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Perturbative formulation and nonadiabatic corrections in adiabatic quantum-computing schemes

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Adiabatic limit is the presumption of the adiabatic geometric quantum computation and of the adiabatic quantum algorithm. But in reality, the variation speed of the Hamiltonian is finite. Here we develop a general formulation of adiabatic quantum computing, which accurately describes the evolution of the quantum state in a perturbative way, in which the adiabatic limit is the zeroth-order approximation. As an application of this formulation, nonadiabatic correction or error is estimated for several physical implementations of the adiabatic geometric gates. A quantum-computing process consisting of many adiabatic gate operations is considered, for which the total nonadiabatic error is found to be about the sum of those of all the gates. This is a useful constraint on the computational power. The formalism is also briefly applied to the adiabatic quantum algorithm.

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Recently a considerable amount of attention has been paid to the idea of using geometric phases accumulated by an adiabatically time-dependent Hamiltonian to realize quantum gates [1–3]. Construction of universal gates by geometric quantum teleportation was studied, with the analysis of errors from imperfect control [4]. On the other hand, adiabatic evolution is also the basis of the so-called adiabatic quantum algorithms [5], for which the speed and the overall time have been analyzed [6].

For these quantum-computing schemes to work, it was supposed that the adiabatic limit is retained. However, in practice, and particularly in the case of quantum computation, where the advantage lies in speedup and the operation time should be shorter than the decoherence time, the evolution is required to be completed in a finite period of time. Therefore, it is important to know the full picture of the evolution of the quantum state and the nonadiabatic correction, which gives rise to error if the adiabatic limit is necessary for the designed quantum-computing scheme. Here we develop a general formulation of adiabatic quantum computing, applicable to the previously proposed quantum-computing schemes and to any slowly varying Hamiltonian. As an adiabatic perturbation theory, it accurately describes the quantum evolution in a perturbative way, in which the adiabatic limit is the zeroth-order approximation. As an application, an examination is made on the nonadiabatic errors in several previously proposed implementations of the adiabatic geometric gates. We also investigate the nonadiabatic error in an entire quantum-computing process consisting of many adiabatic gates, which has not been considered previously. Finally we briefly discuss the adiabatic quantum algorithm, noting that such an algorithm can still be implemented even if the nonadiabatic correction is not vanishingly small.

If the evolution of a time-dependent Hamiltonian is sufficiently slow, the adiabatic theorem tells that *in the adiabatic limit* and under such conditions as continuity, noncrossing, and differentiability, an instantaneous eigenstate at an initial time evolves to a state close to the corresponding instantaneous eigenstate at a later time [7].

In general, using the instantaneous eigenstate $|\phi_n(t)\rangle$, one can always expand the state of the system $|\psi(t)\rangle$ as

$$|\psi(t)\rangle=\sum_n a_n(t)|\phi_n(t)\rangle\exp[i\eta_n(t)], \tag{1}$$

where $\eta_n(t)=-i/\hbar\int_0^t E_n(\tau)d\tau$ is the dynamic phase. Then the Schrödinger equation $i\hbar\partial_t|\psi(t)\rangle=H(t)|\psi\rangle$ leads to

$$\partial_t a_n(t)=-\sum_m a_m(t)\langle\phi_n(t)|\partial_t\phi_m(t)\rangle\exp[i\eta_m(t)-i\eta_n(t)], \tag{2}$$

which together with the initial condition $a_n(0)\equiv\langle\phi_n(0)|\psi(0)\rangle$, determines $|\psi(t)\rangle$.

First suppose $|\psi(0)\rangle$ is a nondegenerate eigenstate $|\phi_n(0)\rangle$. Then in the adiabatic limit, one obtains $|\psi(t)\rangle\approx|\phi_n(t)\rangle\exp[i\gamma_n(t)+i\eta_n(t)]$, where $\gamma_n(t)=\int_0^t\langle\phi_n|\partial_\tau\phi_n\rangle d\tau=\int\langle\phi_n|\partial_\mu\phi_n\rangle dx^\mu$ is the geometric or Berry phase [8].

