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Towards a noncommutative geometric approach to matrix compactification

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In this paper we study generic M(atrix) theory compactifications that are specified by a set of quotient conditions. A procedure is proposed which both associates an algebra to each compactification and leads deductively to general solutions for the matrix variables. The notion of noncommutative geometry on the dual space is central to this construction. As examples we apply this procedure to various orbifolds and orientifolds, including ALE spaces and quotients of tori. While the old solutions are derived in a uniform way, new solutions are obtained in several cases. Our study also leads to a new formulation of gauge theory on quantum spaces. [S0556-2821(98)06314-0]

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I. INTRODUCTION

According to the M(atrix) model proposal [1], M theory in 11-dimensional uncompactified spacetime is microscopically described by the large N limit of the maximally supersymmetric $U(N)$ Yang-Mills quantum mechanics. For finite N the model is conjectured to describe the discrete light cone quantization of M theory [2], in which one light-cone direction is compactified on a circle. An attractive feature of M(atrix) theory is that for the nine transverse directions, the notion of physical space is a derived one in the theory. Since the coordinate variables are valued in the Lie algebra of $U(N)$, the description of space is novel from the beginning.[1]

A well-known generalization of classical (or commutative) geometry for studying novel spaces is the noncommutative geometry pioneered by Connes [4]. By now it is known to be relevant to M(atrix) theory at two different levels. First, a given configuration of the matrix variables for finite N can be identified with a regularized membrane [5], whose world volume is a quantum (or noncommutative) space. For instance, a regularized spherical membrane [5–7] coincides with the quantum sphere defined in various formulations of noncommutative geometry [8]. Interpreted in a different way, the M(atrix) model action can also be thought of as describing the dynamics of N D0-branes in the infinite momentum frame [1]. Previously two of us [9] have shown that this action can be understood as a gauge theory on a discrete noncommutative space consisting of N points.

Accordingly, at the second level, compactification in M(atrix) theory is *a priori* of a noncommutative nature, since compactification implies a certain specification of allowed background configurations. The M(atrix) model compactified on a torus and various orbifolds and orientifolds has already been discussed in the literature. For toroidal compactifications [1,10,11], the original gauge symmetry turns out to give rise to the usual gauge field theory on a dual torus, while the winding modes for one-cycles in the original compactified space become the momentum modes in the dual space. Recently in two interesting papers [12,13] it was shown that M(atrix) theory compactification on a torus can lead to a deformed Yang-Mills theory on the dual space which is a quantum torus, and can be interpreted as an M theory configuration with a nonvanishing three-form background on the compactified light-cone and toroidal directions. This provides a strong physical motivation for studying generic M(atrix) compactifications from the noncommutative point of view.

In this paper we report our recent progress towards a noncommutative geometric approach to a wide class of matrix compactifications, i.e., those on $\mathcal{M}/\mathbf{\Gamma}$, assuming the matrix model on a simply connected space $\mathcal{M}$ is known, with $\mathbf{\Gamma}$ a discrete group acting on $\mathcal{M}$. The compactification is determined by a set of quotient conditions, one for each generator of $\mathbf{\Gamma}$. We will describe a procedure for solving general solutions to the quotient conditions. Before doing this, our procedure naturally associates with each compactification a noncommutative algebra in which the matrix variables take values. It starts from here that the notion of noncommutative geometry using algebras to describe the geometry of quantum spaces comes into play. Furthermore, our procedure leads, in a deductive manner, to solutions which turn out to be gauge theories on dual quantum space. We will use several examples to show how our procedure works in practice. Not only are old solutions, obtained before as classical gauge theories, reproduced by our systematic procedure in a uniform way, new compactification on quantum spaces is also

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[1]Because of supersymmetry, at large distances the space can be approximately classical [3].

derived for several cases, including the Klein bottle, Möbius strip, and asymptotically locally Euclidean (ALE) orbifolds. Two different descriptions [14,15] for dual space in the case of the Klein bottle were thought to be in conflict with each other in the literature; we show that they are both correct, and a continuous interpolation is found between them using quantum spaces.

What we obtain corresponds to the "untwisted" sector, which may be an anomalous gauge theory in some cases. We leave for the future the question about how to derive directly from the M(atrix) model the "twisted" sector that is needed for orbifold and orientifold compactifications to achieve anomaly cancellation.

We first reexamine toroidal compactifications in Sec. II, rederiving the results for the quantum torus with our own procedure. The procedure for generic M(atrix) compactification is described in Sec. III. Then in Secs. IV and V, we will demonstrate how our procedure works for the Klein bottle [14,15] and the ALE space $\mathbf{C}^2/Z_n$ [16,17]. After that some comments on various aspects of M(atrix) compactifications are made in subsequent sections. In the Appendixes we also consider as examples T^2/Z_3, the finite cylinder [18–20], and the Möbius strip [14,15].

II. TOROIDAL COMPACTIFICATION REVISITED

A d-dimensional torus can be defined as the quotient space $\mathbf{R}^d/\mathbf{Z}^d$, where $\mathbf{Z}^d$ is generated by $\{c_1,\ldots,c_d\}$ freely acting on $\mathbf{R}^d$ as

$$c_i:\{x_j\}\to\{x_j+e_{ij}\}, \tag{1}$$

where e_{ij} define a d-dimensional lattice in $\mathbf{R}^d$. The toroidal compactification is defined by the quotient conditions [1,10,11]

$$U_i^\dagger X_j U_i = X_j + e_{ij}, \quad i,j=1,\ldots,d. \tag{2}$$

Standing as a fundamental theory, M(atrix) theory itself should contain the answer for all compactifications described by relations of this type. Although a complete answer including "twisted" sectors is not yet generally known to us, as a first step in this paper we will try to solve these equations for the "untwisted" sector, completely inside the framework of the theory.

One may choose an infinite dimensional matrix representation for the U_i's in Eq. (2), motivated by physical considerations. In our treatment, we prefer to think of them as algebraic elements tensored with an $N\times N$ unit matrix.[2]

To find the underlying algebra for the U_i's, we note that Eq. (2) implies that

$$U_j U_i U_j^\dagger U_i^\dagger X_k U_i U_j U_i^\dagger U_j^\dagger = X_k. \tag{3}$$

For toroidal compactifications, we should not have any additional constraints other than those in Eq. (2). Therefore, if we assume that the only central elements in the algebra of the U_i's are constant times the unity $\mathbf{1}$, we are allowed to impose the following constraints:

$$U_i U_j U_i^\dagger U_j^\dagger = q_{ij}\mathbf{1} \tag{4}$$

or, equivalently,

$$U_i U_j = q_{ij} U_j U_i, \tag{5}$$

with q_{ij} certain phase factors. Different choices of these phases may lead to different solutions, implying that compactification is not completely fixed by the quotient conditions.

The algebra (5) is the same as the algebra of a quantum torus [21]. For $d=2$ the algebra (5) has an $SL(2,\mathbf{Z})$ symmetry

$$U_1\to(U_1)^a(U_2)^b, \quad U_2\to(U_1)^c(U_2)^d, \tag{6}$$

where a,b,c,d are the entries of an $SL(2,\mathbf{Z})$ matrix. It was first pointed out in [12,13] that the phase factors q_{ij} can be related to M theory compactification with a nonzero background three-form field in the compactified null and toroidal directions.

