Suppression of Local Degrees of Freedom of Gauge Fields by Chiral Anomalies

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A path-integral quantization is presented for the chiral Schwinger model on a Riemann surface. Gauge invariance is maintained by integrating over all gauge potentials without the usual gauge fixing. All local degrees of freedom of the gauge field are suppressed after the integration of the anomalous effective action over a gauge orbit. The resulting theory is a topological one for the surviving global gauge excitations. The general implications for consistent quantization of chiral gauge theories are also discussed.

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By now it is well known that gauge fields can have purely global dynamical degrees of freedom,[1] which are best described by nonintegrable phase factors for noncontractible loops.[2] Recently, a lot of interest has been attracted to theories with only such global gauge excitations present. Known examples include discrete gauge theories in continuum space-time[3] and topological Chern-Simons theories.[4,5] In either case the locally propagating gauge-field degrees of freedom are suppressed, at least at large distances. Thus the gauge fields become locally trivial everywhere (except perhaps at isolated points in $D=2+1$ or at stringlike singularities in $D=3+1$), but nonetheless give rise to interesting and observable physical effects, such as discrete quantum hairs of black holes,[6] non-Abelian Aharonov-Bohm effects for cosmic strings,[7] anyon superconductivity,[8] and topological order in the quantum Hall effect and the chiral spin states.[9]

In this Letter we are going to show that suppression of local gauge excitations occurs also in a third wide class of theories, i.e., quantized gauge theories coupled to chiral fermions. More concretely, we will quantize the Euclidean chiral Schwinger model, i.e., $D=2$ QED asymmetrically coupled to left- and right-handed fermions; the action is given by

$$I[A,\psi,\bar{\psi}]=\int_\Sigma d^2x\sqrt{g}\left[\frac{1}{4}F_{\mu\nu}^2+i\bar{\psi}\left(\nabla_\mu-ieA_\mu\frac{1+\gamma_3}{2}\psi\right)\right], \quad (1)$$

e.g., with a left-handed fermion. More generally, one may consider more than one species of chiral fermions with charges e_L, e_R, etc. We note that this model is analogous to (but much simpler than) two-dimensional-induced quantum gravity[10] in that integrating out matter fields leads to an "anomalous" effective action. Because of the generally uncanceled chiral anomaly, we have been careful in employing a path-integral quantization, in which gauge invariance is maintained by integrating over all gauge potentials.[11] We are able to perform the integration of the anomalous effective action over a gauge orbit without the usual gauge fixing. It suppresses all locally propagating degrees of freedom of the gauge field and, on a Riemann surface, results in a sensible topological theory for the surviving global gauge excitations. The general implications of this exercise for consistent quantization of chiral gauge theories will be discussed later.

The chiral Schwinger model (1) on a Minkowskian plane (or on a sphere) has been studied by Jackiw and Rajaraman,[12] using the bosonized action

$$\mathcal{L}_b=-\tfrac{1}{4}F_{\mu\nu}^2+\tfrac{1}{2}(\partial_\mu\phi)^2+e(g^{\mu\nu}-\epsilon_{\mu\nu})\partial_\mu\phi A_\nu + \tfrac{1}{2}e^2aA_\mu^2, \quad (2)$$

where a is a real parameter in regularizing the fermion determinant. They showed that if $a>1$, the theory is unitary and the physical spectrum consists of a massless excitation and a massive vector boson with $m^2=e^2a/(a-1)$; and the $a<1$ case is nonunitary. In our opinion, $a=1$ is a preferred choice in that it corresponds to the consistent anomaly $\epsilon^{\mu\nu}\partial_\mu A_\nu$, and for the left-right-symmetric cases it automatically gives a gauge-invariant effective action. Observe that when $a\to 1$ one has $m^2\to\infty$, thus no propagating gauge excitations can be present. Though this makes the theory trivial on a plane (or sphere), there are global excitations surviving on a Riemann surface Σ with genus $g\geq 1$.