However, when nonadiabatic correction is considered, the exact state should be the solution of Eq. (2). For a slowly varying $H(t)$, using a perturbative approach, one can obtain

$$U(t)|\phi_n(0)\rangle=\exp[i\gamma_n(t)+i\eta_n(t)]\left[|\phi_n(t)\rangle+\hbar\sum_{m\neq n}\frac{|\phi_m(t)\rangle\langle\phi_m(t)|\partial_t\phi_n(t)\rangle}{E_m(t)-E_n(t)}\right]+\cdots. \tag{3}$$

In general, as in quantum computing, $|\psi(0)\rangle=\Sigma_n a_n(0)|\phi_n(0)\rangle$ is a superposition of different eigenstates. Then linearity of quantum evolution implies

$$|\psi(t)\rangle=\sum_n a_n(0)U(t)|\phi_n(0)\rangle, \tag{4}$$

where each $U(t)|\phi_n(0)\rangle$ is as given in Eq. (3). Therefore, $a_n(t)\exp[i\eta_n(t)]=\Sigma_m\langle\phi_n(t)|U(t)|\phi_m(0)\rangle a_m(0)$. From Eq. (3), one obtains $\langle\phi_n(t)|U(t)|\phi_n(0)\rangle\approx\langle\phi_n(t)|U^{(0)}(t)|\phi_n(0)\rangle=\exp[i\gamma_n(t)+i\eta_n(t)]$, while for $n\neq m$, $\langle\phi_n(t)|U(t)|\phi_m(0)\rangle$

$\approx \langle \phi_n(t)|U^{(1)}(t)|\phi_m(0)\rangle = \hbar \exp[i\gamma_m(t) + i\eta_m(t)]\langle \phi_n|\partial_t \phi_m\rangle/[E_n(t) - E_m(t)]$. Here $U^{(k)}$ refers to kth order term. Since $E_n(t) \neq E_m(t)$, $\eta_n(t) \neq \eta_m(t)$. If one implements an all-geometric gate, in which the instantaneous basis states are $|\phi_n(t)\rangle$ and $|\phi_m(t)\rangle$, the difference between $\eta_n(t)$ and $\eta_m(t)$ needs to be cancelled out by using a certain method [1].

In the presence of degeneracy of eigenstates, denote the eigenstates as $|\phi^n_{\alpha_n}(0)\rangle$, where n labels the energy levels, while α_n labels the different eigenstates in the subspace n. As the generalization of Eq. (3), we obtain

$$U(t)|\phi^n_{\alpha_n}(0)\rangle = \exp[i\eta_n(t)]\left[|\chi^n_{\alpha_n}(t)\rangle + \hbar \sum_{m\neq n}\sum_{\beta_m} \frac{|\chi^m_{\beta_m}(t)\rangle\langle\chi^m_{\beta_m}(t)|\partial_t \chi^n_{\alpha_n}(t)\rangle}{E_m(t) - E_n(t)}\right] + \cdots, \tag{5}$$

with $|\chi^n_{\alpha_n}(t)\rangle = \Sigma_{\beta_n} V^n_{\alpha_n\beta_n}(t)|\phi^n_{\beta_n}(t)\rangle$, $V^n(t) = P\exp\int_0^t A^n(\tau)d\tau$, where $A^n_{\alpha_n\beta_n} \equiv \langle \chi^n_{\beta_n}(\tau)|\partial_t \chi^n_{\alpha_n}(\tau)\rangle$ is the connection in the subspace n. V^n may be called nonabelian geometric phase or Wilczek-Zee (WZ) phase [9]. In the zeroth order, $U(t)$ is block diagonal, each block being a WZ phase in the subspace of a set of degenerate eigenstates. In the adiabatic limit, as a unitary transformation, a nonabelian geometric phase, i.e., the first term in Eq. (5), may be used to realize a quantum gate [2].