From the point of view of the covering space, U_i's are translation operators, and so it is natural to write, for $N=1$, $X_j=e_{ij}\tilde{\sigma}_i$ and $U_i=\exp(-\tilde{D}_i)$, where $\tilde{D}_i=\partial/\partial\tilde{\sigma}_i+i\tilde{A}_i$ is the covariant derivative for a $U(1)$ gauge field. By a Fourier transform, the solution in the dual space is [1,10,11]

$$U_i=e^{i\sigma_i}, \quad X_j=-ie_{ij}D_i, \tag{7}$$

$$D_i=\frac{\partial}{\partial\sigma_i}+iA_i(\sigma). \tag{8}$$

In this dual representation, the solution can easily be generalized to $N\times N$ matrices. This is the type of solution we are looking for in the context of M(atrix) compactifications. New physical degrees of freedom reside in X, while the U's are fixed algebraic elements.

A. Classical torus

First we review the commutative case $q_{ij}=1$ [1,10,11]. In this case, the algebra of the U_i's is commutative, and now they are viewed as coordinate functions on the dual (ordinary) torus parametrized by σ_i, and the X's as covariant derivatives. Mathematically Eq. (7) is the general solution of Eq. (2), with the X's and U's being elements in the product of the algebra of differential calculus on a torus and the algebra of $N\times N$ matrices. Physically, the M(atrix) theory compactified on T^d for $d\leqslant 3$ is the $(d+1)$-dimensional supersymmetric Yang-Mills (SYM) theory on the dual torus [1].

Comparing with the uncompactified M(atrix) theory, we are adjoining the new elements $\partial/\partial\sigma_i$ and $\exp(i\sigma_i)$ to the algebra of $N\times N$ matrices for the compactification on a torus. The reason why we are allowed to adjoin these new algebraic elements is that the compactification on a torus introduces

[2]The e_{ij}'s on the right-hand side are understood as proportional to the unity in the algebra tensored with the $N\times N$ unit matrix.

new dynamical degrees of freedom corresponding to the winding string modes that are not present in the uncompactified theory. In general, for compactification on different spaces we need to adjoin different new elements to the algebra of $N \times N$ matrices.

B. Quantum torus

For $q_{ij} \neq 1$, we need to find out the new elements to be adjoined to the algebra of compactification. To define the algebra and to solve for X in this noncommutative case, we first define an auxiliary Hilbert space $\mathcal{H}$, on which the U_i's are represented as operators: It by definition consists of the "vacuum," denoted by $\rangle$, as well as states obtained by acting polynomials of U_i's on the vacuum. For $d=2$ the symmetry (6) induces an $SL(2,\mathbf{Z})$ symmetry on the Hilbert space. This is the S duality of type II B theory.

The Hilbert space $\mathcal{H}$ is spanned by the states $\{U_1^{m_1} \cdots U_d^{m_d}\rangle\}$ with $m_i \in \mathbf{Z}$. This Hilbert space is different from those introduced in [12].[3] For later convenience, we define a set of operators ∂_i by

$$\partial_i U_1^{m_1} \cdots U_d^{m_d}\rangle = m_i U_1^{m_1} \cdots U_d^{m_d}\rangle. \tag{9}$$

It follows that

$$\partial_i U_j = U_j(\partial_i + \delta_{ij}). \tag{10}$$

Thus ∂_i is the (quantum) derivative with respect to the exponent of U_i.

The inner product on $\mathcal{H}$ should be invariant under the group $\hat{\mathcal{G}}(\mathcal{A})$ of gauge transformations of the U_i's which preserve the quotient conditions (2). Since X_i is generic, the only possible such transformation is

$$U_i \rightarrow g_i^\dagger U_i g_i = e^{i\phi_i} U_i, \tag{11}$$

where $g_i = \exp(-i\phi_i\partial_i)$. This implies that the inner product is defined by $\langle f|g\rangle = \langle f^\dagger g\rangle$, where f,g are functions of U and

$$\langle U_i^{m_i} \cdots U_d^{m_d}\rangle = \delta_0^{m_1} \cdots \delta_0^{m_d} \tag{12}$$

up to normalization. Note that the vacuum expectation value $\langle \cdot \rangle$ happens to be equal to the trace over the Hilbert space, which can be determined directly by requiring that it have the property of cyclicity: $\langle fg\rangle = \langle gf\rangle$ for any two functions of U. By a Fourier transform on the basis, $|\sigma\rangle = \Sigma_n \exp(i n_i \sigma_i) U_1^{n_1} \cdots U_d^{n_d}\rangle$, where $\sigma = (\sigma_1, \ldots, \sigma_d)$ and $n = (n_1, \ldots, n_d)$, the trace on $\mathcal{H}$ turns into the integration on a d-torus parametrized by σ. The integration on a quantum torus can be independently defined with respect to the $\hat{\mathcal{G}}(\mathcal{A})$-invariant measure $\Pi_i U_i^\dagger dU_i$ by using Stoke's theorem.

Let the action of X_j on the vacuum be given by

$$X_j\rangle = \hat{A}_j(U)\rangle, \tag{13}$$

where $\hat{A}_j$ is a function of the U_i's. Using Eqs. (2) and (13), we can calculate the action of X_j on any state:

$$\begin{aligned} X_j U_1^{m_1} \cdots U_d^{m_d}\rangle &= U_1^{m_1} \cdots U_d^{m_d}(e_{ij} m_i + \hat{A}_j)\rangle \\ &= (e_{ij}\partial_i + A_j) U_1^{m_1} \cdots U_d^{m_d}\rangle; \end{aligned} \tag{14}$$

i.e., in general $X_j = e_{ij}\partial_j + A_j$, where A_j are functions of $\tilde{U}_i = (\Pi_{j \neq i} q_{ji}^{\partial_j}) U_i$, obtained by replacing U_i's in $\hat{A}_j(U)$ with $\tilde{U}_i$'s and reversing the ordering of a product.

The solutions of the X_i's are functions of operators commuting with all U's, i.e., $\tilde{U}_i U_j = U_j \tilde{U}_i$ for all i,j. The commutation relations among the $\tilde{U}$'s are given by

$$\tilde{U}_i \tilde{U}_j = q_{ij}^{-1} \tilde{U}_j \tilde{U}_i. \tag{15}$$

This is just the algebra for a quantum torus related to that of U by a transformation $q_{ij} \rightarrow q_{ij}^{-1}$ [12]. The Hilbert space is also spanned by $\{\tilde{U}_1^{m_1} \cdots \tilde{U}_d^{m_d}\rangle\}$, and the operators ∂_j act on $\tilde{U}_i$ in the same way as they act on U_i. It is thus natural to think that X_j are the covariant derivatives on the dual quantum torus given by $\tilde{U}_i$.

The same result was obtained in [12] in a different way. They noticed that a generic solution of Eq. (2) is composed of a special solution and a homogeneous solution, and that homogeneous solutions are the elements in the algebra commuting with all the U_i's. Also, they used a Hilbert space different from ours. While the set of U-commuting elements may be found by brute force when the algebra is given, we see that they automatically arise in our procedure. For a different compactification associated with another set of quotient conditions, the trick of using U-commuting operators may no longer work, but we will demonstrate below that the same procedure we used above always works.

Let us now make a remark about the gauge field A_i. As in usual gauge theories, the gauge field A_i does not have to be a well-defined function on the dual quantum torus. Without going into details about the notion of the principal bundle and connection on quantum spaces [4], we simply say that the requirement on A_i is that all quantities invariant under

$$X_j \rightarrow X_j + e_{ij} \tag{16}$$

are well defined. For instance $(-i \log U_i)$ is only defined up to $2n\pi$. Yet $A_j = -im_i \log(U_i) e_{ij}$ with integers m_i is acceptable, because the ambiguity in its value matches precisely the gauge transformation (16). In fact these are the configurations of D-branes wrapping on the torus.

[3] In the notation of [12] our $\mathcal{H}$ superficially corresponds to the case with $p=1,\ q=0$, but p/q appears in some of the relations given by them.

