCCS theory then imposes the level quantization condition, namely the level k has to be an integer; so that the partition function is invariant under a large gauge transformation. This topological argument, which restricts the quantum one-loop shift in k has to be integer too, holds true in the noncommutative case even for the $U(1)$ case [17–19]. Here we see the pivotal importance of the fundamental requirement of gauge invariance in CS theories, ordinary or noncommutative.

However, in our present case the cylindrical spacetime $\mathcal{M} = D \times R$ is not without boundary. Under an infinitesimal gauge transformation, $g = 1 + \varepsilon$, we have

$$\delta_\varepsilon A = A^g - A = -d\varepsilon - [A, \varepsilon] = -D\varepsilon. \tag{8}$$

By using the Bianchi identity

$$DF = dF + [A, F] \equiv 0, \tag{9}$$

we get the gauge variation of the action (4) as

$$\delta_\varepsilon I_{cs} = \frac{ik}{4\pi}\int_{D\times R}\mathrm{Tr}d(\varepsilon dA) = \frac{ik}{4\pi}\int_{D\times R}\mathrm{Tr}d(\varepsilon(F - A^2)). \tag{10}$$

It is obvious now that the NCCS action on $D \times R$ is not gauge invariant any more.

Since any microscopic theory of the quantum Hall effect should be gauge invariant, the gauge non-invariance of the NCCS action (4) on $D\times R$ tells us that *a pure NCCS action in the bulk is not a complete effective description for quantum Hall fluids on a finite geometry with edge.* Since the gauge anomaly term (10) is a total divergence and, therefore, can be written

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as a boundary term, there should be a way to cancel it by adding "boundary" terms. This is the essence of the Callan-Harvey effect [22], where gauge anomalies are canceled between two systems with different dimensionality.

It is known that the NCCS theory is not invariant under time reversal [23]. This is just right, since the microscopic theory of the quantum Hall system involves charged particles in a magnetic field, which explicitly breaks the time reversal symmetry. Now that the motion of a quantum Hall fluid is intrinsically chiral, it is natural to look for a chiral theory on the edge to cancel the gauge anomaly (10). Here we follow the procedure in Ref. [24], to look for a deformed version of a chiral boson action on the edge which, when combined with the bulk NCCS action, makes the total action gauge invariant. With the experience of bosonizing a chiral fermion system, we propose that the following is the desired edge action for a noncommutative chiral boson $h(x, t, \tau) \in G$:

$$S_0 = \frac{ik}{4\pi} \int_{D \times R} dxdtd\tau \mathrm{Tr} \partial_\tau (h^{-1} * \partial_x h * h^{-1} * \partial_x h) \tag{11}$$
$$+ \frac{ik}{2\pi} \int_{D \times R} dxdtd\tau \mathrm{Tr}(h^{-1} * \partial_t h * \partial_x (h^{-1} * \partial_\tau h)).$$

Here x and τ are the angular and radial coordinates on the disc D, while t the time coordinate on R. (See Fig. 1.)

To show such an action is indeed chiral, an elegant way [25] is to connect this action to the noncommutative Wess-Zumino (NCWZ) action in three dimensions:

$$S_{wz} = \frac{1}{12\pi} \int_{D \times R} \mathrm{Tr}(h^{-1}dh)^3. \tag{12}$$

In fact, the above Wess-Zumino term can be rewritten as

$$S_{wz} = \frac{1}{4\pi} \int_{D \times R} dxdtd\tau \mathrm{Tr}([h^{-1} * \partial_x h, h^{-1} * \partial_t h] * h^{-1} * \partial_\tau h), \tag{13}$$

after several integrations by parts and using the identity

$$\partial_\mu (h^{-1} * \partial_\nu h) - \partial_\nu (h^{-1} * \partial_\mu h) + [h^{-1} * \partial_\mu h, h^{-1} * \partial_\nu h] = 0, \tag{14}$$

which is nothing but the flat connection conditions for the one-form $h^{-1}dh$. Finally we write the NCWZ term (12) as

$$S_{wz} = \frac{1}{2\pi} \int_{D \times R} dx dt d\tau \mathrm{Tr}[(h^{-1} * \partial_t h) * \partial_x (h^{-1} * \partial_\tau h)] \tag{15}$$
$$- \frac{1}{4\pi} \int_{D \times R} dx dt d\tau \mathrm{Tr} \partial_\tau (h^{-1} * \partial_x h * g^{-1} * \partial_t h).$$