With the existence of degeneracy of eigenstates, a general superposition state can be written as

$$|\psi(t)\rangle = \sum_n \sum_{\alpha_n} a^n_{\alpha_n}(t)|\chi^n_{\alpha_n}(t)\rangle \exp[i\eta_n(t)]. \tag{6}$$

By choosing an appropriate basis for each degenerate subspace, the initial state can always be expanded in such a way that its projection in each degenerate subspace is a single eigenstate $|\phi^n_{\beta_n}(0)\rangle$, i.e., $|\psi(0)\rangle = \Sigma_n a^n_{\beta_n}(0)|\chi^n_{\beta_n}(0)\rangle$, with $|\chi^n_{\beta_n}(0)\rangle = |\phi^n_{\beta_n}(0)\rangle$. Therefore,

$$|\psi(t)\rangle = \sum_n a^n_{\beta_n}(0)U(t)|\chi^n_{\beta_n}(0)\rangle. \tag{7}$$

Thus $a^n_{\alpha_n}(t)\exp[i\eta_n(t)] = \Sigma_m \Sigma_{\beta_m}\langle \chi^n_{\alpha_n}(t)|U(t)|\chi^m_{\beta_m}(0)\rangle a^m_{\beta_m}(0)$. From Eq. (5),

$$\langle \chi^n_{\alpha_n}(t)|U(t)|\chi^n_{\beta_n}(0)\rangle \approx \langle \chi^n_{\alpha_n}(t)|U^{(0)}(t)|\chi^n_{\beta_n}(0)\rangle = \exp[i\eta_n(t)]\delta_{\alpha_n\beta_n},$$

while for $n \neq m$,

$$\langle \chi^n_{\alpha_n}(t)|U(t)|\chi^m_{\beta_m}(0)\rangle \approx \langle \chi^n_{\alpha_n}(t)|U^{(1)}(t)|\chi^m_{\beta_m}(0)\rangle = \hbar \exp[i\eta_m(t)] \times \langle \chi^n_{\alpha_n}(t)|\partial_t \chi^m_{\beta_m}(t)\rangle/[E_n(t) - E_m(t)].$$

Through this formulation, it becomes clear that the adiabatic quantum computing is based on $\langle \phi_n(t)|U^{(0)}(t)|\phi_n(0)\rangle$ or $\langle \chi^n_{\alpha_n}(t)|U^{(0)}(t)|\chi^n_{\beta_n}(0)\rangle$, with higher-order terms neglected. Besides, while the previous proposals of adiabatic geometric gates are based on closed paths, there is nothing in principle against using open paths as far as the corresponding geometric phases can be detected [10]. Another noteworthy point, which was not pointed out before, is that when the qubits under a gate operation is entangled with other qubits, the linearity of quantum evolution guarantees that the gate operation is still given by Eq. (3) or (5), where the eigenstates are those of this concerned gate; one may include in the coefficients $a_n(0)$ or $a^n_{\alpha_n}(0)$ the states of the other qubits projected in the same branch as the eigenstates of the gated qubits. This is crucial for the possibility that different adiabatic geometric gates can be networked.

There is a significant difference in the uses of (abelian) Berry phase and WZ phase to realize a quantum gate, under adiabatic limit. For a Berry phase gate, it is necessary to have d nondegenerate states, where d is the Hilbert space dimension of the gate. For a WZ phase gate, one intentionally restricts the gate in a single degenerate eigenspace. A quantum gate based on WZ phase is more advantageous than that based on Berry phase, on the aspect that for the former, in the adiabatic limit, the state is always an instantaneous eigenstate of the Hamiltonian, hence there is no dynamical phase difference between the basis states, and it is more stable against environmental perturbation.

The nonadiabatic correction or error at time t is $\epsilon(t) \equiv [U(t) - U^{(0)}(t)]|\psi(0)\rangle = \Sigma_{k=1}^{\infty} U^{(k)}(t)|\psi(0)\rangle \approx U^{(1)}(t)|\psi(0)\rangle$. The adiabatic limit means $|\epsilon(T)| \ll 1$. For a Berry phase gate, with $|\psi(0)\rangle = \Sigma_n a_n(0)|\phi_n(0)\rangle$,

$$\epsilon(t) \approx \sum_n a_n(0) \sum_{m\neq n} \langle \phi_m(t)|U^{(1)}(t)|\phi_n(0)\rangle|\phi_n(0)\rangle. \tag{8}$$

For a WZ phase gate geometric gate at E_n, with $|\psi(0)\rangle = \Sigma_{\alpha_n} a_{\alpha_n}(0)|\phi^n_{\alpha_n}(0)\rangle$,

$$\epsilon(t) \approx \sum_{\alpha_n} a_{\alpha_n}(0) \times \sum_{m\neq n}\sum_{\beta_m} \langle \chi^m_{\beta_m}(t)|U^{(1)}(t)|\chi^n_{\alpha_n}(0)\rangle|\phi^n_{\alpha_n}(0)\rangle. \tag{9}$$