III. GENERIC COMPACTIFICATION

Consider the compactification of M(atrix) theory on the quotient space $\mathcal{M}/\mathbf{\Gamma}$,[4] where $\mathcal{M}$ is a simply connected space $[\pi_1(\mathcal{M})=\mathbf{1}]$ on which the M(atrix) theory is known, and $\mathbf{\Gamma}$ is a discrete group acting on $\mathcal{M}$. If $\mathbf{\Gamma}$ acts freely, it is the fundamental group of the compactified space.

Denote the action of $c\in\mathbf{\Gamma}$ on $\mathcal{M}$ by $\Phi(c)$. Then the compactified M(atrix) theory is obtained by imposing the following constraints: For each element $c\in\mathbf{\Gamma}$,

$$U(c)^\dagger X_a U(c)=\Phi_a(c)(X), \tag{17}$$

where X_a represent all M(atrix) theory variables A_0, X_i, and Ψ. If $\mathbf{\Gamma}$ is generated by a set of elements $\{c_i\}$, one may only need to write down such relations for each generator c_i. We will call these relations "quotient conditions."

For orientifolds, the group $\mathbf{\Gamma}$ is endowed with a $\mathbf{Z}_2$ grading: We associate a number $n(c)=0,1$ with each element $c\in\mathbf{\Gamma}$, and if $c_1c_2=c_3$, then $n(c_1)+n(c_2)=n(c_3)$ (mod 2). The quotient condition (17) is generalized to

$$U(c)^\dagger X_a U(c)=\begin{cases}\Phi_a(c)(X) & \text{if} \quad n(c)=0\\ [\Phi_a(c)(X)]^* & \text{if} \quad n(c)=1.\end{cases} \tag{18}$$

Here the complex conjugation * corresponds to the transpose for Hermitian matrices X, which implies orientation reversal of open strings stretched between D0-branes.

The quotient conditions have to be consistent with the action. Since the action of M(atrix) theory is invariant under gauge transformations, $X\rightarrow U^\dagger XU$, the quotient conditions are consistent with the action only if the action is also invariant under the transformations,

$$X_a\rightarrow\Phi_a(c)(X) \tag{19}$$

for all $c\in\mathbf{\Gamma}$. A function of the X's and their time derivatives is a gauge-invariant physical observable if it is invariant under (19).

We will give below a procedure for solving relations of the type (17) or (18). By this we mean that we shall define the algebra $\mathcal{A}$ in which the relations are understood, and then find the most general solution of X_a as algebraic elements in the algebra $\mathcal{A}$. The physical degrees of freedom of the X_a's reside in the moduli of the solutions to the quotient conditions.

To define the algebra $\mathcal{A}$, first we note that all the U's are considered as fixed elements in $\mathcal{A}$. They form a subalgebra of $\mathcal{A}$ which is constrained by the quotient conditions by requiring that the quotient conditions exhaust all desired constraints on X. If there is a relation $c_1c_2\cdots c_n=\mathbf{1}$ in the group $\mathbf{\Gamma}$, from the quotient conditions for these c's, we will get equations of the form

$$P(U)^\dagger XP(U)=X \quad \text{for all } X\text{'s}, \tag{20}$$

where $P(U)=U(c_1)U(c_2)\cdots U(c_n)$ is the corresponding product of the U's. This relation would impose a new constraint on X unless

$$P(U)=q\mathbf{1}, \tag{21}$$

where q is a phase factor. For orientifolds, let C denote the complex conjugation operator,

$$CaC=a^*, \tag{22}$$

for all $a\in\mathcal{A}$. We have $C^\dagger=C$ and $C^2=\mathbf{1}$. Equation (18) is then equivalent to

$$R(c)^\dagger X_a R(c)=\Phi_a(c)(X), \tag{23}$$

where $R(c)=U(c)C^{n(c)}$. So Eq. (21) is replaced by $P(R)=q\mathbf{1}$. We define the algebra of U, called the U-algebra, by imposing all such relations. We can view nonorientifolds as the special case with $n(c)=0$ for all $c\in\mathbf{\Gamma}$.

It can be shown [22] that these relations can be characterized by a faithful projective representation of $\mathbf{\Gamma}$. Following [12,13], it is natural to suggest that the cohomologically invariant phases in a nontrivial two-cocycle on $\mathbf{\Gamma}$ associated with the projective representation correspond to a nontrivial background field on the compactified space. Accordingly, compactification defined by the quotient conditions is completely characterized by projective representations of the group $\mathbf{\Gamma}$, and the moduli space of the U-algebra (more precisely, the space of cohomologically invariant q parameters in a two-cocycle) may correspond to part of the moduli of M theory compactifications. We take this as a strong motivation for studying M(atrix) theory compactification with nontrivial two-cocycles.

Knowing the U-algebra, we can construct a Hilbert space $\mathcal{H}$ to represent it, which consists of a "vacuum" denoted by $\rangle$ and all polynomials of the $R(c)$'s acting on the vacuum. The algebra $\mathcal{A}$ is then defined as the tensor product of the algebra of operators on $\mathcal{H}$ with the algebra of $N\times N$ matrices. In the action of M(atrix) theory, the total trace is now composed of the trace over $\mathcal{H}$ and the trace over $N\times N$ matrices.

Physically the states in $\mathcal{H}$ correspond to string modes winding around noncontractible one-cycles in the compactified space associated with elements in the group $\mathbf{\Gamma}$. By adjoining this Hilbert space to the space of N-vectors on which the algebra of $N\times N$ matrices is represented, we take care of the new string winding modes arising from the compactification.

For a given algebra $\mathcal{A}$ we define the unitary group $\mathcal{U}(\mathcal{A})$ to be the group of all unitary elements in $\mathcal{A}$. Let $\mathcal{G}(\mathcal{A})$ be the subgroup of $\mathcal{U}(\mathcal{A})$ which preserves the quotient conditions, i.e.,

$$R(c)^\dagger g^\dagger X_a gR(c)=\Phi_a(c)(g^\dagger Xg), \tag{24}$$

[4] In fact we should consider the quotient of a superspace in order to include the fermionic part from the beginning.

for all $g \in \mathcal{G}(\mathcal{A})$. $\mathcal{G}(\mathcal{A})$ can be viewed as the group of gauge transformations on X,[5]

$$X_a \to g^\dagger X_a g, \tag{25}$$

which survives the compactification. As was shown in the previous section, the definition of the dual space may be inferred from the gauge field or, equivalently, from the gauge group $\mathcal{G}(\mathcal{A})$. In general the compactified M(atrix) theory may not be identified with a traditional gauge theory on a classical manifold. We will consider this as a natural generalization of the notion of gauge theories.

On the other hand, $\mathcal{G}(\mathcal{A})$ induces a group of transformations on $R(c)$, denoted by $\hat{\mathcal{G}}(\mathcal{A})$,

$$R(c) \to g R(c) g^\dagger, \quad g \in \mathcal{G}(\mathcal{A}), \tag{26}$$

which preserve the quotient conditions (18). Because we shall allow the most general solution of X, the only possible transformation on $R(c)$ is to multiply them by certain phase factors, and thus the group $\hat{\mathcal{G}}(\mathcal{A})$ is an Abelian group. Since different choices of the $R(c)$'s related by $\hat{\mathcal{G}}(\mathcal{A})$ are equivalent by a gauge transformation, the compactification should be invariant under $\hat{\mathcal{G}}(\mathcal{A})$. Roughly speaking, $\hat{\mathcal{G}}(\mathcal{A})$ is the translation group of the dual space.

The prescription for deriving the general solution for X in the algebra $\mathcal{A}$ was first invented by Zumino [23] to study problems in quantum differential calculi. (Mathematically these two problems are similar in nature.) The prescription is the following.