Now we define the partition function on Σ as

$$Z(\Sigma)=\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A\exp\{-I[A,\psi,\bar{\psi}]\}, \quad (3)$$

where the integral for A is taken over the space of all gauge potentials. To carry out this integration, one needs to know how to parametrize the gauge potentials on Σ and then give the A space a "volume" measure. For simplicity, we assume the U(1) potentials are those on a trivial bundle (i.e., $\int_\Sigma F=0$). Therefore, $A\equiv eA_\mu dx^\mu$ is a one-form on Σ. According to the Hodge theorem,[13] any one-form A can always be uniquely written as

$$A = d\eta + \delta\xi + \gamma, \tag{4}$$

where η is a zero-form, ξ is a two-form, and γ is a harmonic one-form on Σ; d is the exterior derivative, and $\delta \equiv * d *$ with the $*$ the Hodge dual with respect to the metric $g_{\mu\nu}$ on Σ, which one may take to be of constant curvature. According to the de Rham theorem,[13]

$$\gamma = \sum_{i=1}^{g} x^i \alpha_i + \sum_{i=1}^{g} y^i \beta_i . \tag{5}$$

Here x^i, y^i are real numbers and α^i, β^i are canonical harmonic one-forms satisfying

$$\int_{a_j} \alpha_i = 2\pi\delta_{ij}, \quad \int_{b_j} \alpha_i = 0,$$
$$\int_{a_j} \beta_i = 0, \quad \int_{a_j} \beta_i = 2\pi\delta_{ij}, \tag{6}$$

with a_i, b_i $(i = 1, 2, \dots, g)$ a set of canonical one-cycles on Σ (see Fig. 1). The gauge-transformation degrees of freedom hidden in Eq. (4) is

$$ig(x)^{-1} dg(x) = d\eta + \sum_{i=1}^{g} m_i \alpha_i + \sum_{i=1}^{g} n_i \beta_i , \tag{7}$$

where m_i, n_i are integers. $d\eta$ corresponds to small gauge transformations and the remaining represents large ones. Therefore, the gauge-invariant part[14] in the parametrization (4) is

$$\bar{A} = A - ig(x)^{-1} dg(x) = \delta\xi + \sum \bar{x}^i \alpha_i + \sum \bar{y}^i \beta_i , \tag{8}$$

with $0 \le \bar{x}^i, \bar{y}^i < 1$. While the $\delta\xi$ term represents locally propagating degrees of freedom, the harmonic forms always have zero strength and represent global Aharonov-Bohm configurations: $\bar{x}^i$ and $\bar{y}^i$ are nothing but the nonintegrable phases around the holes of Σ.

In the A space, we introduce the gauge and conformal-invariant inner product:

$$\langle \delta A, \delta A' \rangle = \int_\Sigma d^2x \sqrt{g}\, g^{\mu\nu} \delta A_\mu \delta A'_\nu . \tag{9}$$

The three parts in the decomposition (4) are orthogonal to each other,[13] so one has

$$\mathcal{D}A = \mathcal{D}(d\eta)\mathcal{D}(\delta\xi)\mathcal{D}\gamma$$
$$= \mathcal{D}\eta\,\mathcal{D}\zeta \det(\Delta_0)(2\pi)^g \prod_{i=1}^{g} dx^i dy^i . \tag{10}$$

Here we have made a change in the variables: $(d\eta) \to \eta$, $\delta\xi \to \zeta$ with $\zeta \equiv * \xi$ the zero-form dual to ξ. Δ_0 is the Laplacian acting on zero-forms. The measure (10) is independent of $\eta(x)$.

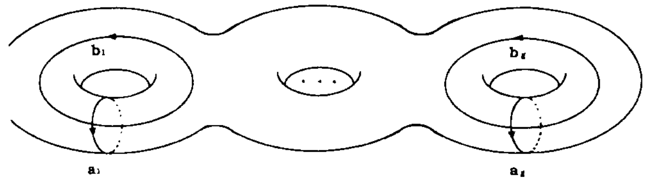

FIG. 1. A set of canonical a and b cycles on Σ.

For how to define chiral fermions and the chiral Dirac operator on a Riemann surface Σ, we refer the reader to Ref. 15. Like Minkowskian chiral fermions, which are functions of only one light-cone coordinate $x_\pm = x_1 \pm x_0$, chiral fermions on Σ can be defined through (anti)holomorphy on the complex coordinate $z = x_1 + ix_2$, with the complex structure that is compatible with the metric on Σ. One has to fix a spin structure on Σ for each species and couple only the A_z $(A_{\bar{z}})$ components to left- (right-) handed fermions.