Note that the first-order correction at time t is determined only by eigenvalues, eigenstates, and their time derivatives at t, hence it is history independent. This simplifies the analysis. The time derivatives do depend on the details of time dependence. However, since only the path is specified [11], without the fine control of the dynamics, numerically it suffices to obtain the order of magnitude. The first-order correction is $\sim \hbar/\Delta T$, where T is the time duration of the gate operation and Δ is the minimum energy gap with other eigenstates. It is the presumption of "slow variation" or perturbative approach that $\hbar/\Delta T < 1$. The kth-order correction is $\sim (\hbar/\Delta T)^k$.

As applications, we now apply the above results to several physical implementations previously proposed. The first pro-

posal, based on Berry phase, uses NMR [1]. The Hamiltonian, in the rotating frame, is $H(t)=\mathbf{R}(t)\cdot\mathbf{I}$, where $\mathbf{R}\equiv(R_x,R_y,R_z)=\hbar(\omega_1\cos\phi,\omega_1\sin\phi,\omega_0-\omega)$, $\mathbf{I}=\frac{1}{2}(\sigma_x,\sigma_y,\sigma_z)$, ω_0 is proportional to the static magnetic field in z direction, ω_1 is proportional to the RF magnetic field in xy plane, ω is its angular frequency, and ϕ is its initial phase. The instantaneous eigenstates is $|1(t)\rangle=1/\sqrt{R}[(R_x-iR_y)/\sqrt{R-R_z}|\uparrow\rangle+\sqrt{R-R_z}|\downarrow\rangle]$, with eigenvalue $R/2$, and $|0(t)\rangle=1/\sqrt{R}[(R_x-iR_y)/\sqrt{R+R_z}|\uparrow\rangle-\sqrt{R+R_z}|\downarrow\rangle]$ with eigenvalue $-R/2$, where $R\equiv|\mathbf{R}|$. From this, one obtains, for $n=0,1$, $U_{nn}(0,t)=\exp[i\gamma_n(t)+i\eta_n(t)]$. $\eta_0(t)=Rt/2$, $\eta_1(t)=-Rt/2$. The Berry phase $\gamma_n(t)$ is, in the case of a cycle path C, $\Omega(C)/2$ for $n=0$ and $-\Omega(C)$ for $n=1$, where $\Omega(C)$ is the solid angel that C subtends at $R=0$. It is straightforward to write down $U_{01}(0,t)$ and $U_{10}(0,t)$. For a gate operation of a period T, the order of magnitude of these two matrix elements, and thus the nonadiabatic correction, is $\hbar/RT=1/\sqrt{\omega_1^2+(\omega_0-\omega)^2}T$. The two-bit gate, of qubits a and b, is effected by addition of the interaction $\hbar JI_{az}I_{bz}$. For the conditional phases of qubit a, ω_{a0} shifted to to $\omega_{a0}+JI_{bz}=\omega_{a0}\pm J/2$, depending on the basis state $|I_{bz}\rangle$ of b. One can obtain the nonadiabatic corrections in the two subspaces, with the substitution of $\omega_{a0}\pm J/2$ for ω_0 in R above. For a gate as in Ref. [1], the gaps are of the order of several hundred Hertz, while T is of the order of second, hence the nonadiabatic corrections are of the order of 10^{-2}.

This method was also applied to a Josephson junction circuit [12]. The effective Hamiltonian is still as that for NMR, now with $\mathbf{R}=[E_J\cos\alpha,-E_J\sin\alpha,E_c(1-2n_{off})]$, where E_J and α are decided by the Josephson couplings of two junctions, E_c is charging energy, and $2en_{off}$ is the offset charge. In the charging regime, as used, $E_c\geqslant E_J$. Thus the nonadiabatic correction is of the order of $\hbar/\sqrt{E_J^2+E_c^2(1-2n_{off})^2}T$. Hence if $1-2n_{off}$ is not too small, the adiabatic condition is $\hbar/E_cT\ll 1$, more relaxed than previously thought [12].