(1) As mentioned above, we define a Hilbert space $\mathcal{H}$ consisting of all polynomials of the $R(c)$'s acting on the vacuum. The inner product on $\mathcal{H}$ has to be fixed to respect the symmetry group $\hat{\mathcal{G}}(\mathcal{A})$. The algebra $\mathcal{A}$ is defined to be the tensor product of the algebra of operators on $\mathcal{H}$ with the algebra of $N \times N$ matrices.

(2) We require the X_a's be operators acting on $\mathcal{H}$ and write the action of X_a on the vacuum as

$$X_a \rangle = \hat{A}_a(R)\rangle, \tag{27}$$

where $\hat{A}_a(R)$ is a function of the $R(c)$'s. All physical degrees of freedom of X_a reside in $\hat{A}_a$, which gives the generalized gauge field. The action of X_a on an arbitrary basis state can be obtained by using the quotient conditions to commute X_a through the $R(c)$'s until it reaches the vacuum and then using Eq. (27).

(3) To find an explicit expression for X_a,[6] one needs to find a set of convenient operators on $\mathcal{H}$. The type of operators (9) used for toroidal compactification are often very useful. As we did in Sec. II B, to write X_a as a function of ∂_i and U_i, one needs to find the action of X_a on a state $U_1^{m_1} \cdots U_d^{m_d}\rangle$ as a function $F(m_1, \ldots, m_d; U)$ acting on the state. Then we can replace F by another function $\tilde{F}$ of ∂ and U.

To gain some insight into the compactified theory, we note that in general we may view the resulting theory as a (deformed) gauge field theory on a dual quantum space. In the spirit of noncommutative geometry, the U-algebra can be viewed as the algebra of functions on the dual quantum space. In addition one may follow the standard procedure used in the study of quantum differential calculus on quantum spaces with quantum group symmetry[7] [24] to define a deformed differential calculus on the U-algebra. Once the derivatives (such as the ∂_i in the previous section) on the dual quantum space are defined, we can use them to express X_a and see that the bosonic X_i's can be thought of as covariant derivatives. In other words, the present approach can be directly used to define gauge theory on a quantum space and is different from most other existing approaches to defining them in the following sense: Given the algebra of functions on a quantum space, usually one will define the gauge field to be a function on the quantum space, but in general our procedure gives a gauge field as an operator, for instance a pseudodifferential operator, on the quantum space.

We will demonstrate below how our above procedure works, for example, for the compactification on the orientifold of Klein bottle and the ALE space of $\mathbf{C}^2/\mathbf{Z}_n$. In the Appendixes, we will also apply the prescription to the following orbifolds and orientifolds: $T^2/\mathbf{Z}_3$, cylinder $(S^1 \times S^1/\mathbf{Z}_2)$, and Möbius strip.

IV. KLEIN BOTTLE

The Klein bottle can be defined as $\mathbf{R}^2/\mathbf{\Gamma}$, where $\mathbf{\Gamma}$ acts on $\mathbf{R}^2$ by

$$c_1 : (x_1, x_2) \to (x_1 + 2\pi R_1, x_2), \tag{28}$$

$$c_2 : (x_1, x_2) \to (-x_1, x_2 + \pi R_2). \tag{29}$$

The group $\mathbf{\Gamma}$ is generated by c_1, c_2 with the commutation relation

$$c_1 c_2 c_1 c_2^{-1} = \mathbf{1}. \tag{30}$$

As an orientifold, its $\mathbf{Z}_2$ grading is defined by $n(c_1) = 0$ and $n(c_2) = 1$.

Thus the quotient conditions are [14,15]

$$U_i^\dagger X_j U_i = X_j + 2\pi \delta_{ij} R_j, \quad i,j = 1,2, \tag{31}$$

$$U_3^\dagger X_1 U_3 = -X_1^*, \tag{32}$$

[5] In fact $\mathcal{G}(\mathcal{A})$ contains more than what we usually call a gauge group on the dual space for it also contains the translation group $\hat{\mathcal{G}}(\mathcal{A})$ to be introduced below.

[6] It is not necessary to have an explicit expression of X_a in terms of other operators as long as X_a is already well defined as an operator on $\mathcal{H}$ as in step (2). But it can be helpful in studying the model.

[7] In our problem the symmetry group is $\hat{\mathcal{G}}(\mathcal{A})$, which is just a classical group. But they play similar roles in this formulation.

$$U_3^\dagger X_2 U_3 = X_2^* + \pi R_2, \tag{33}$$

where $U_1 = U(c_1)$, $U_2 = U(c_2^2)$ and $U_3 = U(c_2)$. Note that since X's are Hermitian, we have $X^T = X^*$. The conditions for U_2 are direct results of Eqs. (32), (33).

Since $R(c) = U(c)C^{n(c)}$, it is easy to verify that the following relations are compatible with the quotient conditions (31)–(33):

$$U_1 U_2 = q_{12} U_2 U_1, \tag{34}$$

$$U_1 U_3 = q_{13} U_3 U_1^T, \tag{35}$$

$$U_2 U_3 = q_{23} U_3 U_2^*, \tag{36}$$

$$U_3 U_3^* = q_3 U_2. \tag{37}$$

We shall rescale U_2 to set $q_3 = 1$. Using Eq. (37) we find $q_{23} = 1$ from Eq. (36). Consistency also requires that $q_{12} = q_{13}^2$. We will denote q_{13} by q. (So the projective representations of the group $\mathbf{\Gamma}$ are labeled only by a phase factor q.)

We will see below that the case studied in [14] corresponds to the case $q = 1$ where the dual space is a cylinder, and the case studied in [15] corresponds to the case $q = -1$ where the dual space is a Klein bottle. We have obtained a one-parameter moduli for this compactification.

The Hilbert space $\mathcal{H}$ is defined to be

$$\mathcal{H} = \{U_1^m (U_3 C)^n\rangle | m,n \in \mathbf{Z}\},$$

or, equivalently, $\{U_1^m U_2^n\rangle, U_1^m U_2^n U_3 C\rangle | m,n \in \mathbf{Z}\}$. We define some operators for later convenience:

$$\partial_1 U_1^m (U_3 C)^n\rangle = m U_1^m (U_3 C)^n\rangle, \tag{38}$$

$$\partial_2 U_1^m U_2^n (U_3 C)^s\rangle = n U_1^m U_2^n (U_3 C)^s\rangle, \tag{39}$$

$$K U_1^m (U_3 C)^n\rangle = U_1^m (U_3 C)^{n+1}\rangle, \tag{40}$$

$$\epsilon U_1^m (U_3 C)^n\rangle = (-1)^n U_1^m (U_3 C)^n\rangle, \tag{41}$$

where $m,n \in \mathbf{Z}$ and $s = 0,1$. It follows that ∂_1, ∂_2 act on U_1, U_2 as derivatives. The commutation relations between the derivatives and functions can easily be derived.