The fermionic integral in Eq. (3) leads to, as usual, the $D = 2$ chiral Dirac determinant:

$$Z_f[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left[-i \int d^2x \sqrt{g}\, \bar{\psi} \not{D}_+ \psi \right] \equiv \det i\not{D}_+ . \tag{11}$$

Because of chiral anomaly, $Z_f[A]$ is not invariant under both small and large gauge transformations. For an infinitesimal gauge transformation $\delta A = d(\delta\eta)$, one has the local consistent anomaly [corresponding to the choice $a = 1$ in Eq. (2)]

$$\delta(\ln Z_f[A]) = \frac{i}{2\pi} \int d^2x\, \delta\eta(x) \tilde{F}(x) , \tag{12}$$

where $\tilde{F} = \frac{1}{2} \epsilon_{\mu\nu} \partial_\mu A_\nu$. (With more than one species, the right-hand side should be multiplied by a constant $\kappa \equiv \sum_L e_L^2 - \sum_R e_R^2$ which we assume is nonzero.) Note that though the chiral Dirac operator depends only on A_z for left-handed fermions, its determinant depends also on $A_{\bar{z}}$, because of the counterterm like the last one in Eq. (2). We emphasize that the local anomaly is of the same form on all Riemann surfaces, since it is local in A and can be calculated by one-loop perturbation theory. Integrating Eq. (12) simply gives

$$Z_f[A] = Z_f[\delta\xi + \gamma] \exp\left[\frac{i}{2\pi} \int d^2x\, \eta(x) \tilde{F}(x) \right] . \tag{13}$$

Substituting Eqs. (10), (11), and (13) into Eq. (3) leads to

$$Z(\Sigma) = \int \mathcal{D}\eta\, \mathcal{D}\zeta\, \mathcal{D}\gamma\, (\det\Delta_0) Z_f[\delta\xi + \gamma] \exp\left[-I_0[F] + \frac{i}{2\pi} \int d^2x\, \eta(x) \tilde{F}(x) \right] . \tag{14}$$

The key observation is that since both the Maxwell action I_0 and $\det\Delta_0$ are independent of $\eta(x)$, it is trivial to perform the $\eta(x)$ integration and obtain the δ functional

$$\prod_{x\in\Sigma}\delta(\tilde{F}(x))=(\det\Delta_0)^{-1}\prod_{x\in\Sigma}\delta(\zeta(x)). \quad (15)$$

It is this δ functional of $\tilde{F}(x)$ that suppresses all local gauge excitations. Thus the partition function (3) is reduced to a finite-dimensional integral

$$Z(\Sigma)=\int\mathcal{D}\gamma Z_f[\gamma] =\int_{-\infty}^{\infty}\prod dx^i dy^i Z_f\left[\sum_i x^i\alpha_i+\sum_i y^i\beta_i\right], \quad (16)$$

leading to a theory for flat U(1) potentials with the chiral determinant $Z_f[\gamma]$ (or a product of such determinants for more than one species) as induced action.

According to Alvarez-Gaume, Moore, and Vafa,[15] the latter is essentially a θ function up to some phase choice. For example, when Σ is a torus T^2, for a left-handed fermion satisfying the antiperiodic (A,A) boundary condition on both the a and b cycles,

$$Z_f[\theta\alpha-\phi\beta]=\eta(\tau)^{-1}\vartheta\begin{bmatrix}\theta\\ \phi\end{bmatrix}(0|\tau), \quad (17)$$

and the complex conjugate, for a right-handed fermion. Here the complex τ parametrizes the flat metric $ds^2=|dx_1+\tau dx_2|^2$ on T^2, $\eta(\tau)$ is the Dedekind eta function, and

$$\vartheta\begin{bmatrix}\theta\\ \phi\end{bmatrix}(0|\tau)=\sum_N \exp\{i\pi(N+\theta)^2\tau+2i\pi(N+\theta)\phi\}.$$

Under a global gauge transformation $\theta\to\theta+m$, $\phi\to\phi-n$, the Z_f transforms as

$$Z_f[(\theta+m)\alpha-(\phi-n)\beta]=\exp(-i2\pi n\theta)Z_f[\theta\alpha-\phi\beta]. \quad (18)$$

Using it we reduce Eq. (16) to an integral over the moduli space of flat potentials:

$$Z(T^2)=2\pi\sum_n\int_0^1 d\theta\int_0^1 d\phi Z_f[\theta\alpha-\phi\beta]\exp(-i2\pi n\theta) =\frac{2\pi}{\eta(\tau)}\int_0^1 d\phi\vartheta\begin{bmatrix}0\\ \phi\end{bmatrix}(0|\tau)=\frac{2\pi}{\eta(\tau)}. \quad (19)$$

Here we have neglected an infinite constant arising from the summation over m, and the sum over n gives a δ function of θ. The chiral determinant for the (P,A), (A,P), and (P,P) spin structures on T^2 can be obtained[15] by shifting $\theta\to\theta+\epsilon_1$ and $\phi\to\phi-\epsilon_2$, with $(\epsilon_1,\epsilon_2)=(\frac{1}{2},0)$, $(0,\frac{1}{2})$, and $(\frac{1}{2},\frac{1}{2})$, respectively. This shifting does not change the expression (19) for $Z(T^2)$ with only one chiral fermion. With more than one species, the partition function remains invariant if one shifts all spin structures by one and the same pair (ϵ_1,ϵ_2).