An implementation of WZ phase gate was proposed for trapped ions [13–15]. The one-bit gates are based on the Hamiltonian $H=\hbar|e\rangle(\omega_0\langle 0|+\omega_1\langle 1|+\omega_a\langle a|+\text{h.c.})$. One can find that the eigenstates are: $|\phi^1\rangle=(\omega|e\rangle+\omega_0^*|0\rangle+\omega_1^*|1\rangle+\omega_a^*|a\rangle)/\sqrt{2}\omega$ with eigenvalue $\hbar\omega$, where $\omega=\sqrt{\omega_0^2+\omega_1^2+\omega_a^2}$, $|\phi_\alpha^0\rangle=(\omega_1|0\rangle-\omega_a|1\rangle)/\sqrt{|\omega_0|^2+|\omega_1|^2}$ and $|\phi_\beta^0\rangle=(\omega_a\omega_0^*|0\rangle+\omega_a\omega_1^*|1\rangle-(|\omega_0|^2+|\omega_1|^2)|a\rangle)/(\omega\sqrt{|\omega_0|^2+|\omega_1|^2})$ with eigenvalue 0, and $|\phi^{-1}\rangle=(-\omega|e\rangle+\omega_0^*|0\rangle+\omega_1^*|1\rangle+\omega_a^*|a\rangle)/\sqrt{2}\omega$ with eigenvalue $-\hbar\omega$. The WZ phase gates are based on U^{00}, in terms of our notation. Using the instantaneous eigenstates and eigenvalues, the nonadiabatic correction is obtained as $\Sigma_{n=-1,1}\Sigma_{x=\alpha,\beta}a_x(0)U_x^{n0}(T)|\phi_x^0(0)\rangle$, whose order of magnitude is of $1/\omega T$. The two-bit gate proposed there is only a Berry phase gate under the Hamiltonian [13] $H_{jk}=(\eta^2/\delta)[-|\Omega_1|^2\sigma_{j1}^{\phi_1}\sigma_{k1}^{\phi_1}+|\Omega_a|^2\sigma_{ja}^{\phi_a}\sigma_{ka}^{\phi_a}]$, where $\sigma_{j\mu}^{\phi_\mu}\equiv e^{i\phi_\mu}|e\rangle_{jj}\langle\mu|+\text{h.c.}$, $\phi_1-\phi_a=\phi/2$, using the notations therein. The eigenstates are $\phi^1=(-|\Omega_1|^2e^{-i\phi}|11\rangle+|\Omega_a|^2|aa\rangle+\sqrt{|\Omega_1|^4+|\Omega_a|^4}|ee\rangle)/\sqrt{2(|\Omega_1|^4+|\Omega_a|^4)}$ with eigenvalue $(\eta^2/\delta)\sqrt{|\Omega_1|^4+|\Omega_a|^4}$, $|\phi^0\rangle=(|\Omega_a|^2|11\rangle+|\Omega_1|^2e^{i\phi}|aa\rangle)/\sqrt{|\Omega_1|^4+|\Omega_a|^4}$ with eigenvalue 0, and $\phi^{-1}=(-|\Omega_1|^2e^{-i\phi}|11\rangle+|\Omega_a|^2|aa\rangle-\sqrt{|\Omega_1|^4+|\Omega_a|^4}|ee\rangle)/\sqrt{2(|\Omega_1|^4+|\Omega_a|^4)}$ with eigenvalue $-(\eta^2/\delta)\sqrt{|\Omega_1|^4+|\Omega_a|^4}$. It was proposed to use $|\phi^0\rangle$ to implement the phase gate. The nonadiabatic correction is of the order of $\hbar/T(\eta^2/\delta)\sqrt{|\Omega_1|^4+|\Omega_a|^4}$.