Thus we can rewrite the action (11) in terms of the NCWZ term as

$$S_0 = \frac{ik}{4\pi} \int_{D \times R} dx dt d\tau \mathrm{Tr} \partial_\tau [h^{-1} * \partial_x h * h^{-1} * (\partial_x + \partial_t) h] + \frac{ik}{12\pi} \int_{D \times R} \mathrm{Tr}(h^{-1} dh)^3. \tag{16}$$

The variation of this action (11) with respect to h gives

$$\delta S_0 = -\frac{ik}{2\pi} \int_{D \times R} dx dt d\tau \mathrm{Tr} \partial_\tau [h^{-1} * \delta h * \partial_x (h^{-1} * \partial_+ h)] \tag{17}$$
$$= -\frac{ik}{2\pi} \int_{D \times R} dx dt d\tau \mathrm{Tr} \partial_\tau [\delta h * h^{-1} * \partial_+ (\partial_x h * h^{-1})].$$

Here $\partial_+ = \partial_x + \partial_t$. Thus the equation of motion is given by

$$\partial_+ (\partial_x h * h^{-1}) = 0. \tag{18}$$

(The exact meaning of the star product in this equation in 1+1 dimensions will be clarified later in this section.) The solution of this equation of motion represents right-going waves. In this way, the action S_0 is the chiral version of the noncommutative deformation of the Wess-Zumino-Witten model.

Before adding the action S_0 to the bulk NCCS action (4), we need to couple the NC chiral boson to the gauge fields A_μ, so that the resulting theory is gauge invariant under the combined variations

$$\delta h = \varepsilon * h, \qquad A \to A + \delta_\varepsilon A. \tag{19}$$

Recalling that the resulting theory has to be chiral, we couple the noncommutative current $\partial_x h * h^{-1}$ to the chiral combination of potentials $A_+ = A_x + A_t$. After some calculations, one can check that under the combined transformation (19), the following action

$$S(h, A) = S_0 + \frac{ik}{2\pi} \int_{D \times R} dx dt d\tau \mathrm{Tr} \partial_\tau [A_+ * \partial_x h * h^{-1}] + \frac{ik}{4\pi} \int_{D \times R} dx dt d\tau \mathrm{Tr} \partial_\tau (A_+ * A_x), \tag{20}$$

has the desired variation

$$\delta_\varepsilon S(h, A) = -\frac{ik}{4\pi} \int_{D \times R} dx dt d\tau \mathrm{Tr} \partial_\tau (\varepsilon dA), \tag{21}$$

$$= -\frac{ik}{4\pi} \int_{D \times R} \mathrm{Tr} d(\varepsilon dA),$$

which is just the opposite of the gauge variation (10) of the bulk NCCS action (4). Therefore, the combined action

$$S_{total} = I_{cs} + S(h, A) \tag{22}$$

is gauge invariant under the combined transformation (19), and thus serves as the complete effective action for a quantum Hall fluid on a disc. The gauge anomaly from the bulk NCCS theory is cancelled by an edge theory described by $S(h, A)$.