To see the topological nature of the resulting field theory, we examine the Wilson loop $\langle W(C)\rangle \equiv\langle\exp(ik\oint_C A)\rangle$ as a gauge-invariant observable:

$$\langle W(C)\rangle=\int\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A\exp\left[ik\oint_C A\right]\exp\{-I[A,\psi,\bar{\psi}]\}\Big/Z(T^2)=\int_{-\infty}^{\infty}d\theta d\phi\exp\left[ik\oint_C A(\theta,\phi)\right]\vartheta\begin{bmatrix}\theta\\ \phi\end{bmatrix}(0|\tau). \quad (20)$$

Decompose the loop C in terms of the a and b cycles: $C=Ma+Nb$ with M,N integers:

$$\exp\left[ik\oint_C A(\theta,\phi)\right]=\exp\{2\pi ik(M\theta-N\phi)\}. \quad (21)$$

It is easy to see that k has to be an integer in order for $W(C)$ to be invariant under larger gauge transformations. Inserting (21) into (20) and proceeding as before, one obtains

$$\langle W(C)\rangle=\exp(i\pi k^2N^2\tau). \quad (22)$$

We note that $\langle W(C)\rangle$ is homotopy invariant, though depending on the conformal moduli parameter τ of the background metric. That the result (22) is asymmetric with respect to the a and b cycles is because of the asymmetric global anomaly (18) implied by the phase choice in the chiral determinant (17). If we multiply the right-hand side of Eq. (17) by $\exp(-i\pi\theta\phi)$, then both (18) and (22) will become symmetric in θ,ϕ or in M,N. Thus the resulting topological theory depends on the phase choice of the chiral determinant.

All of the above results can be generalized to higher genus, again involving the θ functions on Σ. Obviously if the insertions involve the local- and gauge-invariant function $\tilde{F}(x)$, it must be put to zero after the gauge-orbit integration. It would be interesting to calculate the gauge-invariant fermion propagator $\langle\bar{\psi}(x)\exp(i\int_y^x A)\times\psi(y)\rangle$. Also it is possible to consider the cases with a nontrivial bundle (i.e., with $\int_\Sigma F\neq 0$); in such cases $A-A_0$ is always a one-form with A_0 a potential on the same bundle. The details will be published elsewhere.[16]

The intriguing possibility of quantizing a gauge theory with uncanceled chiral anomaly has been advocated in the literature[10,12,17,18] for some time. The first lesson from our above exercise is that the original concern that gauge anomaly would make the partition function vanishing turns out unnecessary. We have seen that the integration of the anomalous effective action over a gauge orbit gives rise to a δ functional in the A space [see Eq. (15) and below Eq. (19)], rather than an identically vanishing factor as usually thought. So at worst, local gauge anomaly renders the theory trivial, but never internally inconsistent. The same is true for a global anomaly on an infinite number of disconnected components of a gauge orbit. Second, in our present example one can see more explicitly why the quantum theory with anomalous effective action can be gauge invariant: Gauge-field configurations for which the effective action Z_f is not gauge invariant are all suppressed, so nothing surviving in the theory is inconsistent with gauge invariance, despite the original presence of chiral anomaly. Finally, we have seen that local gauge anomaly suppresses local gauge-field excitations. Thus, the perturbation theory which presumes the existence of locally propagating dynamical gauge bosons is predestined to introduce internal inconsistency into the theory.[19] On the other hand, anomaly cancellation is still needed for the electroweak theory or any grand unified model to have propagating dynamical gauge bosons.

To conclude, let us mention some future directions. How can one confirm the above new understanding of the role of chiral anomaly in a different quantization scheme, say, in canonical or Becchi-Rouet-Stora-Tyutin or geometric quantization? How about non-Abelian cases or two-dimensional gravity or gauge theory plus gravity? Could the present development shed some light on the yet mysterious equivalence of two-dimensional topological gravity[20] to two-dimensional-induced quantum gravity[10] or the matrix models of noncritical strings?[21] Finally, of course, the most interesting situations would be in higher dimensions, especially in $D=3+1$. We observe that presumably not all local gauge excitations would be suppressed by anomalies, so possibly there are surviving degrees of freedom in a topologically trivial space-time.