Similar proposals were also made in Josephson junction charge qubits [16,17]. For the Hamiltonian used in Ref. [16], there is an eigenstate with eigenvalue $\sqrt{h^2+|J_1|^2+|J_2|^2}$, two degenerate eigenstates with eigenvalue h, two degenerate eigenstates with eigenvalue $-h$, which are used to implement the WZ phase gate, and one ground state with eigenvalue $\sqrt{h^2+|J_1|^2+|J_2|^2}$, where $h=E_c(1-2n_{off})/2$. Thus the nonadiabatic correction is of the order of $\hbar/(\sqrt{h^2+|J_1|^2+|J_2|^2}-h)T$. In the two-bit implementation, the eigenvalues are $-\sqrt{|J_b|^2+(2h)^2}$, $-2h$, 0, $2h$, $\sqrt{|J_b|^2+(2h)^2}$. The eigenstates with eigenvalue $-2h$ are used as the qubit states. The nonadiabatic correction is of the order of $\hbar/\Delta T$, where Δ is the smaller one of $\sqrt{|J_b|^2+(2h)^2}-2h$ and $2h$. Suppose the order of magnitude of Josephson energy is J. Then if $1-2n_{off}$ is close to 1, the energy gap is of the order of J^2/E_c in the single-bit gate, and is of the order of $J^2/4E_c$ in the two-bit gate. Since $E_c\geqslant J$, the energy gap is smaller than J. Hence compared with the case of [12], the adiabatic condition is harder to meet, i.e., the nonadiabatic correction is larger. On the other hand, if $1-2n_{off}$ is tuned to be very small, then the energy gap for the cases of both [16] and [12] are of the order of the Josephson energy.

In the one-bit gate in [17], the energy eigenvalues are $\delta E_c+\sqrt{(\delta E_c)^2+2J^2}$, 0, which is with twofold degeneracy and is used to implement the WZ gate, and $\delta E_c-\sqrt{(\delta E_c)^2+J^2}$, where δE_c is some charging energy difference, $J^2=|J_L|^2+|J_M|^2+|J_R|^2$, using the notations there. Thus the nonadiabatic correction is of the order of $\hbar/\Delta T$, where Δ is the smaller one of $|\delta E_c+\sqrt{(\delta E_c)^2+J^2}|$ and $|\delta E_c-\sqrt{(\delta E_c)^2+J^2}|$. Hence Δ is of the order of δE_c if $\delta E_c>J$, and is of the order of J if $\delta E_c<J$. For the two-bit gate, the three energy eigenvalues are $\sqrt{|J_X|^2+|J_M^{(2)}|^2}/2$, 0, and $-\sqrt{|J_X|^2+|J_M^{(2)}|^2}/2$, where the parameters are as defined there. Hence the nonadiabatic correction is of the order of $2\hbar/\sqrt{|J_X|^2+|J_M^{(2)}|^2}T$. For both one-bit and two-bit gates, the energy gap is at most of the order of Josephson energy. Therefore, though WZ gate has more advantages over Berry gate, the nonadiabatic error for [16,17] is larger than that for [12].

A quantum-computing process in a gate array consists of many gate operations on a large number of qubits, hence a complete estimation of error must include its scaling with the number of gate operations. Suppose from time 0 to T, M_1 adiabatic gates, denoted as $U^j(t)$, $(j=1,\cdots,M_1)$, are in parallel operation, each on a small number of (say, one or two) qubits. For $0\leqslant t\leqslant T$, the entire quantum computer evolves as $|\Psi(t)\rangle=U^1(t)\cdots U^{M_1}(t)|\Psi(0)\rangle=\Sigma_{i_1\cdots i_{M_1}i_r}U^1(t)|\phi_{i_1}^1(0)\rangle\cdots U^{M_1}(t)|\phi_{i_{M_1}}^{M_1}(0)\rangle|\phi_{i_r}^r\rangle$, where $|\phi_{i_j}^j\rangle$ is a basis state of the qubits acted by the jth gate, r denotes the rest qubits, which