Following the prescription described in the last section, we see that the solution is of the form of a gauge field,

$$X_1 = 2\pi R_1 \partial_1 + \frac{1}{2}\hat{A}_1(q^{-N}U_1, K)(1+\epsilon) - \frac{1}{2}\hat{A}_1^*(q^N U_1^{-1}, K)(1-\epsilon), \tag{42}$$

$$X_2 = \pi R_2 N + \frac{1}{2}\hat{A}_2(q^{-N}U_1, K)(1+\epsilon) + \frac{1}{2}\hat{A}_2^*(q^N U_1^{-1}, K)(1-\epsilon), \tag{43}$$

where $N = 2\partial_2 + (1-\epsilon)/2$ acts on $\mathcal{H}$ by

$$N U_1^m (U_3 C)^n\rangle = n U_1^m (U_3 C)^n\rangle. \tag{44}$$

While the Klein bottle is a quotient of the torus, we will see below that the compactification on the former is a gauge theory on a quotient of the dual torus for the latter. We have

$$X_1 = 2\pi R_1 \partial_1 + A_1, \quad X_2 = 2\pi R_2\left(\partial_2 + \frac{1-\epsilon}{4}\right) + A_2. \tag{45}$$

The gauge fields are given by

$$A_i = \frac{1}{2}(A_{i0} + A_{i1}K)(1+\epsilon) + (-1)^i \frac{1}{2}(B_{i0} + B_{i1}K)(1-\epsilon), \tag{46}$$

where A_{ij} and B_{ij} ($i=1,2$ and $j=0,1$) are functions of $\tilde{U}_1, \tilde{U}_2$ with $\tilde{U}_1 = q^{-2\partial_2}U_1$ and $\tilde{U}_2 = q^{2\partial_1}U_2$ satisfying the algebra of the dual torus:

$$\tilde{U}_1 \tilde{U}_2 = q^{-2} \tilde{U}_2 \tilde{U}_1. \tag{47}$$

It is

$$A_{ij}(\sigma_1 - h, -\sigma_2) = B_{ij}^*(\sigma_1, \sigma_2), \quad i=1,2, \ j=0,1, \tag{48}$$

where $q = \exp(ih)$, $\tilde{U}_1 = \exp(i\sigma_1)$, and $\tilde{U}_2 = \exp(i\sigma_2)$. It can be checked that

$$U_3^\dagger A_i U_3 = (-1)^i A_i^*, \quad i=1,2, \tag{49}$$

and all quotient conditions are automatically satisfied.

The condition (48) relates $A_i(\sigma_1^* + h, -\sigma_2^*)$ to $A_i(\sigma_1, \sigma_2)^*$, which is a function of (σ_1^*, σ_2^*). So if the value of A_i at $(\sigma_1 + h, \sigma_2)$ is known, then its value at $(\sigma_1, -\sigma_2)$ is fixed. If $q = \exp[i2\pi/(2k)]$ for an integer k, the fundamental region on which the values of A_i can be freely assigned is a Klein bottle of area $(2\pi)^2/(2k)$. If $q = \exp[i2\pi/(2k+1)]$, the fundamental region is a cylinder of area $(2\pi)^2/2(2k+1)$. In particular, for $q=1$ it is a cylinder, and for $q=-1$ it is a Klein bottle. It was argued in [15,20] that only the latter case gives the area-preserving diffeomorphism group as the gauge group of the model in the large N limit. The gauge group in the bulk of the fundamental region is $U(2N)$ and the gauge group on fixed points of the map $(\sigma_1, \sigma_2) \to (\sigma_1 - h, -\sigma_2)$ is $O(2N)$.

K and ϵ can be represented by 2×2 matrices. Let

$$K = e^{i\sigma_2/2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \epsilon = \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{50}$$

$$A_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} = \begin{pmatrix} A_{i0} & (-1)^i B_{i1} e^{i\sigma_2/2} \\ A_{i1} e^{i\sigma_2/2} & (-1)^i B_{i0} \end{pmatrix}. \tag{51}$$

The results above can then be rewritten as

$$\left.\begin{pmatrix} \delta_i & \gamma_i \\ \beta_i & \alpha_i \end{pmatrix}\right|_{(\sigma_1+h,\sigma_2)} = (-1)^i \left.\begin{pmatrix} \alpha_i^* & \beta_i^* \\ \gamma_i^* & \delta_i^* \end{pmatrix}\right|_{(\sigma_1,-\sigma_2)}. \tag{52}$$

The 2×2 unit matrix and K (for "fixed" σ_2) generate the algebra of functions on $\mathbf{Z}_2$, and ϵ is a derivative on $\mathbf{Z}_2$ in the sense of noncommutative geometry [4]. Thus X_i can be viewed as covariant derivatives on the dual space which is the product of a classical space parametrized by σ_1,σ_2 and a quantum space of two points ($\mathbf{Z}_2$). The Hilbert space can also be written as a column of two functions of $\tilde{U}_1$ and $\tilde{U}_2$. Thus it is natural to say that the dual space has coordinates $\tilde{U}_1$, $\tilde{U}_2$, and K, where $\tilde{U}_i$ satisfy the same algebra as U_i ($i=1,2$) with $q\to q^{-1}$.

The trace over $\mathcal{H}$ is equivalent to the composition of the integration over (σ_1,σ_2) and the trace over the 2×2 representation of K and ϵ. The integration has the cyclicity property so that the M(atrix) theory action is gauge invariant.

As was noted in [12], the algebra of the dual quantum torus (47) can be realized on functions on a classical torus as the star product:

$$(f*g)(\sigma)=q^{\partial_2\partial_1'-\partial_1\partial_2'}f(\sigma)g(\sigma')|_{\sigma'=\sigma}. \tag{53}$$

Therefore the action of M(atrix) theory appears to be the action for a field theory defined on T^2 with higher derivative terms. It is yet to be studied how to make sense of such theories.

As a side remark we note that the calculation above can be done with a little more ease if we impose the reality conditions $U_1^*=U_1^{-1}$, $U_2^*=U_2$, and $U_3^*=U_3$, which are consistent with the U-algebra. The result is independent of such conditions.

So far we have ignored the transverse bosonic and fermionic fields in the M(atrix) theory. The quotient conditions on them are [14,15]

$$U_i^\dagger A_0 U_i = A_0, \quad U_3^\dagger A_0 U_3 = -A_0^*, \tag{54}$$

$$U_i^\dagger X_a U_i = X_a, \quad U_3^\dagger X_a U_3 = X_a^*, \tag{55}$$

$$U_i^\dagger \Psi U_i = \Psi, \quad U_3^\dagger \Psi U_3 = \Gamma_{01}\Psi^*, \tag{56}$$

where $i=1,2$, $a=3,\ldots,9$, and Ψ is in the Majorana representation. It is straightforward to solve these relations in the same way. These quotient conditions can be determined by required surviving supersymmetry (SUSY) or by their consistency with the M(atrix) theory Lagrangian [1]

$$L=\mathrm{Tr}\left(\frac{1}{2}(D_0X_i)^2+\frac{1}{4}[X_i,X_j]^2-\frac{1}{2}\Psi^\dagger D_0\Psi - \frac{1}{2}\Psi\Gamma^i[X_i,\Psi]\right), \tag{57}$$

where $D_0=\partial/\partial t+iA_0$.

The dynamical SUSY transformation of M(atrix) theory is [1]

$$\delta X_\mu = i\bar{\epsilon}\Gamma_\mu\Psi, \quad \mu=0,\ldots,9, \tag{58}$$

$$\delta\Psi=(D_0X_i)\Gamma^{0i}\epsilon+\frac{i}{2}[X_i,X_j]\Gamma^{ij}\epsilon, \quad i,j=1,2,\ldots,9, \tag{59}$$

and the kinetic SUSY transformation is

$$\tilde{\delta}X_\mu=0, \quad \tilde{\delta}\Psi=\tilde{\epsilon}. \tag{60}$$

One-half of the dynamical SUSY is preserved by the compactification on a Klein bottle.