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[1]Y. Aharonov and D. Bohm, Phys. Rev. **119**, 485 (1959).

[2]C. N. Yang, Phys. Rev. Lett. **33**, 445 (1974); T. T. Wu and C. N. Yang, Phys. Rev. D **12**, 3845 (1975).

[3]L. M. Krauss and F. Wilczek, Phys. Rev. Lett. **62**, 1221 (1989).

[4]R. Jackiw and S. Templeton, Phys. Rev. D **23**, 2291 (1981); J. Schonfeld, Nucl. Phys. **B185**, 157 (1981); C. R. Hagen, Ann. Phys. (N.Y.) **157**, 342 (1987).

[5]E. Witten, Commun. Math. Phys. **117**, 353 (1988); **121**, 351 (1989); A. Schwarz, Lett. Math. Phys. **2**, 247 (1978).

[6]M. Alford and F. Wilczek, Phys. Rev. Lett. **62**, 1071 (1989); M. Alford, J. March-Russell, and F. Wilczek, Harvard University Report No. HUTP 89/A040 (to be published); J. Preskill and L. Krauss, Caltech Report No. CALT-TH-68-1601 (to be published).

[7]M. Alford, J. March-Russell, and F. Wilczek, Nucl. Phys. **B328**, 140 (1989); M. Alford, K. Benson, S. Coleman, J. March-Russell, and F. Wilczek, Harvard University Reports No. HUTP 89/051 and No. HUTP 89/052 (to be published); F. Wilczek and Y. S. Wu, Institute for Advanced Study, Princeton University, Report No. IASSNS-HEP-90/02 (to be published).

[8]R. B. Laughlin, Phys. Rev. Lett. **60**, 2677 (1989); Y . H. Chen, F. Wilczek, E. Witten, and. B. I. Halperin, Int. J. Mod. Phys. B **3**, 1001 (1989); X. G. Wen and A. Zee, Phys. Rev. B **41**, 240 (1990).

[9]X. G. Wen, Int. J. Mod. Phys. B **4**, 239 (1990); Phys. Rev. B **40**, 7387 (1989); X. G. Wen and Q. Niu, Phys. Rev. B **41**, 9337 (1990).

[10]A. M. Polyakov, Phys. Lett. **103B**, 207 (1981); Mod. Phys. Lett. A **2**, 893 (1987); V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, Mod. Phys. Lett. A **3**, 1213 (1988).

[11]K. Harada and I. Tsutsui, Phys. Lett. B **183**, 311 (1987); O. Babelon, F. A. Schaposnik, and C. M. Viallet, Phys. Lett. B **177**, 385 (1986).

[12]R. Jackiw and R. Rajaraman, Phys. Rev. Lett. **54**, 1219 (1985).

[13]For a textbook readable for most physicists see, e.g., S. Goldberg, *Curvature and Cohomology* (Dover, New York, 1962).

[14]See Wu and Yang, Ref. 2.

[15]L. Alvarez-Gaume, G. Moore, and C. Vafa, Commun. Math. Phys. **106**, 1 (1986).

[16]H. C. Lee, Y. S. Wu, and W. D. Zhao (to be published).

[17]L. D. Faddeev, in *Proceedings of the Nuffield Workshop on Supersymmetry and Its Applications—Superstrings, Anomalies and Supergravity,* edited by G. W. Gibbons, S. W. Hawking, and P. K. Townsend (Cambridge Univ. Press, Cambridge, 1986); L. D. Faddeev and S. L. Shatashvili, Phys. Lett. **167B**, 225 (1986).

[18]Y. S. Wu, in *Proceedings of the Twenty-Third International Conference on High-Energy Physics, Berkeley, 1986,* edited by S. C. Loken (World Scientific, Singapore, 1987), Vol. 1, p. 526; A. Niemi, G. Semenoff, and Y. S. Wu, Nucl. Phys. **B276**, 173 (1986).

[19]D. Gross and R. Jackiw, Phys. Rev. D **6**, 477 (1972).

[20]E. Witten, Institute for Advanced Study, Princeton University, Report No. IASSNS-HEP-89/66 (to be published); R. Dijkgraaf and E. Witten, Institute for Advanced Study, Princeton University, Report No. IASSNS-HEP-90/18 (to be published).

[21]E. Brezin and V. Kazakov, Phys. Lett. B **236**, 144 (1990); M. Douglas and S. Shenker, Nucl. Phys. **B335**, 635 (1990); D. J. Gross and A. Migdal, Phys. Rev. Lett. **64**, 127 (1990).