are not operated by any gate during this period. We know that $U^j(t)|\phi_{i_j}^j(0)\rangle = U_0^j(t)|\phi_{i_j}^j(0)\rangle + \epsilon^j(t)$, where $U_0^j(t)$ represents the adiabatic limit of jth gate, while $\epsilon^j(t) < 1$ is its nonadiabatic correction. The state of the quantum computer at T is then $|\Psi(T)\rangle = |\Psi_0(T)\rangle + \sigma(0,T)$, where $|\Psi_0(T)\rangle = U_0^1(T)\cdots U_0^{M_1}(T)|\Psi(0)\rangle$, $\sigma(0,T) \approx \Sigma_{j=1}^{M_1}\epsilon^j(T)$ is the first-order error of the entire quantum computer accumulated from time 0 to T. Afterwards, during $T < t < 2T$, the quantum computer is operated in parallel by M_2 gates, labeled as $M_1+1,\ldots,M_1+M_2$, to which the qubits are allocated in a way usually different from the period $0 < t < T$. Using a derivation similar to the above, it can be obtained that $|\Psi(2T)\rangle = U_0^{M_1+1}(T)\cdots U_0^{M_1+M_2}(T)|\Psi(T)\rangle + \sigma(T,2T)$, where $\sigma(T,2T) \approx \Sigma_{j=M_1+1}^{M_1+M_2}\epsilon^j(T)$ is the error of the entire quantum computer accumulated from T to $2T$. Therefore, $|\Psi(2T)\rangle = U_0^{M_1+M_2}(T)\cdots U_0^{M_1+1}(T)\cdots U_0^{M_1}(T)\cdots U_0^1|\Psi(0)\rangle + \sigma(0,2T)$, where $\sigma(0,2T) \approx \sigma(0,T) + \sigma(T,2T) \approx \Sigma_{j=1}^{M_1+M_2}\epsilon^j(T)$ is the total error at $2T$. Therefore, for a quantum-computing process consisting of many gate operations, no matter how they are arranged in space and time, the total nonadiabatic error is, to the first order, just the sum of the errors of all these gates.

Suppose for each adiabatic gate, the time duration $\leqslant T$, and the minimum energy gap with other eigenstates $\leqslant \Delta$. Thus the *lower bound* of the nonadiabatic error for each gate is $||\epsilon(T)|| \sim \hbar/\Delta T$. Hence the lower bound of the total error is $\sigma \approx M\epsilon(T)$, where M is is total number of gate operations. For a quantum-computing process to make sense, it is constrained that $||\sigma|| < 1$. Therefore, $M < 1/||\epsilon(T)|| = \Delta T/\hbar$. In Shor's algorithm, to factor a number N, $M \sim 300(\log_{10} N)^3$ [18]. Therefore, adiabatic quantum computing can at most factor $N \approx 10^{(1/300||\epsilon(T)||)^{1/3}}$. For $||\epsilon(T)||$ of the order 10^{-2}, $N \approx 10$.

Let us switch to the adiabatic quantum algorithm [5], which is based on adiabatically varying the Hamiltonian from a beginning Hamiltonian H_b at $t=0$ to a final one H_p at $t=T$. Under the adiabatic limit, if the system starts with the ground state of H_b, it ends up as the ground state of H_p, which gives the solution to an optimization problem. For a finite varying rate of the Hamiltonian, to the first order, the more accurate state is as given in Eq. (3). Hence to the first order approximation, the nonadiabatic correction at time T is of the order of $1/\Delta_p T$, where Δ_p is the energy gap of H_p, *which is independent of the specific path in which H_b is evolved to H_p.*

According to Eq. (3), as far the perturbative approach is valid, i.e., $|\langle\phi_0(t)|\partial_t\phi_n(t)\rangle/(E_n - E_0)| < 1$, the measurement shows that one of the eigenstates appears with probability clearly the largest. Then one can know that this state corresponds to the ground state and thus the solution to the problem. This is consistent with Ref. [6].

To summarize, we developed a general, perturbative, formulation of the adiabatic quantum-computing schemes, which perturbatively describe the accurate evolution of the state. It leads to a deeper understanding of the related issues. The formalism is applied to analyze both the adiabatic geometric quantum computation and the adiabatic quantum algorithm. The order of magnitude of the first-order nonadiabatic error is the inverse of the executing time times the minimum gap with other eigenstates. Several proposed physical implementations of the former are considered from this point of view. Different proposals based on charging Josephson junctions are compared. We also consider an entire quantum-computing process consisting of many adiabatic gates, obtaining the lower bound of the nonadiabatic error, as an interesting constraint on the power of the quantum computation based on adiabatic geometric gates. One needs to enlarge the energy gap in order to reduce the nonadiabatic error and thus improve the computational power. For the adiabatic quantum algorithm, it is noted that it can be realized as far as the perturbative approach, rather than the rigorous adiabatic limit, is valid, hence the computational time may be appropriately shortened.

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