Before ending this section, we need to clarify more precisely the meaning of the star product in eq. (18) on the edge. Note that originally the star product is defined in the bulk of the disc. In eq. (18) we first do the star product as if in the bulk and then restrict the value of τ to be that on the boundary: i.e. $\tau = R$ with R the radius of the disc. We observe that (τ, x) are actually the polar coordinates on the disc, while originally we have $[x^1, x^2] = i\theta$ in terms of Cartesian coordinates. So the restriction of eq. (2) leads to the following explicit expression on the boundary

$$(f * g)(x, t) = \exp\{\frac{i}{2}\theta[\cos x(\frac{\partial}{\partial \tau})_f - \frac{\sin x}{\tau}(\frac{\partial}{\partial x})_f][\sin x(\frac{\partial}{\partial \tau})_g + \frac{\cos x}{\tau}(\frac{\partial}{\partial x})_g]$$
$$-(f \leftrightarrow g)\} f(\tau, x, t) g(\tau, x, t)|_{\tau = R = 1}. \tag{23}$$

Where we have set the radius of disc as unity.

In ordinary effective CS theory, the chiral boson field h lives only on the edge. However in the NCCS theory the star product on the edge is inherited from that in the bulk, so the chiral boson field h must be extended to the bulk. Thus, the edge action (20), though consisting of integrals on the boundary, actually depends on the extension of the chiral boson field h from the edge to the bulk, because the star product (23) involves infinitely many higher order derivatives in the radial direction. So rigorously speaking, the chiral boson field h is not restricted to the edge only. But its bulk action, the NCWZ term in

the action (16), like its commutative counterpart, is a topological action [18,19]. Hence its dynamics is described only by the boundary terms in the actions (16) and (20). It is in this (weaker) sense that we say the additional action (20) describes a dynamical theory on the edge. We note that this agrees with the present physical picture in condensed matter community about the edge states in the quantum Hall effect: Namely, the edge excitations described by the above chiral boson are actually quasiparticles, which do exist in the bulk as gapful excitations, but become gapless and dynamical at low energy while on the edge.

III. DOUBLE-LAYER QUANTUM HALL SYSTEM

In this section, we are going to construct a gauge invariant action for a quantum Hall double-layer system using NCCS effective field theory. We first briefly review Wen and Zee's formulation [6], in which a double-layer quantum Hall system is described by the following Lagrangian for two ordinary $U(1)$ CS fields, one for each layer:

$$S = \frac{1}{4\pi} \int \sum_{I,J} K_{IJ} A^I d A^J, \tag{24}$$

where $I, J = 1, 2$ are layer indices. If the symmetric K-matrix does not have zero eigenvalue, then it describes an incompressible quantum Hall fluid, with the total filling factor $\nu = \sum_{I,J}(K)^{-1}_{IJ}$. On the other hand, if the matrix K has a zero eigenvalue, then some linear combination of the gauge fields becomes massless and the Hall fluid is compressible. In particular, when all the matrix elements take the same value, say k, such effective theory describes a system with the total filling factor $\nu = \frac{1}{k}$ [21,26]. It is easy to check that the Lagrangian (24) is gauge invariant on a compact space under the $U(1) \times U(1)$ gauge transformations

$$A^I \to A'^I = A^I + d\lambda^I, \tag{25}$$

where $I = 1, 2$. Namely, the fields at different layers transform independently. In this formulation, gauge invariance of the CS theory does not require the symmetric matrix K to

be quantized. However, as pointed out by Wen and Zee, the sources of gauge potential A_I are vortex-like quasi-particles in the fluid; it is their circulation quantization that fixes the elements of the K matrix to be integers.

Now we proceed to generalizing Wen-Zee's Lagrangian (24) for multiple $U(1)$ CS fields to the noncommutative case. Some new features are noted immediately. First, instead of eq. (25), the noncommutative $U(1)$ gauge transformation law in an individual layer is modified to

$$\delta A^I = d\lambda^I + [A^I, \lambda^I], \tag{26}$$

(no summation over I). Therefore, noncommutative gauge invariance requires that the $U(1)$ NCCS action for individual layers, i.e. the terms with $I = J$ in the action (24), be supplemented by cubic terms, as those shown in eq.(4). For the sake of generality, it is natural to introduce also new cubic cross-terms involving both A^1 and A^2, whose coefficients are to be determined by gauge invariance. These considerations lead to the following suggestion for the most general form of the NCCS action for a double-layer system:

$$S_{double} = K_{11} \int (A^1 dA^1 + \frac{2}{3}A^1 A^1 A^1) + K_{22} \int (A^2 dA^2 + \frac{2}{3}A^2 A^2 A^2) \tag{27}$$
$$+ K_{12} \int (A^1 dA^2 + L_{122}A^1 A^2 A^2) + K_{21} \int (A^2 dA^1 + L_{211}A^2 A^1 A^1).$$

Obviously, if $\theta = 0$, all the cubic terms automatically vanish and we return to the action (24).