V. $\mathbf{C}^2/\mathbf{Z}_N$

The quotient condition for $\mathbf{C}^2/\mathbf{Z}_n$ is

$$U^\dagger Z_a U = qZ_a, \quad a=1,2, \tag{61}$$

where $Z_1=X_1+iX_2$, $Z_2=X_3+iX_4$, and $q=\exp(2\pi i/n)$. It follows that $U^{-n}Z_aU^n=Z_a$. Following our procedure, the U-algebra is given by $U^n=p\mathbf{1}$, where p is a phase. Rescaling U by $p^{1/n}$, we find

$$U^n=1. \tag{62}$$

The Hilbert space is $\mathcal{H}=\{U^m\rangle|m=0,1,\ldots,n-1\}$. Let $Z_a\rangle=A_a(U)\rangle$, where $A_a(U)=\Sigma_{m=0}^{n-1}\alpha_{am}U^m$. The action of Z on $\mathcal{H}$ is

$$Z_aU^m\rangle=q^mU^mA_a\rangle \tag{63}$$

$$=A_a(U)q^MU^m\rangle, \tag{64}$$

where M is defined by $MU^m\rangle=mU^m\rangle$. The solution of Z_a is thus $Z_a=A_a(U)q^M$. Instead of M, one can also use V defined by $UV=qVU$ and $V\rangle=\rangle$. Thus Z can also be expressed as

$$Z_a=A_a(U)V^{-1}. \tag{65}$$

U and V can be realized as $n\times n$ matrices:

$$U_{ij}=\delta_{i,(j-1)}, \quad V_{ij}=q^i\delta_{ij}, \tag{66}$$

where U_{ij} is nonvanishing only if $i=j-1$ (mod n). We find

$$(Z_a)_{ij}=\sum_m \alpha_{am}q^{-j}\delta_{i,(j-m)}, \quad i,j=0,1,\ldots,n-1. \tag{67}$$

This is exactly what one would expect through the same line of reasoning Taylor used [10] for toroidal compactifications. The coefficient a_m represents the string stretched between D0-branes separated by m copies of the fundamental region.

In the representation (66), U is viewed as an operator that shifts one point in $\mathbf{Z}_n$ to the next point. In a dual representation where $U_{ij}=q^{-i}\delta_{ij}$, U can be viewed as the generator of the algebra of functions on the dual quantum space $\mathbf{Z}_n$, and V becomes the shift operator. Thus we see that the dual of $\mathbf{Z}_n$ is also $\mathbf{Z}_n$.

The group $\mathcal{G}(\mathcal{A})$ is generated by U and V. A unitary function $g(U)$ induces a gauge transformation $A(U) \to g^\dagger(U)A(U)g(qU)$. In the dual representation where U is diagonal, it is easy to see that the gauge group of this theory is $U(N)^n$. The fields A_a are now diagonal blocks of $N\times N$ matrices with each block transforming in the fundamental and antifundamental representations under two adjacent $U(N)$ factors [17].

The gauge transformation by V^k is $A(U)\to A(q^kU)$, which is in fact a translation (cyclic permutation) on the dual space $\mathbf{Z}_n$. This also corresponds to the only nontrivial elements in $\hat{\mathcal{G}}(\mathcal{A})$: $U\to q^kU$. Requiring its invariance under $\hat{\mathcal{G}}(\mathcal{A})$, the inner product on $\mathcal{H}$ is fixed to be $\langle U^k\rangle = \delta_0^k$ for $k=0,1,\ldots,n-1$.

Note that in M(atrix) theory it is only the field strength defined by $[X_i,X_j]$ (for flat space) and other gauge-invariant quantities that need to be well defined on the dual space. For instance, $U^{1/n}$ is only defined up to an integral power of q. But it is acceptable to have $A(U)=U^{m/n}F(U)$ with $m=0,1,\ldots,n-1$, where $F(U)$ is a polynomial of U. The reason is that this ambiguity is precisely of the form of a gauge transformation on X, and so all gauge-invariant quantities are still well defined.

Denote $X_0=A_0$. The rest of the quotient conditions are

$$U^\dagger X_\mu U = X_\mu, \quad \mu=0,5,\ldots,9, \tag{68}$$

$$U^\dagger \Psi U = \Lambda\Psi, \tag{69}$$

where $\Lambda=\exp[-\pi(\Gamma^{12}+\Gamma^{34})/n]$. Because $\Lambda^n=-\mathbf{1}$, Eq. (62) should be replaced by $U^n=(-1)^F$, where F is the fermion number operator. It is easy to see that A_0, X_μ, and Ψ are in the adjoint representation of $U(N)^n$.

It is easy to see that the quotient conditions for $\mathbf{C}^2/\mathbf{Z}_n$ preserve one-half of the dynamical SUSY and one-half of the kinetic SUSY.

VI. NONCOMMUTATIVE GEOMETRY AND T DUALITY

Let us recall how the notion of noncommutative geometry naturally arises as a generalization of classical geometry. We know that if a classical space is given, one can immediately define the algebra of functions on that space. According to the Gelfand-Naimark theorem, the converse is also true: any commutative C^* algebra is isomorphic to the algebra of functions (vanishing at infinity) on a locally compact Hausdorff space, which can be constructed as the space of maximal ideals of the algebra. The notions of the algebra of functions and that of the underlying space are dual to each other via the Gelfand map. This motivates the generalization of classical spaces to quantum spaces. A quantum space is simply defined as the underlying space of a noncommutative algebra.

The dual space for a M(atrix) compactification can thus be roughly viewed as the underlying space on which the M(atrix) theory is defined as a field theory. When the U-algebra is noncommutative, the dual space is a quantum space. Thus in a sense T duality naturally introduces the ideas of a noncommutative geometry into M(atrix) theory.

For the compactifications on $\mathcal{M}/\mathbf{\Gamma}$ with $\mathcal{M}$ simply connected, we have seen in the above examples that for a factor of $\mathbf{Z}$ in $\mathbf{\Gamma}$ there is a factor of S^1 in the dual space. (Note that this statement is more general than the statement that the dual space of a circle is a circle, because there can be different compactifications with the same group $\mathbf{\Gamma}$. They lead to different field theories on the same dual space.) In the above we also see that for a factor of $\mathbf{Z}_n$ in $\mathbf{\Gamma}$ there is a factor of the dual $\mathbf{Z}_n$ in the dual space. It would be useful to know more about the correspondence between the group $\mathbf{\Gamma}$ and the dual space.

VII. COMMENTS AND DISCUSSION

Finally we make a few remarks.

To be treated as a fundamental theory by itself, M(atrix) theory needs to know everything without consulting string theory or supergravity. Since the notion of spacetime is from the very beginning noncommutative in M(atrix) theory, *a priori* one is allowed to consider compactifications on spaces which are exotic from a classical point of view. The criterion for an admissible compactification is only whether the corresponding generalized gauge theory on the dual space can make sense.

For compactifications on a classical d-torus, the fundamental group is commutative and is d dimensional; thus it results in a d-dimensional dual space. For compactifications on Riemann surfaces of higher genus, the fundamental group is noncommutative and therefore the dual space must be a quantum space.

A Riemann surface of genus $g>1$ can be obtained as a quotient of the Lobachevskian disk which is simply connected. The quotient conditions are of the form

$$U_i^\dagger Z U_i = \frac{a_i Z + b_i \mathbf{1}}{c_i Z + d_i \mathbf{1}}, \quad i=1,\ldots,2g, \tag{70}$$

where $\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$ are $SU(1,1)$ matrices and $|Z|<1$. It is a challenge to find the solution for Z.

For two classical compactifications, it is possible that there is a family of compactifications on nonclassical spaces with sensible dual theories interpolating them. Such interpolation may help our understanding of the various dualities [25].

Obviously there are a lot of important issues we need to clarify before we can proceed further. If the solution of the quotient conditions gives us an anomalous gauge theory, what we have obtained in this paper is only the so-called untwisted sectors in M(atrix) theory. To view M(atrix) theory as a fundamental theory, we also need to learn how to determine the twisted sectors for anomaly cancellation without consulting with string theory. On the other hand, for the consideration of quantum spaces to be physically relevant, it is urgent to look for more correspondence between M(atrix) compactification on quantum space and the moduli space of M theory compactification.

