Magnetization of Ising Model in Nonzero Magnetic Field*

Daniel C. Mattis and Michael Plischke
Belfer Graduate School of Science, Yeshiva University, New York, New York 10033

(Received 10 February 1969)

Knowing only the zero-field magnetization (e.g., Yang's result) of the Ising model in any number of dimensions, one can construct a lower bound on $m(h)$, the magnetization in finite field. Knowledge of u, the internal energy per bond, enables a more efficient lower bound to be constructed. Both are applications of the Griffiths inequality, as recently generalized by Kelly and Sherman, and should prove useful in the lattice gas problem where it is essential to know $m(h)$.

We present nontrivial *lower* bounds on the magnetization in finite magnetic field $|m(h)|$ of the Ising model. An "upper upper" bound is $|m(h)| = 1$. Subsequently we hope to derive an improved upper bound which, together with the present result, should help constrain the true $m(h)$ fairly well.

Consider an isotropic $M \times N$ lattice, with a spin at every site and periodic boundary conditions, and a Hamiltonian

$$H(h) = -J \sum_{(ij)} \sigma_i \sigma_j - h \sum_i \sigma_i . \tag{1}$$

The partition function $Z(M, N, h, \beta)$ is:

$$Z(M, N, h, \beta) \equiv \mathrm{Tr}\,\{e^{-\beta H}\} = Z(M, N, h', \beta)\langle e^{\beta(h-h')\Sigma_i \sigma_i}\rangle_{h'}, \tag{2}$$

where $\langle\ \rangle_{h'}$ indicates "thermodynamic average w.r.t. $H(h')$." Expanding:

$$\langle e^{\beta(h-h')\Sigma_i\sigma_i}\rangle_{h'} = \cosh^{MN} \beta(h - h')\left\langle \prod_1^{MN} (1 + \sigma_i t)\right\rangle_{h'}, \tag{3}$$

in which $t \equiv \tanh \beta(h - h')$.

We factor the product into pairs and apply the generalized Griffiths inequality[1] due to Kelly and Sherman.[2]

$$\left\langle \prod_1^{MN} (1 + \sigma_i t)\right\rangle_{h'} \geq \prod_1^{\frac{1}{2}MN} \langle[1 + t(\sigma_i + \sigma_{i+1}) + t^2\sigma_i\sigma_{i+1}]\rangle_{h'}. \tag{4}$$

By translational invariance, all factors are equal, and the rhs of (4) is

$$(1 + 2tm(h') + t^2 |u(h')|)^{\frac{1}{2}MN}. \tag{5}$$

In the limit $MN \to \infty$ followed by $h' \to 0$, Griffiths has shown that m is positive and obeys

$$\lim_{h'\to 0} \lim_{MN\to\infty} m(h') \geq m_Y , \tag{6}$$

where m_Y is the (positive) magnetization calculated by Yang.[3] Similarly, the limit:

$$\lim_{h'\to 0} \lim_{MN\to\infty} |u(h')| = |u| \tag{7}$$

is the zero-field short-range correlation function—i.e., the absolute value of the internal energy per bond. Thus,

$$Z(M, N, h, \beta) \geq Z(M, N, 0, \beta)(\cosh^{MN} \beta h) \times (1 + 2tm_Y + t^2 |u|)^{\frac{1}{2}MN}. \tag{8}$$

On the lhs, we have

$$Z(M, N, h, \beta) \equiv \exp\left[MN\beta\int_0^h dh'' m(h'')\right] Z(M, N, 0, \beta). \tag{9}$$

Because $m(h'')$ is a nondecreasing function of its argument,

$$hm(h) \geq \int_0^h dh'' m(h''). \tag{10}$$

Combine (10) and (8) to obtain

$$m(h) \geq (h\beta)^{-1}\{\log \cosh \beta h + \tfrac{1}{2} \log (1 + 2tm_Y + t^2 |u|)\}. \tag{11}$$

We illustrate this result in Fig. 1, plotting the rhs of (11) for one temperature above T_c (curve A), one at T_c (B), and two below T_c (C and D).

A lower bound, which is somewhat less efficient above T_c but almost as good as (11) below it, can be obtained with far less numerical work; according to Refs. (1) and (2), $|u| \geq m_Y^2$, therefore using this on the rhs of (11) we find

$$m(h) \geq (\beta h)^{-1} \log (\cosh \beta h + m_Y \sinh \beta h). \tag{12}$$

Above or at T_c, $m_Y = 0$, and the resultant lower bound is shown as the dotted curve in Fig. 1. Below

* This research supported by the United States Air Force, AFOSR grant No. 69-1642.

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[2] D. G. Kelly and S. Sherman, J. Math. Phys. **9**, 466 (1968).

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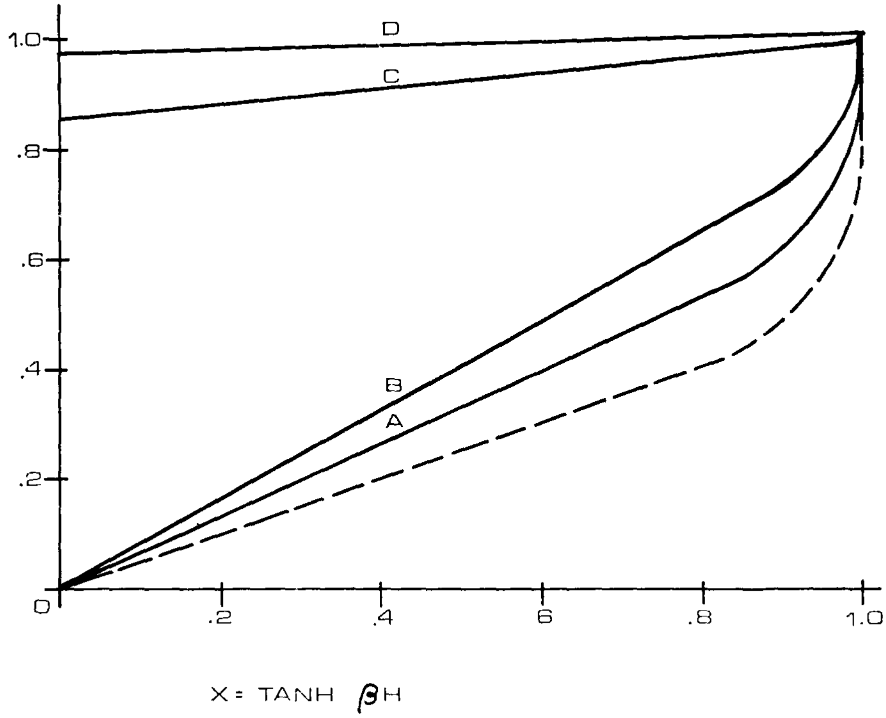

FIG. 1. Lower bounds to the magnetization at finite field $m(h)$ plotted vs $x = \tan \beta h$ at various temperatures. For A, $T/T_c = 1.83$; for B, $T = T_c$; for C, $T/T_c = 0.927$; and for D, $T/T_c = 0.61$; all using inequality (11). Dotted curve is inequality (12) at all $T \geq T_c$.

T_c, the lower bound (12) rapidly approaches (11) and would be indistinguishable from curves C and D at the temperatures we have chosen, on the scale of our graph.

It is hoped that the present results might be useful in lattice gas theory as well as in magnetism. It should be noted that they are not at all restricted to two dimensions; once a variational estimate of m in zero field is known for three dimensions, it can be used forthwith in Eq. (12).