To examine gauge invariance of the action (27), we first write the variation of the action (27), under generic δA^1 and δA^2, as a sum of four terms:

$$\delta S^{11}_{double} = 2K_{11} \int \delta A^1 F^1; \qquad \delta S^{22}_{double} = 2K_{22} \int \delta A^2 F^2, \tag{28}$$

$$\delta S^{12}_{double} = K_{12} \int \delta A^1 (dA^2 + L_{122}A^2 A^2) + K_{12} \int \delta A^2 [dA^1 + L_{122}(A^2 A^1 + A^1 A^2)], \tag{29}$$

$$\delta S^{21}_{double} = K_{21} \int \delta A^2 (dA^1 + L_{211}A^1 A^1) + K_{21} \int \delta A^1 [dA^2 + L_{211}(A^1 A^2 + A^2 A^1)]. \tag{30}$$

Now let us assume that A^1 and A^2 transform independently, as given by eqs. (25) with λ_1 and λ_2 independent of each other. Then the gauge variation of the diagonal terms δS^{11}_{double}

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and δS_{double}^{22} are separately zero, while that of the cross terms, $\delta S_{double}^{12} + \delta S_{double}^{21}$ vanishes only if

$$K_{12} + K_{21} = 0, \qquad L_{122} = L_{211} = 0. \tag{31}$$

Substituting these back to the action (27), we see that the cross terms are identically zero. This leads to completely decoupled two layers, not interesting at all.

In a double-layer quantum Hall system, the two layers can couple to each other either due to interlayer tunneling or due to Coulomb interactions between charged particles in different layers. This will naturally lead to the appearance of the cross terms in the effective theory. The result (31) tells us that in order for an action like eq. (27) to describe a *correlated* double-layer quantum Hall system, one can not assume the two $U(1)$ NCCS fields transform independently. This is an amazing consequence of gauge invariance in a multiple $U(1)$ NCCS theory.

Then we ask: what kind of modification of the gauge transformations will be allowed that is compatible with NCCS gauge invariance? We need cancellations in the gauge variations of the cross terms. So this motivates us to propose that when the NCCS field in one layer transforms like a gauge potential, that in the other layer transforms covariantly like an adjoint matter:

$$\delta_J A^I = \delta_J^I \delta \lambda^I + [A^I, \lambda^J], \tag{32}$$

where $I, J = 1, 2$ and no summation is assumed for repeated indices. First let us take $J = 1$ in the combined gauge transformation (32), then

$$\delta A^1 = D\lambda^1, \qquad \delta A^2 = [A^2, \lambda^1]. \tag{33}$$

Consequently, the gauge variations in eq. (28) are evaluated as

$$\delta S_{double}^{11} = 0; \qquad \delta S_{double}^{22} = -2K_{22} \int d\lambda^1 A^2 A^2, \tag{34}$$

$$\delta S_{double}^{12} = K_{12} L_{122} \int d\lambda^1 A^2 A^2 + K_{12} \int \lambda^1 d(A^2 A^1 + A^1 A^2), \tag{35}$$

$$\delta S_{double}^{21} = K_{21}(1 - L_{211}) \int \lambda^1 d(A^1 A^2 + A^2 A^1). \tag{36}$$