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APPENDIX A: $T^2/\mathbf{Z}_3$

The quotient conditions for $T^2/\mathbf{Z}_3$ are

$$U_1^\dagger Z U_1 = Z + \mathbf{1}, \tag{A1}$$

$$U_2^\dagger Z U_2 = Z + \tau, \tag{A2}$$

$$U_3^\dagger Z U_3 = qZ, \tag{A3}$$

where $\tau = q = \exp(2\pi i/3)$ and $Z = (X_1 + iX_2)/R_1$.

The U-algebra is given by

$$U_1U_2 = q_{12}U_2U_1, \tag{A4}$$

$$U_1U_3 = q_{13}U_3U_1^\dagger U_2^\dagger, \tag{A5}$$

$$U_3U_1 = q_{31}U_2U_3, \tag{A6}$$

$$U_3U_2 = q_{32}U_1^\dagger U_2^\dagger U_3, \tag{A7}$$

$$U_3^3 = q_3\mathbf{1}, \tag{A8}$$

where q_{12}, q_{13}, q_{31} are phases and consistency requires $q_{32} = q_{13}q_{31}^{-1}$. By rescaling the U's we can set all the q factors to 1 except that q_{12} is still arbitrary.

The Hilbert space $\mathcal{H}$ is $\{U_1^m U_2^n U_3^s\rangle | m,n \in \mathbf{Z}, s = 0,1,2\}$. Define the operators ∂_i, Δ_s, K by

$$\partial_1 U_1^m U_2^n U_3^s\rangle = m U_1^m U_2^n U_3^s\rangle, \tag{A9}$$

$$\partial_2 U_1^m U_2^n U_3^s\rangle = n U_1^m U_2^n U_3^s\rangle, \tag{A10}$$

$$\Delta_s U_1^m U_2^n U_3^{s'}\rangle = \delta_{ss'} U_1^m U_2^n U_3^{s'}\rangle, \tag{A11}$$

$$K U_1^m U_2^n U_3^s\rangle = U_1^m U_2^n U_3^{s+1}\rangle, \tag{A12}$$

where $\delta_{ss'} = 1$ if $s - s' = 0$ (mod 3), and vanishes otherwise.

Let $Z\rangle = \hat{A}(U)\rangle$ and $\hat{A}(U) = \Sigma_{mns}\alpha_{mns}U_1^m U_2^n U_3^s$. Then

$$\begin{aligned} ZU_1^m U_2^n U_3^s\rangle &= U_1^m U_2^n U_3^s(m + \tau n + q^s\hat{A})\rangle \\ &= (\partial_1 + \tau\partial_2 + A)U_1^m U_2^n U_3^s\rangle, \end{aligned} \tag{A13}$$

where

$$A = \sum_{m,n \in \mathbf{Z}; s=0,1,2} \alpha_{mns}K^s \times \left(\sum_{s'=0,1,2} (U_3^{s'}\tilde{U}_2 U_3^{-s'})^n (U_3^{s'}\tilde{U}_1 U_3^{-s'})^m q^{s'}\Delta_{s'} \right), \tag{A14}$$

where $\tilde{U}_1 = q_{12}^{-\partial_2}U_1$ and $\tilde{U}_2 = q_{12}^{\partial_1}U_2$. It is not hard to calculate $U_3\tilde{U}_1U_3^{-1} = \tilde{U}_2$, $U_3^2\tilde{U}_1U_3^{-2} = U_3\tilde{U}_2U_3^{-1} = \tilde{U}_1^{-1}\tilde{U}_2^{-1}$ and $U_3^2\tilde{U}_2U_3^2 = \tilde{U}_1$. The solution of Z is thus

$$Z = \partial_1 + \tau\partial_2 + A. \tag{A15}$$

To put the result in a more amiable form, let $U_1 = \exp(i\sigma_1)$ and $U_2 = \exp(i\sigma_2)$. Also let $U_3 = P \bigcirc U$, where U is given by Eq. (66) for $n = 3$ and P is an algebraic operation defined by

$$P\sigma_1P^{-1} = \sigma_2, \quad P^2\sigma_1P^{-2} = -\sigma_1 - \sigma_2, \tag{A16}$$

$$P\sigma_2P^{-1} = -\sigma_1 - \sigma_2, \quad P^2\sigma_2P^{-2} = \sigma_1. \tag{A17}$$

Then it is easy to see that Eq. (A15) can be rewritten as

$$Z = \left(-i\frac{\partial}{\partial\sigma_1} - i\tau\frac{\partial}{\partial\sigma_2} \right)\mathbf{1} + A(\sigma_1, \sigma_2), \tag{A18}$$

where $\mathbf{1}$ is the 3×3 unit matrix and A is a 3×3 matrix of functions of (σ_1, σ_2) satisfying

$$A_{i-1,j-1}(\sigma_1, \sigma_2) = qA_{ij}(\sigma_2, -\sigma_1 - \sigma_2 + \pi/3), \tag{A19}$$

where the indices are defined modulo 3. The dual space is again $T^2/\mathbf{Z}_3$.

The rest of the quotient conditions are fixed by the Lagrangian (57) to be

$$U^\dagger X_\mu U = X_\mu, \quad \mu = 0,3,\ldots,9, \tag{A20}$$

$$U^\dagger \Psi U = \Lambda_3\Psi, \tag{A21}$$

where $\Lambda_3 = \exp(-\pi\Gamma^{12}/3)$. Because $\Lambda_3^3 = -\mathbf{1}$, strictly speaking Eq. (A8) should be replaced by $U_3^3 = (-1)^F$, where F is the fermion number operator. All the SUSY is broken in this case.

APPENDIX B: FINITE CYLINDER

Matrix compactification on the orientifold $S^1\times S^1/\mathbf{Z}_2$ is related to the heterotic string theory [18,19]. The quotient conditions are[8] [18,19]

$$U_i^\dagger X_j U_i = X_j + 2\pi\delta_{ij}R_j, \quad i,j = 1,2, \tag{B1}$$

[8]In general there can be an additional term of $2k\pi R_1$ for any integer k in Eq. (32), but it can be absorbed in a shift of X_1 by $X_1 \to X_1 + k\pi R_1$.

$$U_3^\dagger X_1 U_3 = -X_1^*, \tag{B2}$$

$$U_3^\dagger X_2 U_3 = X_2^*. \tag{B3}$$

The U-algebra is

$$U_1 U_2 = q_{12} U_2 U_1, \tag{B4}$$

$$U_1 U_3 = q_{13} U_3 U_1^T, \tag{B5}$$

$$U_2 U_3 = q_{23} U_3 U_2^*, \tag{B6}$$

$$U_3 U_3^* = q_3 \mathbf{1}. \tag{B7}$$

Consistency of the U-algebra imposes constraints on the parameters q_{ij}'s. Taking the complex conjugation of Eq. (B7), we find $q_3 = \pm 1$. Equation (B7) and the transpose of Eq. (B5) imply that $q_{13} = \pm 1$. Rescaling U_2 can give $q_{23} = 1$. The U-algebra is therefore parametrized by a phase $q = q_{12}$, $q_{13} = \pm 1$ and $q_3 = \pm 1$. For $q = q_{13} = q_3 = 1$ we get the same algebra as in [18,19].