Thus the gauge invariance, $\delta S_{double} = 0$, requires that

$$K_{12}L_{122} = 2K_{22}; \qquad K_{12} + K_{21} = K_{21}L_{211}. \tag{37}$$

Similarly, we take $J = 2$ and derive from gauge invariance the equations

$$K_{21}L_{211} = 2K_{11}; \qquad K_{12} + K_{21} = K_{12}L_{122}. \tag{38}$$

Solving this two sets of equations, we get

$$K_{11} = K_{22} = \frac{K_{12} + K_{21}}{2}; \tag{39}$$

$$L_{122} = \frac{K_{12} + K_{21}}{K_{12}}; \qquad L_{211} = \frac{K_{12} + K_{21}}{K_{21}}.$$

It is easy to show that we can always take the K-matrix to be symmetric, so it is constrained to be of the form

$$K = k \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{40}$$

Moreover, we have

$$L_{122} = L_{211} = 2. \tag{41}$$

These are constraints required by the gauge invariance under the combined transformations (32).

In summary, we have obtained a gauge invariant double $U(1)$ NCCS action as follows:

$$S_{double} = \frac{k}{4\pi} \int (A^1 dA^1 + \frac{2}{3}A^1 A^1 A^1) + (A^2 dA^2 + \frac{2}{3}A^2 A^2 A^2) \tag{42}$$

$$+ \frac{k}{4\pi} \int (A^1 dA^2 + 2A^1 A^2 A^2) + (A^2 dA^1 + 2A^2 A^1 A^1).$$

The gauge transformation laws are those given in eqs. (32). We suggest that this NCCS theory describes the Halperin (kkk) state for a double-layer quantum Hall system.

Incidentally, we note that if we include in the integral sign the trace in group space as well, then the above result (42) applies to the double U(N) NCCS theory too. If we further

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take the NC parameter $\theta = 0$, then it reduces to a gauge-invariant action for coupling two single-layer non-Abelian Chern-Simons fluids, with $SU(N)$ gauge group, into a correlated double-layer system. To our knowledge, such a construction did not exist before even for ordinary CS theory.

Returning to the double $U(1)$ NCCS action (42), one immediate question coming to our mind is whether the level k should be quantized or not? In the commutative counterpart, there is no topological argument for the quantization of k; but Wen and Zee [6] have argued that the elements of the K-matrix should be integers, based on the circulation quantization of the vortex-like quasiparticles in the theory. Here we conjecture that there should be a one-loop quantum shift of the level parameter k in the double $U(1)$ NCCS theory, and its value should be unity, just like the single-layer $U(1)$ NCCS theory as computed by us in Ref. [17]. The quantized one-loop shift of k is perhaps constrained by a topological argument, which generalizes a similar argument in the single-layer NCCS case [18,19] and leads to the quantization of level k as well.

IV. CONCLUSIONS AND DISCUSSIONS

In this paper we have generalized the construction of the $U(N)$ NCCS theory from a single-layer plane to a single-layer disc and to a double-layer plane. Gauge invariance is the main issue we addressed.

In the first case, it is necessary to introduce a group-valued bosonic field h with chiral boundary terms, whose gauge variation cancels that of the bulk NCCS action. Mathematical steps of this construction proceed much the same way as in the commutative counterpart [24]. But some new distinct features appear due to noncommutativity in the bulk. Essentially now the boundary action becomes non-local both in the edge and in the radial direction of the disc, in contrast to the ordinary case where the boundary action describes local dynamics on the edge. Since the boundary terms depend on the extension of the group-valued h-field in the bulk, it can no longer be viewed as degrees of freedom that only live on the boundary,

though its bulk action is still a topological one (the NCWZ term). Mathematically these two new features arise due to the restriction to the edge of the noncommutative star product in the bulk, which involves infinitely many higher order derivatives in both the angular and the radial directions.

When the gauge group is $U(1)$, the construction gives a noncommutative deformation of some known theory for edge states of quantum Hall (QH) fluids (see Refs. [6,24]). It would be interesting to see what new physics the aforementioned non-local features of our construction would bring to the QH edge theory. Obviously the non-locality of the boundary terms would make the interactions along the edge and the bulk-edge connections more complicated and intriguing than we thought before. Of course, one of the key questions is whether or not and, if yes, how the chiral Luttinger-liquid exponents could be affected by all these complications. An analysis based on renormalization group (RG) is needed to answer this question. Indeed an RG analysis for the noncommutative Landau-Ginsburg theory on a plane has been done already in Ref. [27]; however, the case at hand is quite different and presents several new features. We hope to have chance to return to this problem in the future.