The Hilbert space is $\mathcal{H} = \{U_1^m U_2^n (U_3 C)^s\rangle | m,n \in \mathbf{Z}, s = 0,1\}$. Define ∂_i, K and ϵ by

$$\partial_i U_1^{m_1} U_2^{m_2} (U_3 C)^s\rangle = m_i U_1^{m_1} U_2^{m_2} (U_3 C)^s\rangle, \tag{B8}$$

$$K U_1^m U_2^n (U_3 C)^s\rangle = U_1^m U_2^n (U_3 C)^{s+1}\rangle, \tag{B9}$$

$$\epsilon U_1^m U_2^n (U_3 C)^s\rangle = (-1)^s U_1^m U_2^n (U_3 C)^s\rangle. \tag{B10}$$

To follow Zumino's prescription, we consider

$$\begin{aligned} X_i U_1^{m_1} U_2^{m_2}\rangle &= U_1^{m_1} U_2^{m_2} [2\pi m_i R_i + \hat{A}_i(U_1, U_2, U_3)]\rangle \\ &= [2\pi R_i \partial_i + A_i(\tilde{U}_1, \tilde{U}_2, K)] U_1^{m_1} U_2^{m_2}\rangle. \end{aligned} \tag{B11}$$

If

$$\hat{A}_i = \Sigma_{mns} \alpha_{mns}^i U_1^m U_2^n (U_3 C)^s,$$

then

$$A_i = \Sigma_{mns} \alpha_{mns}^i \tilde{U}_2^n \tilde{U}_1^m K^s,$$

where

$$\tilde{U}_1 = q^{-\partial_2} U_1, \quad \tilde{U}_2 = q^{\partial_1} U_2.$$

Similarly,

$$\begin{aligned} X_i U_1^{m_1} U_2^{m_2} U_3 C\rangle &= U_1^{m_1} U_2^{m_2} U_3 C [2\pi m_i R_i + (-1)^i \hat{A}_i(U_1, U_2, U_3)]\rangle \\ &= [2\pi R_i \partial_i + (-1)^i A_i^*(q_{13}\tilde{U}_1^{-1}, \tilde{U}_2, K)] U_1^{m_1} U_2^{m_2} U_3\rangle. \end{aligned} \tag{B12}$$

Therefore we get

$$\begin{aligned} X_i = 2\pi R_i \partial_i &+ \frac{1}{2} A_i(\tilde{U}_1, \tilde{U}_2, K)(1+\epsilon) \\ &+ (-1)^i \frac{1}{2} B_i(\tilde{U}_1, \tilde{U}_2, K)(1-\epsilon), \end{aligned} \tag{B13}$$

where $B_i(\sigma_1, \sigma_2, K) = A_i^*(\sigma_1 - h_{13}, -\sigma_2, K)$ with $\tilde{U}_1 = e^{i\sigma_1}$, $\tilde{U}_2 = e^{i\sigma_2}$, and $q_{13} = e^{ih_{13}}$ ($h_{13} = 0, \pi$). The fundamental region on which the gauge field can be freely assigned is a dual cylinder: $\sigma_1 \in [0, 2\pi)$, $\sigma_2 \in [0, \pi]$ for $q_{13} = 1$. For $q_{13} = -1$ it is a dual Klein bottle.

Let $A_i = A_{i0}(\tilde{U}_1, \tilde{U}_2) + A_{i1}(\tilde{U}_1, \tilde{U}_2) K$ and similarly for B_i. The Hermiticity of A_i implies that

$$A_{i0}^\dagger = A_{i0}, \quad B_{i0}^\dagger = B_{i0}, \quad A_{i1}^\dagger = (-1)^i q_3 B_{i1}. \tag{B14}$$

Clearly, ∂_1, ∂_2 are derivatives on the dual space. In fact K can also be viewed as a function on $\mathbf{Z}_2$ and ϵ as the derivative on $\mathbf{Z}_2$ in the sense of a noncommutative geometry [4]. Hence the dual quantum space is the product of the dual cylinder with $\mathbf{Z}_2$. Furthermore, the form of X resembles the covariant derivative on the dual quantum space as defined in [4,26]. A similar construction was used for rewriting the standard model as a gauge theory on a noncommutative space [26].

The algebra on the $\mathbf{Z}_2$ factor of dual space can be represented by Pauli matrices. For instance, $K = \tau_1$ and $\epsilon = \tau_3$ for $q_3 = 1$. From Eq. (B13), $X_i = -i2\pi R_i \partial/\partial\sigma_i + \mathcal{A}_i(\sigma_1, \sigma_2)$, where

$$\mathcal{A}_i = \begin{pmatrix} A_{i0} & (-1)^i B_{i1} \\ A_{i1} & (-1)^i B_{i0} \end{pmatrix} \tag{B15}$$

is a Hermitian matrix. Each entry of the 2×2 matrices is an $N \times N$ matrix.

The quotient conditions for other coordinates for the compactification on a cylinder are [18,19]

$$U_i^\dagger A_0 U_i = A_0, \quad U_3^\dagger A_0 U_3 = -A_0^*, \tag{B16}$$

$$U_i^\dagger X_a U_i = X_a, \quad U_3^\dagger X_a U_3 = X_a^*, \tag{B17}$$

$$U_i^\dagger \Psi U_i = \Psi, \quad U_3^\dagger \Psi U_3 = \Gamma_{01} \Psi^*, \tag{B18}$$

where $i=1,2$, and $a=3,\dots,9$. The M(atrix) theory on a cylinder is related to the heterotic string theory [18,19]. It is a gauge theory with the gauge group $U(2N)$ in the bulk of the dual cylinder but with the gauge group $O(2N)$ $(q_{13}=1)$ or $USp(2N)$ $(q_{13}=-1)$ on the boundary [18]. One-half of the dynamical SUSY is preserved.

APPENDIX C: MÖBIUS STRIP

The quotient conditions for a Möbius strip [14,15] are Eq. (B1) and

$$U_3^\dagger X_1 U_3 = X_2^*, \tag{C1}$$

$$U_3^\dagger X_2 U_3 = X_1^*. \tag{C2}$$

The U-algebra is

$$U_1 U_2 = q_{12} U_2 U_1, \tag{C3}$$

$$U_1 U_3 = q_{13} U_3 U_2^*, \tag{C4}$$

$$U_2 U_3 = q_{23} U_3 U_1^*, \tag{C5}$$

$$U_3 U_3^* = q_3 \mathbf{1}. \tag{C6}$$

Considerations similar to those in the previous sections lead to $q_3=\pm 1$ and $q_{13}=q_{23}=1$. The phases $q_{12}=q$ and $q_3=\pm 1$ label two one-parameter families of compactifications.

The Hilbert space and the operators ∂_i, K, ϵ can be defined similarly as in the previous section. We get the solution for X_1, X_2 as

$$X_i = 2\pi R_i \partial_i + \frac{1}{2} A_i(\tilde{U}_1, \tilde{U}_2, K)(1+\epsilon) + \frac{1}{2} B_i(\tilde{U}_1, \tilde{U}_2, K)(1-\epsilon), \tag{C7}$$

where the A's and B's are functions of $(\tilde{U}_1, \tilde{U}_2) = (q^{-\partial_2} U_1, q^{\partial_1} U_2) = (e^{i\sigma_1}, e^{i\sigma_2})$. It is

$$A_i(-\sigma_2, -\sigma_1) = B_j^*(\sigma_1, \sigma_2), \tag{C8}$$

where $(i,j)=(1,2)$ or $(2,1)$. From Eqs. (C1), (C2), (C4), and (C5), the fundamental region is the dual Möbius strip and the compactified M(atrix) theory is a field theory on the dual Möbius strip.

The quotient conditions for A_0 and X_a $(a=3,\dots,9)$ are the same as those for a cylinder. Those for Ψ can also be obtained:

$$U_i^\dagger \Psi U_i = \Psi, \tag{C9}$$

$$U_3^\dagger \Psi U_3 = \Gamma_\perp \Psi^*, \tag{C10}$$

where $\Gamma_\perp = (1/\sqrt{2})\Gamma_0(\Gamma_1 - \Gamma_2)$. One-half of the dynamical SUSY is preserved.

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