In the case of a double-layer system, gauge invariance does not constrain ordinary multiple $U(1)$ CS action, while the constraints in the non-Abelian $U(N)$ CS case have never been worked out. In this paper we have found that for an NCCS theory on a *correlated* double-layer, gauge invariance indeed severely constrains the form of the K-matrix in the action: Namely all elements of the K-matrix have to be the same integer. Experimentally, there is a state in the double-layer quantum Hall system, the Halperin (kkk) state, which is believed to be described, at least, by the ordinary double $U(1)$ CS theory with a similar K-matrix. It would be interesting to see whether our noncommutative deformation could be used to describe this state as we have suggested. In particular, there are new (cubic) terms appearing in the cross action that mixes the CS fields at different layers, as required by gauge invariance. What would be the physical effects of these terms? We also note that if we take the NC parameter $\theta = 0$, our result (42) gives a new action that generalizes multiple $U(1)$ ordinary CS theory to multiple *non-Abelian $U(N)$*.

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One question that puzzles us is the following: Many of the K-matrix in double $U(1)$ ordinary CS theory that are discussed in the literature on the QH edge states are *not* of the very restrictive form (40) as derived in this paper. The question is whether it is possible to get NC deformation of CS theory with those K-matrices? On one hand, we feel it should be possible. On the other hand, we do not know how to do it. The point is that we have made a specific assumption (32) on how one CS field transforms under the gauge transformation of the other. Could we relax this assumption to get the NC deformation of multiple CS theory with more general K-matrix?

Finally let us mention that in relation to string theory, we think it could be interesting to work out the brane realization in the spirit of Ref. [28,20], or the CS matrix realization in the spirit of Ref. [12,14], of the NCCS theories we discussed above.

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FIG. 1. Disc Geometry
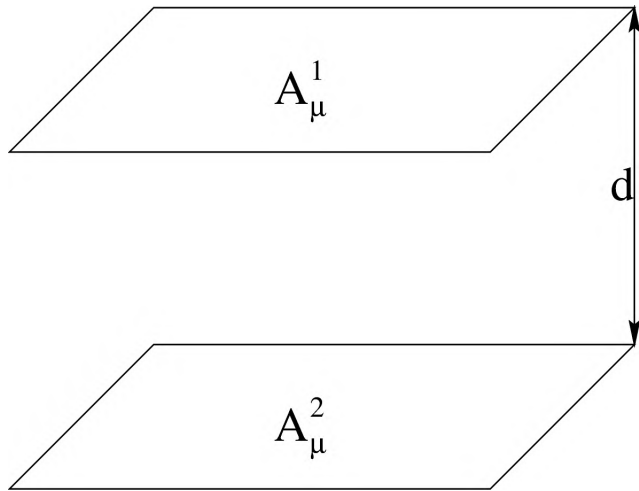
FIG. 2. Double-layer geometry
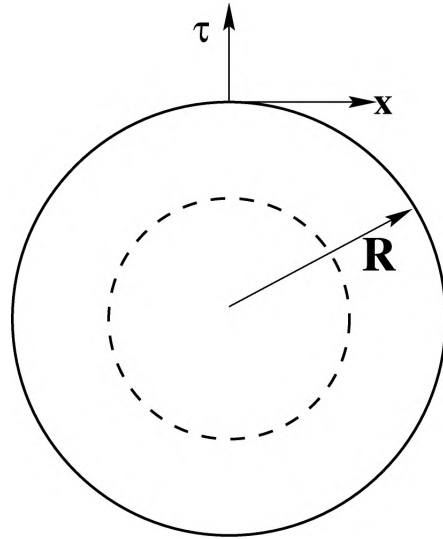
Fig. 2 Double-layer geometry

Fig. 1 Disc gemometry with extension